SAND2007-0806C

Time Harmonic Scar Statistics In Two Dimensional Cavities

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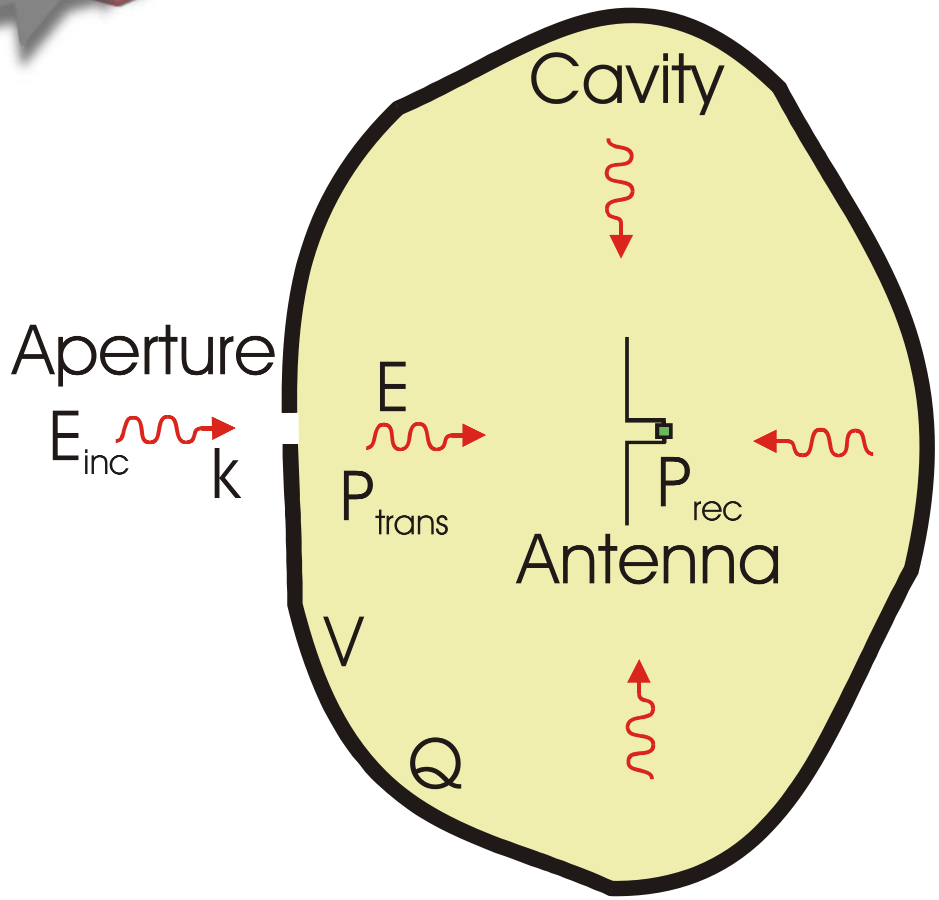
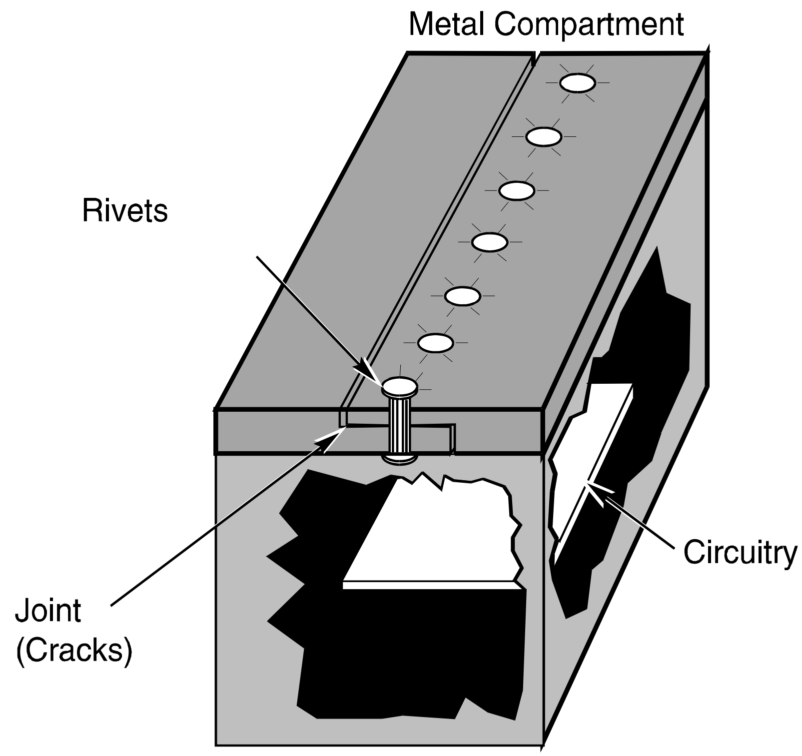
How this Work Fits Into Overall Picture

- Our interests
  - Shielding effectiveness of well shielded cavities
  - Pin level voltage
- Aperture coupling into cavity
  - Realistic slots (overlap depth and large susceptance, losses and Q)
  - Power bounds (matched load – worst case)
- Cavity
  - Frequency regimes
    - Fundamental
    - Undermoded
    - Overmoded
  - Statistical modal description
    - Modal frequencies and field distributions
  - Source interactions (antennas and apertures))
    - Quantities of interest (impedance, effective height, cross section)
  - Localized Enhancements
    - Stable
    - Unstable scars
- Cable coupling
  - Braid penetration
  - Interior modes and loads

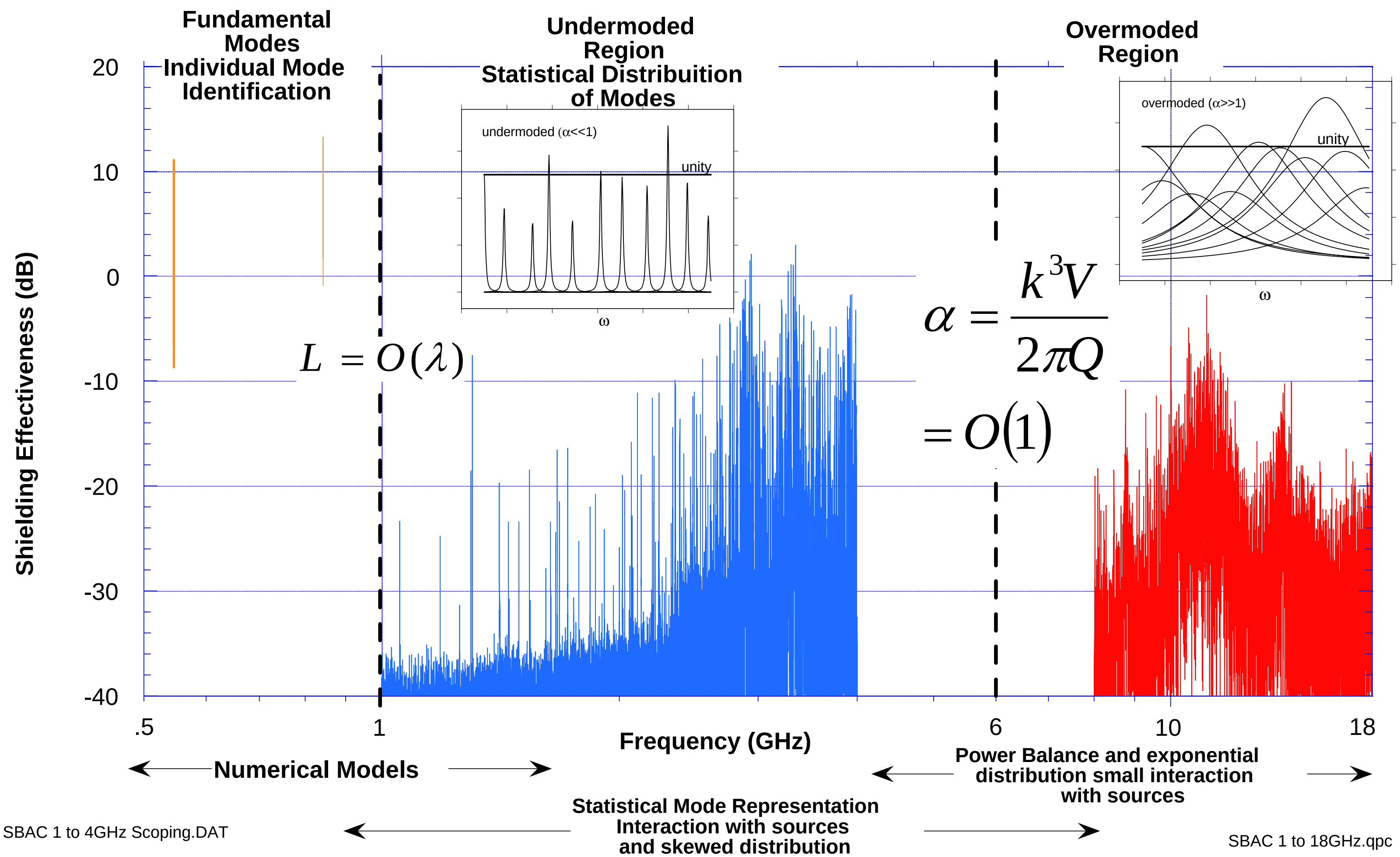
Overall Problem Description





- **High frequency and bounding approach**

Frequency Regimes



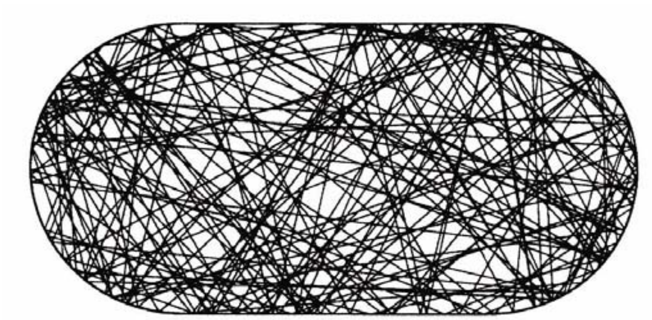
High Frequency Chaotic Behavior in Stadium Cavity

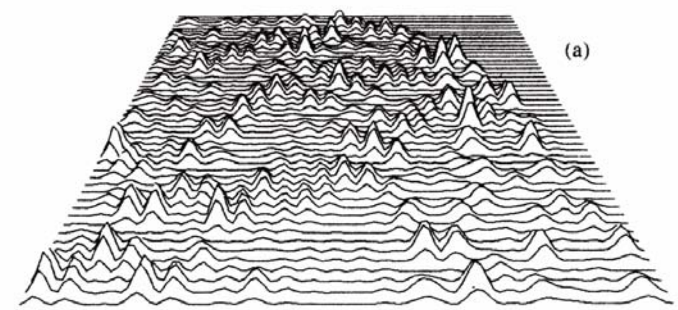
- **Undermoded to overmoded region**

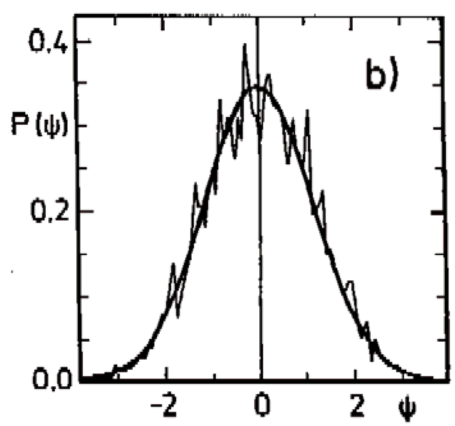
$$\Delta\omega_m = \omega_{m+1} - \omega_m = \langle \Delta\omega_m \rangle \mathrm{w} \qquad f(w) = (\pi w/2)e^{-w^2\pi/4} \qquad \langle \Delta\omega_m \rangle \sim 2\pi c^2/(\omega_m A) \Leftrightarrow \pi^2 c^3/(V\omega_m^2)$$

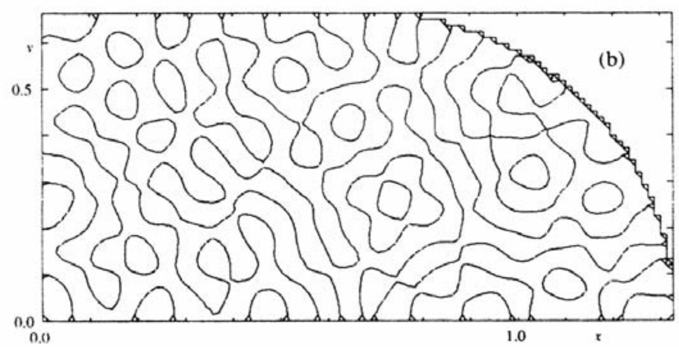
  - **Eigenvalue spacing (Wigner, Deus, et. al.)**
  - **Gaussian (Berry, McDonald & Kaufmann, Lehman)**
  - **Vector correlation (Eckhardt, et. al., Hill, Warne & Lee)**

$$\langle E_{mz}(\underline{r}) E_{mz}(\underline{r}') \rangle = (6\pi/k)\,\mathrm{Im}(\Gamma_{zz}) \qquad E_{mz} = u \Leftrightarrow u/\sqrt{3} \qquad f(u) = e^{-u^2/2}/\sqrt{2\pi}$$



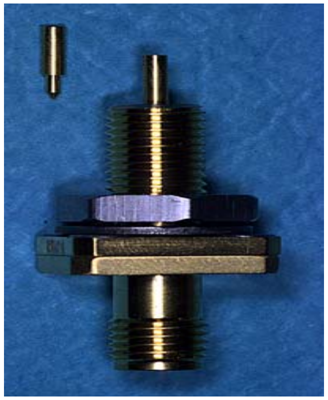


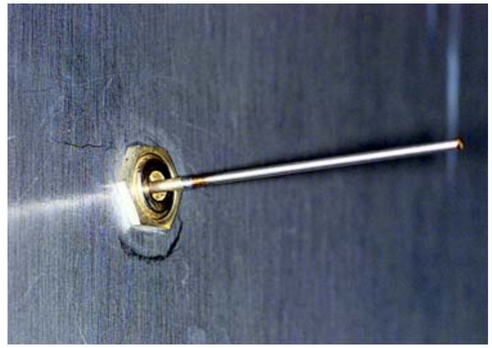


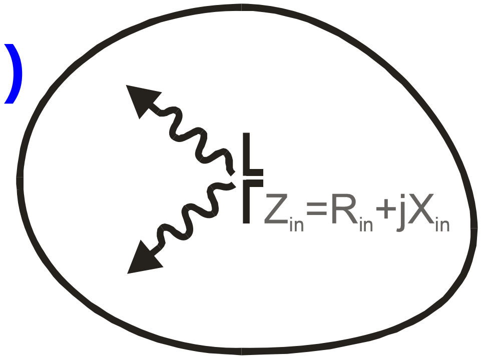


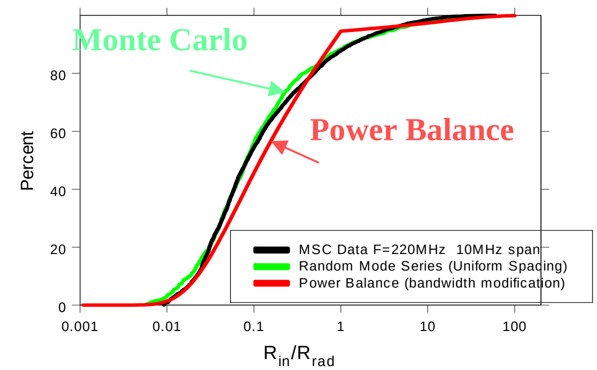
Quantities of Interest

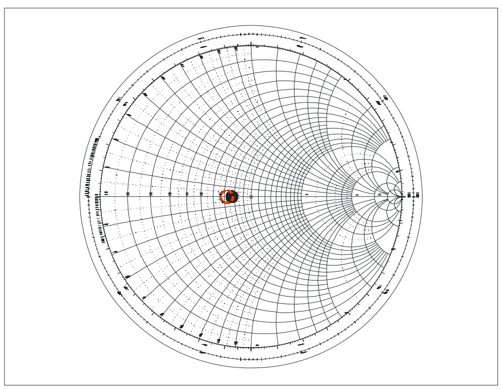
- **Impedance (Warne, et. al., Zheng, et. al.)**

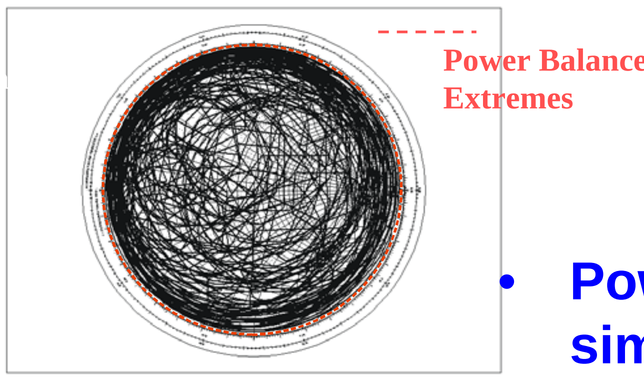




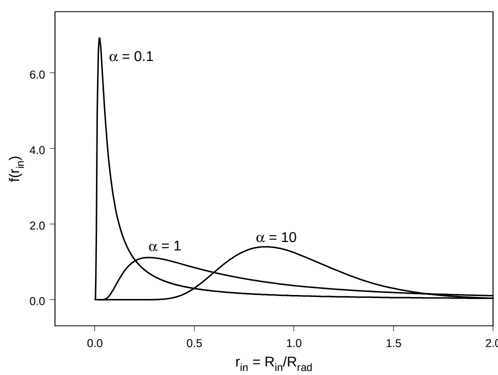


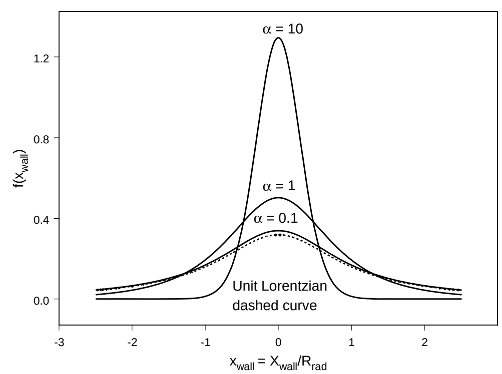






- **Power balance derivation of simple engineering formulas**





$$\alpha' - \sqrt{\alpha'^2 - 1} < R_{in} / R_{rad} < \alpha' + \sqrt{\alpha'^2 - 1}$$

$$-\sqrt{\alpha'^2 - 1} < X_{in} / R_{rad} < +\sqrt{\alpha'^2 - 1}$$

$$\alpha' = 1 + 3^2 (2/\alpha)$$

$$\alpha = k^3 V / (2\pi Q)$$

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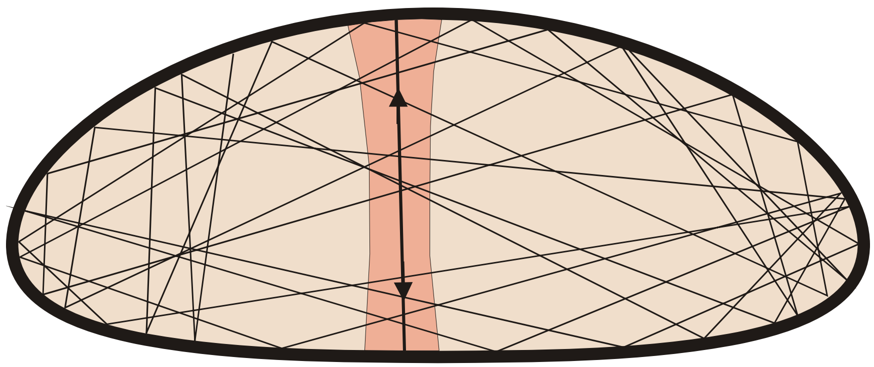
Localized Enhancements

- **Objective**

  Determine connection between high frequency modal amplitude distribution and cavity boundary topology; in particular the connection between scarring (modal enhancements along periodic classical paths) and local curvature of the boundary surface.
- **Reason - near worst case vulnerability assessments**

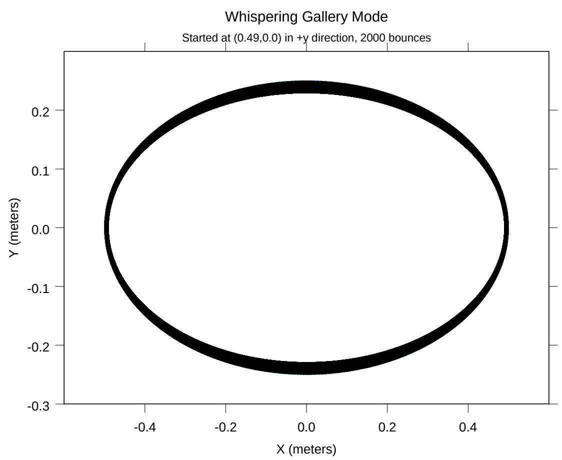


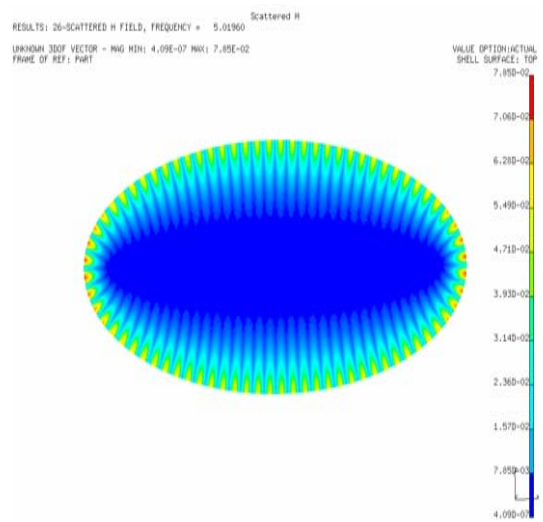
- **Approach**

  Develop high frequency ray method to give local modal amplitude enhancements (above chaotic background levels) in terms of boundary topology

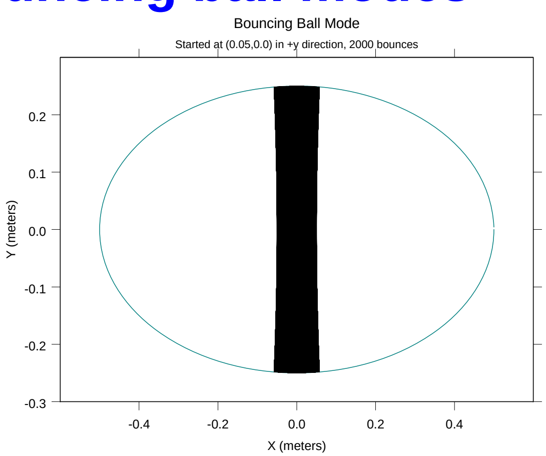
Ray Tracing of Stable Modes

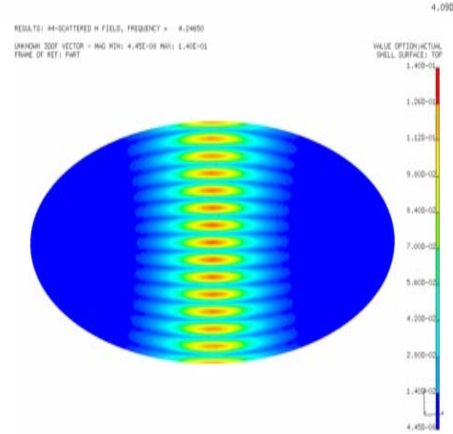
- **Geometric optics identifies periodic orbit resonant frequencies (includes phase jumps through foci) and stability**
  - **Stable orbit boundary layer construction of wave amplitudes (Vaynshteyn, Babic)**
- **Whispering gallery modes (example in the ellipse)**





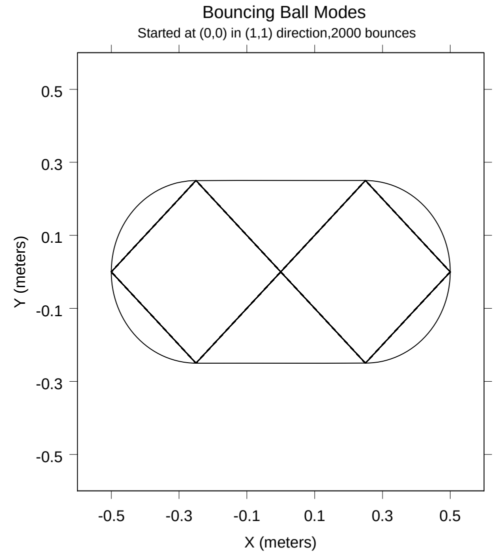
- **Bouncing ball modes**

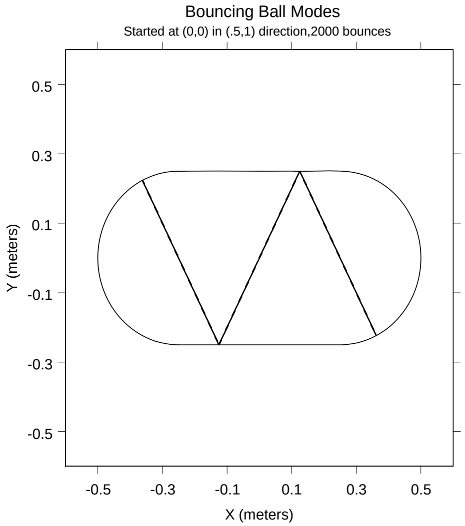




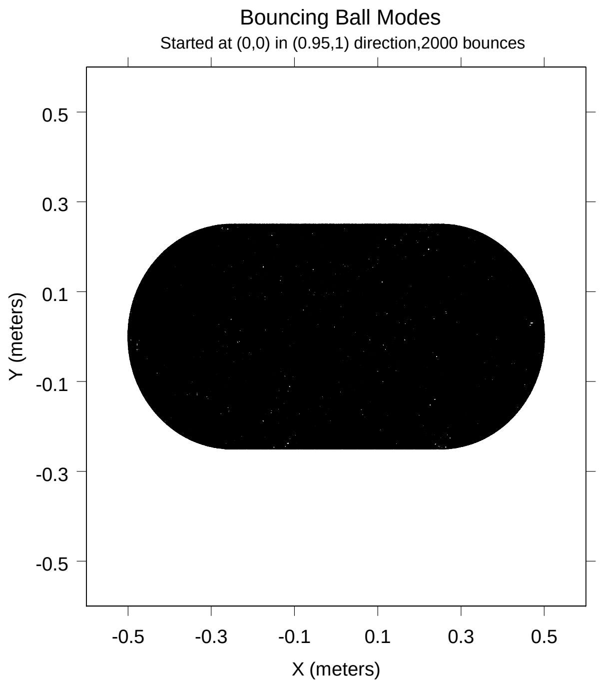
Ray Tracing of Unstable Modes

- **Unstable periodic orbits**
  - **Resonant frequencies**
  - **Stability exponents**





- **Chaotic ray behavior**

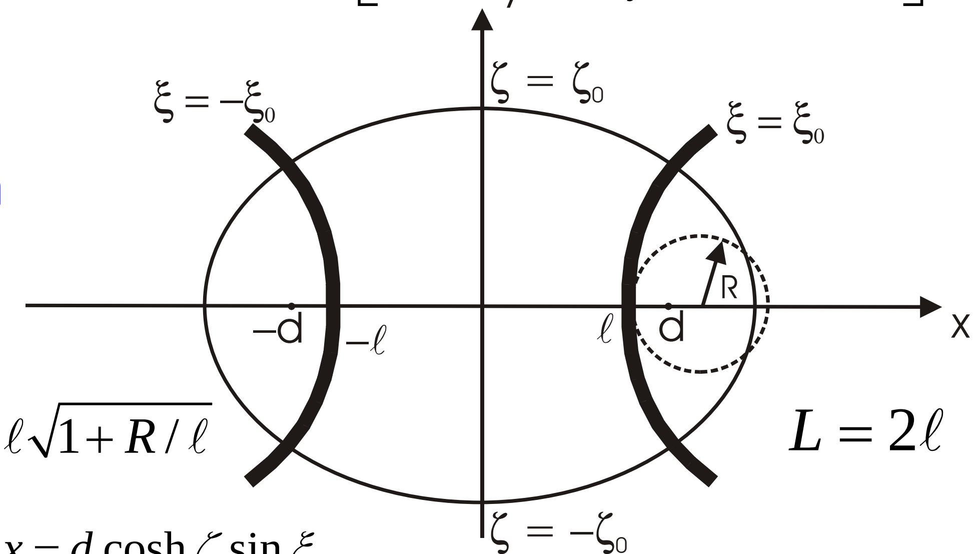


Bow Tie Scar Theory
Convex Boundary

$$\left\langle A u_r^2 \right\rangle = 1$$

$$u_r = \lim_{N\to\infty} \mathrm{Re}\left[\sqrt{2/(AN)}\sum_{j=1}^{N} a_j e^{i\alpha_j + i\underline{k}\bullet\underline{r}}\right]$$

- **Unstable scarred orbit statistics**
  - **Enhancements in fields relative to chaotic random plane wave background (Antonsen, et., al.)**
  - **Vaynshteyn high frequency construction with Antonsen random phase**



$$d = \ell\sqrt{1 + R/\ell}$$

$$L = 2\ell$$

$$x = d\cosh\zeta\sin\xi$$

$$y = d\sinh\zeta\cos\zeta$$

$$\left(\nabla^2 + k^2\right)u = 0 = \frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\zeta^2} + k^2 d^2\left(\cosh^2\zeta - \sin^2\xi\right)u$$

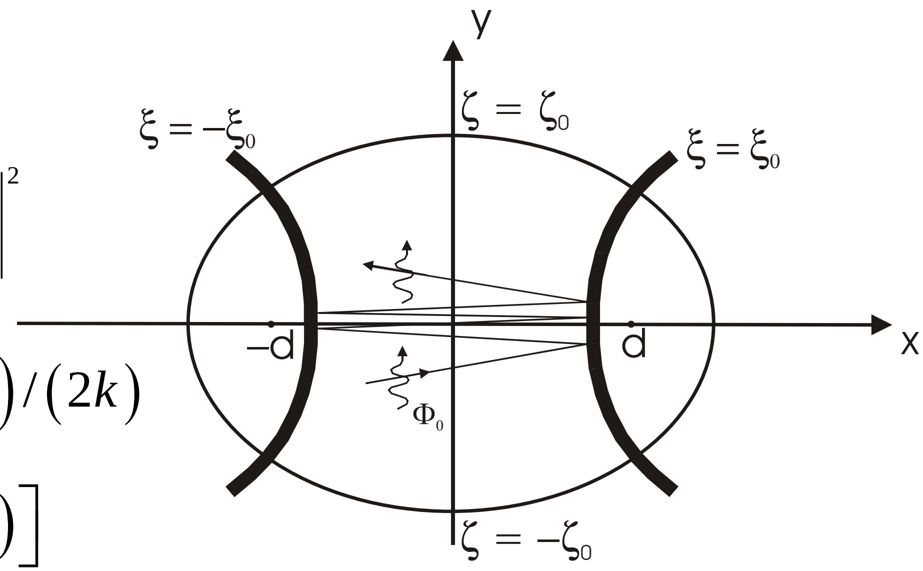
$$E_z = u \sim W e^{ikd\sin\xi}$$

$$\frac{\partial^2 W}{\partial\zeta^2} + 2ikd\cos\xi\frac{\partial W}{\partial\xi} + \left(k^2 d^2\zeta^2 - ikd\sin\xi\right)W = 0$$

Bow Tie Scar Field and Normalization

- **Scar frequency and stability exponent**

$$k_p \ell = (p - 1/2)\pi \qquad \Lambda = \left| \frac{d+\ell}{d-\ell} \right|^2$$



- **Solution**

$$s = 4(k - k_p)\ell / \ln\Lambda \sim \lambda_1 = \sqrt{\ell R}\left(k^2 - k_p^2\right)/(2k)$$

$$\psi(s,\tau) = c\,\mathrm{Re}\left[U_+(s,\tau) + e^{i\Phi_0}U_+^*(s,\tau)\right]$$

$$u_p \sim 2\psi\left(s, \zeta\sqrt{2kd}\right)\cos\left[kd\sin\xi - s\,\mathrm{Arcsin}(\tan\xi)\right]/\sqrt{\cos\xi}$$

- **Normalization (Antonsen, et. al.)** $\quad U_+(s,\tau) = e^{-\pi(s+i/2)/4}U\left(-is, \tau e^{-i\pi/4}\right)$

$$\oint_S \left(\frac{\partial \underline{E}}{\partial\omega}\times\underline{H}^* + \underline{E}^*\times\frac{\partial\underline{H}}{\partial\omega}\right)\cdot\underline{n}\,dS = i\int_V\left(\mu_0\underline{H}\cdot\underline{H}^* + \varepsilon_0\underline{E}\cdot\underline{E}^*\right)dV - \int_V\left(\frac{\partial\underline{E}}{\partial\omega}\cdot\underline{J}^* + \underline{E}^*\cdot\frac{\partial\underline{J}}{\partial\omega}\right)dV$$

$$\mu_0\varepsilon_0\int_A |u|^2\,dS = \int_{-\xi_0}^{\xi_0} u\,\frac{1}{\omega}\frac{\partial}{\partial\omega}\left(\frac{\partial u_p}{\partial\zeta}\right)d\xi \;,\quad K_z = \frac{i2}{\omega\mu_0}\frac{\partial u_p}{\partial y}(x,0)$$

$$1 = 2c^2\left(\frac{d\Phi_0}{dk^2}\right)\frac{\mathrm{Im}^2\left[U_+'(s,0)\right]}{\left|U_+'(s,0)\right|^2}\sqrt{2kd}\,\ln\Lambda$$

$$dk^2/d\Phi_0 \sim v^2\Delta k^2/(2\pi) \quad,\quad f(v) = e^{-v^2/2}/\sqrt{2\pi}$$

$$\Delta k^2 \sim 16\pi/A\ (\text{even-even spacing})$$

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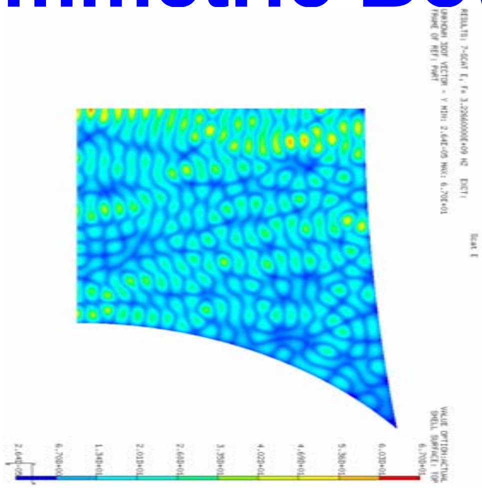
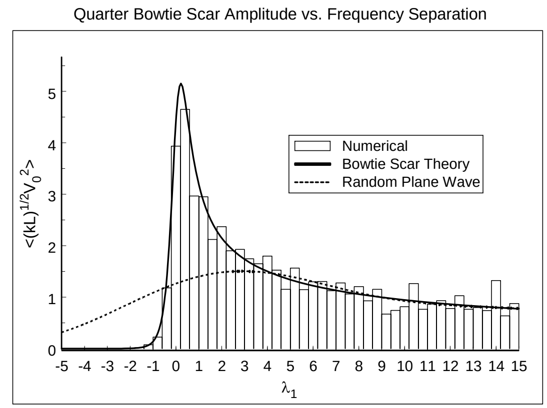
Symmetric Bow Tie

- **Projection along scar**

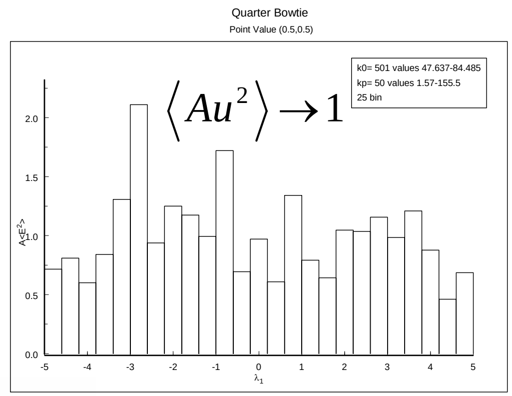
$$V_p = \int_{-\ell}^{\ell} \cos(k_p x)\, u(x,0)\, dx$$

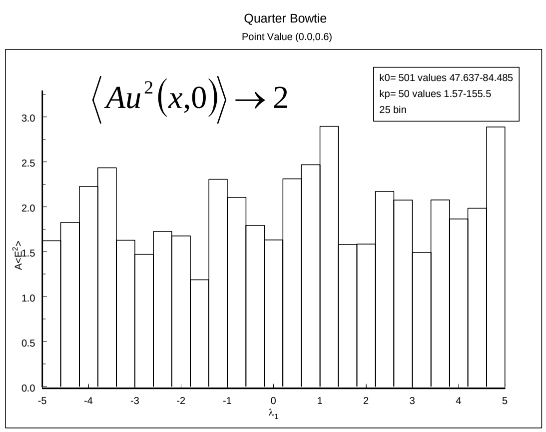


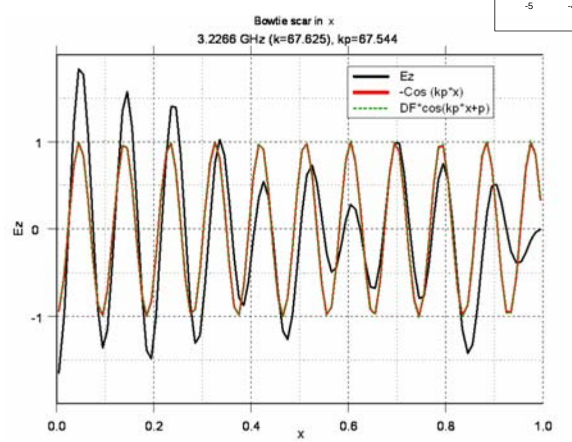


- **Point statistics in volume & on axis (scar decomposition)**

$$u \sim \sum_p^P u_p + u_r - \sum_p^P c_{rp} u_p$$





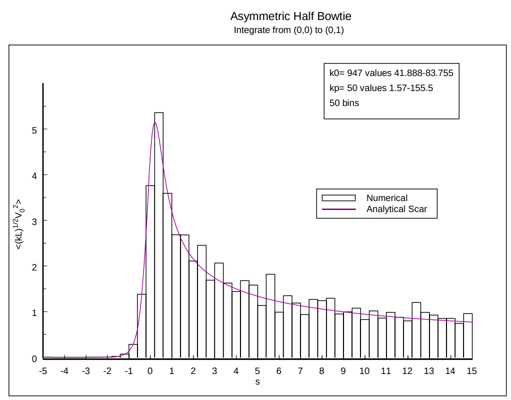


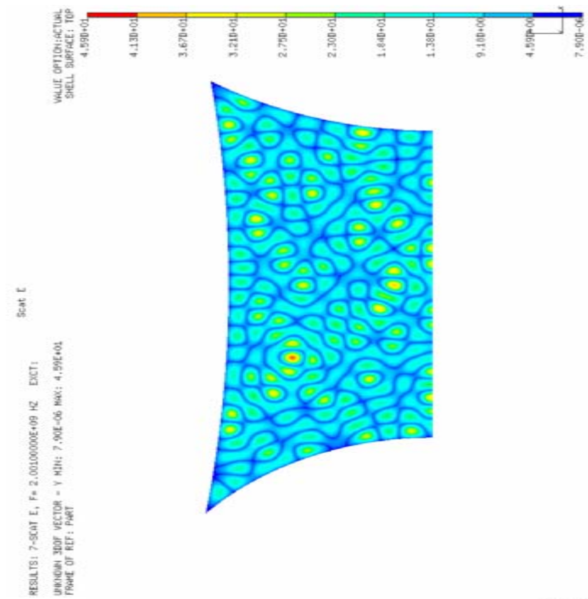
Asymmetric Bow Tie

- **Projections same as even and odd cases**

  - **Even projection**

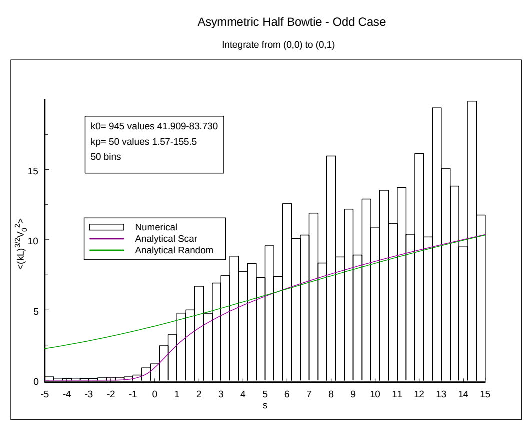
$$V_p = \int_{-\ell}^{\ell} \cos(k_p x)\, u(x,0)\, dx$$

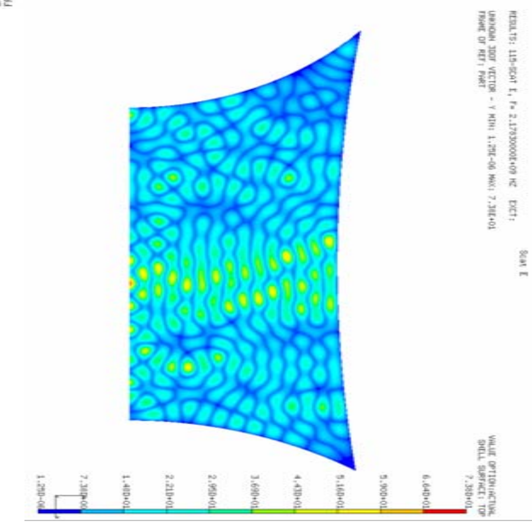




  - **Odd projection**

$$V_p = \int_{-\ell}^{\ell} \cos(k_p x) \frac{\partial u}{k \partial n}(x,0)\, dx$$





Stadium Scar Theory
Concave Boundary

- **Interior foci treated**
  - **Elliptic paths (Vaynshteyn)**
  - **Multiple regions along scar**
  - **EM energy theorem normalization**
  - **Sub-wavelength focal point shift required to perform leading order phase match**

$$u \sim W e^{ikd\cosh\zeta} \quad , \ \text{Region 2}$$

$$\frac{\partial^2 W}{\partial \xi'^2} + 2ikd\sinh\zeta\frac{\partial W}{\partial \zeta} + \left(k^2 d^2 \xi'^2 + ikd\cosh\zeta\right)W = 0$$

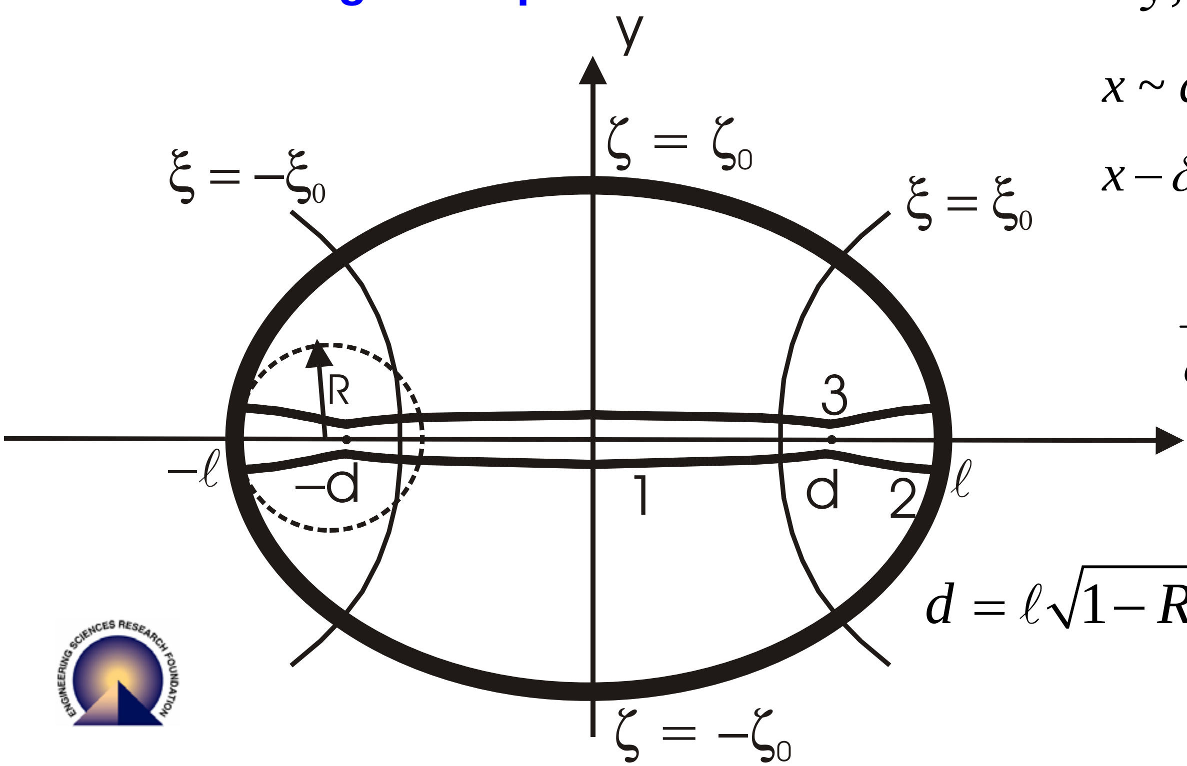
$$\xi' = \pi/2 - \xi$$

$$\hat{\zeta}, \hat{\xi}' << 1 \ , \ \text{Region 3}$$

$$x \sim d\left(1 + \zeta^2/2 - \xi'^2/2\right) \ , \ \ y \sim d\zeta\xi'$$

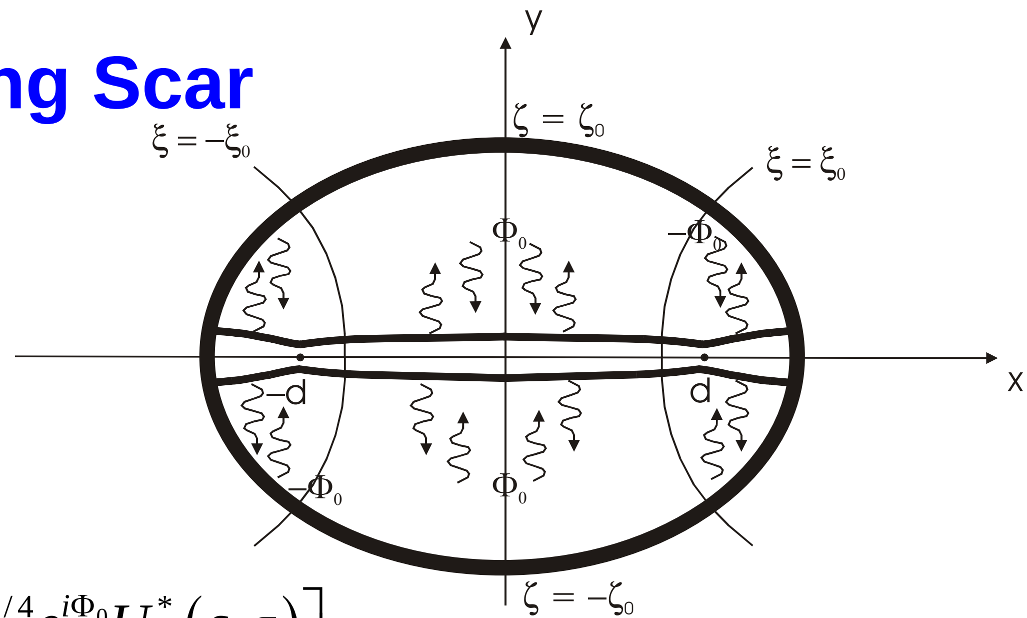
$$x - \delta \sim d\left(1 + \hat{\zeta}^2/2 - \hat{\xi}'^2/2\right) \ , \ \ y \sim d\hat{\zeta}\hat{\xi}'$$

$$\frac{\partial^2 u}{\partial \hat{\xi}'^2} + \frac{\partial^2 u}{\partial \hat{\zeta}^2} + k^2 d^2\left(\hat{\zeta}^2 + \hat{\xi}'^2\right)u = 0$$



$$d = \ell\sqrt{1 - R/\ell}$$

Field Along Scar



- **Region 1** $0 \le x < d - O(\lambda)$

$$\psi_1(s,\tau) = c\,\mathrm{Re}\left[e^{-i\pi/4}U_+(s,\tau) + e^{i\pi/4}e^{i\Phi_0}U_+^*(s,\tau)\right]$$

$$u_p \sim 2\psi_1\left(s,\zeta\sqrt{2kd}\right)\cos\left[kd\sin\xi - s\,\mathrm{Arcsin}(\tan\xi)\right]/\sqrt{\cos\xi}$$

- **Region 2** $d + O(\lambda) < x \le \ell$

$$\psi_2(-s,\tau') = c\,\mathrm{Re}\left[U_+(-s,\tau') + e^{-i\Phi_0}U_+^*(-s,\tau')\right]$$

$$u_p \sim 2\psi_2\left(-s,\xi'\sqrt{2kd}\right)\cos\left[kd\cosh\zeta + s\ln\{\tanh(\zeta/2)\} - \pi/4\right]/\sqrt{\sinh\zeta}$$

- **Region 3** $|x - d - \delta| << d, \ell$

$$u_p \sim (-1)^n (2kd)^{1/4}\sec(\Phi_0/2)\psi_1\left(s,\hat{\zeta}\sqrt{2kd}\right)\psi_2\left(-s,\hat{\xi}'\sqrt{2kd}\right)$$

Matching and Normalization

- Scar frequency and symmetry conditions (focal point phase jump)

$$k_p \ell = (p - 1/4)\pi$$

$$s = 4(k - k_p)\ell / \ln \Lambda \qquad \Lambda = \left| \frac{\ell + d}{\ell - d} \right|^2$$

$$\mathrm{Re}\left[U_+'(-s,0) + e^{-i\Phi_0} U_+^{*}{}'(-s,0)\right] = 0 = \mathrm{Re}\left[e^{-i\pi/4} U_+'(s,0) + e^{i\pi/4} e^{i\Phi_0} U_+^{*}{}'(s,0)\right]$$

- Matching condition (simulations indicate n is taken to make shift positive)

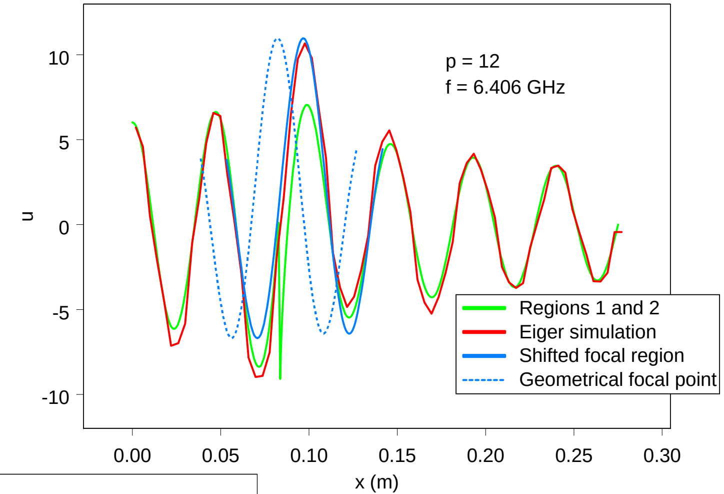
$$k(d + \delta) = k d_e = s \ln\left(2\sqrt{2kd}\right) - \Phi_0 / 2 + n\pi$$

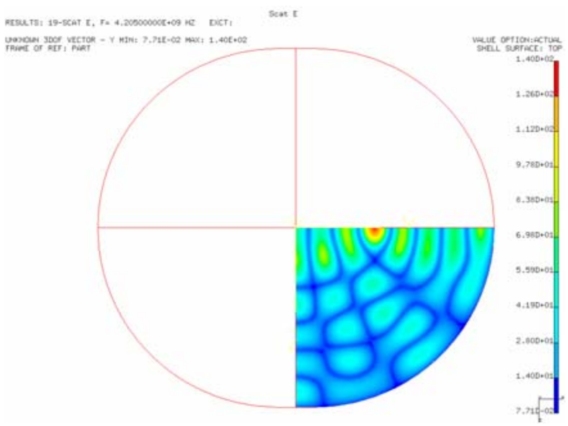
- Normalization
  - Region 3 gives principal value interpretation of energy integral

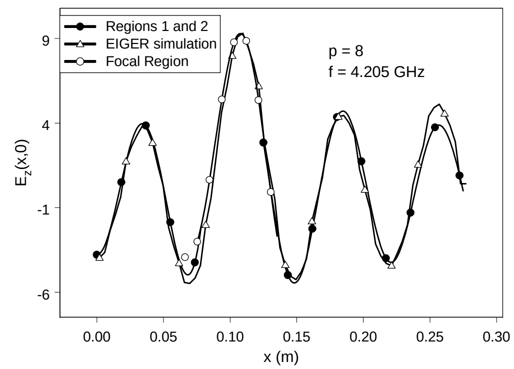
$$1 = 2c^2 \left(\frac{d\Phi_0}{dk^2} \right) \frac{\mathrm{Im}^2\left[e^{-i\pi/4} U_+'(s,0)\right]}{\left|U_+'(s,0)\right|^2} \sqrt{2kd} \ln \Lambda$$

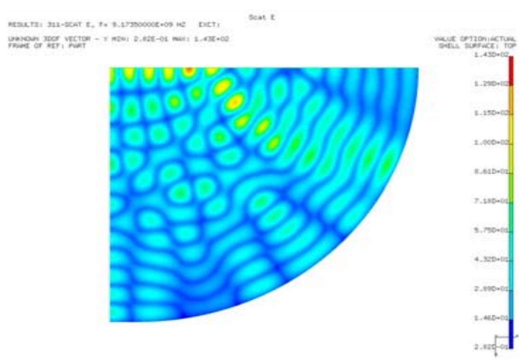
Spatial Comparisons

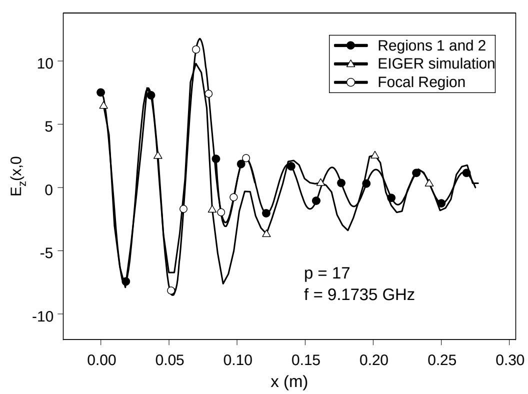
- **Spatial comparisons with EIGER simulations (overall amplitude adjusted)**





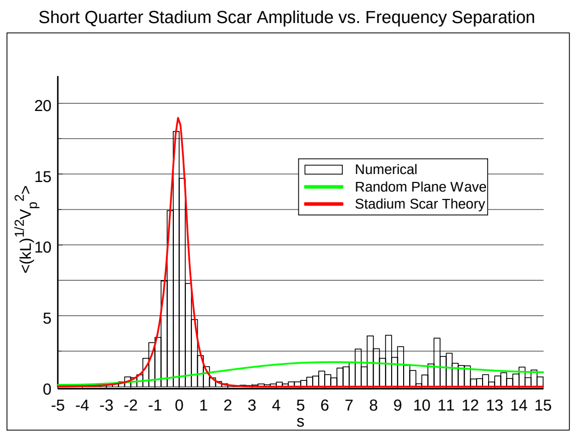


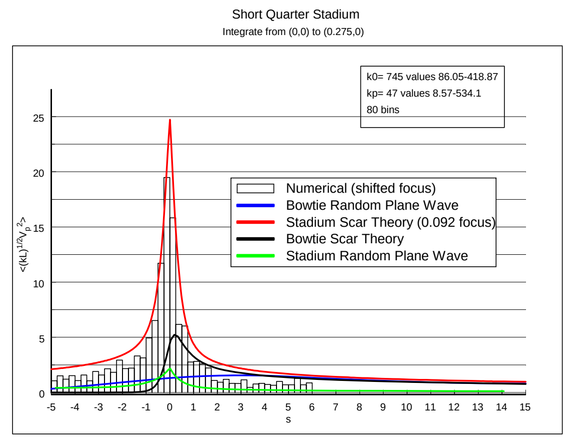




Stadium Projection and Point Statistics

- **Larger deviations from random plane wave background**
  - **Projection (trigonometric & scar function orthogonality)**
  - **Point statistics (focal point & volume)**





$$V_p^T = 2\int_0^d u(x,0)\cos(k_p x)\,dx$$

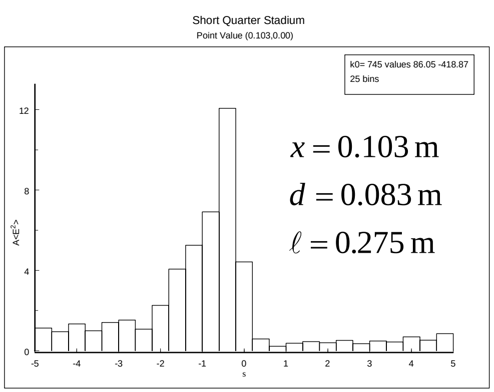
$$+2\int_d^\ell u(x,0)\cos(k_p x - \pi/4)\,dx$$

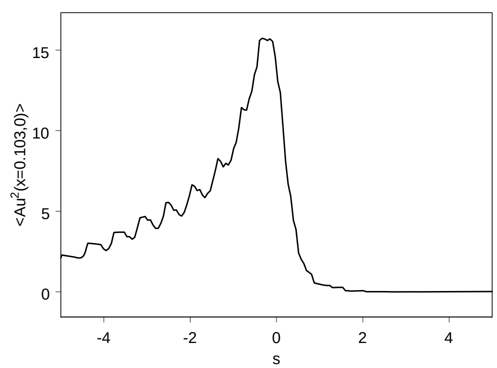
$$d \to \langle d_e \rangle$$

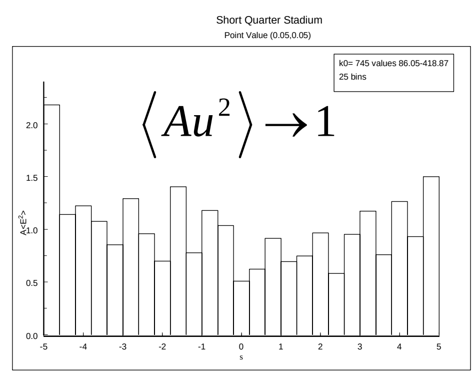
$$e^{\pi|s|/4}V_p = e^{\pi s/4}2\int_0^d u(x,0)\left(1-x^2/d^2\right)^{-1/4}\cos\left(k_p x + p(x,s)\right)dx$$

$$+e^{-\pi s/4}2\int_d^\ell u(x,0)\left(x^2/d^2-1\right)^{-1/4}\cos\left(k_p x - \pi/4 + p(x,s)\right)dx$$

$$p(x,s) = \frac{1}{2}s\left[\ln\left|\frac{\ell+d}{\ell-d}\right|(x/\ell) - \ln\left|\frac{d+x}{d-x}\right|\right]$$







Conclusions

- Combined elliptic coordinate and random phase reflection approach for time harmonic treatment of unstable scar field enhancements
- Concave walls and interior foci in 2D
  - Multiple region (focal region shift)
  - Large enhancements relative to chaotic background
  - Results in terms of stability exponents, modal frequency versus scar frequency difference, observation point versus focus location, orbit length, cavity area
- 3D axisymmetric case similar