SAND2015-5690C

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Simulation of Elastic Wave Propagation using Cellular Automata and Peridynamics

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Outline

- Introduction to Cellular Automata
- Introduction to Peridynamics
- Model Geometry
- Homogeneous Results
- Apply Disorder
- Heterogeneous Results
- Conclusions

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Introduction to Cellular Automata (CA)

- Local computational method
  - Each element is dependent on elements that share an edge or corner.
- Mathematically equivalent to finite difference method of classical elasticity
  - Avoids derivation of governing partial differential equations

$$\rho \dot{v}_y(i,j) = \frac{\lambda + 2\mu}{\Delta y}\left(\frac{u_y(i,j+1) - u_y(i,j)}{\Delta y} - \frac{u_y(i,j) - u_y(i,j-1)}{\Delta y}\right)$$
$$+ \frac{\lambda}{2\Delta y}\left(\frac{u_x(i+1,j+1) - u_x(i-1,j)}{2\Delta x} - \frac{u_x(i+1,j-1) - u_x(i-1,j-1)}{2\Delta x}\right)$$
$$+ \frac{\mu}{\Delta x}\left(\frac{u_y(i+1,j) - u_y(i,j)}{\Delta x} - \frac{u_y(i,j) - u_y(i-1,j)}{\Delta x}\right)$$
$$+ \frac{\mu}{2\Delta x}\left(\frac{u_x(i+1,j+1) - u_x(i+1,j-1)}{2\Delta y} - \frac{u_x(i-1,j+1) - u_x(i-1,j-1)}{2\Delta y}\right)$$

Introduction to Peridynamics (PD)

- Non-local continuum mechanics formulation
- Integro-Differential governing equation

$$\rho(\boldsymbol{x})\ddot{\boldsymbol{u}}(\boldsymbol{x},t) = \int_{H_x} \boldsymbol{f}(\boldsymbol{u}' - \boldsymbol{u}, \boldsymbol{x}' - \boldsymbol{x}, t)\, dV + \boldsymbol{b}(\boldsymbol{x},t)$$

  - Restatement of Newton’s Second Law
  - No spatial derivatives
    - Designed to handle fracture problems
- Difficult to solve analytically
  - Some solutions exist (Silling 2003)

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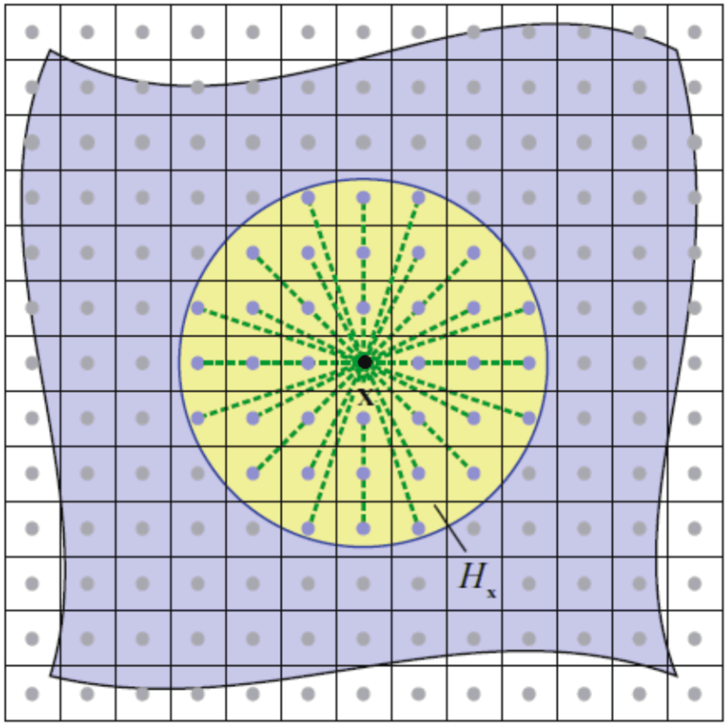
Discretize Eqn. of Motion

- Apply square mesh over domain
- Assume mass is concentrated at center
- Integral goes to sum:



$$\rho(\boldsymbol{x})\ddot{\boldsymbol{u}}(\boldsymbol{x},t) = \int_{H_x} \boldsymbol{f}(\boldsymbol{u}' - \boldsymbol{u}, \boldsymbol{x}' - \boldsymbol{x})\, dV + \boldsymbol{b}(\boldsymbol{x},t)$$

$$\rho(\boldsymbol{x}_i)\ddot{\boldsymbol{u}}_i(\boldsymbol{x}_i,t) = \sum_{p=1}^{n} \boldsymbol{f}(\boldsymbol{u}_p - \boldsymbol{u}_i, \boldsymbol{x}_p - \boldsymbol{x}_i)\ dV + \boldsymbol{b}_i(\boldsymbol{x}_i,t)$$

- Can be interpreted as non-local spring-mass model

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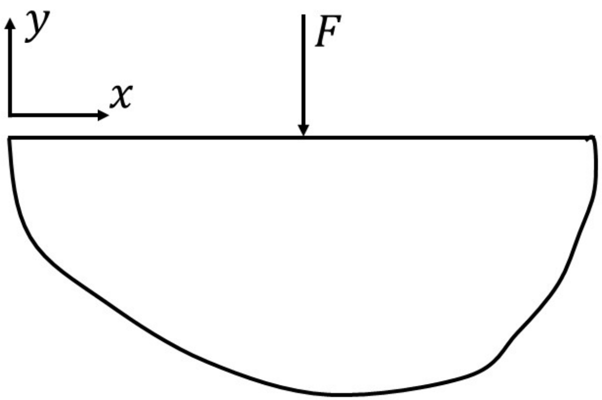
Model Geometry

- Half-Plane subject to load (Lamb's Problem)
  - Normal, impulse load
  - Two-Dimensional Problem



- Motivation
  - Surface structures subject to earthquakes
  - Underground structures

Validation – Homogeneous Case

- Comparison with experiments (J.W. Dally 1967)
  - CR-39
    - Photoelastic Material
    - Elastic Modulus – 3.85 GPa, Poisson Ratio – 1/3, Density – 1300 kg/m^3
  - Response of surface at different times
    - Pressure Wave (P)
    - Shear Wave (S)
    - Surface or Rayleigh Wave (R)
  - Explosive charge used as input
- Analytical Results
  - Classical Elasticity (Partial Differential Equations)
  - Triangular impulse load

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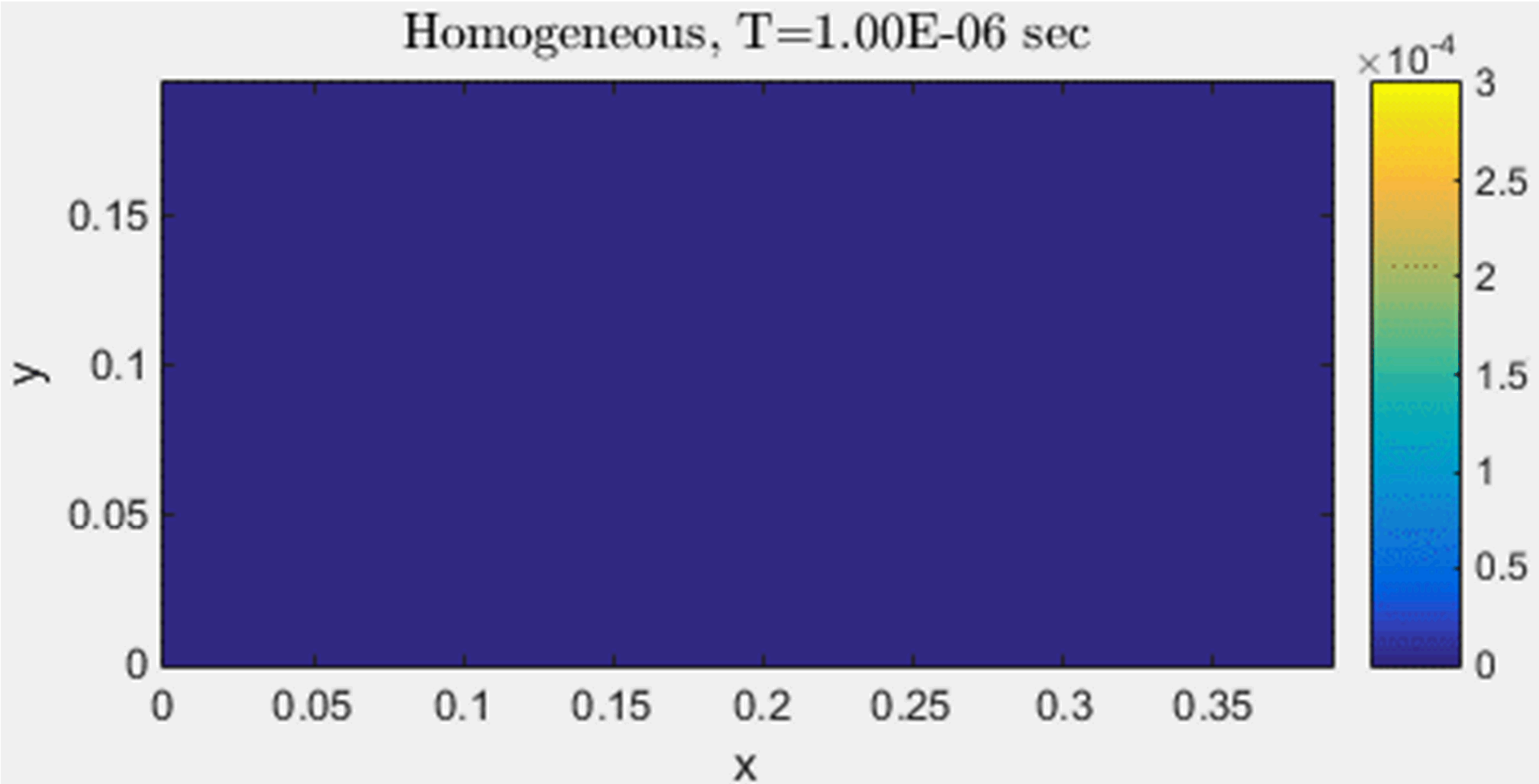
Results – CA Video



Plot of displacement magnitude

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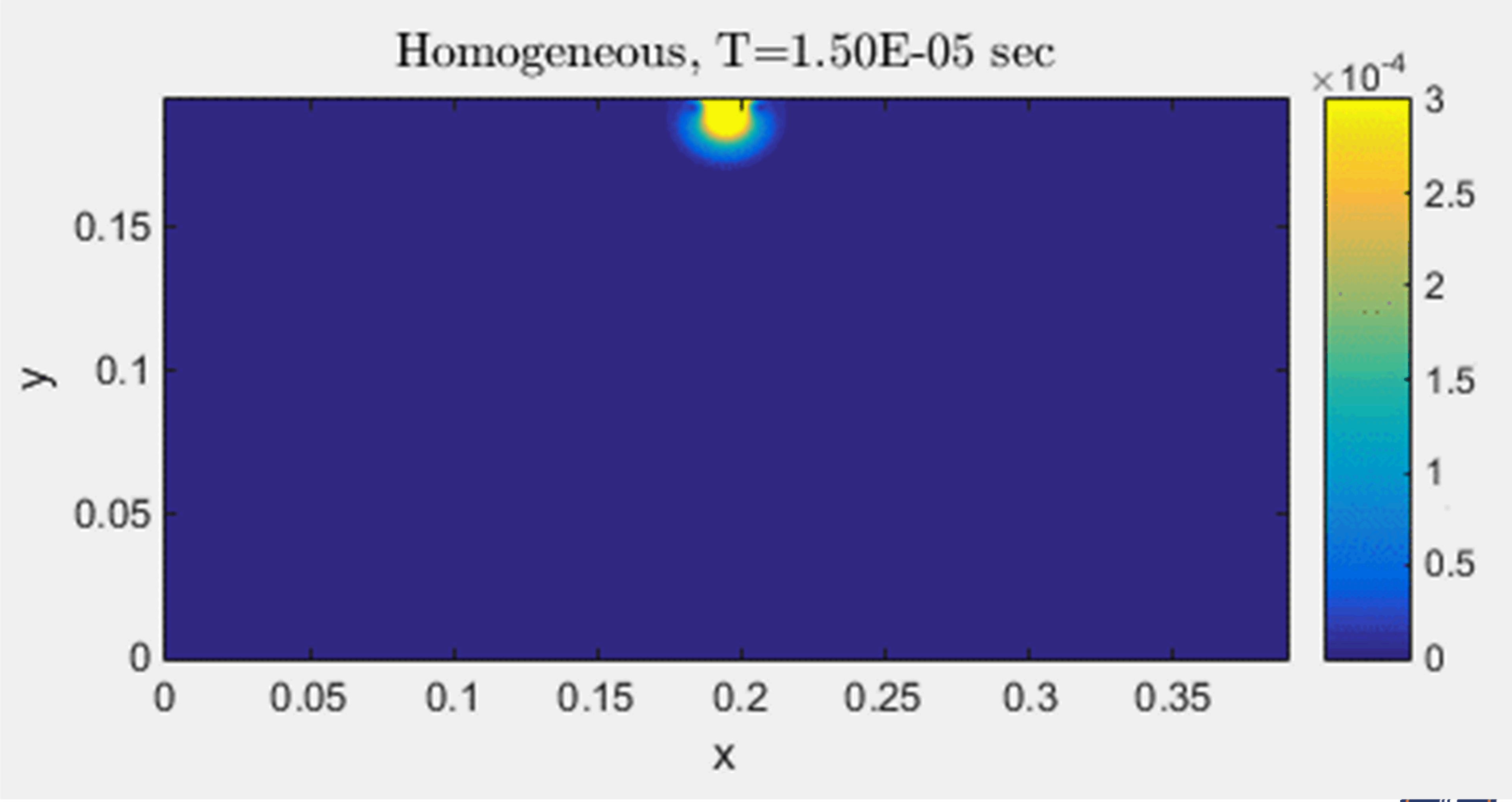
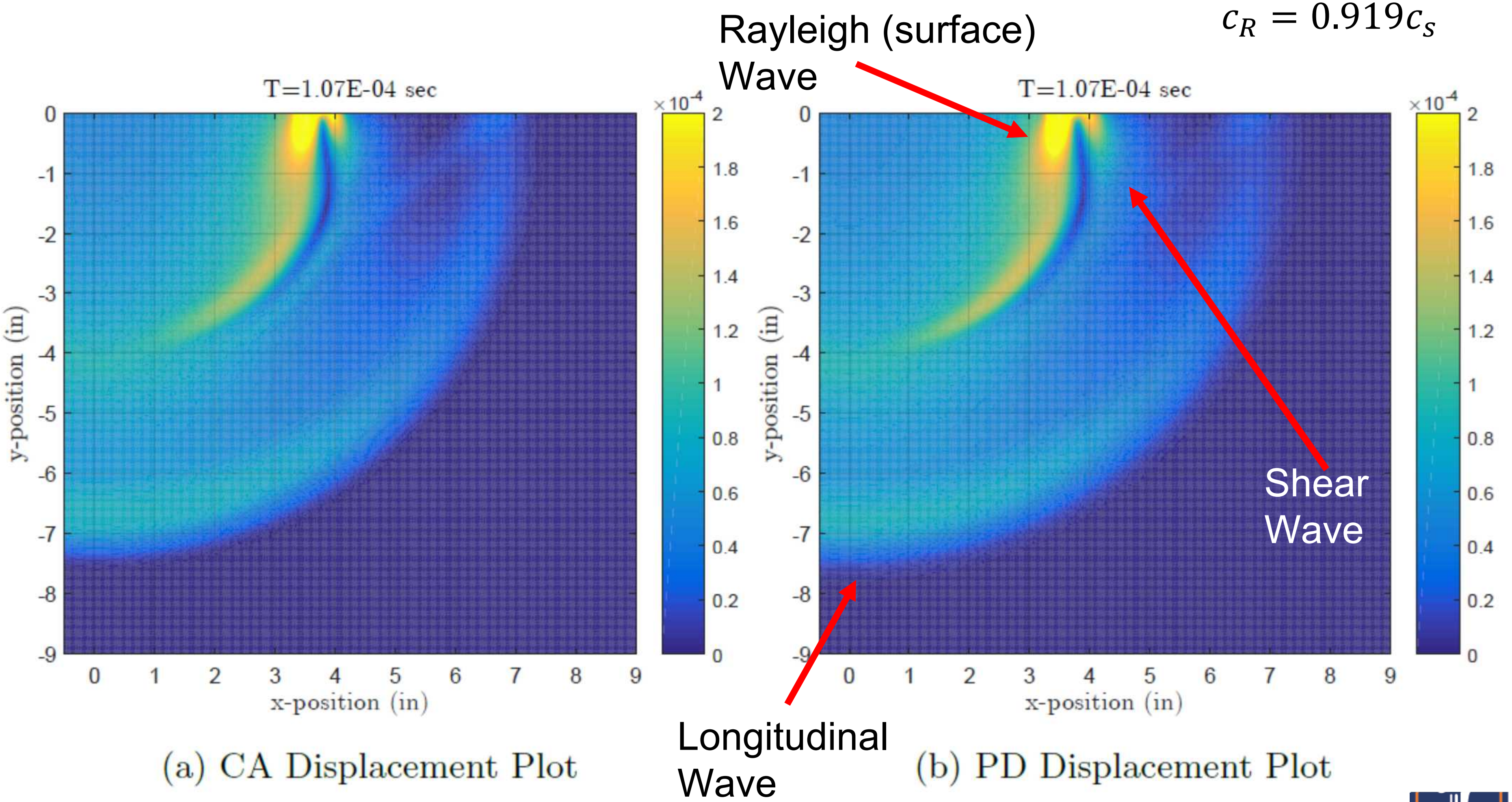
Results – PD Video



Plot of displacement magnitude

Comparing CA and PD

$c_R = 0.919 c_s$



(a) CA Displacement Plot

(b) PD Displacement Plot

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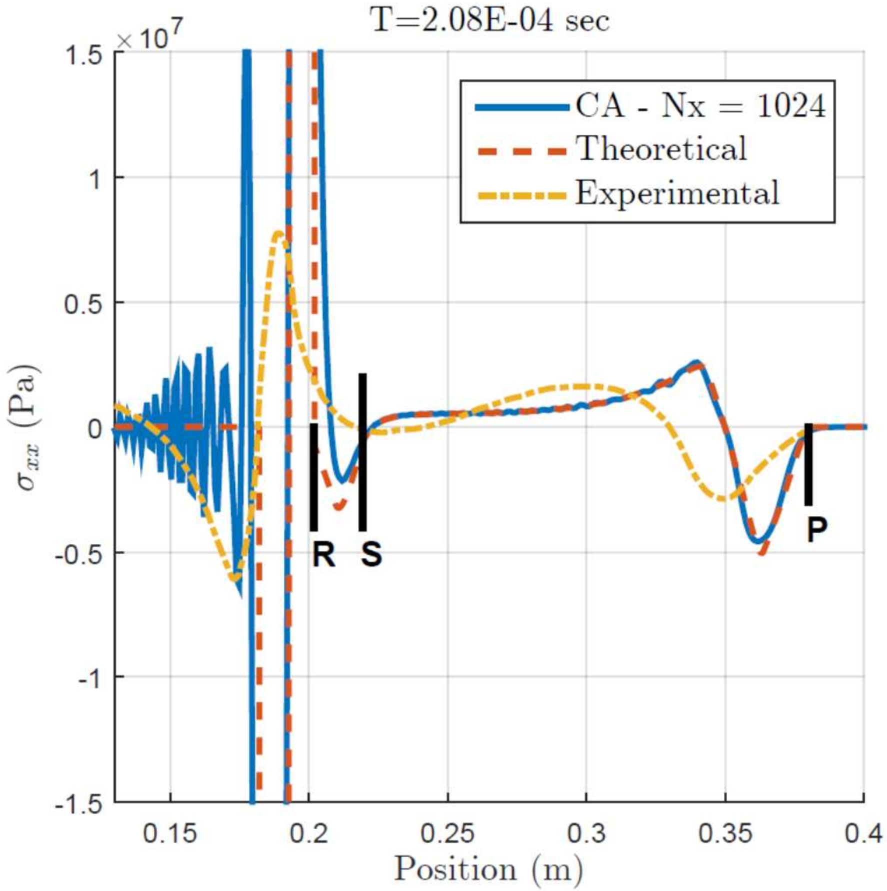
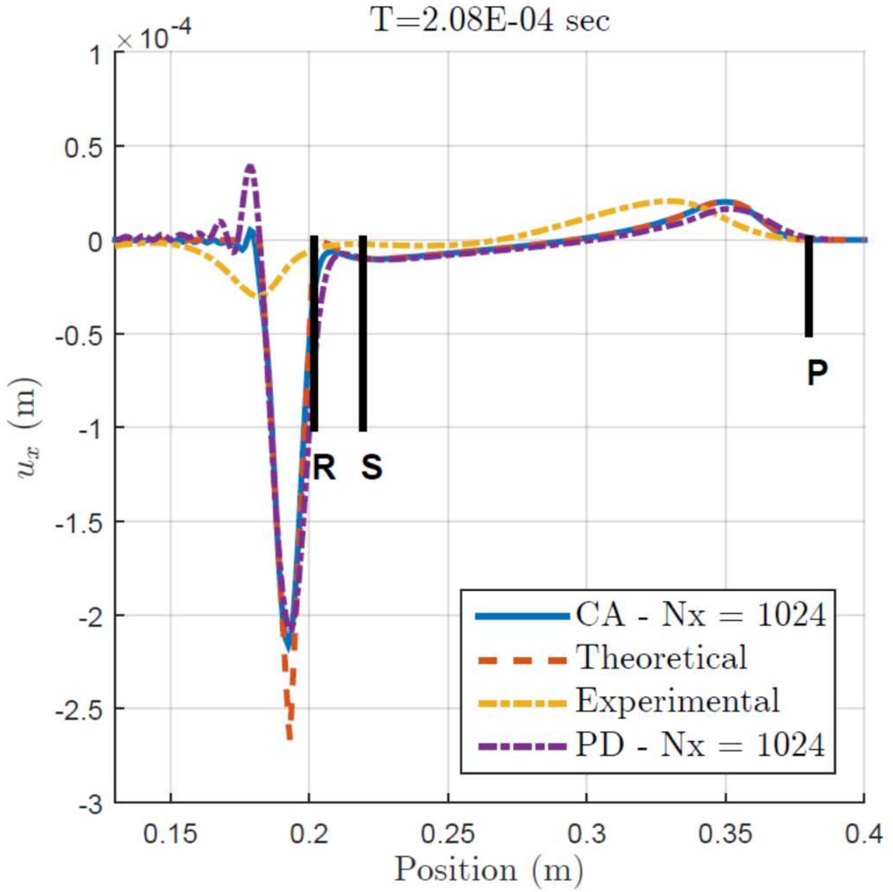
Results – Cont.





*Note that Peridynamics does not have an equivalent measurement to Cauchy Stress of classical elasticity. Therefore, for Peridynamics, we compare displacements.

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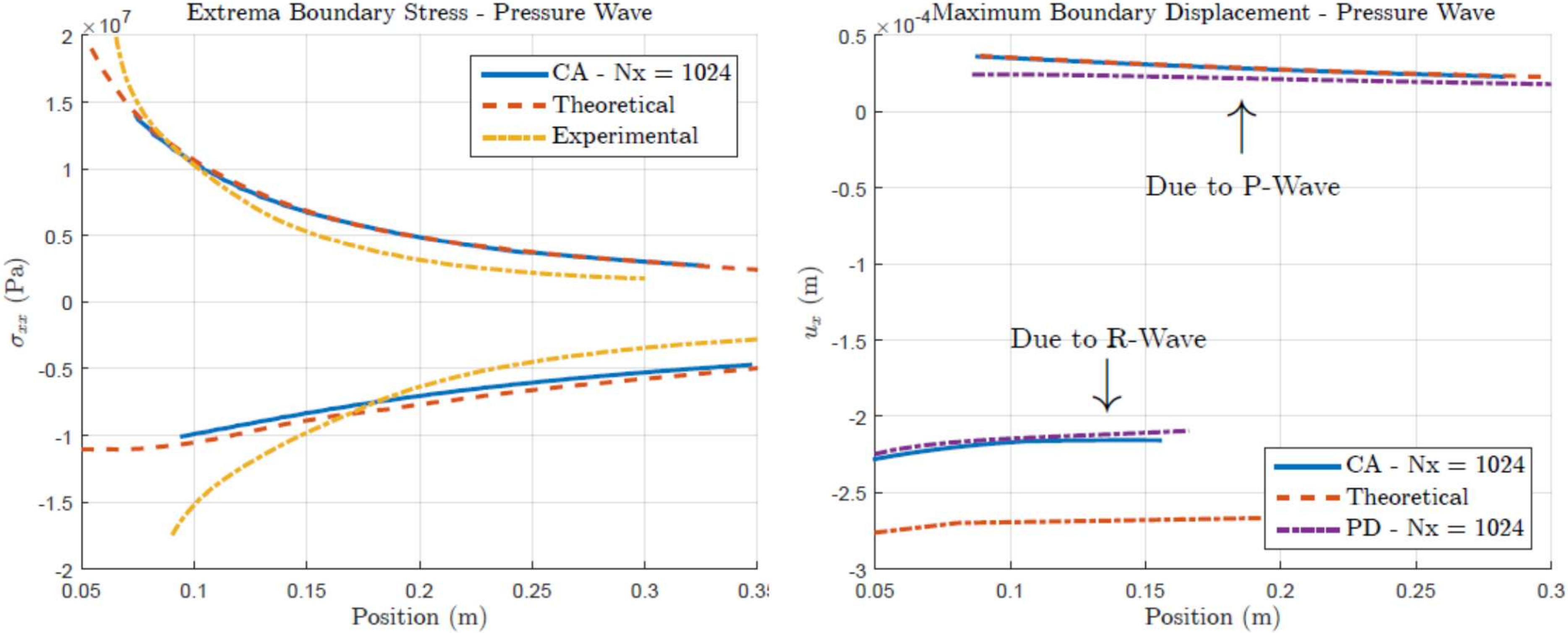
Results – Amplitude Decay



(a) Comparing CA with Experimental and Analytical Results

(b) Comparing PD and CA Displacement Results

Background on Distributions – Hurst Effect

- Hurst Parameter
  - Long Term Memory
    - Negative autocorrelation (An increase followed by a decrease)

$$0 < H < 0.5$$

    - Positive autocorrelation (A decrease followed by a decrease)

$$0.5 < H < 1$$

    - Random Walk

$$H = 0.5$$

  - Heavy-tail behavior of covariance function (Joseph Effects)
    - Covariance decays slower than exponential function

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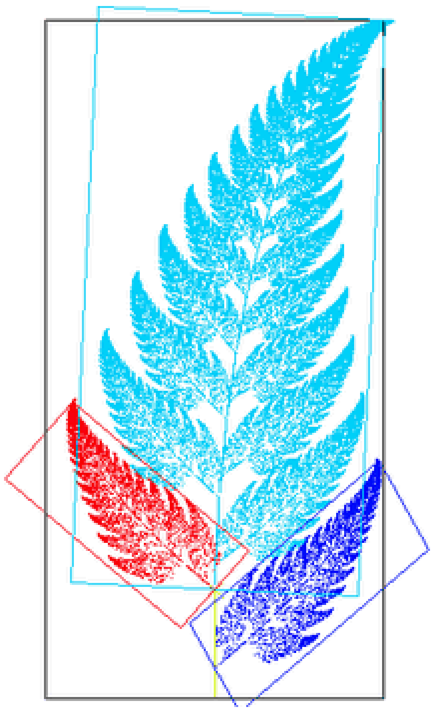
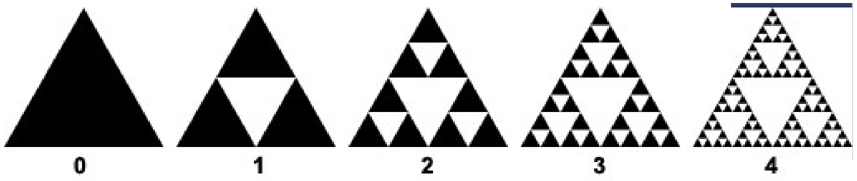
Background on Distributions - Fractals

- Fractals (Mandelbrot 1975)
  - "Self Similarity"
    - Geometry Repeats at smaller scale
  - Found in Nature
  - Fractal Dimension is given by: $D = -\frac{\log N}{\log r}$
    - N – Number of Segments, r – the scaling factor
  - Example: Sierpinski Triangle
    - Triangle reduced by 1/2 ($r = 1/2$), requires 3 triangles ($N = 3$), $D \approx 1.58$





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Statistical Definitions

- Expected (mean) value

$$E[Z(x)] = \int_{-\infty}^{\infty} z f(z; x)\, dz$$

  - $Z = Z(x;\ \omega); \omega \in \Omega$ is a random process or field
  - f is the probability density function of Z

- Covariance

$$C(\boldsymbol{x}_1, \boldsymbol{x}_2) = E[\{Z(\boldsymbol{x}_1) - E[Z(\boldsymbol{x}_1)]\}\{Z(\boldsymbol{x}_2) - E[Z(\boldsymbol{x}_2)]\}]$$
$$= R(\boldsymbol{x}_1, \boldsymbol{x}_2) - E[Z(\boldsymbol{x}_1)]E[Z(\boldsymbol{x}_2)]$$

  - $Z(\boldsymbol{x}_{\boldsymbol{1}})$ and $Z(\boldsymbol{x}_2)$ are random variables, R is the variance
  - Measures the strength of the correlation
  - If $Z(\boldsymbol{x}_1)$ and $Z(\boldsymbol{x}_2)$ are uncorrelated, then $C = 0$

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Random Fields

- Scalar Random Field
  - Mass density

$$\rho(\boldsymbol{x}, \omega); \omega \in \Omega$$

- Wide-Sense Stationary (WSS)
  - Random process $Z(\boldsymbol{x})$ is WSS if its mean is independent of $\boldsymbol{x}$ and its autocorrelation depends only on separation $\boldsymbol{x}$:

$$R(\boldsymbol{x}_1, \boldsymbol{x}_2) = R(\boldsymbol{x}), \qquad \boldsymbol{x} = \boldsymbol{x}_2 - \boldsymbol{x}_1$$

  - Isotropic Random Field

$$C(\boldsymbol{x}) = C(\|\boldsymbol{x}\|) = C(h)$$

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Mass Density Field Disorder

- For an Isotropic, WSS, random process $\rho = Z(x)$, with zero mean, covariance is: $C(h) = E[Z(x_1)Z(x_1 + h)]$
- Wide variety of models exist
  - White Noise: $C(h) = \delta(h)$
  - Gaussian: $C(h) = e^{-h^2}$
  - Matern: $C(h) = h^{\nu}\kappa_{\nu}(h)$, where κ is the modified Bessel function
  - Cauchy Distribution: $C(h) = (1 + h^{\alpha})^{-\beta/\alpha}, \alpha \in (0,2], \beta > 0$
  - Dagum Distribution: $C(h) = 1 - \left(1 + h^{-\beta}\right)^{-\frac{\alpha}{\beta}}, \beta \in (0,1], \alpha \in (0,1)$
    - Cauchy and Dagum Distributions can model the fractal dimension and Hurst parameter – not only model but decouple!
    - White noise, Gaussian and Matern cannot model fractal dim. or Hurst
- Dagum Distribution is considered here

$$D = n + 1 - \frac{\alpha}{2}, \qquad H = 1 - \frac{\beta}{2}$$

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Dagum Random Fields

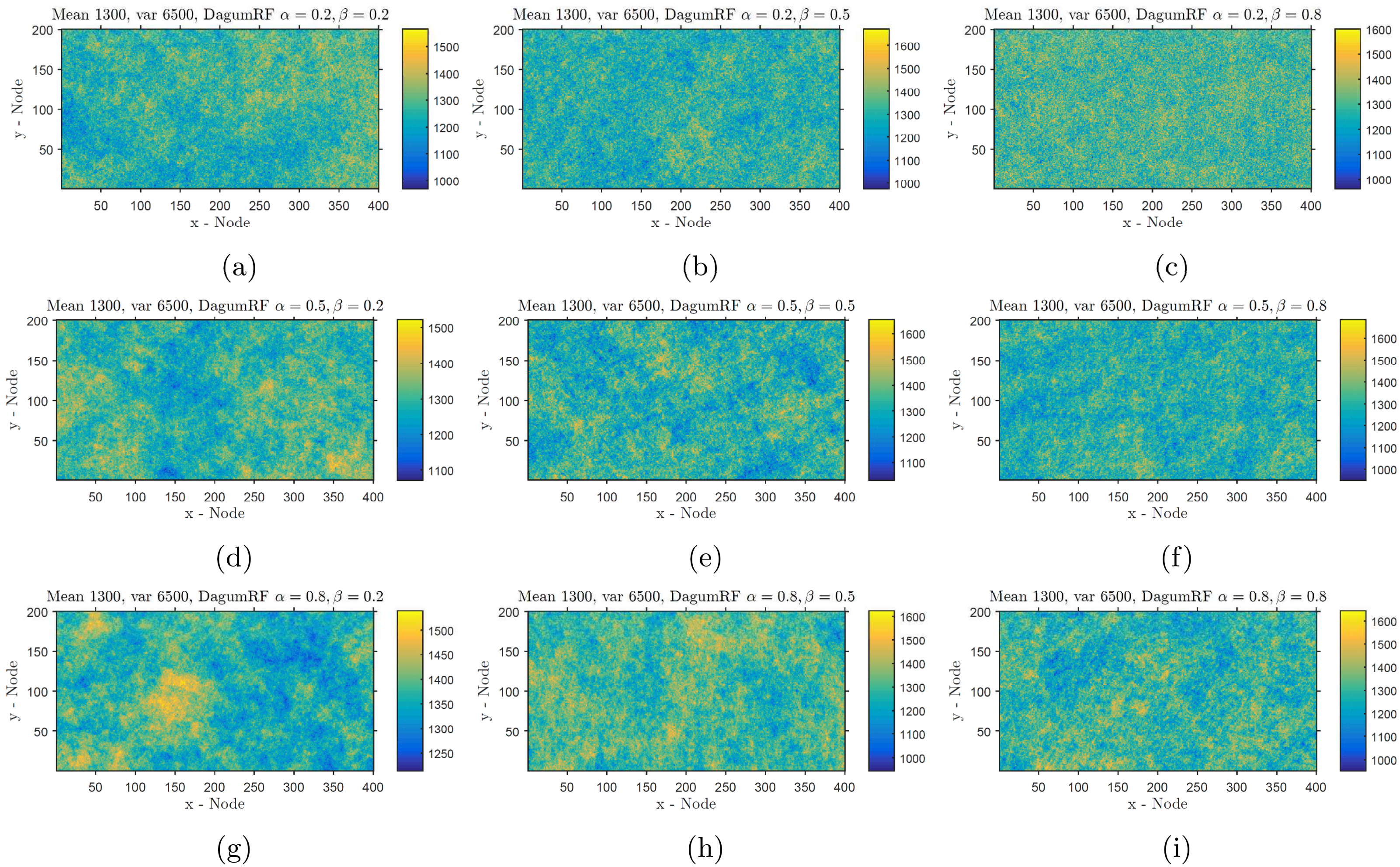
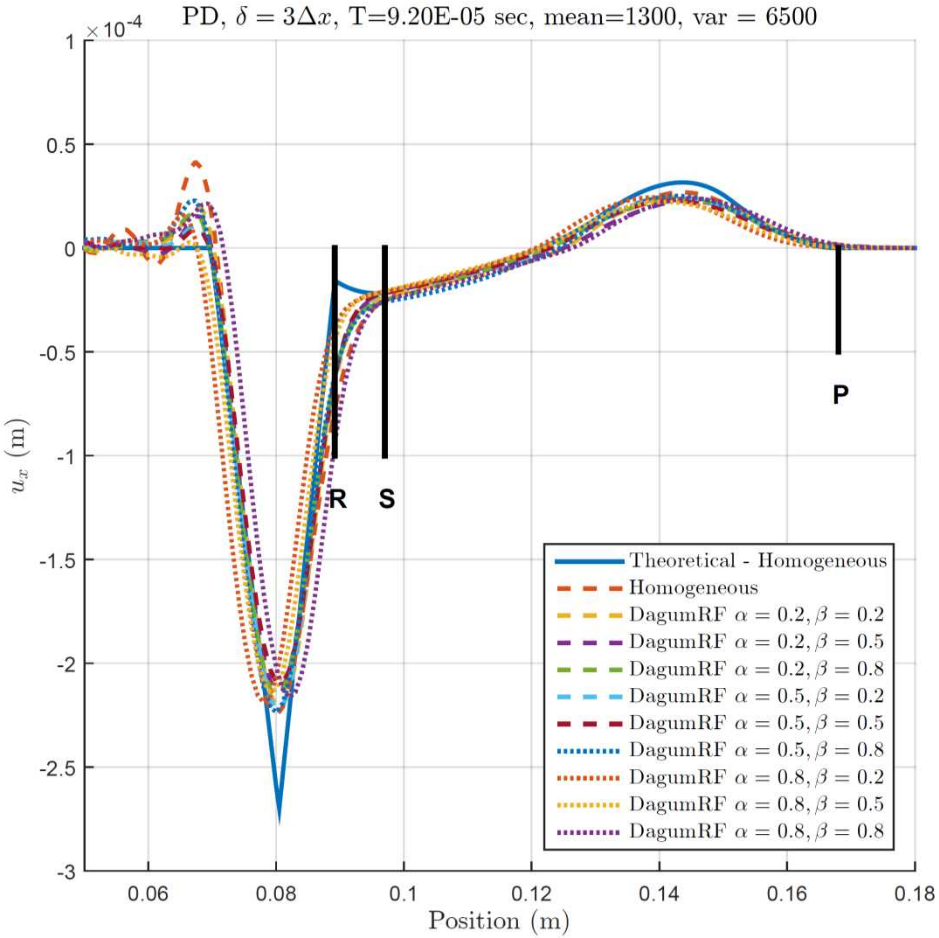
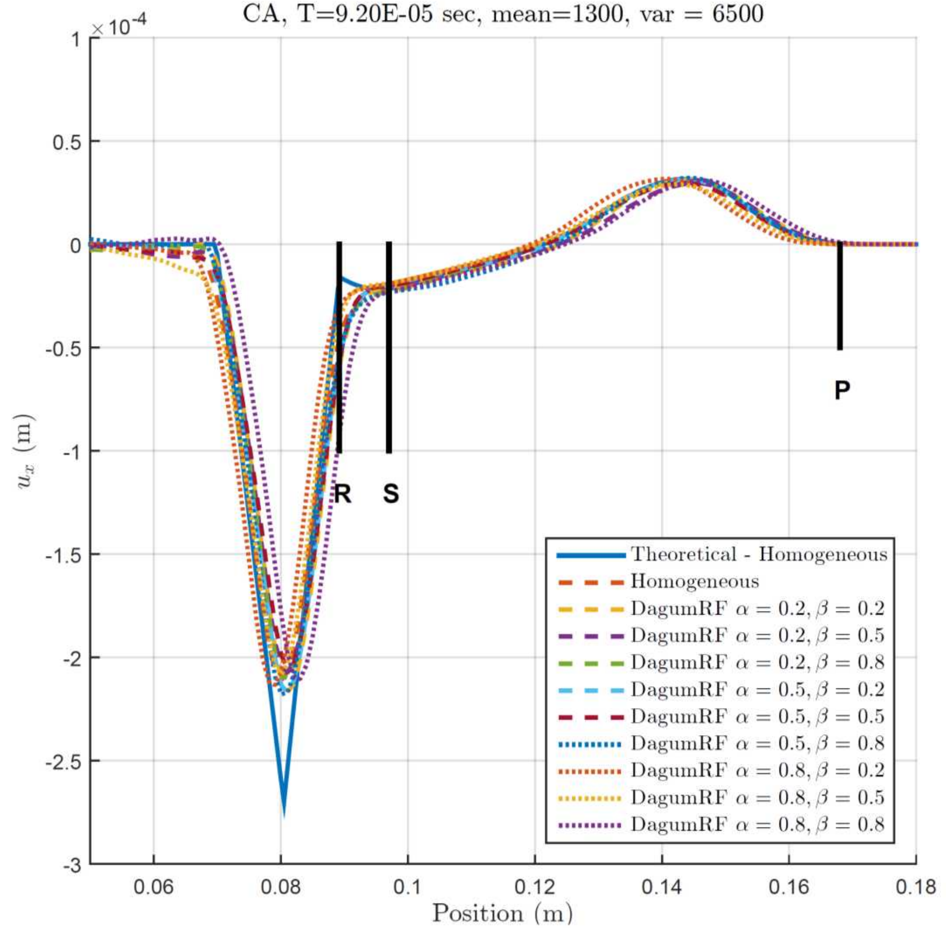


Figure 11: Left Column: $\beta = 0.2(H = 0.9)$, Center Column: $\beta = 1.0(H = 0.5)$, Right Column: $\beta = 0.8(H = 0.1)$. Top Row: $\alpha = 0.2(D = 2.9)$, Middle Row: $\alpha = 0.5(D = 2.75)$, Bottom Row: $\alpha = 0.8(D = 2.6)$

Variance = 6500, std dev = 80.6

Dagum Results – CA and PD





Variance = 6500, std dev = 80.6

Dagum Random Fields – Higher Variance

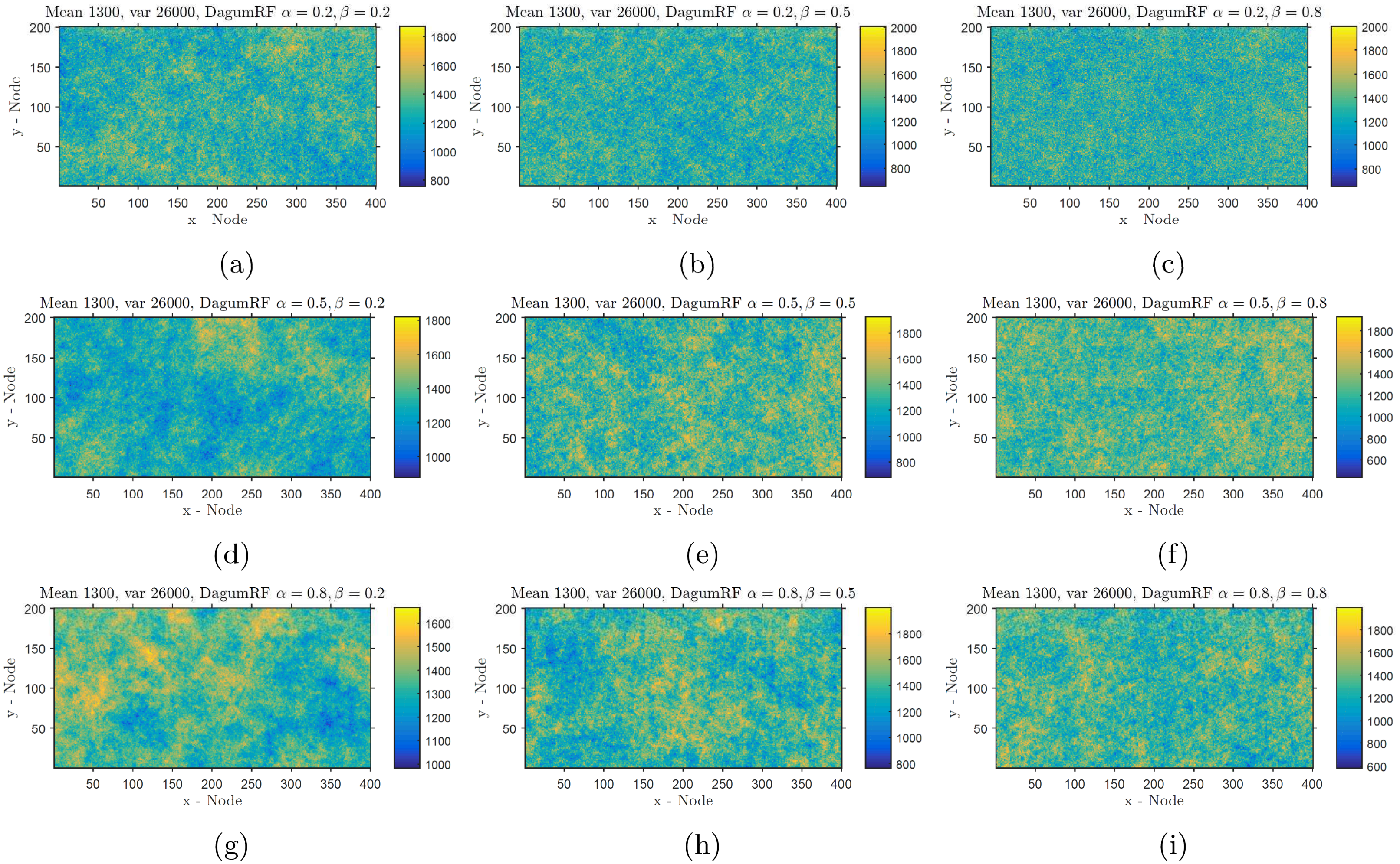
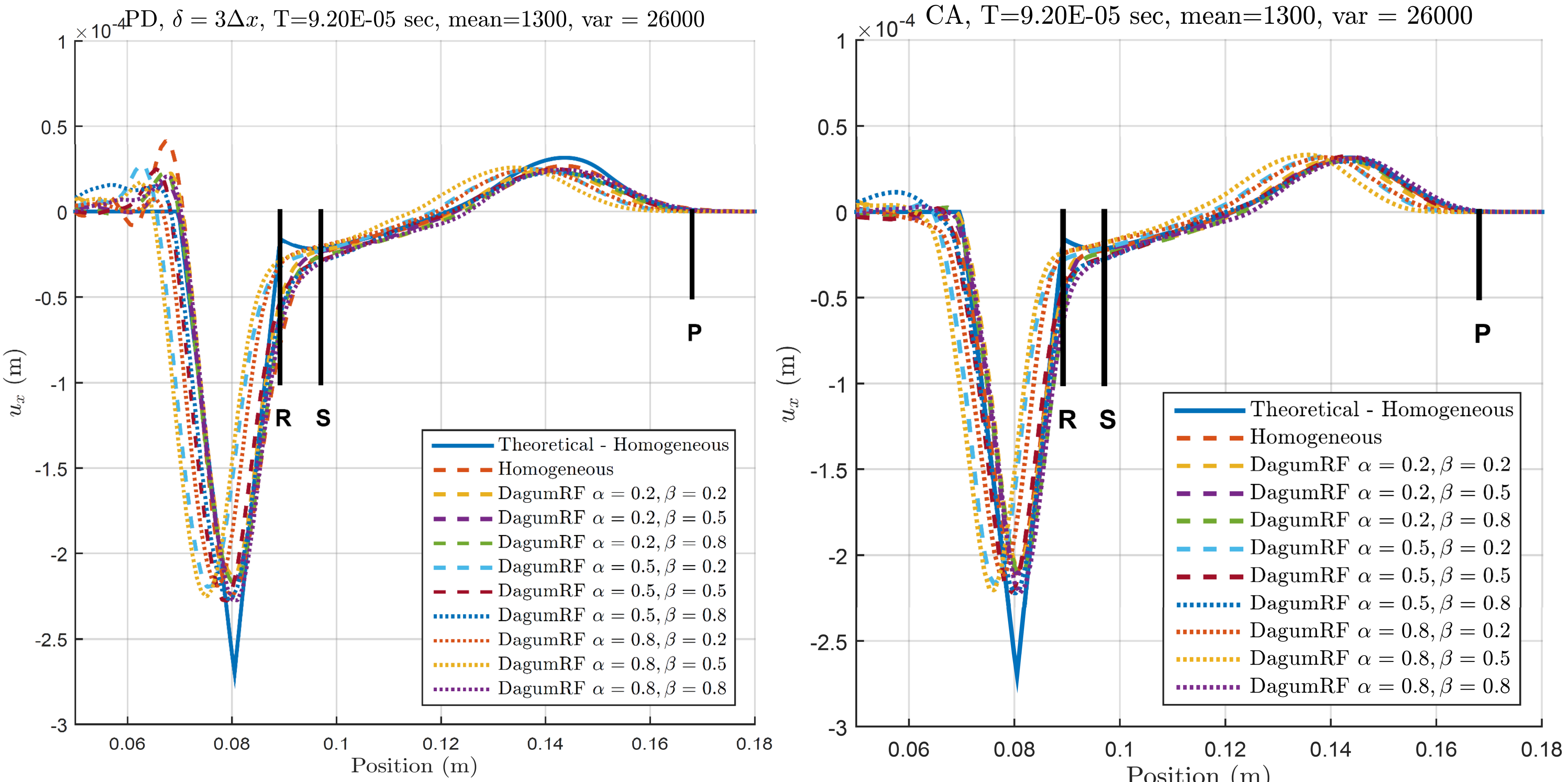


Figure 17: Left Column: $\beta = 0.2 (H = 0.9)$, Center Column: $\beta = 1.0 (H = 0.5)$, Right Column: $\beta = 0.8 (H = 0.1)$. Top Row: $\alpha = 0.2 (D = 2.9)$, Middle Row: $\alpha = 0.5 (D = 2.75)$, Bottom Row: $\alpha = 0.8 (D = 2.6)$

Variance = 26000, std dev = 161.2

Dagum RF– CA & PD – Higher Variance



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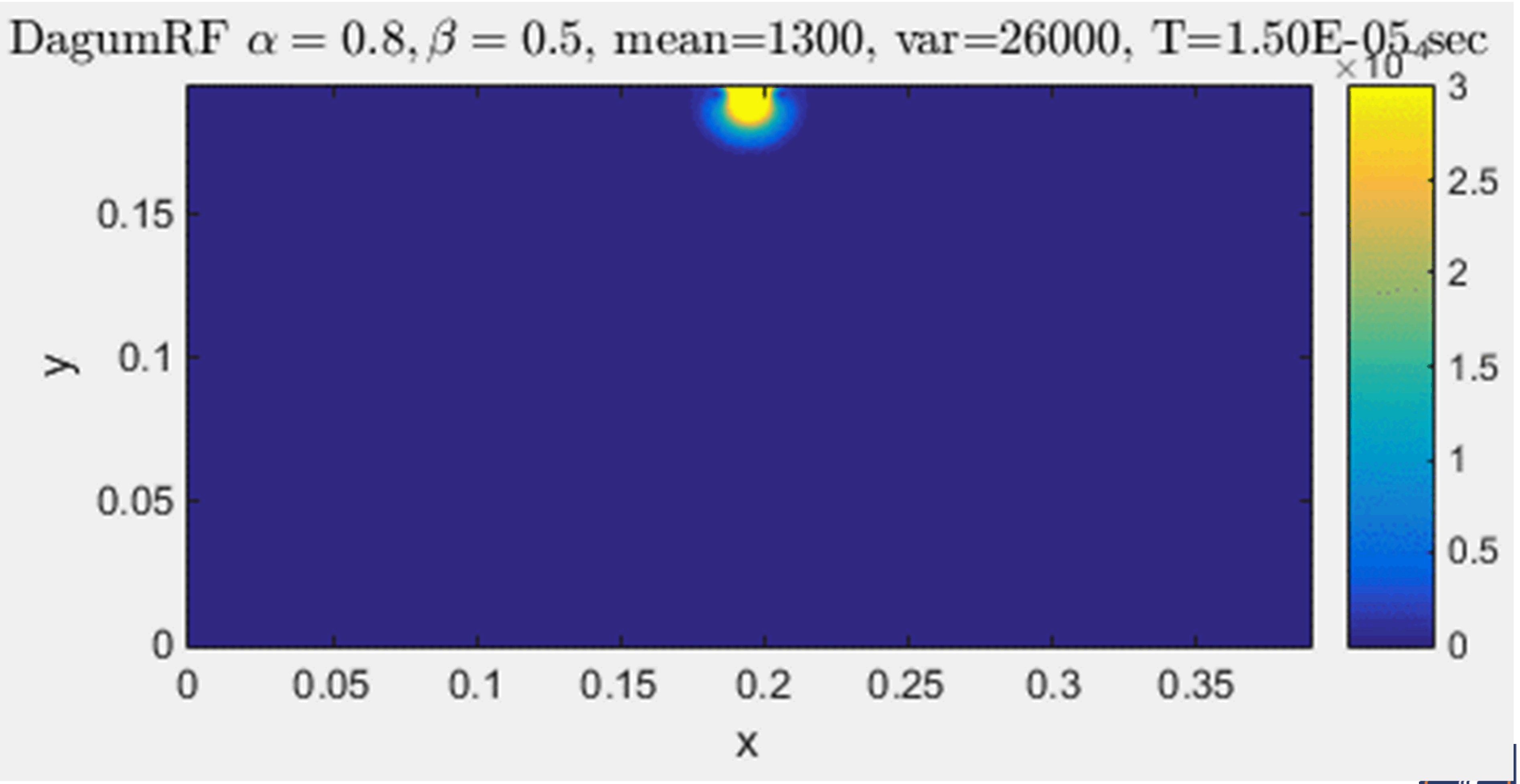
Variance = 26000, std dev = 161.2

Results – PD Video



Plot of displacement magnitude

Conclusions

- Homogeneous
  - Theoretical, CA and PD results follow very well
  - Neither method follows experimental results
    - Input excitation may not reflect explosive input
    - Simulation is 2D
    - Dissipation and friction
- Heterogeneous Mass-Density Field
  - Negligible difference between CA and PD results
    - Unexpected!
  - Small β (High H), shows the strongest deviation from homogeneous result

This work was partially supported by the NSF under grants CMMI-1030940 and IIP-1362146

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QUESTIONS

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BACKUP SLIDES

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Definitions of Peridynamics

- Undeformed position: $\boldsymbol{x}$, Displacements: $\boldsymbol{u}$, Deformed position $\boldsymbol{y}$.

$$\boldsymbol{y}(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x}, t) + \boldsymbol{x}$$

- Stretch between $\boldsymbol{x}$ and $\boldsymbol{x}'$:

$$s(\boldsymbol{u}' - \boldsymbol{u}, \boldsymbol{x}' - \boldsymbol{x}) = \frac{|\boldsymbol{y}' - \boldsymbol{y}| - |\boldsymbol{x}' - \boldsymbol{x}|}{|\boldsymbol{x}' - \boldsymbol{x}|}$$

- Force response function:

$$\boldsymbol{f}(\boldsymbol{u}' - \boldsymbol{u}, \boldsymbol{x}' - \boldsymbol{x})$$

  - The force that $\boldsymbol{x}'$ exerts on $\boldsymbol{x}$ per unit volume squared.

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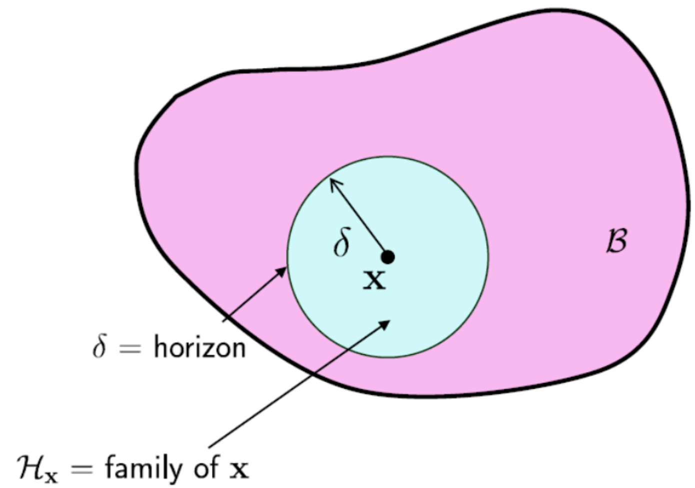
Peridynamic Horizon

- Non-locality in PD is defined by the ‘horizon’
- Horizon – Defines the sphere of influence around a particle.
- Why is non-locality important?
  - Significant at small scales
  - Predicts finite stress at crack tip (Eringen 1974a, b)

$$\boldsymbol{\sigma}(\boldsymbol{x}) = \int_V A(\boldsymbol{x}, \boldsymbol{x}')\boldsymbol{C}:\boldsymbol{\varepsilon}(\boldsymbol{x}')\,dV'$$



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Balance Laws

- Equation of Motion

$$\rho(\boldsymbol{x})\ddot{\boldsymbol{u}}(\boldsymbol{x},t) = \int_{H_x} \boldsymbol{f}(\boldsymbol{u}' - \boldsymbol{u}, \boldsymbol{x}' - \boldsymbol{x}, t)\, dV + \boldsymbol{b}(\boldsymbol{x},t)$$

  - Newton's Laws

$$\boldsymbol{f}(\boldsymbol{u}' - \boldsymbol{u}, \boldsymbol{x}' - \boldsymbol{x}) = -\boldsymbol{f}(\boldsymbol{u} - \boldsymbol{u}', \boldsymbol{x} - \boldsymbol{x}')$$

- Balance of Linear Momentum

$$\int_{\boldsymbol{V}}\int_{\boldsymbol{V}} \boldsymbol{f}(\boldsymbol{u}' - \boldsymbol{u}, \boldsymbol{x}' - \boldsymbol{x})\, \boldsymbol{dV'}\, \boldsymbol{dV} = \int_{\boldsymbol{V}}\int_{\boldsymbol{V}} \boldsymbol{f}(\boldsymbol{u} - \boldsymbol{u}', \boldsymbol{x} - \boldsymbol{x}')\, \boldsymbol{dV}\, \boldsymbol{dV'}$$

  - Net internal force

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Balance Laws – cont.

- Balance of Angular Momentum

$$(\boldsymbol{y}' - \boldsymbol{y}) \times \boldsymbol{f}(\boldsymbol{u}' - \boldsymbol{u}, \boldsymbol{x}' - \boldsymbol{x}) = \mathbf{0}$$

  - $(\boldsymbol{y}' - \boldsymbol{y})$ and $\mathbf{f}(\boldsymbol{u}' - \boldsymbol{u}, \boldsymbol{x}' - \boldsymbol{x})$ must be parallel

- Pairwise response function:

$$\boldsymbol{f}(\boldsymbol{u}' - \boldsymbol{u}, \boldsymbol{x}' - \boldsymbol{x}) = c \cdot s(\boldsymbol{u}' - \boldsymbol{u}, \boldsymbol{x}' - \boldsymbol{x}) \frac{\boldsymbol{y}' - \boldsymbol{y}}{|\boldsymbol{y}' - \boldsymbol{y}|}$$

  - c is called the “bond constant”
  - s is stretch

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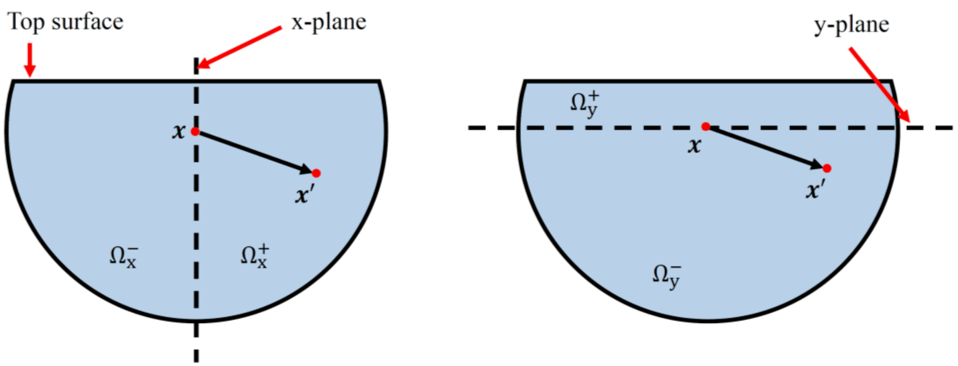
Surface Correction

- Reduced number of bonds for particles near surface
- Normalize strain-energy
- Equation of motion becomes

$$\rho(\boldsymbol{x})\ddot{\boldsymbol{u}}(\boldsymbol{x},t) = \int_{H_x} g * \boldsymbol{f}(\boldsymbol{u}' - \boldsymbol{u}, \boldsymbol{x}' - \boldsymbol{x}, t)\, dV + \boldsymbol{b}(\boldsymbol{x},t)$$

  - Correction factor is denoted by g



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