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An Approach to Peridynamic Theory for Fractal Media

Paul N. Demmie[1]
Sandia National Laboratories
Albuquerque, New Mexico
pndemmi@sandia.gov

Martin Ostoja-Starzewski
University of Illinois
Urbana-Champagne, IL
martinos@illinois.edu

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Background

- Benoît Mandelbrot introduced the term fractal to denote an object that is broken or fractured in space or time.
  - However, there is no universally accepted definition of a fractal.
- Fractals provide models for many media over some finite range of length scales with lower and upper cutoffs.
- Tarasov introduced a field theory for fractal media [1-3].
  - He developed continuum-type equations of conservation of mass, linear and angular momentum, and energy for fractals, and studied several fluid mechanics and wave problems.
  - Tarasov's approach relies on dimensional regularization of fractal objects through fractional integrals in Euclidean space, a technique with its roots in quantum mechanics.

[1] Tarasov, V.E.: Continuous medium model for fractal media. *Phys. Lett. A* **336,** 167–174 (2005)
[2] Tarasov, V.E.: Fractional hydrodynamic equations for fractal media. *Ann. Phys.* **318**(2), 286–307 (2005)
[3] Tarasov, V.E.: Wave equation for fractal solid string. *Mod. Phys. Lett. B* **19**15), 721–728 (2005)

Background

- Whereas the formulation of Tarasov is based on the Riesz measure, and thus more suited to isotropic fractal media, a model that is based on a product measure was introduced by Ostoja-Starzewski and Li [4,5].
  - This measure has different fractal dimensions in different directions.
  - It grasps the anisotropy of fractal geometry better than the original formulation for a range of length scales between the lower and upper cutoffs.
- Demmie and Ostoja-Starzewski developed conservation equations for fractal media including an energy equation [6] and applied it to a study of waves in fractal media.
  - This formulation is the basis of our approach to peridynamic theory for fractal media.

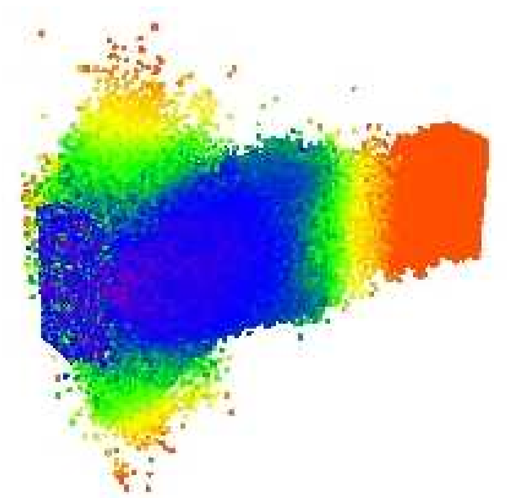
[4] Ostoja-Starzewski, M. and J. Li: Fractal materials, beams and fracture mechanics. *Z. Angew. Math. Phys*. **60**, 1–12 (2009)

[5] Li, J. and Ostoja-Starzewski, M.: Fractal solids, product measures and fractional wave equations. *Proc. R. Soc. A* **465**, 2521–2536 (2009) doi:10.1098/rspa.2009.0101. Errata (2010) doi:10.1098/rspa.2010.0491

[6] Demmie, P. N. and Ostoja-Starzewski, M., "Waves in Fractal Media", *Journal of Elasticity*, 104(1-2), 187–204, 2011

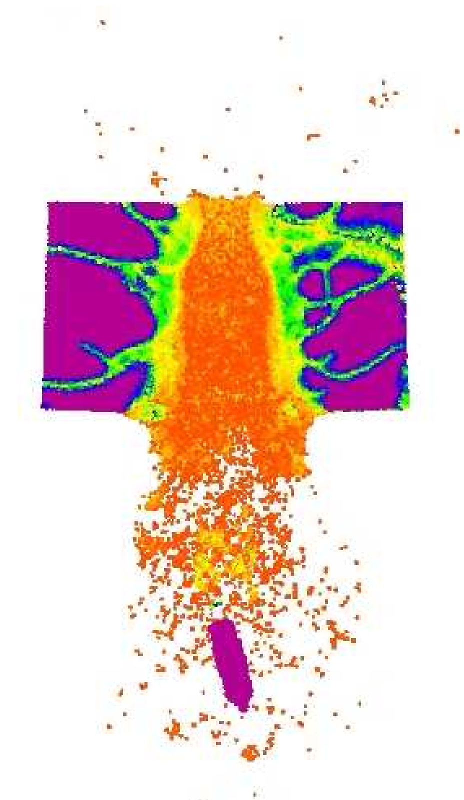
What is Peridynamic Theory?

- ***Peridynamic theory*** **(PD)** is a theory of continuum mechanics that uses integro-differential equations without spatial derivatives rather than partial differential equations.
  - Bond-Based Peridynamics [7]
  - State-Based Peridynamics [8]



Peridynamic means "near force".

1. PD is a reformulation of continuum mechanics that applies everywhere regardless of discontinuities.
2. PD provides a consistent formulation of both deformation and failure of materials.

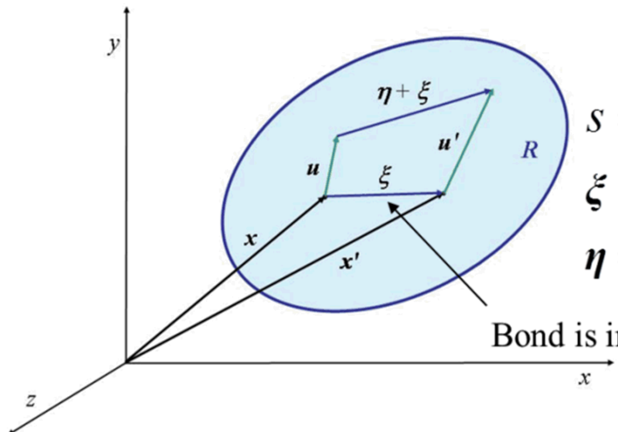


[7] Silling, "Reformulation of elasticity theory for discontinuities and long-range forces", in *Journal of the Mechanics and Physics of Solids*, 48 (2000) , pp. 175-209.

[8] S..A. Silling et al. "Peridynamic States and Constitutive Modeling", in *J Elasticity*, 88 (2007), pp. 151–184.

Bond-Based Peridynamic Theory

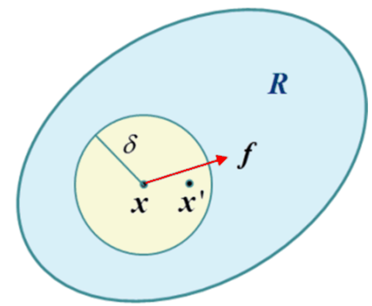
- In bond-based peridynamics the force state at a point is given by a functional over the pairwise interactions with all other points in the continuum.



$$s = \left(|\boldsymbol{\eta} + \boldsymbol{\xi}| - |\boldsymbol{\xi}|\right) / |\boldsymbol{\xi}|$$

$$\boldsymbol{\xi} = \boldsymbol{x}' - \boldsymbol{x}$$

$$\boldsymbol{\eta} = \boldsymbol{u}(\boldsymbol{x}', t) - \boldsymbol{u}(\boldsymbol{x}, t)$$



$$\rho(\boldsymbol{x}) \, \frac{d^2}{dt^2} \, \boldsymbol{u}(\boldsymbol{x}, t) = \iiint_{\boldsymbol{R}} \boldsymbol{f}(\boldsymbol{\eta}, \boldsymbol{\xi}) dV' + \boldsymbol{b}(\boldsymbol{x}, t)$$

where

***x*, *x*´** are position vectors,
t is the time,
ρ (***x***) is the density at ***x***,
u is the displacement vector,
R is the domain of the body,
f is the pairwise force function (*PFF*),
η and ***ξ*** are defined in the figure,
b is the body-force density, and
δ is the horizon.

Material Modeling in Peridynamics

- The integrand in the fundamental equation of peridynamics, ***f***, is called the ***pairwise force function*** (*PFF*) and is a force per unit volume squared between particles located at two points.
  - Peridynamic interaction between two points is called a ***bond***.
  - Constitutive properties of materials are given by specifying the *PFF*.
  - Bond properties are derivable from measured material properties.
- Material Failure
  - A bond fails when its stretch exceeds an input value called the ***critical stretch***.
- Newton's third law and conservation of angular momentum imply that the *PFF* is of the form

$$\boldsymbol{f}(\boldsymbol{\eta}, \boldsymbol{\xi}) = F(\boldsymbol{\eta}, \boldsymbol{\xi})(\boldsymbol{\eta} + \boldsymbol{\xi}) \text{ for all } \boldsymbol{\eta}, \boldsymbol{\xi}$$

where the scalar function such that $F(-\boldsymbol{\eta},-\boldsymbol{\xi}) = F(\boldsymbol{\eta},\boldsymbol{\xi})$.

Mass Power Law and Fractional Integrals

- By a *fractal medium*, we mean a medium with a pre-fractal geometric structure.
  - A pre-fractal geometry is fractal-like at some scale with cutoffs.
- In order to deal with general anisotropic, fractal media, we use a power law relation with respect to each coordinate and the mass is specified via a product measure as

$$m(W) = \oiiint_W \rho(x_1, x_2, x_3) dl_1(x_1)\, dl_2(x_2)\, dl_3(x_3)$$

- The length measure in each coordinate is provided using the transformation coefficients

$$dl_\mu(x_\mu) = c_1^{(\mu)}(\alpha_\mu, x_\mu) dx_\mu, \quad \mu = 1, 2, 3 \quad \text{(no sum)}$$

Transformation Coefficients

- We adopt the modified Riemann-Liouville fractional integral of Jumarie [8,9] for the transformation coefficients

$$c_1^{(\mu)} = \alpha_\mu \left(\frac{l_\mu - x_\mu}{l_{\mu 0}}\right)^{\alpha_\mu - 1}, \qquad \mu = 1, 2, 3 \text{ (no sum)}$$

where l_μ is the total length (integral interval) along x_μ and $l_{\mu 0}$ is the characteristic length in the given direction, like the mean pore size.

- In the product-measure formulation, the resolution length scale is given by

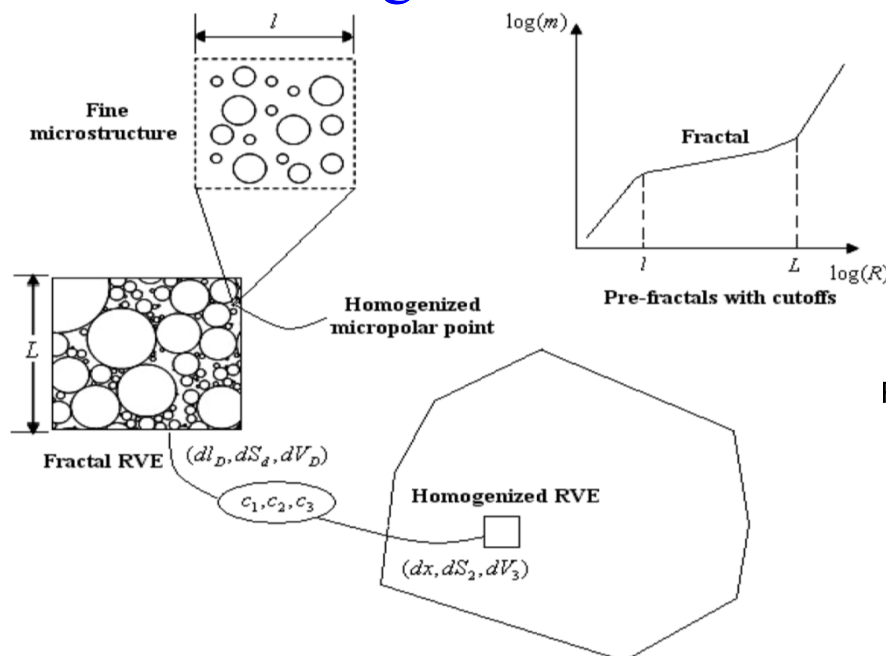
$$R = \sqrt{l_\mu l_\mu}$$

[8] Jumarie, G.: On the representation of fractional Brownian motion as an integral with respect to *(dt)a* . *Appl. Math. Lett*. **18**, 739–748 (2005)

[9] Jumarie, G.: Table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions. *Appl. Math. Lett.* **22**(3), 378–385 (2009)

Continuum Mechanics for Fractal Media

- We extend continuum thermomechanics to a fractal medium characterized by a mass (or spatial) fractal dimension, a surface fractal dimension, and a resolution length scale.



RVE (representative volume element)

- The continuum theory is based on dimensional regularization, in which global mass, momentum, and energy laws employ fractional integrals.

Homogenization Process for Fractal Media

$$dV_D = c_3(\boldsymbol{x})dV_3 \qquad n_\mu dS_D = c_\mu^2(x_\upsilon, x_\sigma)dS_2$$

$$c_3(\boldsymbol{x}) = c_\mu^1(x_\mu)\, c_\mu^2(x_\upsilon, x_\sigma) \qquad \mu \neq \upsilon, \mu \neq \sigma, \upsilon \neq \sigma$$

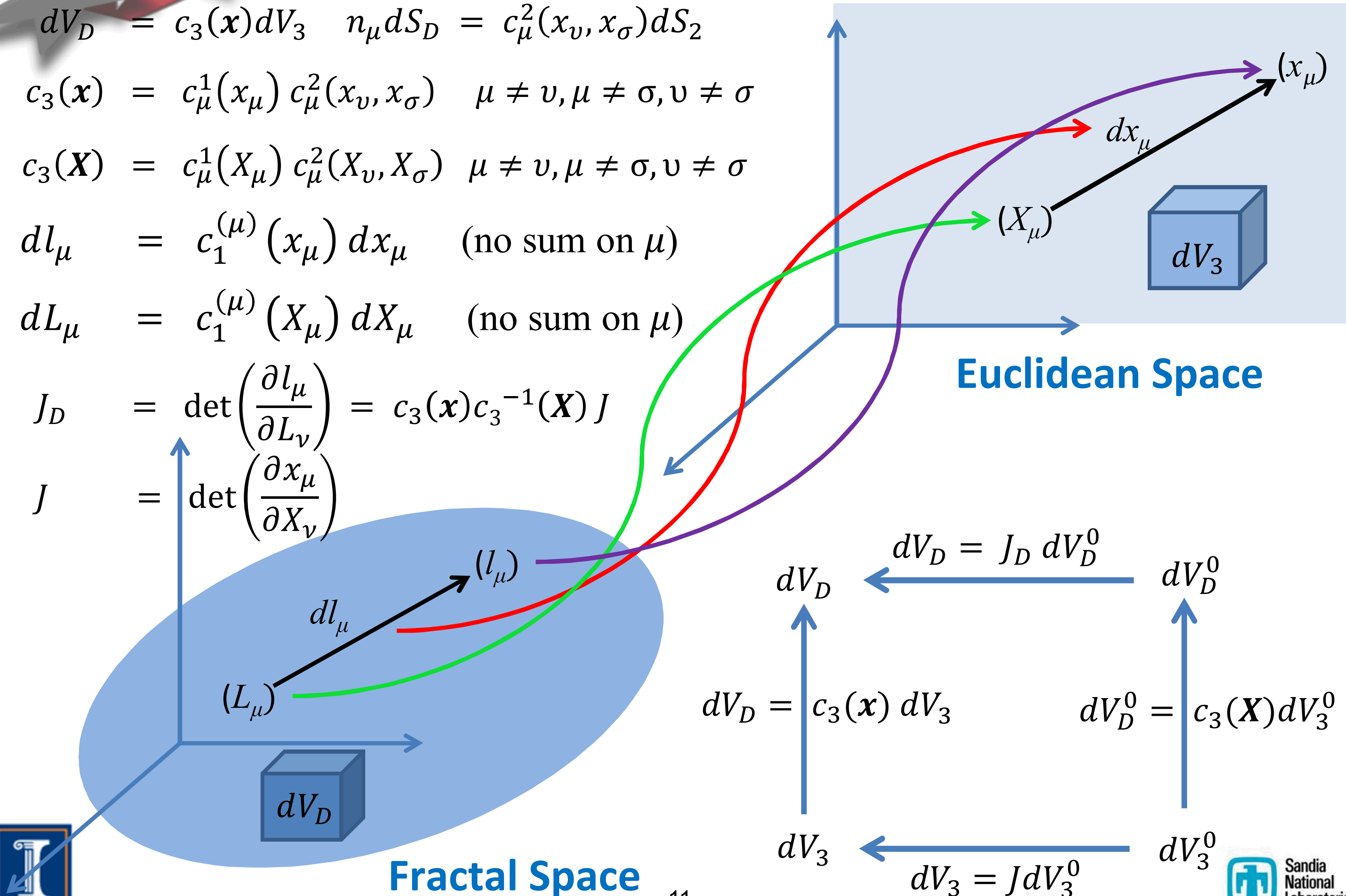
$$c_3(\boldsymbol{X}) = c_\mu^1(X_\mu)\, c_\mu^2(X_\upsilon, X_\sigma) \qquad \mu \neq \upsilon, \mu \neq \sigma, \upsilon \neq \sigma$$

$$dl_\mu = c_1^{(\mu)}(x_\mu)\, dx_\mu \qquad \text{(no sum on } \mu)$$

$$dL_\mu = c_1^{(\mu)}(X_\mu)\, dX_\mu \qquad \text{(no sum on } \mu)$$

$$J_D = \det\left(\frac{\partial l_\mu}{\partial L_\nu}\right) = c_3(\boldsymbol{x})c_3^{-1}(\boldsymbol{X})\, J$$

$$J = \det\left(\frac{\partial x_\mu}{\partial X_\nu}\right)$$



Velocities and Time Derivatives

- From the expression,

$$dl_\mu = c_1^{(\mu)}(x_\mu)\, dx_\mu \quad \text{(no sum on } \mu\text{)}$$

we get the following relationship between the velocities in fractal and Euclidean spaces,

$$v_\mu^D = \frac{dl_\mu}{dt} = c_1^{(\mu)}(x_\mu)\frac{dx_\mu}{dt} = c_1^{(\mu)}(x_\mu)\, v_\mu \quad \text{(no sum on } \mu\text{)}$$

- In terms of the fractal velocity and derivative,

$$\frac{dP}{dt} = \frac{\partial P}{\partial t} + v_\mu \frac{\partial P}{\partial x_\mu} = \frac{\partial P}{\partial t} + v_\mu^D \nabla_\mu^D P = \left(\frac{dP}{dt}\right)_D$$

Fractional Integral Theorems and Fractal Derivatives

- Using the conventional Gauss theorem and noting that each $c_2^{(k)}$ does not depend on the coordinate x_k, we obtain

$$\oiint_{\partial W} \boldsymbol{f} \cdot \boldsymbol{n} \; dS_D = \iiint_W \left[f k c_2^{(\mu)}\right]_{,\mu} dV_3 = \iiint_W \frac{f_{\mu,\mu}}{c_1^{(\mu)}} dV_D$$

- Based on this expression, we define the ***Fractal Derivative***, ∇_μ^D *as*

$$\nabla_\mu^D = \frac{1}{c_1^{(\mu)}(x_\mu)} \frac{\partial}{\partial x_\mu} = \frac{\partial x_\mu}{\partial l_\mu} \frac{\partial}{\partial x_\mu} = \frac{\partial}{\partial l_\mu} \quad (no\ sum)$$

- Based on this definition, ***Gauss Theorem for Fractal Media*** becomes

$$\oiint_{\partial W} \boldsymbol{f} \cdot \boldsymbol{n} \; dS_D = \iiint_W \left(\nabla_\mu^D f_\mu\right) dV_D = \iiint_W \left(\boldsymbol{\nabla}^D \cdot \boldsymbol{f}\right) dV_D$$

Fractional Integral Theorems and Fractal Derivatives

- We generalize the Reynold's Transport Theorem as follows:

$$\frac{d}{dt}\oiiint_W P\,dV_D = \frac{d}{dt}\oiiint_W PJ_D\,dV_D^0 = \oiiint_W \frac{d}{dt}(PJ_D)\,dV_D^0 =$$

$$\oiiint_W \left[\left(\frac{dP}{dt}\right)J_D + P\left(\frac{dJ_D}{dt}\right)\right]dV_D^0 = \oiiint_W \left[\left(\frac{\partial P}{\partial t} + \nabla_\mu^D\left(v_\mu^D P\right)\right)\right]J_D\,dV_D^0 =$$

$$\oiiint_W \left[\frac{\partial P}{\partial t} + \nabla_\mu^D\left(v_\mu^D P\right)\right]dV_D$$

- The ***Reynold's Transport Theorem for Fractal Media*** is:

$$\frac{d}{dt}\oiiint_W P\,dV_D = \oiiint_W \left[\frac{\partial P}{\partial t} + \nabla_\mu^D\left(v_\mu^D P\right)\right]dV_D = \oiiint_W \left(\frac{dP}{dt}\right)_D dV_D$$

Continuum Mechanics for Fractal Media

- We specify the relationship between surface force, $\boldsymbol{F}^S$, and the Cauchy stress tensor, $\boldsymbol{\sigma}_{\mu\nu}$, using fractional integrals as

$$F_\mu{}^S = \iint_{S_D} \sigma_{\mu\nu} n_\nu \, dS_D = \int_{S_2} \sigma_{\mu\nu} n_\nu \, c_2{}^{(\nu)} \, dS_2$$

- For small deformations, we define the strain, $\boldsymbol{\varepsilon}_{\mu\nu}$, in terms of the displacement $\boldsymbol{u}$ as

$$\varepsilon_{\mu\nu} = \frac{1}{2}\left(\nabla_\nu{}^D u_\mu + \nabla_\mu{}^D u_\nu\right) = \frac{1}{2}\left[\frac{1}{c_1{}^{(j)}} u_\mu + \frac{1}{c_1{}^{(i)}} u_\nu\right]$$

- The fractal equations for continuity, momentum, and energy follow from the balance laws for mass, momentum, and energy.

Fractal Continuity Equation

- Consider the equation for conservation of mass in a region W

$$\frac{d}{dt}\iiint_W \rho\, dV_D = 0$$

- Using the fractional Reynolds transport theorem, we obtain

$$\frac{d}{dt}\iiint_W \rho\, dV_D = \iiint_W \left[\frac{\partial\rho}{\partial t} + \nabla_\mu^D\left(\rho v_\mu^D\right)\right] dV_D = 0$$

- Since W is arbitrary, the ***Fractal Continuity Equation*** is

$$\frac{\partial\rho}{\partial t} + \nabla_\mu^D\left(\rho v_\mu^D\right) = 0 \quad or \quad \frac{d\rho}{dt} + \rho\nabla_\mu^D\left(v_\mu^D\right) = 0$$

Fractal Momentum Equation

- Consider the balance law of linear momentum for W, with $\boldsymbol{F}^B$ the body force, and $\boldsymbol{F}^S$ the surface force,

$$\frac{d}{dt} \oiiint_W \rho v_\mu^D \, dV_D = \boldsymbol{F}^B + \boldsymbol{F}^S$$

- Using Reynold's transport and Gauss theorems and the continuity equation, we obtain

$$\frac{d}{dt} \oiiint_W \rho v_\mu^D \, dV_D = \oiiint_W \rho \frac{d v_\mu{}^D}{dt} dV_D = \oiiint_W b_\mu dV_D + \oiint_{\partial W} \sigma_{\mu\nu} n_\mu dS_D =$$

$$\oiiint_W b_\mu dV_D + \oiiint_W \left(\nabla_\nu{}^D \sigma_{\mu\nu}\right) dV_D$$

- Since W is arbitrary, the ***Fractal Linear Momentum Equation*** is

$$\rho \frac{d v_\mu^D}{dt} = b_\mu + \nabla_\mu^D \sigma_{\mu\nu}$$

Fractal Energy Equation

- The most general form of balance (conservation) of energy must be used to obtain the fractal energy equation.
  - *The time rate of change of the kinetic energy K plus the internal energy E of a region, W, in a continuum equals the sum of the rate of work performed on W by external agencies plus the flux Q of all other energies supplied to or removed from W by external agencies across the boundary of W.*
- In terms of the kinetic energy K, specific internal energy e, and flux per unit area $\boldsymbol{q}$,

$$K = \oiiint_W \rho v_\mu{}^D v_\mu{}^D dV_D \,, \quad E = \oiiint_W \rho e dV_D \,, \quad Q = -\iint_{\partial W} q_\mu n_\mu dS_D$$

and using the fractal momentum equation, we obtain

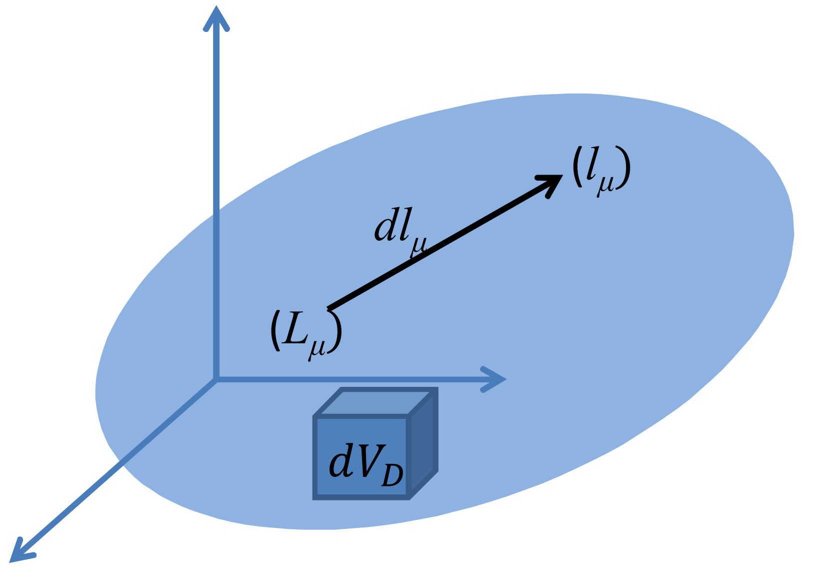
$$\oiiint_W \rho \frac{de}{dt} dV_D = \oiiint_W \sigma_{\mu\nu} (\nabla_\mu{}^D v_\nu) dV_D - \oiiint_W (\nabla_\mu{}^D q_\nu) dV_D$$

- Since W is arbitrary, the ***Fractal Energy Equation*** is

$$\rho \frac{de}{dt} = \sigma_{\mu\nu} \left(\nabla_\mu^D v_\nu^D \right) - \nabla_\mu^D q_\mu$$

Peridynamic Theory for Fractal Media

- We obtain the bond-based peridynamics momentum equation by replacing the divergence term in the fractal momentum equation by the regularized integral from the fundamental equation of peridynamics.



$$\rho \frac{d\boldsymbol{v}^D}{dt} = \iiint_{\boldsymbol{R}} \boldsymbol{f}^{\boldsymbol{D}}(\boldsymbol{\xi}^D, \boldsymbol{\eta}^D) dV_D{}' + \boldsymbol{b}$$

Summary and Conclusions

- We extended continuum thermomechanics to fractal media.
  - The continuum theory is based on dimensional regularization, in which we employ fractional integrals to state global balance laws.
  - We derived fractal continuity, linear momentum, and energy equations, which through dimensional regularization can be cast into equations in E^3.
- We extended this continuum thermomechanics to peridynamic theory for fractal media.
- Future work includes implementation of this extension in a computer code, specification of appropriate force states, and failure predictions for fractal media subject to high-impulse loading.

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