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Exergy and Irreversible Entropy Production Thermodynamic Concepts for Control Design: Nonlinear Regulator Systems

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***Abstract*— This paper[1] develops a novel control system design methodology that uniquely combines: concepts from thermodynamic exergy and entropy; Hamiltonian systems; Lyapunov's direct method and Lyapunov optimal analysis; electric AC power concepts; and power flow analysis. Relationships are derived between exergy/entropy and Lyapunov optimal functions for Hamiltonian systems. The methodology is demonstrated with two fundamental numerical simulation examples: 1) a Duffing oscillator/Coulomb friction nonlinear model that employs PID regulator control and 2) a van der Pol nonlinear oscillator system. The control system performances and/or appropriately identified terms are partitioned and evaluated based on exergy generation and exergy dissipation terms. This novel nonlinear control methodology results in both necessary and sufficient conditions for stability of nonlinear systems.**

I. Introduction

Today's engineering systems sustain desirable performance by using well-designed control systems based on fundamental principles and mathematics. Many engineering breakthroughs and improvements in sensing and computation have helped to advance the field. Control systems currently play critical roles in many areas, including automation, manufacturing, electronics, communications, transportation, computers, and networks, as well as many commercial and military systems [1]. Traditionally, almost all modern control design is based on forcing the nonlinear systems to perform and behave like linear systems, thus limiting its maximum potential. In this paper a novel nonlinear control design methodology is introduced that overcomes this limitation.

Several of the popular advanced nonlinear control system approaches are based in passivity and dissipative control theories. Initially, Moylan [2] discussed the implications of passivity for a broad class of nonlinear systems, a connection is established between the input-output property of passivity and a set of constraints on the state equations for the system. Later, Wyatt, et.al. [3], [4] clarified the meaning of passivity and losslessness as understood in nonlinear circuit theory, and their counterparts in classical physics. Most recently, Ortega, Jiang, and Hill [5] reviewed recent results on the stabilization of nonlinear systems using a passivity approach. Passivity properties play a vital role in designing asymptotically stabilizing controllers for nonlinear systems where the nonlinear versions of the Kalman-Yacubovitch-Popov lemma are used as key testing tools. The dissipative characteristics of dynamical systems has its origins in work by Willems [6] with further specifics given by Hill and Moylan [7]. In [7], a technique is introduced for generating Lyapunov functions for a broad class of nonlinear systems represented by state equations. The system, for which a Lyapunov function is required, is assumed to have a property called dissipativeness. In other words, the system absorbs more energy from the external world than it supplies. Different types of dissipativeness can be considered depending on how the "power input" is selected. Dissipativeness is shown to be characterized by the existence of a computable function which can be interpreted as the "stored energy" of the system. Under certain conditions, this energy function is a Lyapunov function which establishes stability, and in some cases asymptotic stability, of the isolated system. It was shown that for a certain class of nonlinear systems, that an "energy" approach was useful in analyzing stability. Kokotovic and Arcak [8] provide a recent discussion about the historical perspective of constructive nonlinear control theories. Structural properties of nonlinear systems and passivation-based designs exploit the connections between passivity and inverse optimality, and between Lyapunov functions and optimal value functions. Recursive design procedures, such as backstepping and forwarding, achieve certain optimal properties for important classes of nonlinear systems. Some of the more popular nonlinear control system designs [9], [10], [11] have their fundamental foundations built upon these concepts.

In other engineering disciplines, Alonso and Ydstie [12] connect thermodynamics and the passivity theory of nonlinear control. The storage function is derived from the convexity of the entropy and is closely related to thermodynamic avail-

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ability. Dissipation is related to positive entropy production. In this form the supply function is a product of force and flow variation variables. Results are discussed in relationship to heat conduction and reaction diffusion equation problems. Anthony [13] suggests that non-equilibrium thermodynamics of irreversible processes may be included into the framework of a Lagrangian formalism. This formalism presents a unified method for reversible and irreversible processes. A straightforward procedure allows for the incorporation of both the first and second laws of thermodynamics into the Lagrangian. The theory is illustrated in three representative examples which include; material flow, heat conduction, diffusion and chemical reactions.

The main contribution of this paper is to present a novel nonlinear control design methodology that is based on thermodynamic exergy and irreversible entropy production concepts. Relationships are developed between exergy, irreversible entropy production, Hamiltonian systems, Lyapunov optimal functions, electric AC power concepts, and power flow, for control system design. Both necessary and sufficient conditions for stability are determined for nonlinear systems. By combining the first and second laws of thermodynamics, an exergy analysis approach is developed to construct Lyapunov optimal functions for Hamiltonian systems. The first time derivative of the Lyapunov functions, based on exergy, irreversible entropy production rate, and power flow is partitioned into either exergy dissipative or exergy generative terms.

This paper is divided into eight sections. Sections II and III provide the preliminary thermodynamics and Hamiltonian mechanics definitions. Section IV develops the relationships and connections between thermodynamics and Hamiltonian mechanics. Section V defines the necessary and sufficient conditions for stability of nonlinear systems. Section VI shows how, with simplifications to this novel control theory, conventional Lyapunov optimal and passivity control design methodologies are recovered. Section VII presents regulator control design examples that include; 1) a PID control regulator for a nonlinear Duffing oscillator/Coulomb friction dynamic system and 2) a van der Pol nonlinear oscillator system. Numerical simulations resulted in the demonstration of both performance and stability criteria. Finally, section VIII summarizes the results with concluding remarks.

II. Thermodynamic Concepts

In this section the first and second laws of thermodynamics are used to define exergy. One interpretation of the first law of thermodynamics states energy is conserved (see Fig. 1-left). The second law of thermodynamics implies that the entropy of the universe always increases. The first law is a conservation equation while the second law is an inequality. Mathematically, a result of the first law can be written in terms of it's time derivatives or energy rate for a system [14] as

$$\dot{\mathcal{E}} = \sum_i \dot{\mathcal{Q}}_i + \sum_j \dot{\mathcal{W}}_j + \sum_k \dot{m}_k \left(h_k + ke_k + pe_k + \ldots \right). \quad (1)$$

The term on the left represents the rate at which energy is changing within the system. The heat entering or leaving the system is given by $\dot{\mathcal{Q}}_i$ and the work entering or leaving the system is given by $\dot{\mathcal{W}}_j$. Next, material can enter or leave the system by $\dot{m}_k$ that includes enthalpy, h, kinetic and potential energies, ke, pe, etc. In addition, each term is "summed" over an arbitrary number of entry and exit locations i, j, k.

The second law or entropy rate equation for a system [14] is given as

$$\dot{\mathcal{S}} = \sum_i \frac{\dot{\mathcal{Q}}_i}{T_i} + \sum_k \dot{m}_k s_k + \dot{\mathcal{S}}_i = \dot{\mathcal{S}}_e + \dot{\mathcal{S}}_i. \quad (2)$$

Where the left hand term is the rate entropy changes within the system and the right hand terms represent, in order, the rate heat conducts entropy to and from the system and the rate material carries it in or out. These two terms can be combined into one term $\dot{\mathcal{S}}_e$, the entropy exchanged (either positive or negative) with the environment and $\dot{\mathcal{S}}_i$ is the irreversible entropy production rate within the system. Figure 1 (middle) shows the entropy exchanges and production within the system [15].

The irreversible entropy production rate can be written as the sum of the thermodynamic forces and the thermodynamic flows [15], [16]

$$\dot{\mathcal{S}}_i = \sum_k \mathcal{F}_k \dot{\mathcal{X}}_k \geq 0 \quad (3)$$

where the entropy change is the sum of all the changes due to the irreversible flows $\dot{\mathcal{X}}_k$ with respect to each corresponding thermodynamic force $\mathcal{F}_k$.

Next, for systems with a constant environmental temperature, a thermodynamic quantity called the availability function which has the same form as the Helmholtz free energy function is defined as [15]

$$\Xi = \mathcal{E} - T_o \mathcal{S} \quad (4)$$

where T_o is the reference environmental temperature. The availability function is described as the maximum theoretically available energy that can do work which we call exergy. Exergy is also known as negative-entropy [14], [17]. By taking the time derivative of the availability function (4) and substituting in the expressions for (1) and (2) results in the exergy rate equation

$$\begin{aligned} \dot{\Xi} \;=\; & \sum_i \left(1 - \frac{T_o}{T_i}\right) \dot{\mathcal{Q}}_i \\ & + \sum_j \left(\dot{\mathcal{W}}_j - p_o \frac{d\bar{V}}{dt} \right) + \sum_k \dot{m}_k \zeta_k^{flow} - T_o \dot{\mathcal{S}}_i . \end{aligned} \quad (5)$$

Where $\dot{\Xi}$ is the rate at which exergy stored within the system is changing. The terms on the right, in order, define the rate exergy is carried in/out by; i) heat, ii) work (less any work the system does on the environment at constant environmental pressure p_o if the system volume $\bar{V}$ changes), and iii) by the material (or quantity known as flow exergy). The final term, $T_o \dot{\mathcal{S}}_i$, is the rate exergy is destroyed within the system.

III. Hamiltonian Mechanics

The derivation of the Hamiltonian [18] begins with the Lagrangian for a system defined as

$$\mathcal{L} = \mathcal{T}(q,\dot{q},t) - \mathcal{V}(q,t) \tag{6}$$

where
t = time explicitly
q = N-dimensional generalized coordinate vector
$\dot{q}$ = N-dimensional generalized velocity vector
$\mathcal{T}$ = Kinetic energy, and
$\mathcal{V}$ = Potential energy.

The Hamiltonian is defined in terms of the Lagrangian as

$$\mathcal{H} \equiv \sum_{i=1}^{n} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i - \mathcal{L}(q,\dot{q},t) = \mathcal{H}(q,\dot{q},t). \tag{7}$$

The Hamiltonian in terms of the canonical coordinates (q,p) is

$$\mathcal{H}(q,p,t) = \sum_{i=1}^{n} p_i\dot{q}_i - \mathcal{L}(q,\dot{q},t) \tag{8}$$

where the canonical momentum is defined as

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}. \tag{9}$$

Then Hamilton's canonical equations of motion become

$$\begin{aligned} \dot{q}_i &= \frac{\partial \mathcal{H}}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial \mathcal{H}}{\partial q_i} + Q_i \end{aligned} \tag{10}$$

where Q_i is the generalized force vector. Next taking the time derivative of (8) gives

$$\dot{\mathcal{H}} = \sum_{i=1}^{n}\left(\dot{p}_i\dot{q}_i + p_i\ddot{q}_i - \frac{\partial \mathcal{L}}{\partial t} - \frac{\partial L}{\partial q_i}\dot{q}_i - \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i\right). \tag{11}$$

Then substitute (10) into (11) and simplifying gives

$$\dot{\mathcal{H}} = \sum_{i=1}^{n} Q_i\dot{q}_i - \frac{\partial \mathcal{L}}{\partial t}. \tag{12}$$

Hamiltonians for most natural systems are not explicit functions of time (or $\partial\mathcal{L}/\partial t = 0$). Then for

$$\mathcal{L} = \mathcal{L}(q,\dot{q}) \tag{13}$$

the power (work/energy) equation becomes

$$\dot{\mathcal{H}}(q,p) = \sum_{i=1}^{n} Q_i\dot{q}_i. \tag{14}$$

IV. Thermo-Mechanical Relationships

A. Conservative Mechanical Systems

A system is conservative if

$$\dot{\mathcal{H}} = 0 \quad \text{and} \quad \mathcal{H} = \text{constant}.$$

A force is conservative if

$$\oint F\cdot dx = \oint F\cdot v dt = \oint Q_j\dot{q}_j dt = 0$$

where F is the force, dx the displacement, and v the velocity. Basically, all of the forces can be modeled as potential force fields which are storage devices.

B. Reversible Thermodynamic Systems

A thermodynamic system is reversible if

$$\begin{aligned} d\mathcal{S} &= \frac{d\mathcal{Q}}{T} \\ \oint d\mathcal{S} &= \oint \frac{d\mathcal{Q}}{T} = 0 \\ \oint d\mathcal{S} &= \oint\left[d\mathcal{S}_i + d\mathcal{S}_e\right] = \oint\left[\dot{\mathcal{S}}_i + \dot{\mathcal{S}}_e\right]dt = 0 \end{aligned}$$

which implies that $\dot{\mathcal{S}}_e = \dot{\mathcal{Q}}/T$ since by definition the second law gives $\dot{\mathcal{S}}_i = 0$.

C. Irreversible Thermodynamic Systems

For

$$\oint d\mathcal{S} = \oint\left[\dot{\mathcal{S}}_i + \dot{\mathcal{S}}_e\right]dt = 0$$

then $\dot{\mathcal{S}}_e \leq 0$ and $\dot{\mathcal{S}}_i \geq 0$.

D. Analogies and Connections

Now the connections between thermodynamics and Hamiltonian mechanics are investigated.

1) The irreversible entropy production rate can be expressed as

$$\dot{\mathcal{S}}_i = \sum_k \mathcal{F}_k\dot{\mathcal{X}}_k = \frac{1}{T_o}\sum_k Q_k\dot{q}_k \geq 0. \tag{15}$$

2) The time derivative of the Hamiltonian is equivalent to the exergy rate

$$\begin{aligned} \dot{\mathcal{H}} &= \textstyle\sum_k Q_k\dot{q}_k \\ \dot{\Xi} &= \dot{W} - T_o\dot{\mathcal{S}}_i = \textstyle\sum_{j=1}^{N} Q_j\dot{q}_j - \sum_{l=N}^{M-N} Q_l\dot{q}_l \end{aligned} \tag{16}$$

Where N is the number of generators, $M-N$ the number of dissipators, and let $\dot{W} = \sum_j \dot{\mathcal{W}}_j$. The following assumptions apply when utilizing the exergy rate equation (5) for *Hamiltonian systems*:

a) No substantial heat flow:

$$\dot{\mathcal{Q}}_i \approx 0.$$

b) No substantial exergy flow or assume T_i is only slightly greater than T_o:

$$1 - \frac{T_o}{T_i} \approx 0.$$

c) No $p_o\bar{V}$ work on the environment:

$$p_o\frac{d\bar{V}}{dt} = 0.$$

d) No mass flow rate:

$$\sum_k \dot{m}_k\zeta_k^{flow} = 0.$$

e) Then define:

$$\begin{aligned} \dot{W} &\geq 0 \quad \text{power input/generated} \\ T_o\dot{\mathcal{S}}_i &\geq 0 \quad \text{power dissipated.} \end{aligned}$$

3) A conservative system is equivalent to a reversible system when

$$\dot{\mathcal{H}} = 0 \quad \text{and} \quad \dot{\mathcal{S}}_e = 0$$

then

$$\dot{\mathcal{S}}_i = 0 \quad \text{and} \quad \dot{W} = 0.$$

4) For a system that "appears to be conservative", but is not reversible is defined as:

$$\begin{aligned} \dot{\mathcal{H}}_{ave} &= P_{ave}(\text{over a cycle}) = 0 \\ &= \tfrac{1}{\tau}\oint[\dot{W} - T_o\dot{\mathcal{S}}_i]dt \\ &= (\dot{W})_{ave} - (T_o\dot{\mathcal{S}}_i)_{ave} \\ &= \tfrac{1}{\tau}\oint[\textstyle\sum_j^N Q_j\dot{q}_j - \sum_{l=N}^{M-N} Q_l\dot{q}_l]dt \end{aligned}$$

where τ is the period of the cycle. To be more specific about the average power calculations, the AC power factor [19] provides an excellent example. For the general case of alternating current supplied to a complex impedance the voltage and current differ in phase by an angle θ. For

$$\begin{aligned} \dot{W} &= P = Q\dot{q} = v\,i = \sqrt{2}\bar{v}\cos(\omega t + \theta)\cdot\sqrt{2}\bar{i}\cos\omega t \\ &= \bar{v}\bar{i}\,[\cos\theta + \cos(2\omega t + \theta)] \end{aligned}$$

where P is power, v is voltage ($\bar{v}$), i is current ($\bar{i}$), θ is the phase angle, and ω is the frequency. Integrating over a cycle gives

$$(\dot{W})_{ave} = \bar{v}\bar{i}\cos\theta$$

where for the second term

$$\oint \cos(2\omega t + \theta)dt = 0.$$

This is an important set of conditions that will be used in the next section to find the generalized stability boundary.

5) Finally, the power terms are sorted into three categories:
   a) $(\dot{W})_{ave}$ - power generators; $(Q_j\dot{q}_j)_{ave} > 0$,
   b) $(T_o\dot{\mathcal{S}}_i)_{ave}$ - power dissipators; $(Q_l\dot{q}_l)_{ave} < 0$,
   c) $(T_o\dot{\mathcal{S}}_{rev})_{ave}$ - reversible/conservative exergy storage terms; $(Q_k\dot{q}_k)_{ave} = 0$.

These three categories are fundamental terms in the following design procedures.

V. Necessary and Sufficient Conditions for Stability

The Lyapunov function is defined as the total energy which for most mechanical systems is equivalent to an appropriate Hamiltonian function

$$V = \mathcal{H} \tag{17}$$

which is positive definite. The time derivative is

$$\begin{aligned} \dot{V} &= \dot{\mathcal{H}} = \textstyle\sum_k Q_k\dot{q}_k = \sum_j^N Q_j\dot{q}_j - \sum_{l=N}^{M-N} Q_l\dot{q}_l \\ &= \dot{W} - T_o\dot{\mathcal{S}}_i. \end{aligned} \tag{18}$$

A. Stability and Instability Theorems

To describe a nonlinear system's behavior two theorems [20] help to characterize the essential features of its motion. In addition, by bounding the Lyapunov function between these Theorems, both necessary and sufficient conditions are a result of the transition of the time derivative of the Lyapunov function from stable to unstable.

1) **Lyapunov Theorem for Stability** Assume that there exists a scalar function V of the state x, with continuous first order derivatives such that

$$\begin{aligned} &V(x) \quad \text{is positive definite} \\ &\dot{V}(x) \quad \text{is negative definite} \\ &V(x) \quad \to\infty \quad \text{as} \quad \|x\|\to\infty \end{aligned}$$

Then the equilibrium at the origin is globally asymptotically stable.

2) **Chetaev Theorem for Instability** Considering the equations of disturbed motion, let V be zero on the boundary of a region R which has the origin as a boundary point, and let both V and $\dot{V}$ be positive-definite in R; then the undisturbed motion is unstable at the origin.

B. Stability Lemma for Nonlinear Systems

Based on the relationship between thermodynamic exergy and Hamiltonian systems a fundamental stability Lemma can be formulated.

Fundamental Stability Lemma for Hamiltonian Systems The stability of Hamiltonian systems is bounded between Theorems 1 and 2. Given the Lyapunov derivative as a decomposition and sum of exergy generation rate and exergy dissipation rate then:

$$\dot{V} = \dot{W} - T_o\dot{\mathcal{S}}_i = \sum_{j=1}^{N} Q_j\dot{q}_j - \sum_{l=N}^{M-N} Q_l\dot{q}_l \tag{19}$$

that is subject to the following general necessary and sufficient conditions:

$$\begin{aligned} T_o\dot{\mathcal{S}}_i &\geq 0 \quad \text{Positive semi-definite, always true} \\ \dot{W} &\geq 0 \quad \text{Positive semi-definite; exergy pumped in.} \end{aligned}$$

The following corollaries encompass both stability and instability for Hamiltonian systems which utilize AC power concepts [19]:

Cor 1: For $(T_o\dot{\mathcal{S}}_i)_{ave} = 0$ and $(\dot{W})_{ave} = 0$ then $\dot{V} = 0$ the Hamiltonian system is neutrally stable, conservative and reversible.

Cor 2: For $(T_o\dot{\mathcal{S}}_i)_{ave} = 0$ and $(\dot{W})_{ave} > 0$ then $\dot{V} > 0$ the Hamiltonian system is unstable.

Cor 3: For $(T_o\dot{\mathcal{S}}_i)_{ave} > 0$ and $(\dot{W})_{ave} = 0$ then $\dot{V} < 0$ the Hamiltonian system is asymptotically stable and a passive system in the general sense (passivity controllers).

Cor 4: Given apriori $(T_o\dot{\mathcal{S}}_i)_{ave} > 0$ and $(\dot{W})_{ave} > 0$ then the Hamiltonian system is further subdivided into:

4.1: For $\left(T_o\dot{\mathcal{S}}_i\right)_{ave} > \left(\dot{W}\right)_{ave}$ with $\dot{V} < 0$ yields asymptotic stability.

4.2: For $\left(T_o\dot{\mathcal{S}}_i\right)_{ave} = \left(\dot{W}\right)_{ave}$ with $\dot{V} = 0$ yields neutral stability.

4.3: For $\left(T_o\dot{\mathcal{S}}_i\right)_{ave} < \left(\dot{W}\right)_{ave}$ with $\dot{V} > 0$ yields an unstable system.

The bottom line is that stability is defined in terms of power flow which determines whether the system is moving toward or away from its minimum energy and maximum entropy state.

VI. Lyapunov Optimal and Passivity Control

Present day robotic and aerospace applications use feedback controller designs that are *Lyapunov Optimal* [21]. A control law is *Lyapunov Optimal* if it minimizes the first time derivative of the Lyapunov function over a space of admissible controls. In general, a set of feedback gains are optimized by minimizing the regulating and/or tracking error of the feedback controller while regulating to zero and/or tracking a desired reference input. The Lyapunov function is the total error energy which for most mechanical systems is equivalent to an appropriate Hamiltonian function

$$V = \mathcal{H}. \tag{20}$$

Then the concept of Lyapunov Optimal [21] follows directly from setting $\dot{W} = 0$ in (19) and maximizing $T_o\dot{\mathcal{S}}_i$ for which the time derivative of the Lyapunov function (Hamiltonian) or the modified power (work/energy) equation is written as

$$\dot{V} = \dot{\mathcal{H}} = -T_o\dot{\mathcal{S}}_i = -\sum_{j=1}^{N} Q_j\dot{q}_j = -\sum_{j=1}^{N} F_j\dot{R}_j \tag{21}$$

which is independent of system dynamics and is a *kinematic quantity* that applies to any system. Note that F_j denotes a set of forces acting on a mechanical system and $\dot{R}_j$ denotes the inertial linear velocity of the point where F_j is applied.

Passivity control [9] for robotic systems follows directly from setting $\dot{W} = 0$ in (19).

VII. Regulator Control Design Examples

Two nonlinear dynamic systems are investigated to demonstrate exergy/entropy control design analogies for control design theory and to provide unique insights as well. These examples are based on 1) a PID regulator control for nonlinear Duffing oscillator/Coulomb friction dynamic system and 2) a van der Pol nonlinear system.

A. Duffing Oscillator/Coulomb Friction with PID Control System

This example is the design of a control law for a single degree of freedom nonlinear oscillator. The Duffing oscillator/Coulomb friction dynamic model (see Fig. 1 - right) is defined as

$$M\ddot{x} + C\dot{x} + C_{NL}\ \text{sign}(\dot{x}) + Kx + K_{NL}x^3 = u \tag{22}$$

where $M, C, K,$ and u are the mass, damper, stiffness coefficients and external force input terms, respectively. The nonlinear stiffness and Coulomb friction coefficients are K_{NL} and C_{NL}, respectively. The PID controller is defined as

$$u = -K_P x - K_I\int_0^t x d\tau - K_D\dot{x} \tag{23}$$

where K_P, K_I, and K_D are the proportional, integral and derivative controller gains, respectively.

Initially, the nonlinear Duffing oscillator is investigated as a neutrally stable, reversible conservative system or

$$M\ddot{x} + Kx + K_{NL}x^3 = -K_P x$$

subject to the initial condition $x(0) = x_o = 1.0$. Now apply exergy/entropy control design and the derivative of the Lyapunov function/Hamiltonian becomes

$$\dot{V} = \dot{\mathcal{H}} = \dot{W} - T_o\dot{\mathcal{S}}_i = \sum_{j=1}^{N} Q_j\dot{q}_j - \sum_{l=N}^{M-N} Q_l\dot{q}_l$$

which yields

$$\begin{aligned} T_o\dot{\mathcal{S}}_i &= 0 \\ \dot{W} &= 0 \\ (T_o\dot{\mathcal{S}}_{rev})_{ave} &= (M\ddot{x}\cdot\dot{x} + (K+K_P)x\cdot\dot{x} + K_{NL}x^3\cdot\dot{x})_{ave} \\ &= 0. \end{aligned}$$

Numerical simulations are performed with the numerical values listed in Table I. Note that for all cases that $M = 10.0$ kg, $K = 10.0$ N/m, and $K_{NL} = 100.0$ N/m^3. For this initial Case 1 the phase plane plot and the potential and kinetic energy rate plots are shown in Fig. 2 (top row). This run demonstrates Corollary 1 and a stable orbit for the nonlinear system with offsetting potential and kinetic energy rates responses.

Next, consider the additional PID, linear, and Coulomb friction effects applied to the Duffing oscillator and partition into exergy generation and exergy dissipation terms. Now apply the exergy/entropy control design and the derivative of the Lyapunov function/Hamiltonian becomes

$$\dot{V} = \dot{\mathcal{H}} = \dot{W} - T_o\dot{\mathcal{S}}_i = \sum_{j=1}^{N} Q_j\dot{q}_j - \sum_{l=N}^{M-N} Q_l\dot{q}_l$$

which yields

$$\begin{aligned} T_o\dot{\mathcal{S}}_i &= (C+K_D)\dot{x}\cdot\dot{x} + C_{NL}\ \text{sign}(\dot{x})\cdot\dot{x} \\ \dot{W} &= -K_I\textstyle\int_o^t x d\tau\cdot\dot{x} \\ (T_o\dot{\mathcal{S}}_{rev})_{ave} &= (M\ddot{x}\cdot\dot{x} + (K+K_P)x\cdot\dot{x} + K_{NL}x^3\cdot\dot{x})_{ave} \\ &= 0. \end{aligned}$$

To determine the nonlinear stability boundary from the exergy/entropy control design

$$\dot{V} = \dot{\mathcal{H}} = \dot{W} - T_o\dot{\mathcal{S}}_i$$

which gives

$$(\dot{W})_{ave} = (T_o\dot{\mathcal{S}}_i)_{ave}.$$

Substituting the actual terms yields the following:

$$\left[-K_I\int_o^t x d\tau\cdot\dot{x}\right]_{ave} = [(C+K_D)\dot{x}\cdot\dot{x} + C_{NL}\ \text{sign}(\dot{x})\cdot\dot{x}]_{ave} \tag{24}$$

which is the *nonlinear stability boundary*. To best understand how the boundary is determined, concepts and analogies from electric AC power have been introduced earlier. Essentially, when the average power$_{in}$ is equivalent to the average power$_{dissipated}$ over a cycle, then the system is operating at the stability boundary. Later, in the exergy and exergy rate responses for the nonlinear system, one may observe that the area under the curves for the exergy rate generation and the exergy rate dissipation are equivalent and for the corresponding exergy responses the slopes will be equal and opposite. This helps to explain why PID control works well for nonlinear systems.

Numerical simulations are performed to demonstrate where the nonlinear stability boundary lies for the Duffing oscillator/Coulomb friction dynamic model subject to PID control. Three separate cases are conducted with the numerical values listed in Table I. The nonlinear system is subject to an initial condition of $x_0 = 1.0$. For Case 2 the integral of position, position, velocity, and acceleration responses along with the exergy and exergy rate responses are plotted in Fig. 2 (second row from top). For this case, the dissipative term is greater than the generative term. This is observed from the decaying system responses. In Case 3 the system responses along with the exergy and exergy rate responses are shown in Fig. 2 (third row from top). In this case, the average exergy slopes and integrated power areas for the dissipative and generative terms are equivalent which demonstrates (24). This results in system responses that do not decay, displaying constant nonlinear oscillatory behavior. In final Case 4, the system responses along with the exergy and exergy rate responses are shown in Fig. 2 (bottom row). In this case, the dissipative term is less than the generative term which results in a system response with increasing nonlinear oscillatory behavior. In conclusion, Fig. 3 shows the responses for the total exergy with respect to each case along with the phase plane plot for the nonlinear system. For Case 3 the nonlinear stability boundary (or neutral stability) is characteristic of an average zero output for the total exergy response or validation of (24). For the phase plane plot, Case 2 demonstrates an asymptotically stable decaying response, Case 3 a neutrally stable orbital response, and Case 4 an asymptotically unstable increasing orbit response.

The last three cases for the PID control regulator Duffing oscillator/Coulomb friction dynamic system demonstrates the three subcases for Corollary 4: Given apriori $(T_o\dot{\mathcal{S}}_i)_{ave} > 0$ and $(\dot{W})_{ave} > 0$ then the nonlinear system showed the following:

i. Case 2 yielded $(T_o\dot{\mathcal{S}}_i)_{ave} > (\dot{W})_{ave}$; asymptotic stability; damped stable nonlinear response and demonstration of Corollary 4.1.
ii. Case 3 yielded $(T_o\dot{\mathcal{S}}_i)_{ave} = (\dot{W})_{ave}$; neutral stability; and demonstration of Corollary 4.2. This case is the nonlinear stability boundary where dissipation and generation terms cancel each other out on the average.
iii. Case 4 yielded $(T_o\dot{\mathcal{S}}_i)_{ave} < (\dot{W})_{ave}$; increasingly unstable towards another orbit; and demonstration of Corollary 4.3.

B. Van der Pol Nonlinear System

The classic van der Pol's equation [22] is analyzed using the techniques of this section. Originally, the "van der Pol equation" is credited to van der Pol, and is a model of an electronic circuit for early radio vacuum tubes of a triode electronic oscillator [22]. The tube acts like a normal resistor when the current is high, but acts as a negative resistor if the current is low. The main feature is that electrical circuits that contain these elements pump up small oscillations due to a negative resistance when currents are small, but drag down large amplitude oscillations due to positive resistance when the currents are large. This behavior is known as a *relaxation oscillation*, as each period of the oscillation consists of a slow buildup of energy ('stress phase') followed by a phase in which energy is discharged ('relaxation phase'). This particular system has played a large role in nonlinear dynamics and has been used to study limit cycles and self-sustained oscillatory phenomena in nonlinear systems.

Consider the van der Pol equation with mass (m) and stiffness (k) values other than unity and a nonlinear damping term (μ) to be defined as:

$$m\ddot{x} + \mu(1 - x^2)\dot{x} + kx = 0.$$

The appropriate Hamiltonian/Lyapunov function is defined as:

$$\mathcal{H} = V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 > 0.$$

Then the corresponding time derivative of the Lyapunov function/Hamiltonian becomes

$$\begin{aligned} \dot{V} &= [m\ddot{x} + kx]\,\dot{x} \\ &= \left[\mu\dot{x}(1 - x^2)\right]\dot{x} \\ &= \mu\dot{x}^2 - \mu x^2\dot{x}^2. \end{aligned}$$

Next identifying the generator and dissipator terms yields

$$\begin{aligned} T_o\dot{\mathcal{S}}_i &= \mu x^2\dot{x}^2 \\ \dot{W} &= \mu\dot{x}^2 \\ (T_o\dot{\mathcal{S}}_{rev})_{ave} &= (m\ddot{x}\cdot\dot{x} + kx\cdot\dot{x})_{ave} = 0. \end{aligned}$$

The *nonlinear stability boundary* can be determined as

$$\begin{aligned} \left[\dot{W}\right]_{ave} &= \left[T_o\dot{\mathcal{S}}_i\right]_{ave} \\ \left[\mu\dot{x}^2\right]_{ave} &= \left[\mu x^2\dot{x}^2\right]_{ave} \end{aligned}$$

By investigating several initial conditions both inside, on, and outside the limit cycle then three separate regions can be observed. Figure 4 shows these conditions with the corresponding numerical values given in Table I. The responses are plotted on the Hamiltonian 3D surface (top) with the projection onto the phase plane shown on the 2D plot (middle). For the case outside the limit cycle, the dissipator term dominates and for the case inside the limit cycle the generator term dominates. For both cases inside and outside the limit cycle, the system migrates back to the stability boundary. For the case already on the limit cycle then the system is already at neutral stability. The neutral exergy-rate and exergy plots are shown in

Fig. 4 (bottom). The cycle is defined at approximately $\tau = 3.5$ seconds. For the neutral pair the terms cancel each other out at the end of the cycle or $[\dot{W}]_{ave} = [T_o\dot{S}_i]_{ave}$. For the generator case then $[\dot{W}]_{ave} > [T_o\dot{S}_i]_{ave}$ and for the dissipator case then $[\dot{W}]_{ave} < [T_o\dot{S}_i]_{ave}$, respectively. Eventually, given enough cycles both the generator and dissipator cases will converge to the neutral case.

VIII. Summary and Conclusions

A novel control system design methodology was developed that uniquely combined: concepts from thermodynamic exergy and entropy; Hamiltonian systems; Lyapunov's direct method and Lyapunov optimal analysis; electric AC power concepts; and power flow analysis. Relationships were derived between exergy/entropy and Lyapunov optimal functions for Hamiltonian systems. The methodology is demonstrated with two fundamental numerical simulation examples: 1) a Duffing oscillator/Coulomb friction nonlinear model that employs PID regulator control and 2) a van der Pol nonlinear oscillator system. The control system performance results and/or appropriately identified terms were partitioned and evaluated based on exergy generation and exergy dissipation terms. These numerical results showed the stability boundaries for each nonlinear system. This novel nonlinear control methodology resulted in both necessary and sufficient conditions for stability of nonlinear systems. In the near future, this novel control system design methodology will be extended to tracking and adaptive control of multi-input/multi-output nonlinear systems. This methodoloy is applicable to a large class of nonlinear systems.

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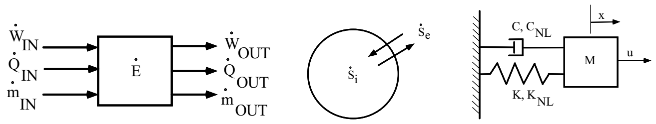

Fig. 1. **Energy flow control volume (left), second law entropy with flux exchange system (middle), and Duffing oscillator/Coulomb friction system (right)**

TABLE I

Duffing oscillator/Coulomb friction model and PID control system gains

Case	K_P	K_I	K_D	C	C_{NL}
No.	(kg/s^2)	(kg/s^3)	(kg/s)	(kg/s)	(N)
1	10.0	0.0	0.0	0.0	0.0
2	10.0	20.0	2.0	0.1	5.0
3	10.0	40.05	2.0	0.1	5.0
4	10.0	80.0	2.0	0.1	5.0

TABLE II

Van der Pol model numerical values

Case	x_o	$\dot{x}_o$	μ	m	k
	(m)	(m/s)	(kg/s)	(kg)	(kg/s^2)
generate	0.1	-0.1	1.5	1.0	1.0
neutral	1.0	-1.0	1.5	1.0	1.0
dissipate	2.0	-2.0	1.5	1.0	1.0

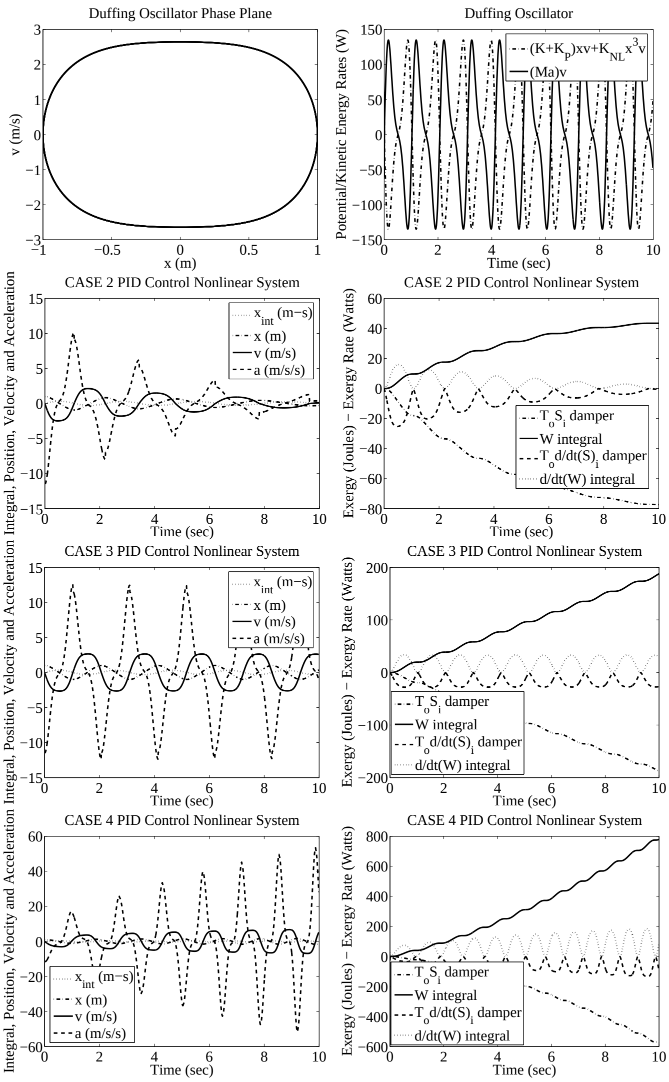

Fig. 2. **Cases 1-4: Duffing oscillator/Coulomb friction with PID control numerical results**

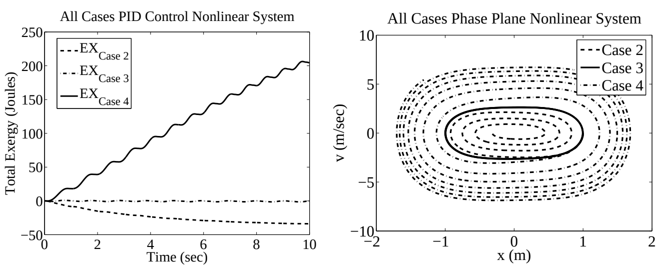

Fig. 3. **Cases 2:4 - Duffing oscillator/Coulomb friction numerical results**

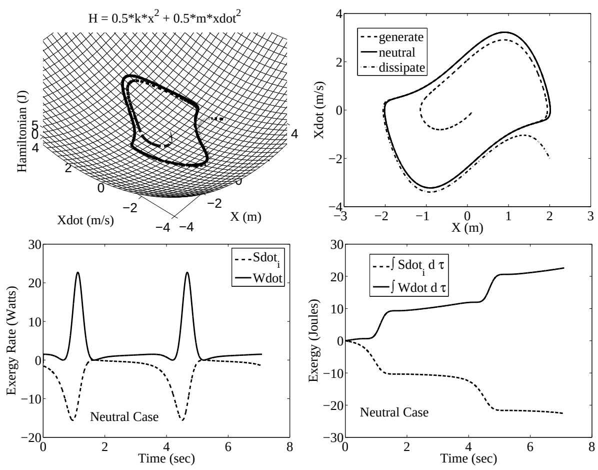

Fig. 4. **Van der Pol responses - 3D Hamiltonian, phase plane plot (top), and exergy-rate and exergy plots (bottom)**