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A Set of Manufactured Solutions for Coupled Radiation (SPn) and Conduction Problems

John Tencer, Tolulope Okusanya, Adam Hetzler

U.S. DEPARTMENT OF ENERGY

NNSA National Nuclear Security Administration

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Outline

- Method of manufactured solutions background
- Problem details
- Relevant analytical solutions
- Set of manufactured solutions
- Summary
- Questions

Method of Manufactured Solutions (MMS)

- Generally, when solving a system of equations $L(u)=g$ the operator L (including any parameters) and the source g are known. The solution, $u=L^{-1}(g)$ is found numerically.
- For some special cases, the solution may be found analytically. However, these cases often neglect terms failing to fully test the implementation.
- The method of manufactured solutions (MMS) involves assuming a solution and deriving the corresponding source term.
- Ideally, u is chosen so that all of the terms in L are exercised

MMS allows for verification of complicated multi-physics codes

- Coupled (nonlinear) physics and complicated boundary conditions make analytical solutions impossible or impractical even for simple geometries

- MMS allows us to look at convergence rates while exercising all of the relevant terms

Governing Equations

- Energy Equation:

$$-\vec{\nabla} \bullet \left(k \vec{\nabla} T \right) = \sigma_A \left(\int I\left(\vec{\Omega}\right) d\Omega - 4\sigma T^4 \right)$$

- Radiative Transfer Equation (RTE):

$$\vec{\Omega} \bullet \vec{\nabla} I\left(\vec{\Omega}\right) + \sigma_T I\left(\vec{\Omega}\right) = \sigma_A \frac{\sigma T^4}{\pi} + \frac{\sigma_S}{4\pi} \int I\left(\vec{\tilde{\Omega}}\right) d\tilde{\Omega}$$

- The steady-state gray RTE is partial differential equation for the directional intensity in 5 dimensions. The angular integration is particularly costly and many approximations have been developed to approximately solve the RTE

SPn Approximation

- RTE is approximated using the simplified spherical harmonics (SPn) approximation

$$-\vec{\nabla}\bullet\left(\frac{\mu_n^2}{\sigma_T}\vec{\nabla}I_n\right)+\sigma_T I_n=\frac{\sigma_S}{4\pi}G+\sigma_A\frac{\sigma T^4}{\pi}$$

$$G=4\pi\sum_{m=1}^{(N+1)/2}w_m I_m$$

- Diffuse gray Mark boundary condition:

$$-\frac{\mu_n}{\sigma_T}\vec{\nabla}I_n\bullet\vec{n}=\frac{\varepsilon}{2-\varepsilon}\left(I_n-\frac{\sigma T^4}{\pi}\right)+$$

$$\frac{1-\varepsilon}{2-\varepsilon}\left[\frac{\sum\left(I_k-\frac{\mu_k}{\sigma_T}\vec{\nabla}I_k\bullet\vec{n}\right)\mu_k w_k}{\sum\mu_k w_k}-I_n+\frac{\mu_n}{\sigma_T}\vec{\nabla}I_n\bullet\vec{n}\right]$$

Exercise All Terms

$$-\vec{\nabla}\bullet\left(k\vec{\nabla}T\right)=\sigma_A\left(G-4\sigma T^4\right)$$

$$-\vec{\nabla}\bullet\left(\frac{\mu_n^2}{\sigma_T}\vec{\nabla}I_n\right)+\sigma_T I_n=\frac{\sigma_S}{4\pi}G+\sigma_A\frac{\sigma T^4}{\pi}$$

$$-\frac{\mu_n}{\sigma_T}\vec{\nabla}I_n\bullet\vec{n}=\frac{\varepsilon}{2-\varepsilon}\left(I_n-\frac{\sigma T^4}{\pi}\right)+$$

$$\frac{1-\varepsilon}{2-\varepsilon}\left[\frac{\sum\left(I_k-\frac{\mu_k}{\sigma_T}\vec{\nabla}I_k\bullet\vec{n}\right)\mu_k w_k}{\sum\mu_k w_k}-I_n+\frac{\mu_n}{\sigma_T}\vec{\nabla}I_n\bullet\vec{n}\right]$$

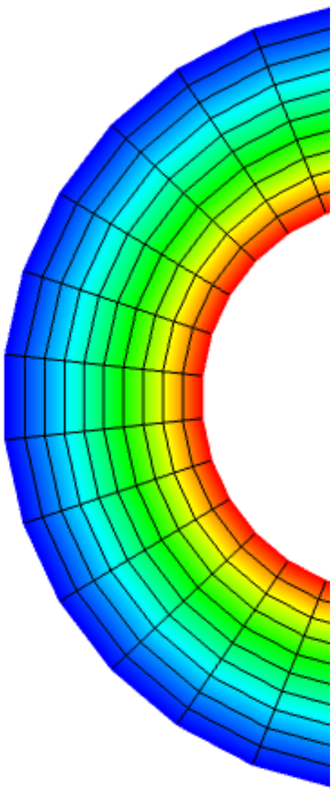
- *T* twice differentiable
- Absorption $\sigma_A\neq 0$
- Intensities twice differentiable
- Total opacity not constant
- Scattering $\sigma_S\neq 0$
- Absorbing boundary $\varepsilon\neq 0$
- Reflecting boundary $\varepsilon\neq 1$
- Emitting boundary $T\neq 0$
- Higher order $N>1$
- No lumping of opacity

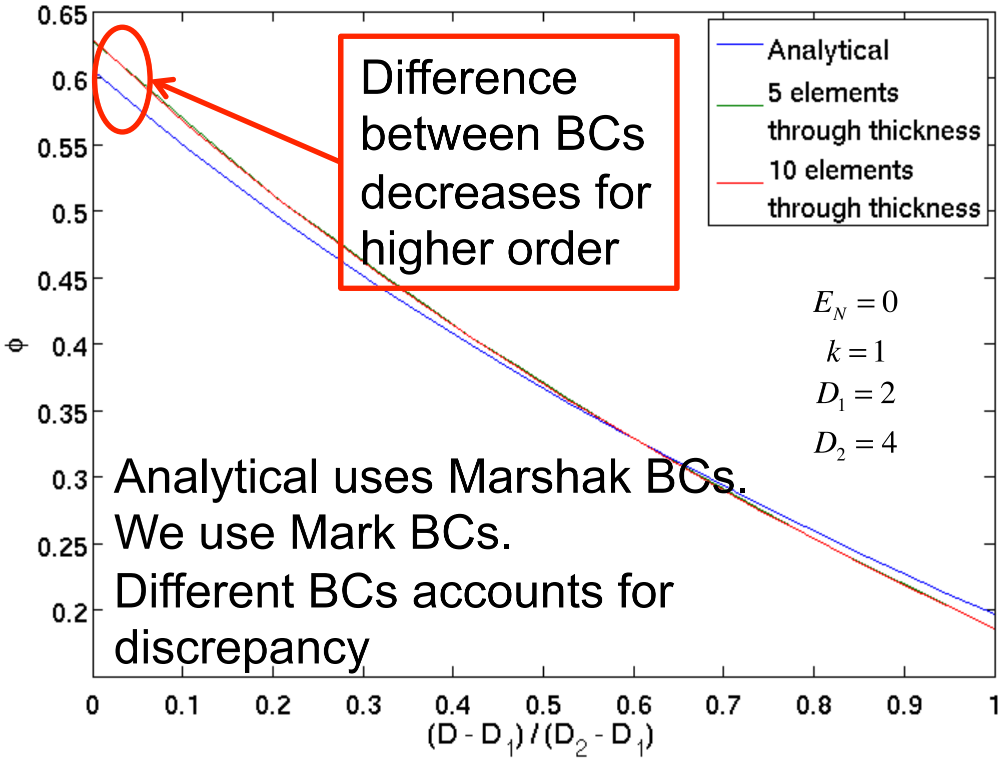
EXAMPLE ANALYTICAL SOLUTIONS

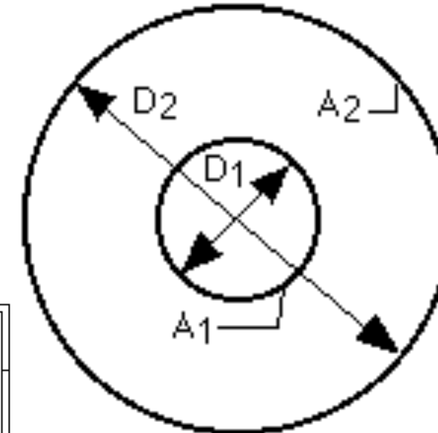
Infinitely long concentric cylinders (SP_1)

$$\psi = \frac{1}{\frac{3}{8}KD_1\, ln\left(\frac{D_2}{D_1}\right) + \left(E_1 + \frac{1}{2}\right) + \frac{D_1}{D_2}\left(E_2 + \frac{1}{2}\right)}$$

$$\phi(D) = \psi\left[-\frac{3}{8}KD_1\, ln\left(\frac{D}{D_1}\right) + \frac{D_1}{D_2}\left(E_2 + \frac{1}{2}\right)\right]$$







$$E_N = \frac{1 - \varepsilon_{wN}}{\varepsilon_{wN}}$$

$$\psi = \frac{Q_1}{A_1\sigma\left(T_{w1}{}^4 - T_{w2}{}^4\right)}$$

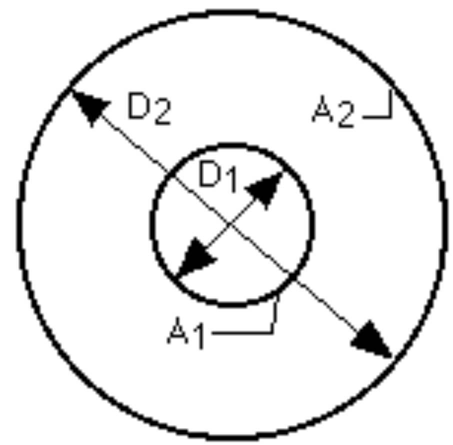
$$\phi = \frac{\left(T^4 - T_{w2}{}^4\right)}{\left(T_{w1}{}^4 - T_{w2}{}^4\right)}$$

$$K = \sigma_T = \sigma_a + \sigma_s$$

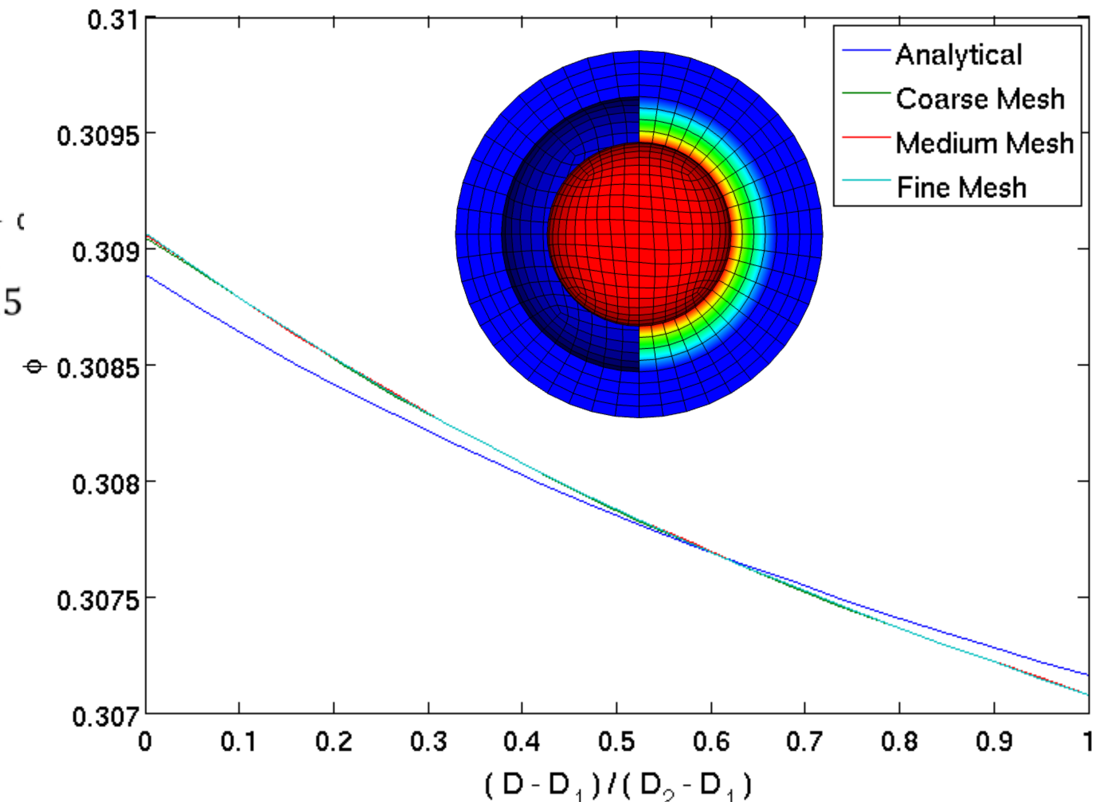
Concentric spheres (SP_1)

$$\psi = \frac{1}{\frac{3}{8}KD_1\left(1-\frac{D_1}{D_2}\right)+\left(E_1+\frac{1}{2}\right)+\left(\frac{D_1}{D_2}\right)^2\left(E_2+\frac{1}{2}\right)}$$

$$\phi(D) = \psi\left[-\frac{3}{8}KD_1\left(\frac{D_1}{D_2}-\frac{D_1}{D}\right)+\left(\frac{D_1}{D_2}\right)^2\left(E_2+\frac{1}{2}\right)\right]$$



$E_N = 0$
$K = 0.1$
$D_1 = 0.1$
$D_2 = 0.15$



$$E_N = \frac{1-\varepsilon_{wN}}{\varepsilon_{wN}}$$

$$\psi = \frac{Q_1}{A_1\sigma\left({T_{w1}}^4-{T_{w2}}^4\right)}$$

$$\phi = \frac{\left(T^4-{T_{w2}}^4\right)}{\left({T_{w1}}^4-{T_{w2}}^4\right)}$$

$$K = \sigma_T = \sigma_a + \sigma_s$$

Restrictions of Analytical Solutions

- Pure radiation. No conduction.
- Only SP1. Many terms in the expansion drop out for SP1. SP3 would exercise those terms.
- Marshak BCs. Mark BCs are less common and solutions which include them are unavailable (although they could easily be derived).

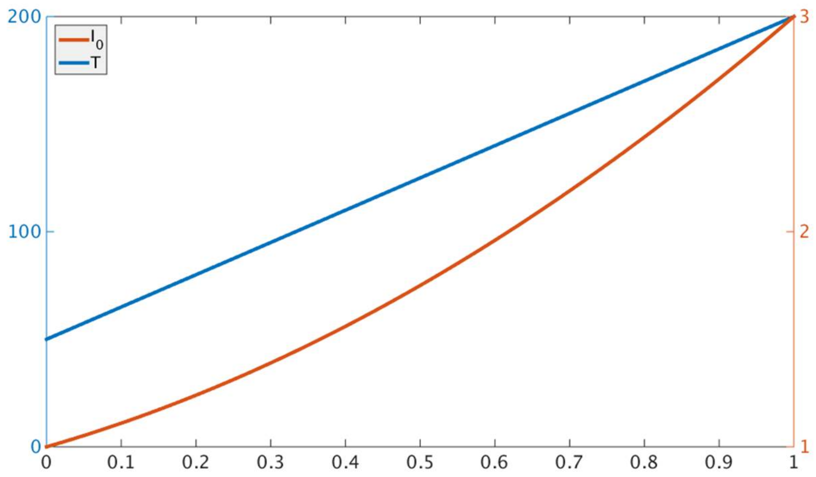
MANUFACTURED SOLUTIONS

MMS (SP_1, Quasi-1D)

$$-\vec{\nabla}\bullet\left(k\vec{\nabla}T\right)=4\pi\sigma_A\left(I_0-\frac{\sigma T^4}{\pi}\right)+S_E$$

$$-\vec{\nabla}\bullet\left(\frac{1}{3\sigma_T}\vec{\nabla}I_0\right)=-\sigma_A\left(I_0-\frac{\sigma T^4}{\pi}\right)+S_T$$

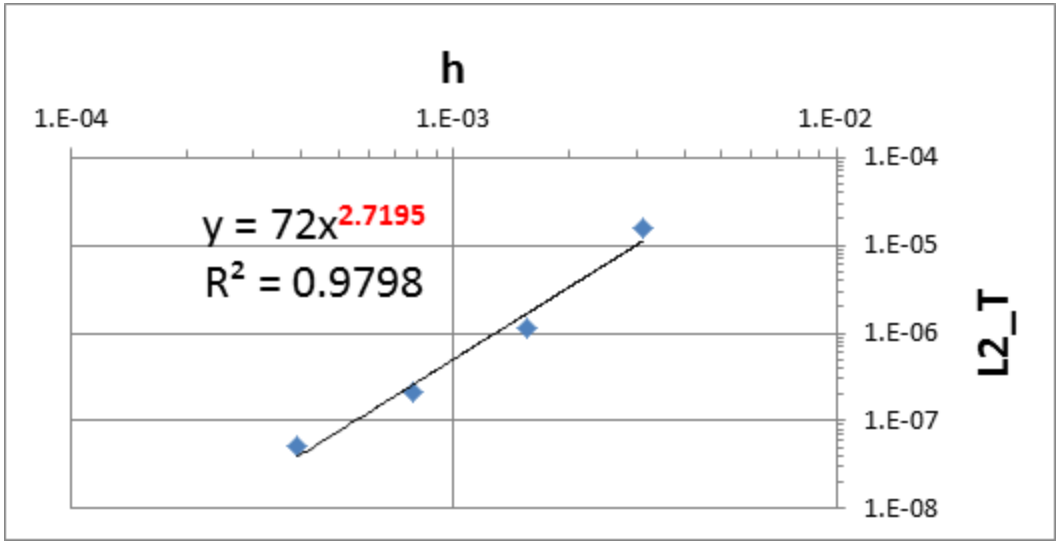
$$-\frac{1}{\sigma_T\sqrt{3}}\vec{\nabla}I_0\bullet\vec{n}=\frac{\varepsilon}{2-\varepsilon}\left(I_0-\frac{\sigma T^4}{\pi}\right)$$



$I_0 = x^2 + x + 1$ $\quad \varepsilon(0) = 0.5 \quad \sigma_a = \sigma_s = \sigma_T/2$

$T = 50 + 150x$ $\quad \varepsilon(1) = 1.0 \quad k = 1$

$$\sigma_T(x) = \left(\frac{\sqrt{3}}{1-\sigma 50^4/\pi}\right)(1-x) + \left(\frac{\sqrt{3}}{\sigma 200^4/\pi - 3}\right)x$$



MMS (SP_1 , 2D X-Y)

$$-\vec{\nabla}\bullet\left(k\vec{\nabla}T\right)=4\pi\sigma_A\left(I_0-\frac{\sigma T^4}{\pi}\right)+S_E$$

$$-\vec{\nabla}\bullet\left(\frac{1}{3\sigma_T}\vec{\nabla}I_0\right)=-\sigma_A\left(I_0-\frac{\sigma T^4}{\pi}\right)+S_T$$

$$-\frac{1}{\sigma_T\sqrt{3}}\vec{\nabla}I_0\bullet\vec{n}=\frac{\varepsilon}{2-\varepsilon}\left(I_0-\frac{\sigma T^4}{\pi}\right)$$

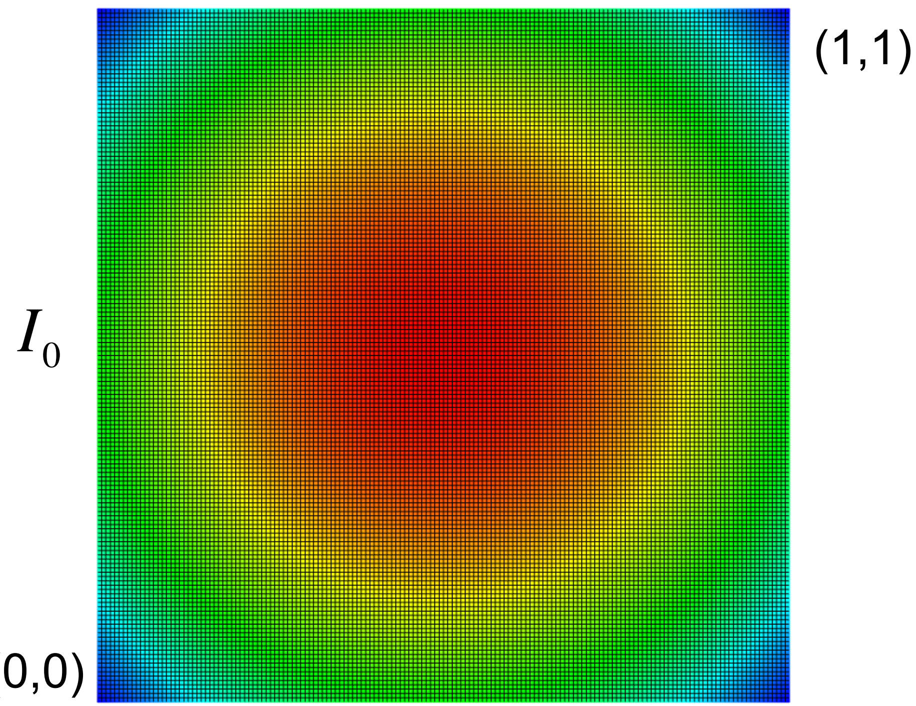
$$I_0 = 5 - x^2 - y^2$$

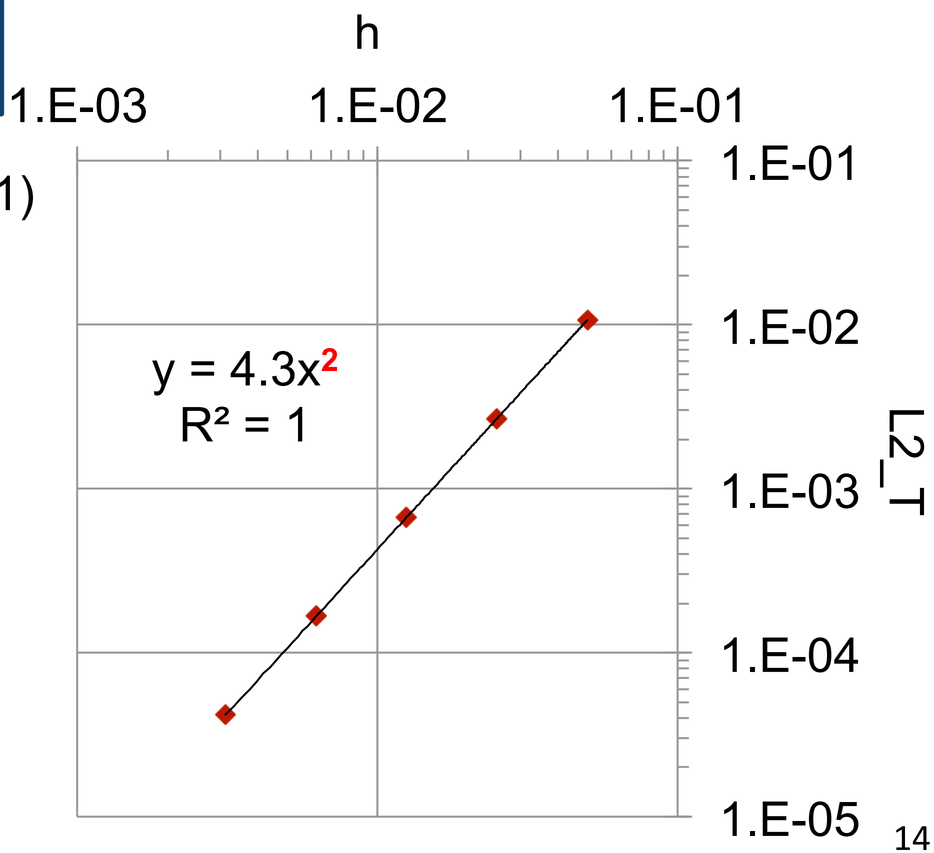
$$T = \sqrt[4]{\frac{\pi}{\sigma}\left(5 - \frac{\sqrt{3}}{3} - x^2 - y^2\right)}$$

$\sigma_T = 1 \quad k = 1$

$\sigma_a = 0.5 \quad \varepsilon = 1$

S_T added exclusively for verification





MMS (SP_1 , 2D Axially Symmetric)

$$-\vec{\nabla}\bullet\left(k\vec{\nabla}T\right)=4\pi\sigma_A\left(I_0-\frac{\sigma T^4}{\pi}\right)+S_E$$

$$-\vec{\nabla}\bullet\left(\frac{1}{3\sigma_T}\vec{\nabla}I_0\right)=-\sigma_A\left(I_0-\frac{\sigma T^4}{\pi}\right)+S_T$$

$$-\frac{1}{\sigma_T\sqrt{3}}\vec{\nabla}I_0\bullet\vec{n}=\frac{\varepsilon}{2-\varepsilon}\left(I_0-\frac{\sigma T^4}{\pi}\right)$$

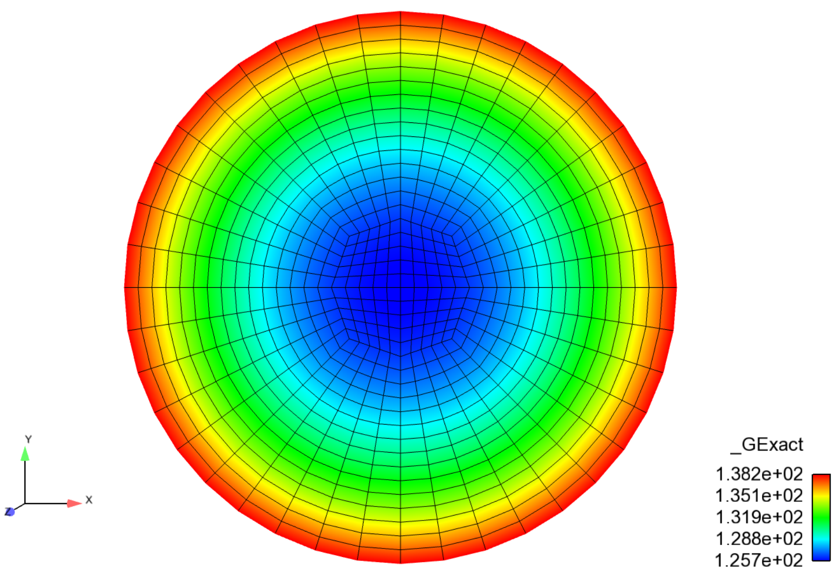
$I_0 = r^2 + 10$

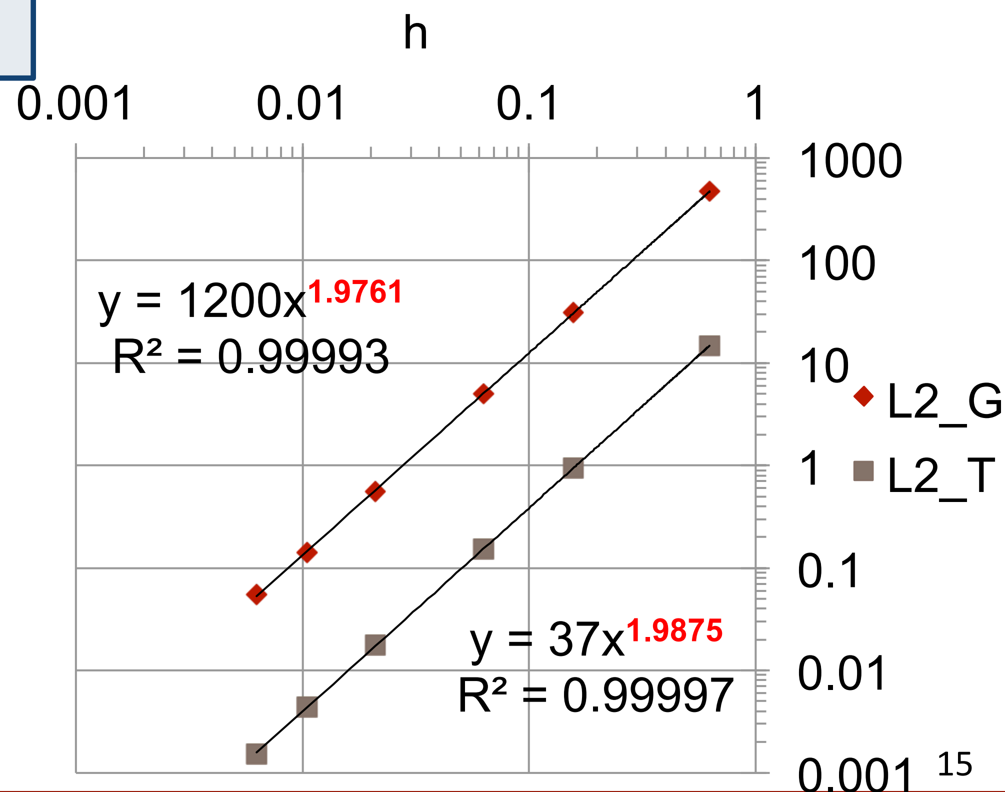
$T = 100r^2 + 300$

$\sigma_T \approx 0.00384 \qquad \sigma_A = 4\sigma_T/5$

$k = 1 \qquad \varepsilon = 0.8$

$r \in [0,1]$





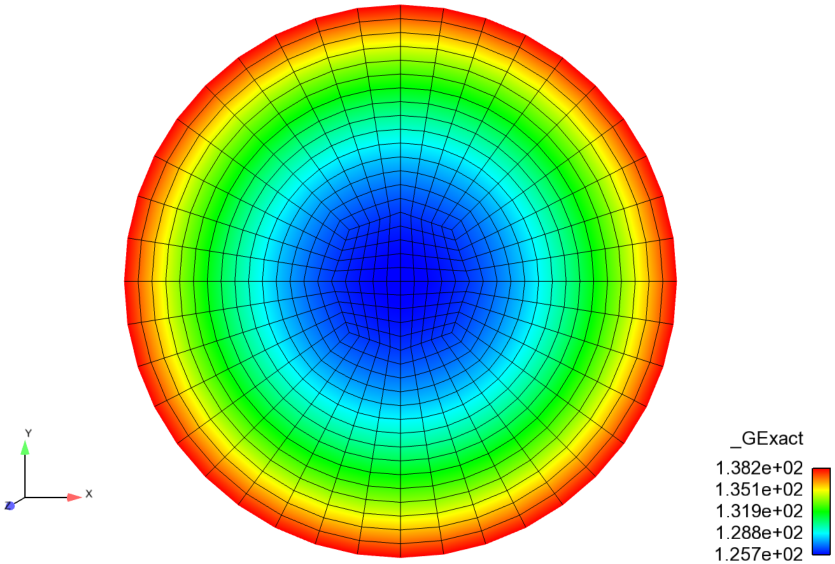
MMS (SP$_1$, 3D Axially Symmetric)

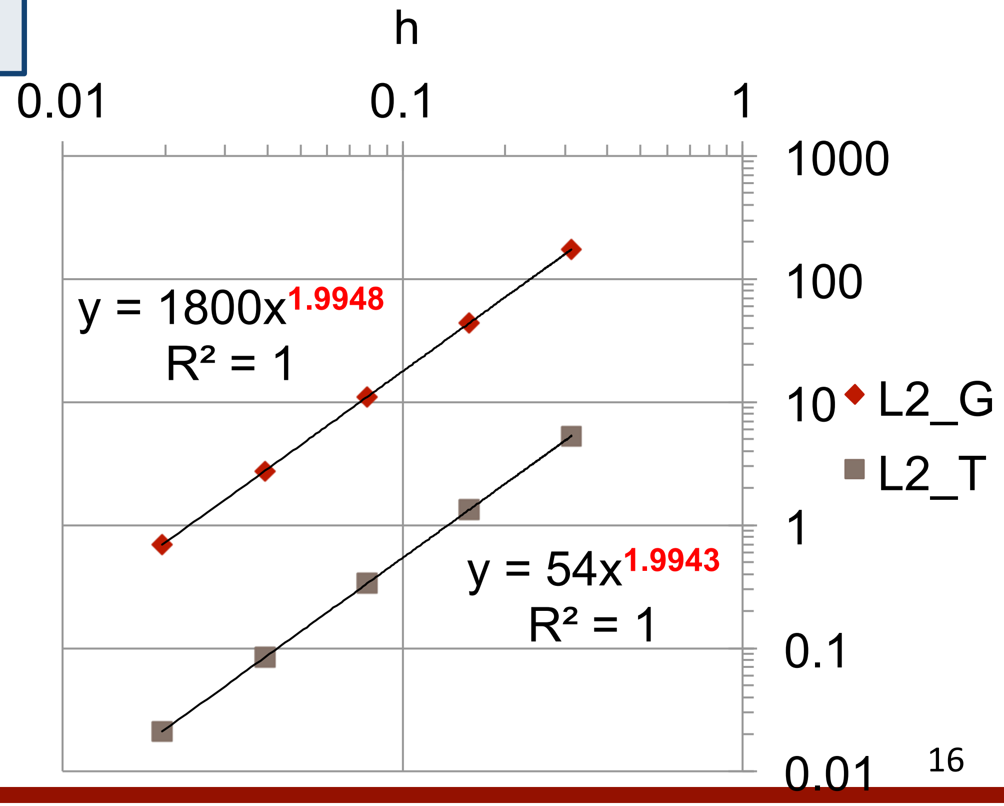
$$-\vec{\nabla}\cdot\left(k\vec{\nabla}T\right)=4\pi\sigma_A\left(I_0-\frac{\sigma T^4}{\pi}\right)+S_E$$

$$-\vec{\nabla}\cdot\left(\frac{1}{3\sigma_T}\vec{\nabla}I_0\right)=-\sigma_A\left(I_0-\frac{\sigma T^4}{\pi}\right)+S_T$$

$$-\frac{1}{\sigma_T\sqrt{3}}\vec{\nabla}I_0\cdot\vec{n}=\frac{\varepsilon}{2-\varepsilon}\left(I_0-\frac{\sigma T^4}{\pi}\right)$$

$$I_0=r^2+10$$
$$T=100r^2+300$$
$$\sigma_T\approx 0.00384 \qquad \sigma_A=4\sigma_T/5$$
$$k=1 \qquad \varepsilon=0.8$$
$$r\in[0,1]$$





MMS (SP_3 , 2D Axially Symmetric)

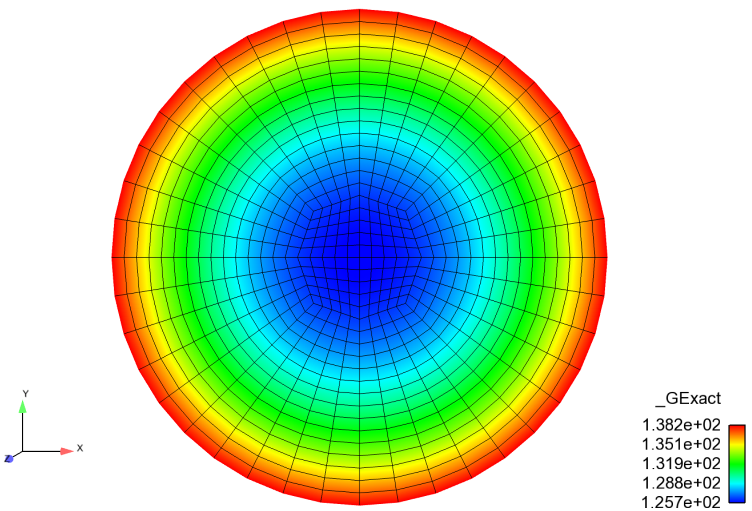
$$-\vec{\nabla}\cdot\left(k\vec{\nabla}T\right) = 4\pi\sigma_A\left(I_0 - \frac{\sigma T^4}{\pi}\right) + S_E$$

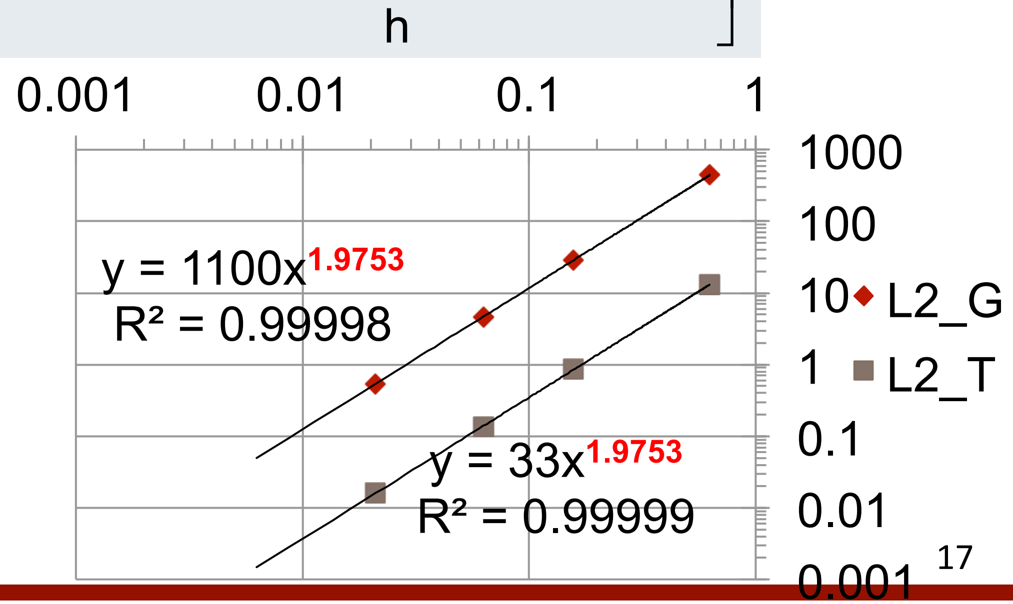
$$-\vec{\nabla}\cdot\left(\frac{\mu_0^2}{\sigma_T}\vec{\nabla}I_0\right) + \sigma_T I_0 = \frac{\sigma_S}{4\pi}\left(w_0 I_0 + w_1 I_1\right) + \sigma_A\frac{\sigma T^4}{\pi}$$

$$-\vec{\nabla}\cdot\left(\frac{\mu_1^2}{\sigma_T}\vec{\nabla}I_1\right) + \sigma_T I_1 = \frac{\sigma_S}{4\pi}\left(w_0 I_0 + w_1 I_1\right) + \sigma_A\frac{\sigma T^4}{\pi}$$

$$-\frac{\mu_n}{\sigma_T}\vec{\nabla}I_n\bullet\vec{n} = \frac{\varepsilon}{2-\varepsilon}\left(I_n - \frac{\sigma T^4}{\pi}\right) + \frac{1-\varepsilon}{2-\varepsilon}\left[\frac{\sum\left(I_k - \frac{\mu_k}{\sigma_T}\vec{\nabla}I_k\bullet\vec{n}\right)\mu_k w_k}{\sum\mu_k w_k} - I_n + \frac{\mu_n}{\sigma_T}\vec{\nabla}I_n\bullet\vec{n}\right]$$

$I_0 = 10r^2 + 10$	$I_1 = 100r^2 + 500$
$T = 100r^2 + 300$	$r \in [0,1]$
$\sigma_T \approx 3.591$	$\sigma_A = 0.8$
$\varepsilon \approx 0.4137$	$k = 1$





MMS (SP_1 , 3D Radially Symmetric)

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$$-\vec{\nabla} \cdot \left(k \vec{\nabla} T \right) = 4\pi\sigma_A \left(I_0 - \frac{\sigma T^4}{\pi} \right) + S_E$$

$$-\vec{\nabla} \cdot \left(\frac{1}{3\sigma_T} \vec{\nabla} I_0 \right) = -\sigma_A \left(I_0 - \frac{\sigma T^4}{\pi} \right) + S_T$$

$$-\frac{1}{\sigma_T \sqrt{3}} \vec{\nabla} I_0 \bullet \vec{n} = \frac{\varepsilon}{2-\varepsilon} \left(I_0 - \frac{\sigma T^4}{\pi} \right)$$

$I_0 = (100r)^2 + 20$ $T = 5(100r)^2$

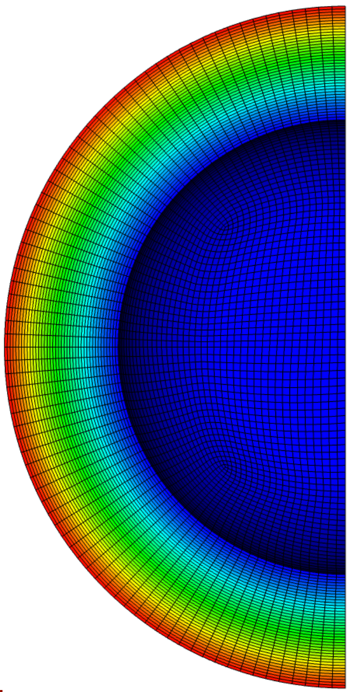
$\sigma_T = 25$

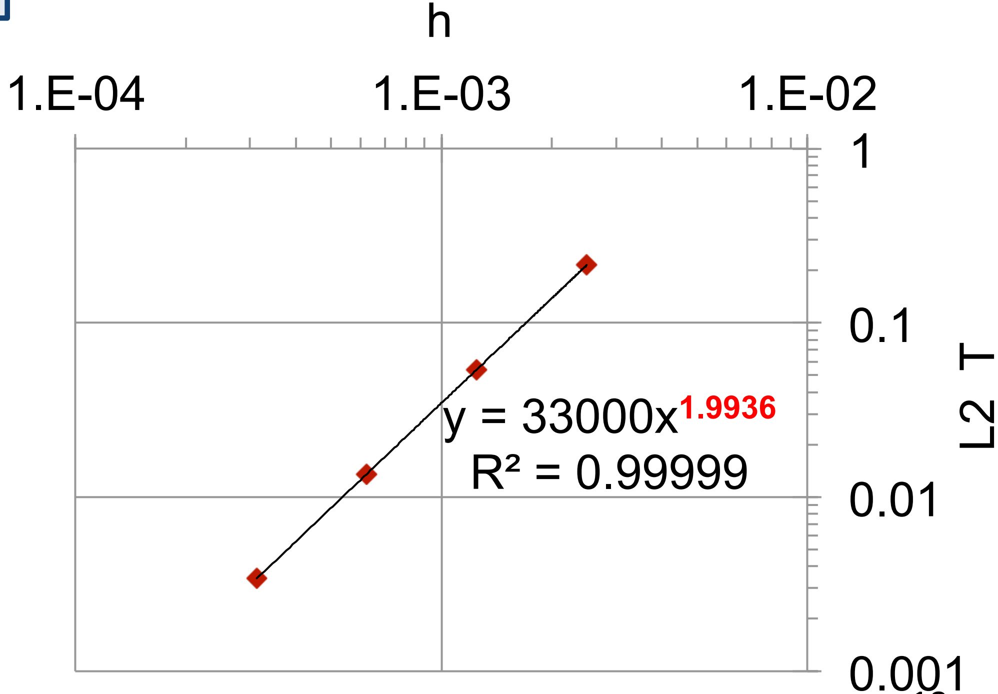
$\sigma_A = 1/3$

$k = 1$

$\varepsilon(0.05) = 0.725$

$\varepsilon(0.075) = 0.971$





Summary

- A set of manufactured solutions for coupled conduction / participating media radiation is presented
- SPn approximation with Mark boundary conditions is used to represent the radiative transport
- All relevant terms are exercised
- Expected quadratic convergence rate is observed in all cases

QUESTIONS?