SAND2015-3770C

Approximate Block Factorization Preconditioners for Scalable Solution of Multiphysics MHD Systems.

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Outline

- **General Scientific and Mathematical/Computational Motivation**
- **Motivation for Fully Implicit Newton – Krylov Solution Methods**
- **Brief Overview of 3D Resistive MHD Equations and Numerical Approximation**
- **Motivation for Approximate Block Factorization (ABF) Preconditioners**
    - **indicate Scaling of Fully-coupled AMG for Systems (Stabilized FE MHD)**
    - **Scaling of ABF Preconditioners (Stabilized FE MHD)**
    - **Scaling of ABF for Mixed Integration and Edge-element Formulations**
- **Conclusions**

Motivation: Science/Technology and Mathematical / Computational

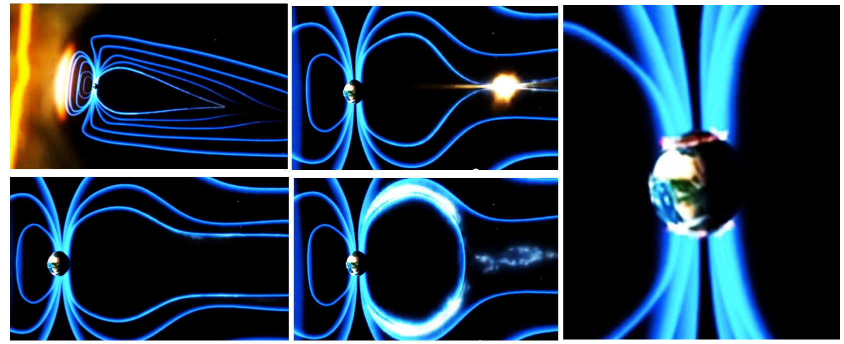
Science / Technology Motivation:

Resistive and extended MHD models are used to study important plasma physics systems

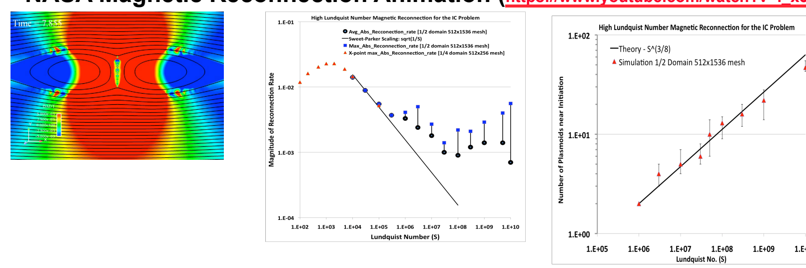
- Astrophysics: Magnetic reconnection, solar flares, ..
- Planetary-physics: Earth's magnetospheric sub-storms, Aurora, geo-dynamo, planetary-dynamos
- Fusion: Magnetic Confinement [MCF] (e.g. ITER), Inertial Conf. [ICF] (e.g. NIF, Z-pinch)

Mathematical/Computational Motivation:

Achieving Scalable Predictive Simulations of Complex Highly Nonlinear Multiphysics Systems to Enable Scientific Discovery and Engineering Design/Optimization



NASA Magnetic Reconnection Animation (https://www.youtube.com/watch?v=i_x3s8ODaKg)



Magnetic Reconnection: S = 1e+9 (left), Reconn. Rate vs. SP theory (right)

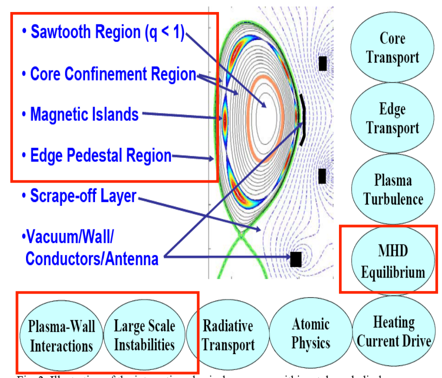
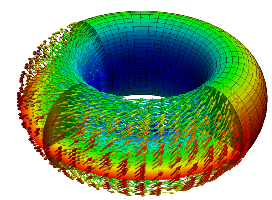
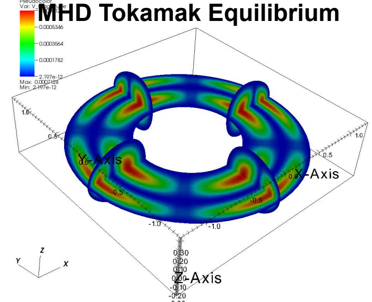


Fig. 2: Illustration of the interacting physical processes within a tokamak discharge.



MHD Tokamak Equilibrium



MHD Equilibrium Instability

What are multi-physics systems? (A multiple-time-scale perspective)

These systems are characterized by a myriad of complex, interacting, nonlinear multiple time- and length-scale physical mechanisms.

These mechanisms:

- **can be dominated by one, or a few processes, that drive a short dynamical time-scale consistent with these dominating modes,**
- **consist of a set of widely separated time-scales that produce a stiff system response,**
- **nearly balance to evolve a solution on a dynamical time-scale that is long relative to the component time scales,**
- **or balance to produce steady-state behavior.**

Mathematical Approach - develop:

- Stable & higher-order accurate fully-implicit / IMEX formulations allowing resolution of dynamical time-scale of interest
- Stable and accurate spatial discretizations for complex geom.,
  Options enforcing key mathematical properties (e.g. positivity), and structure-preserving-forms (e.g. div B = 0)
- Robust and efficient fully-coupled nonlinear/linear iterative solution methods based on Newton-Krylov (NK) methods
- Scalable and efficient preconditioners utilizing multi-level (AMG) methods (Fully-coupled AMG, physics-based, approx. block factorization)

=> Also enables beyond forward simulation & integrated UQ

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3D Resistive MHD Equations: Lagrange-Multiplier Formulation and VMS

Resistive MHD Model in Residual Notation

$$\mathbf{R_v} = \frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot [\rho \mathbf{v} \otimes \mathbf{v} - (\mathbf{T} + \mathbf{T}_M)] + 2\rho\Omega \times \mathbf{v} - \rho \mathbf{g} = \mathbf{0}$$

$$\mathbf{T} = -[P - \frac{2}{3}\mu(\nabla \cdot \mathbf{v})]\mathbf{I} + \mu[\nabla \mathbf{v} + \nabla \mathbf{v}^T]$$

$$\mathbf{T_M} = \frac{1}{\mu_0}\mathbf{B} \otimes \mathbf{B} - \frac{1}{2\mu_0}\|\mathbf{B}\|^2 \mathbf{I}$$

$$R_P = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$R_e = \frac{\partial (\rho e)}{\partial t} + \nabla \cdot [\rho \mathbf{v} e + \mathbf{q}] - \mathbf{T} : \nabla \mathbf{v} - \eta \|\frac{1}{\mu_0} \nabla \times \mathbf{B}\|^2 = 0$$

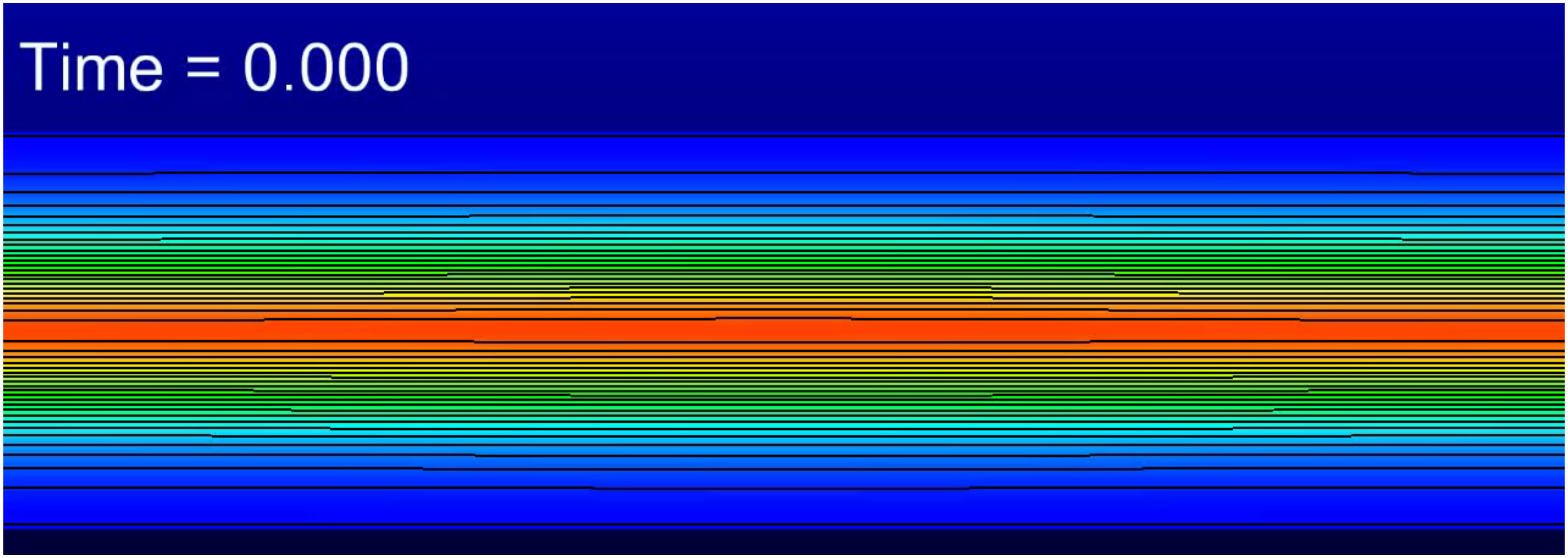
$$\mathbf{R_B} = \frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot \left[\mathbf{B} \otimes \mathbf{v} - \mathbf{v} \otimes \mathbf{B} - \frac{\eta}{\mu_0}\left(\nabla \mathbf{B} - (\nabla \mathbf{B})^T\right) + \psi \mathbf{I}\right] = \mathbf{0}$$

$$R_\psi = \nabla \cdot \mathbf{B} = 0$$

$$\mathcal{R}(\mathbf{u}) = \mathcal{L}(\mathbf{u}) - \mathbf{f} = \mathbf{0}$$

- Divergence free involution enforced as elliptic constraint with a Lagrange multiplier. (Dedner et. al. 2002; Codina et. al. 2006, 2011)
  - Only weakly divergence free in FE implementation (stabilization of B - ψ coupling)
- Can show relationship with projection (e.g. Brackbill and Barnes 1980) when 1st order-splitting is used.
- Issue for using C^0 FE for domains with re-entrant corners / soln singularities (Costabel et. al. 2000, 2002, Codina, 2011, see Badia et. al. 2013 for stabilized FE that is unconditionally stable on on appropriate meshes)

Multiple-time-scale systems: E.g. 2D Tearing Mode
Low Mach number compressible; M ~ 10^{-4}; Fully-implicit (BDF2), IMEX (SSP3)



<u>Approx. Computational Time Scales:</u>

- **Divergence Constraint ($\nabla \cdot B = 0$):1/∞ = 0**
- **Fast Magnetosonic Wave (c_f): 10^{-4} to 10^{-2}**
- **Alfven Wave (c_a): 10^{-4} to 10^{-2}**
- **Slow Magnetosonic Wave (c_s): 10^{-2} to 10^{-1}**
- **Sound Wave (c): 10^{-2}**

- **Advection (c_v): ∞ to 10^1**
- **Diffusion: 10^{-4} to 10^{-2}**
- **Macroscopic Tearing Mode: 10^2**

Fully-implicit (BDF2) / IMEX SSP3
Max CFL:

$CFL_{div} = \infty$
$CFL_{cf} \sim 10^5$ to 10^4
$CFL_{cA} \sim 10^5$ to 10^4
$CFL_{cs} \sim 10^3$ to 10^2
$CFL_{c} \sim 10^3$ to 10^2
$CFL_{cv} \sim 1$ to 0.25

<u>Wave speeds</u>

$$\|\mathbf{u}\|, \|\mathbf{u}\| \pm c_s, \|\mathbf{u}\| \pm c_a, \|\mathbf{u}\| \pm c_f, \pm c_h$$

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Summary of Structure of Linear Systems Generated in Newton's Method

$$\mathcal{J}\,\Delta\mathbf{x} = -\mathcal{F}$$

VMS Stabilize produces SUPG like terms, stabilizing terms for inf-sup condition (all Q1 interpolants), physics cross-coupling terms and discontinuity Capturing type operators

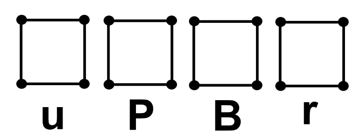
$$\mathcal{J} = \begin{bmatrix} F & B_p^T & Z & \\ B_p & C_u & & \\ Y & & D & B_r^T \\ & & B_r & C_B \end{bmatrix}$$

$$\mathbf{x} = [\mathbf{v}, P, \mathbf{B}, r]^T$$

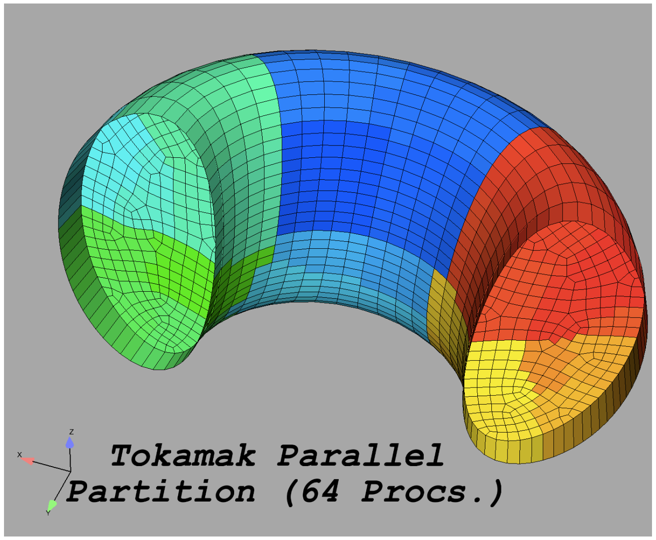
$$\mathcal{F} = [\mathbf{F}_{\mathbf{v}}, F_P, \mathbf{F}_{\mathbf{B}}, F_r]^T$$

$$C_u = \sum_e \int_{\Omega^e} \rho \tau_m \nabla\Phi \cdot \nabla\Phi \; d\Omega$$

$$C_B = \sum_e \int_{\Omega^e} \tau_B \nabla\Phi \cdot \nabla\Phi \; d\Omega$$



Preconditioning



Three variants of preconditioning

1. Domain Decomposition (Trilinos/Aztec & IFPack)

- 1 –level Additive Schwarz DD
- ILU(k) Factorization on each processor
  (with variable levels of overlap)
- High parallel eff., non-optimal algorithmic scalability

2. Multilevel Methods for Systems: ML pkg (Tuminaro, Sala, Hu, Siefert, Gee)

Fully-coupled Algebraic Multilevel methods

- Consistent set of DOF-ordered blocks at each node (e.g. Stabilized FE)
- Uses block non-zero structure of Jacobian
- Aggregation techniques and rates can be chosen
- Jacobi, GS, ILU(k) as smoothers
- Can provide optimal algorithmic scalability

(See Paul Lin Talk)

3. Approximate Block Factorization / Physics-based (Teko package)

- Applies to mixed interpolation (FE), staggered (FV), physics compatible discretization approaches using segregated unknown blocking
- Applies to systems where coupled AMG is difficult or might fail
- Enables specialized AMG, e.g. H(grad), H(curl) to be applied to distinct discretizations.
- Can provide optimal algorithmic scalability for coupled systems

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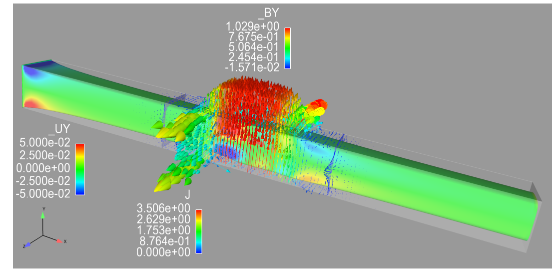
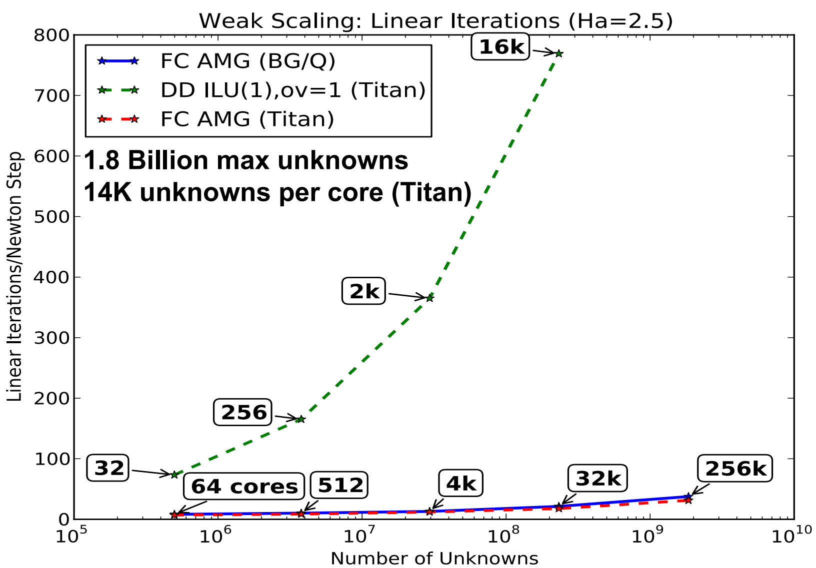
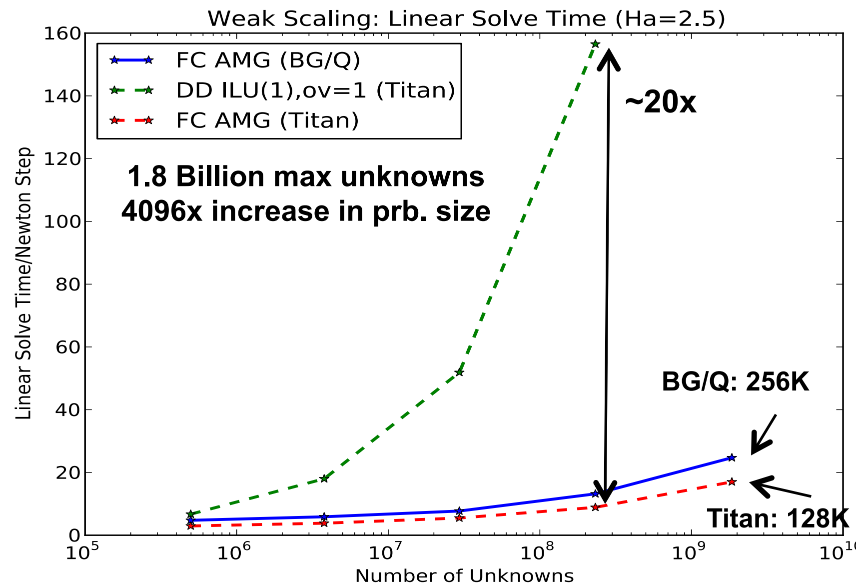
SFE Initial Scaling Studies for Cray XK7 AND BG/Q. 3D MHD Generator [Re = 500, Re_m = 1, Ha = 2.5]







Details: See Paul Lin's Talk Afternoon Session

(DOE/ORNL Titan Cray XK7: Joule Metric)

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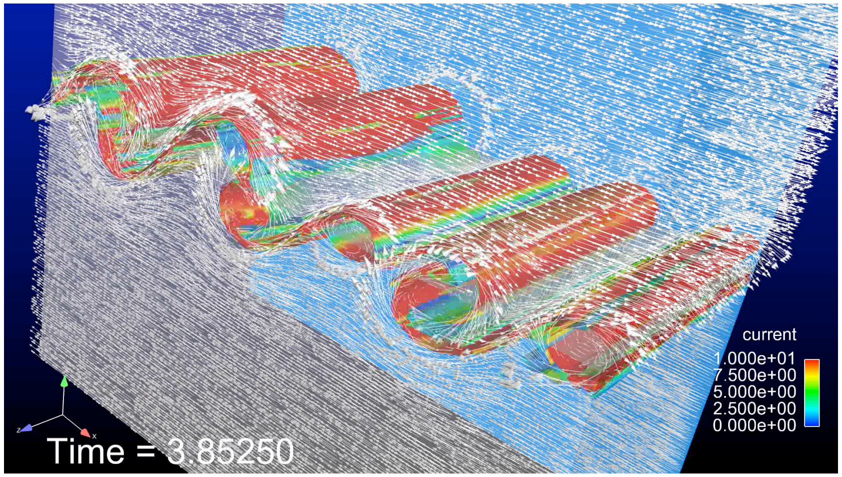
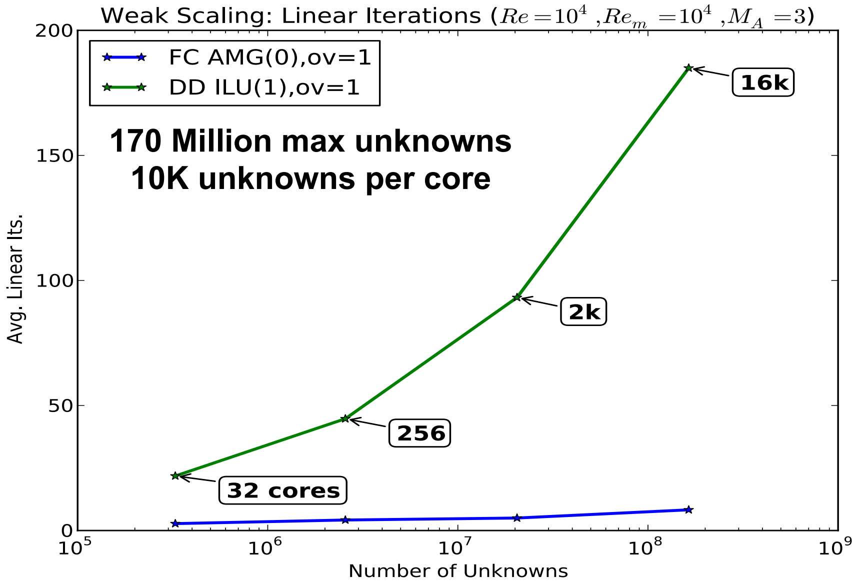
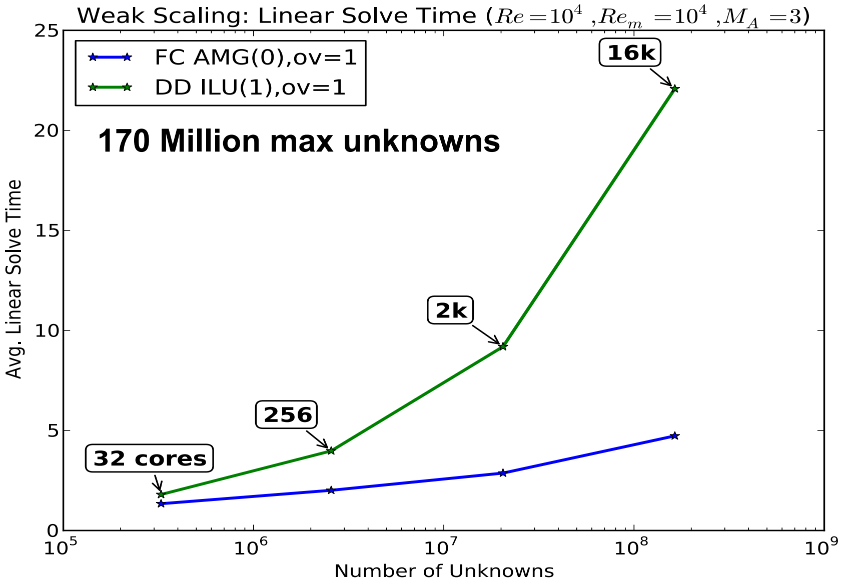
Initial Scaling Study for Cray XK7.
3D Hydromagnetic Kelvin-Helmholtz Instability
[$Re = 10^4$, $Re_m = 10^4$, $M_A = 3$; CFL_{max} ~5]







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Physics-based and Approximate Block Factorizations: Coercing Strongly Coupled Off-Diagonal Physics / Disparate Discretizations and Scalable Multigrid to play well together

Physics-based (Parabolization):

$$\partial_t u = \partial_x v \ , \ \partial_t v = \partial_x u.$$

$$u^{n+1} = u^n + \Delta t \partial_x v^{n+1}, v^{n+1} = v^n + \Delta t \partial_x u^{n+1}.$$

$$(I - \Delta t^2 \partial_{xx}) u^{n+1} = u^n + \Delta t \partial_x v^n$$

Schur Complement, (Approximate) Block Factorization:

$$\begin{bmatrix} I & -\Delta t C_x \\ -\Delta t C_x & I \end{bmatrix} \begin{bmatrix} u^{n+1} \\ v^{n+1} \end{bmatrix} = \begin{bmatrix} u^n - \Delta t C_x v^n \\ v^n - \Delta t C_x u^n \end{bmatrix}$$

$$\begin{bmatrix} D_1 & U \\ L & D_2 \end{bmatrix} = \begin{bmatrix} I & U D_2^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} D_1 - U D_2^{-1} L & 0 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ D_2^{-1} L & I \end{bmatrix}$$

The Schur complement is then

$$D_1 - U D_2^{-1} L = (I - \Delta t^2 C_x C_x) = (I - \Delta t^2 \partial_{xx})$$

Result:

1) Stiff (large-magnitude) off-diagonal hyperbolic type operators (blocks) are now combined onto diagonal parabolic type operator (block).

2) Coupled physics is segregated into partitioned sub-systems to which specialized solvers can be applied (optimal AMG methods e.g. H(grad), H(curl))

3) Effective Schur complement approximations need to preserve strong cross-coupling physics and time scales.

Knoll, Chacon et. al. JFNK Methods for accurate time integration of stiff-wave systems, Journal of Scientific Computing, 2005
L. Chacon, ``An optimal, parallel, fully implicit Newton-Krylov solver for three-dimensional visco-resistive magnetohydrodynamics," Phys. Plasmas, 2008
Elman, Howle, Shadid, and Tuminaro, "A Parallel Block Multi-level Preconditioner for the Three-Dimensional Incompressible Navier-Stokes", JCP, 2003
Elman, Howle, Shadid, Shuttleworth, Tuminaro,"A Taxonomy of Parallel Mulit-level Block Preconditioners for the Incompressible Navier-Stokes", JCP, 2008
Cyr, Shadid, Tuminaro, Pawlowski, Chacon, "A new approximate block factorization preconditioner for 2D incompressible (reduced) resistive MHD," SISC, 201

Incompressible Resistive MHD a Nested Schur Complement Approach

Block LU factorization gives

$$\begin{bmatrix} F & B^T & Z \\ B & C & 0 \\ Y & 0 & D \end{bmatrix} = \begin{bmatrix} I & & \\ BF^{-1} & I & \\ YF^{-1} & -YF^{-1}B^TS^{-1} & I \end{bmatrix} \begin{bmatrix} F & B^T & Z \\ & S & -BF^{-1}Z \\ & & P \end{bmatrix}$$

$$S = C - BF^{-1}B^T$$

$$P = D - YF^{-1}(I + B^TS^{-1}BF^{-1})Z$$

- **3x3 system leads to embedded Schur complements**
- **Embedding is independent of ordering (C^{-1} doesn't need to exist!)**
- **How is *P* approximated?**
- **Chacon & Knoll (2004,..) explored compressible flow ($\frac{\partial \rho}{\partial t}$ included in C) and incompressible flow using stream-function vorticity to simplify factors (i,e, eliminate $\nabla \cdot \mathbf{v} = 0$ elliptic constraint).**
- **Can we simplify nested structure? E.g. Operator split prec.**

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Operator split / Residual-based Defect-Correction ABF Preconditioner

1) **Residual defect-correction factorization procedure strongly couples operators producing the pressure wave (infinite speed) and Alfven wave (finite speed) and reduces to two 2x2 blocks for the ABF:**

$$\begin{bmatrix} F & B^T & Z \\ B & C & 0 \\ Y & 0 & D \end{bmatrix} \approx \begin{bmatrix} F & & Z \\ & I & \\ Y & & D \end{bmatrix} \begin{bmatrix} F^{-1} & & \\ & I & \\ & & I \end{bmatrix} \begin{bmatrix} F & B^T & \\ B & C & \\ & & I \end{bmatrix} = \begin{bmatrix} F & B^T & Z \\ B & C & \\ Y & \boxed{YF^{-1}B^T} & D \end{bmatrix}$$

2) **3x3 -> two 2x2 sub-systems that contain dominate coupling (pressure wave, Alfven wave)**

$$\mathcal{S} = C_u - B\hat{F}^{-1}B^T$$

<u>Relation to pressure wave (elliptic constraint)</u>
Formally assume
- **Stable mixed integration**
- **Transient with small time-step-size**

$$\hat{\mathbf{S}} = -\mathbf{B}[\Delta t \mathbf{Q}_L^{-1}]\mathbf{B}^T \approx -\Delta t \tilde{\mathbf{L}}_p$$

$$\mathcal{P} = D - Y\hat{F}^{-1}Z$$

<u>Relation to Alfven wave</u>
Formally assume a stiff wave linearized analysis (with $\mathbf{u}_0 = \mathbf{0} \ \ \& \ \ \mathbf{B}_0 = \text{const.}$)

$$\frac{\partial^2 \mathbf{B}_1}{\partial t^2} - \frac{1}{\rho\mu_0}\nabla\times([(\nabla\times\mathbf{B}_1)\times\mathbf{B}_0]\times\mathbf{B}_0) = 0.$$

$$\frac{\partial^2 A_{z1}}{\partial t^2} - \frac{\|\mathbf{B}_0\|^2}{\rho\mu_0}\nabla\cdot\nabla A_{z1} = 0.$$

Consider NS Schur complement methods (e.g. Pressure Proj., SIMPLEC), Press-Conv-Diff (PCD) and Least Squares comutator (LSC) type approaches)

See e.g. Elman, Howle, S., Shuttleworth, Tuminaro,"A Taxonomy of Parallel Mulit-level Block Preconditioners for the Incompressible Navier-Stokes Equations", JCP, v. 227, 3, pp 1790 - 1808, 2008

Cyr, S., Tuminaro, Pawlowski, and Chacon, "A new approximate block factorization preconditioner for 2D incompressible (reduced) resistive MHD," SIAM Journal on Scientific Computing, 35:B701-B730, 2013

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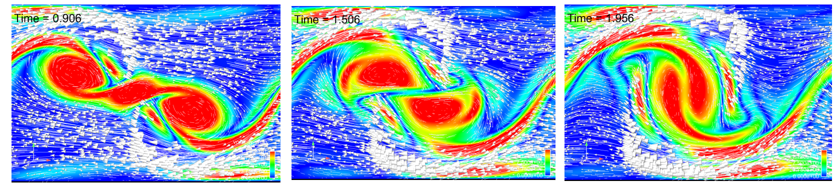
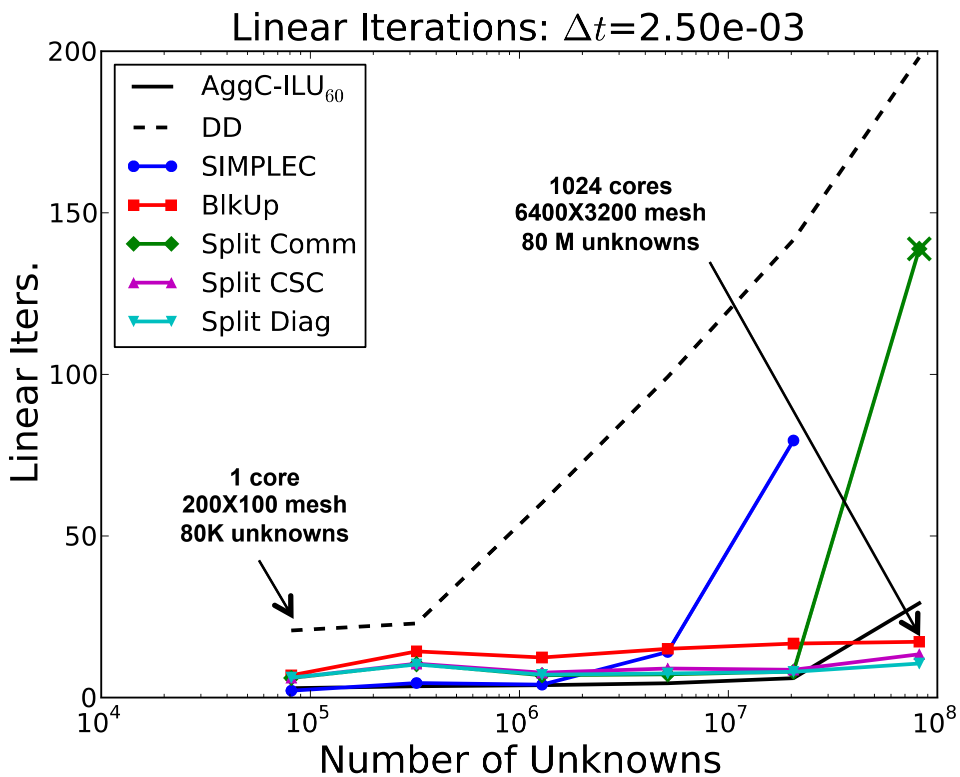
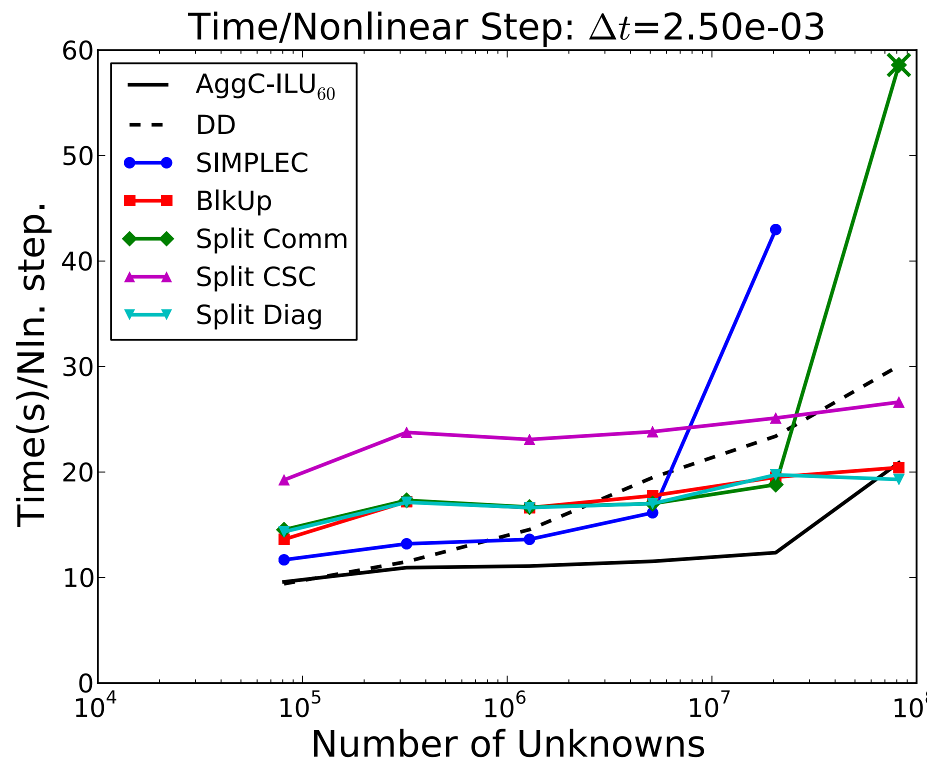
Transient 2D Hydromagnetic Kelvin-Helmholtz Problem, SFE

Re = 5e+3, S = 1e+3; M_A = 1.5; CFL_{max} ~ 10







Comm – comutator; CSC – continuous Schur comp.; Diag. – diagonal approx of inverse in Schur comp.

Quad-core Nehalems with Infini-band SNL Red Sky

Cyr, S., Tuminaro, Pawlowski, and Chacon, "A new approximate block factorization preconditioner for 2D incompressible (reduced) resistive mhd," SIAM Journal on Scientific Computing, 35:B701-B730, 2013

Cyr, S., and Tuminaro, "Teko an abstract block preconditioning capability with concrete example applications to Navier-Stokes and resistive MHD," Submitted to SISC.

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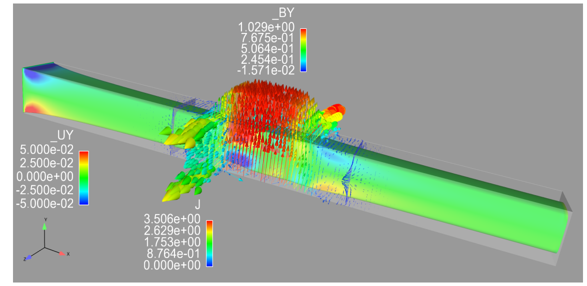
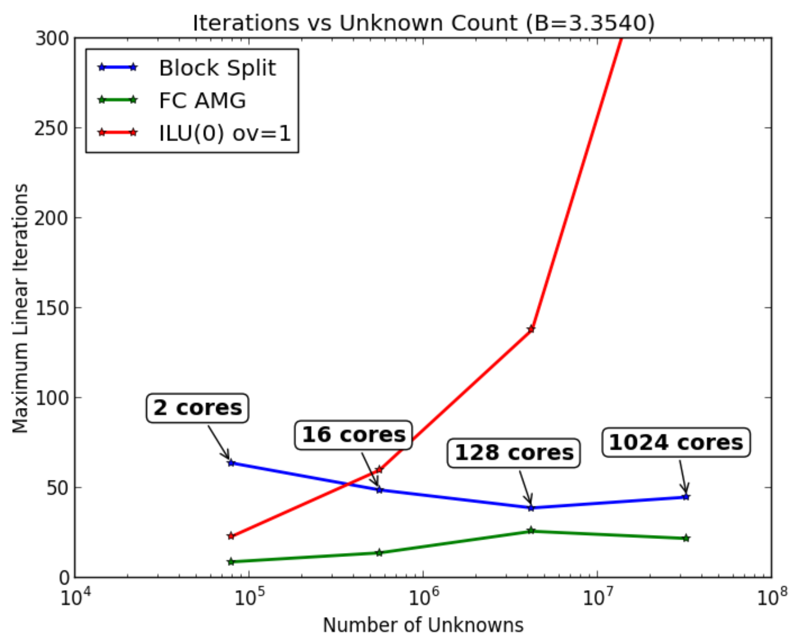
Extensions to 3D: Initial Approximate Block Preconditioning
3D MHD Generator [Re = 500, Re_m = 1, Ha = 2.5], SFE

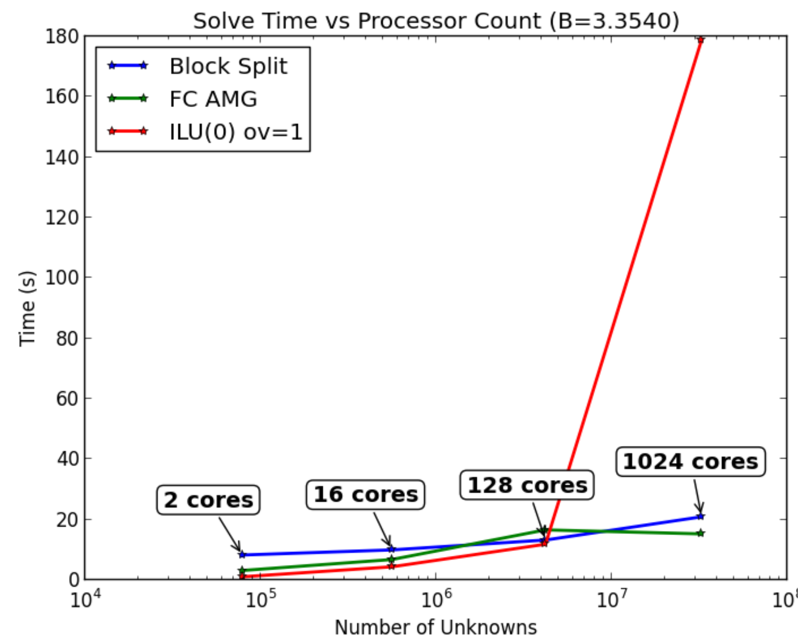


$$\mathcal{J} = \begin{bmatrix} F & B_p^T & Z & \\ B_p & C_u & & \\ Y & & D & B_r^T \\ & & B_r & C_B \end{bmatrix} \overset{\Rightarrow}{\equiv} \begin{bmatrix} F & B_p^T & \hat{Z} \\ B_p & C_u & \\ \hat{Y} & & \hat{D} \end{bmatrix}$$

$$\mathcal{J} \approx \mathcal{M}_{Split} = \begin{bmatrix} F & & Z \\ & I & \\ Y & & \hat{D} \end{bmatrix} \begin{bmatrix} F^{-1} & & \\ & I & \\ & & I \end{bmatrix} \begin{bmatrix} F & B^T & \\ B & C & \\ & & I \end{bmatrix}$$

$$S = C - BF^{-1}B^T$$
$$\hat{P} = \hat{D} - YF^{-1}Z$$





Weak scaling of FC-AMG and block preconditioners reasonable to 1024 cores
Both suffer some performance degradation on this capacity machine (Redsky)

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New residual defect-correction ABF strongly couples Alfven wave operators and reduces to three 2x2 blocks

$$M_1 \tilde{x} = b \; ; \quad M_2(\hat{x} - \tilde{x}) = (b - \mathcal{J}\tilde{x}) \; ; \text{leads to this ABF} \quad \hat{x} = M_2^{-1}(M_1 + M_2 - \mathcal{J}) M_1^{-1} b$$

$$\begin{bmatrix} F_m & B^T & Z & 0 \\ B & C_P & 0 & 0 \\ Y & 0 & F_B & B^T \\ 0 & 0 & B & C_\psi \end{bmatrix} \approx \begin{bmatrix} F_m & 0 & Z & 0 \\ 0 & I & 0 & 0 \\ Y & 0 & F_B & I \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} F_m{}^{-1} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & F_B{}^{-1} & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} F_m & B^T & 0 & 0 \\ B & C_P & 0 & 0 \\ 0 & 0 & F_B & B^T \\ 0 & 0 & B & C_\psi \end{bmatrix}$$

$$= \begin{bmatrix} F_m & B^T & Z & \boxed{Z F_B^{-1} B^T} \\ B & C_P & 0 & 0 \\ Y & \boxed{Y F_m^{-1} B^T} & F_B & B^T \\ 0 & 0 & B & C_\psi \end{bmatrix}$$

- **Order-of-magnitude analysis of structural error terms for ABF and previous work on 2D and 3x3 systems suggests diagonal, and comutator approaches should be workable in appropriate parameter regimes.**

- **Reduction to 2 problem types that are similar to what we have studied and developed Schur complement approaches for**
  - **Saddle point systems** $S_m = C_P - B\hat{F}_m^{-1}B^T \; ; \quad S_B = C_\psi - B\hat{F}_B^{-1}B^T$
  - **Momentum-magnetics coupling** $P = F_B - Y\hat{F}_m^{-1}Z$

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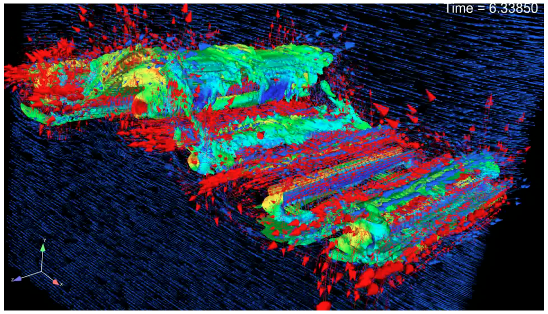
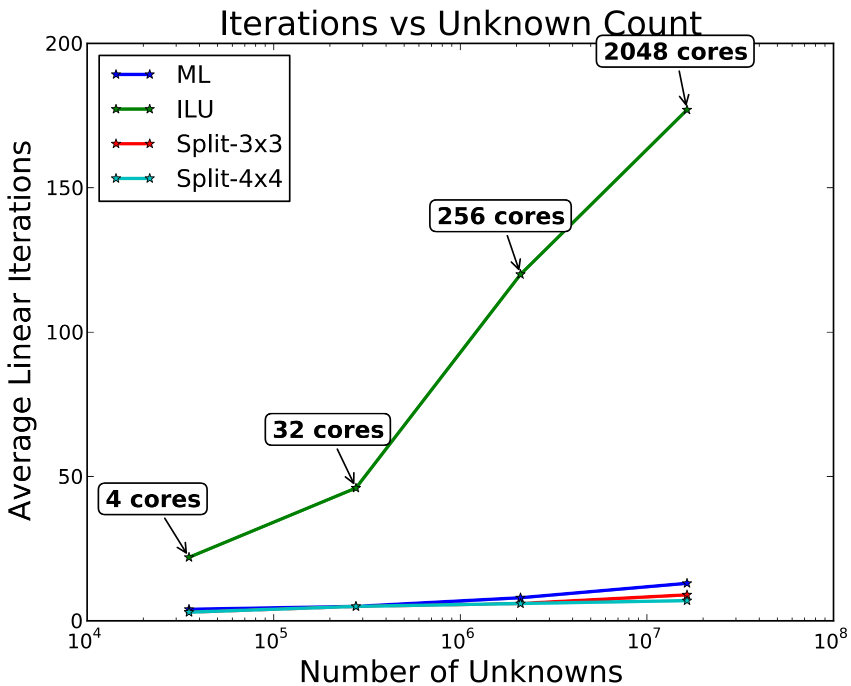
Extensions to 3D: Initial Approximate Block Preconditioning

3D HMKH [Re $=10^4$, Rem$=10^4$, $M_A = 3$; CFL ~0.125], SFE

FC-AMG – ILU(0), V(3,3); 3x3, 4x4 SIMPLEC and Gauss-Seidel

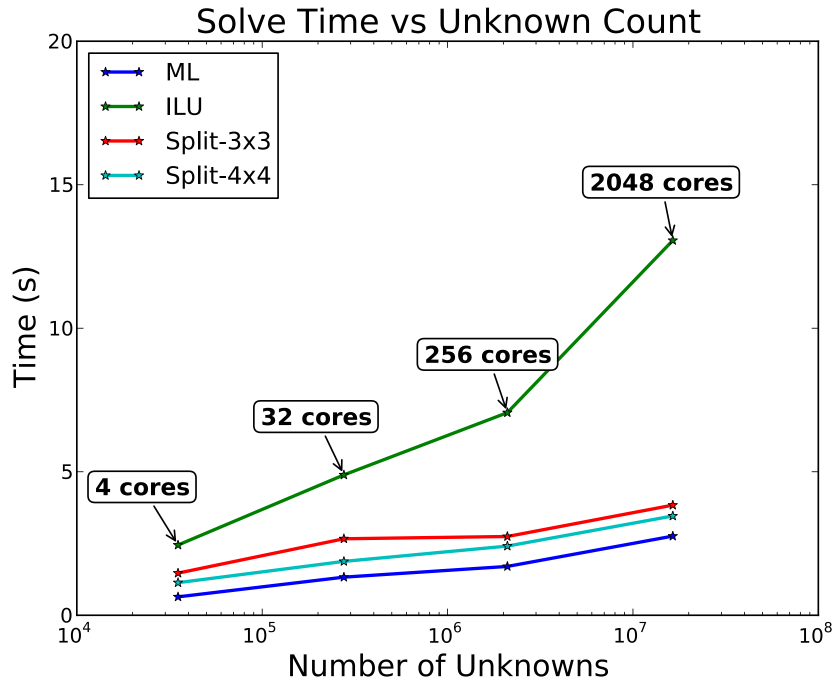




<u>Fully coupled Algebraic</u>

ML: Uncoupled AMG with repartitioning

DD: Additive Schwarz Domain Decomposition



<u>Block Preconditioners</u>

<u>Split-3x3</u>: 3x3 (SIMPLEC everywhere)

<u>Preliminary Split-4x4</u>: 4x4

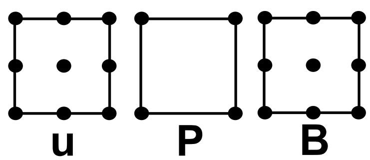
ABF preconditioners scale algorithmically, more relevant for mixed and physics-compatible discretizations

General Structure of Newton System: $\mathcal{J}\Delta\mathbf{x} = -\mathcal{F}$

$$\mathcal{J} = \begin{bmatrix} F & B_p^T & Z \\ B_p & 0 & 0 \\ Y & 0 & D \end{bmatrix} \qquad \begin{array}{l} \mathbf{x} = [\mathbf{v}, P, \mathbf{B} \quad]^T \\ \mathcal{F} = [\mathbf{F_v}, F_P, \mathbf{F_B} \quad]^T \end{array}$$

Exact Penalty Formulation: (Q2/Q1 Navier-Stokes, Q2 B field; see e.g. Gunzburger et. al. 1991, Phillips et. al. 2014)

Mixed basis*:



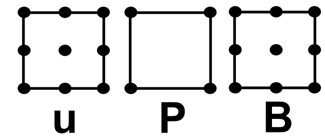
Weak correspondence to parabolic divergence cleaning method for $\nabla \cdot \mathrm{B}$ errors (see e.g. Dedner 2002).

Physics-based and Approximate Block Factorizations: Coercing Strongly Coupled Off-Diagonal Physics / Disparate Discretizations and Scalable Multigrid to play well together

(w/ H. Elman, UMD)

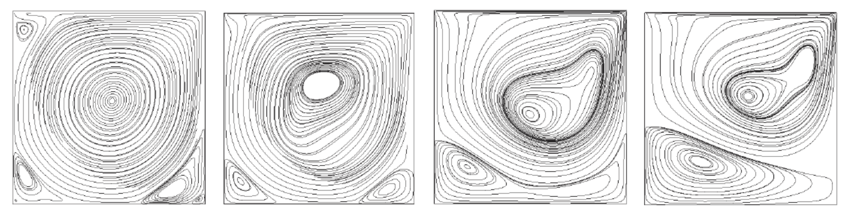
Exact Penalty Formulation: **(Q2/Q1 Navier-Stokes, Q2 B field; see e.g. Gunzburger)**

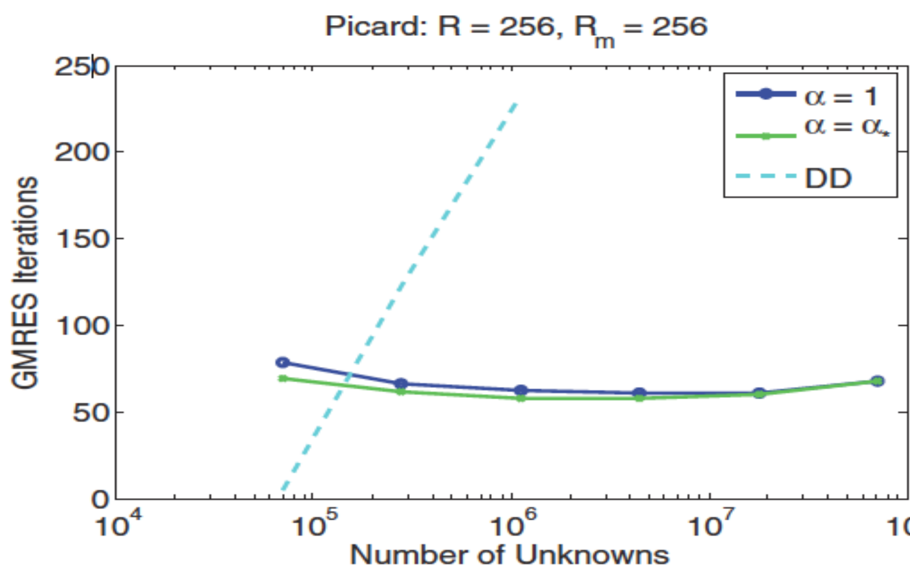
Mixed basis*:

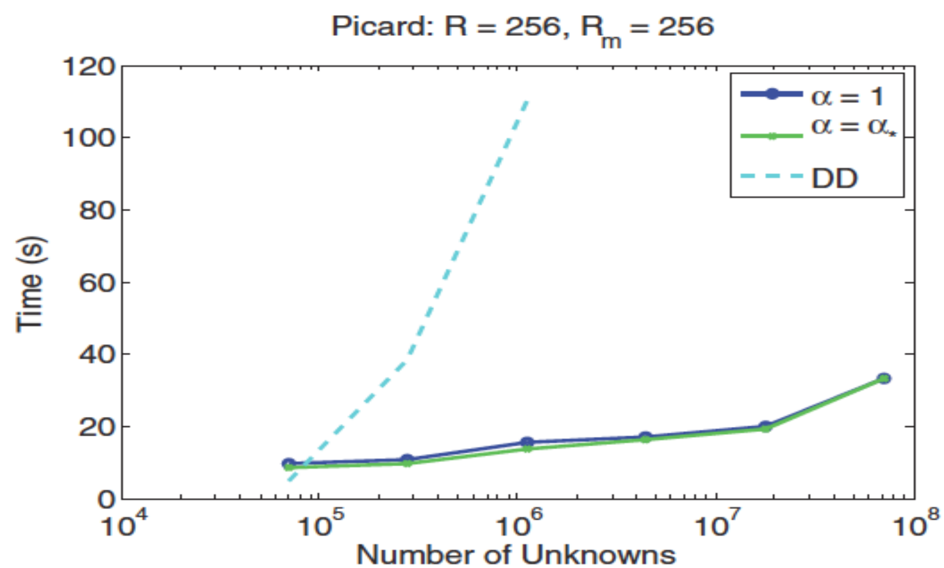


$$\mathcal{A}_P = \begin{pmatrix} F & B^t & Z \\ B & 0 & 0 \\ -Z^t & 0 & A \end{pmatrix} \qquad \mathcal{P}_{P,\alpha} = \begin{pmatrix} \hat{A} & -Z^t & 0 \\ 0 & \hat{X} & B^t \\ 0 & 0 & \hat{Y}_\alpha \end{pmatrix} \qquad \begin{aligned} X &= F + ZA^{-1}Z^t, \\ Y &= -BX^{-1}B^t. \end{aligned}$$

$$\mathbf{x} = [\mathbf{u}, P, \mathbf{B}]^T \qquad \mathbf{x} = [\mathbf{B}, \mathbf{u}, P]^T$$







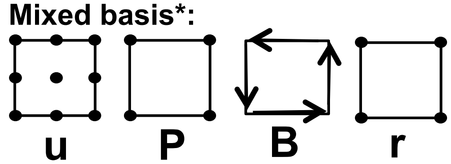
Phillips, Elman, Cyr, S., Pawlowski, A Block Preconditioner for an Exact Penalty Formulation for Stationary MHD, SISC 2014

General Structure of Newton System: $\mathcal{J}\Delta\mathbf{x} = -\mathcal{F}$

$$\mathcal{J} = \begin{bmatrix} F & B_p^T & Z & \\ B_p & 0 & & \\ Y & & D & B_r^T \\ & & B_r & 0 \end{bmatrix} \qquad \begin{aligned} \mathbf{x} &= [\mathbf{v}, P, \mathbf{B}, r]^T \\ \mathcal{F} &= [\mathbf{F}_{\mathbf{v}}, F_P, \mathbf{F}_{\mathbf{B}}, F_r]^T \end{aligned}$$

Shotzau Formulation: **(Q2/Q1 Navier-Stokes, B -edge, Q1 Lagrange Multiplier, see e.g. Shotzau 2004)**

Mixed basis*:



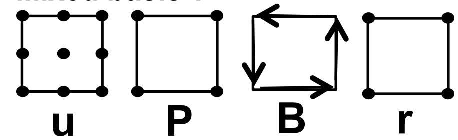
$\nabla \cdot \mathbf{J} = 0$ **to machine precision.**

Drekar – Element types implemented with
***Intrepid (PI-Bochev, Ridzal, Peterson)**

Shotzau Formulation: (Q2/Q1 Navier-Stokes, B -edge, Q1 Lagrange Multiplier)

Structure of preconditioner and Maxwell ABF (w/ H. Elman, UMD)

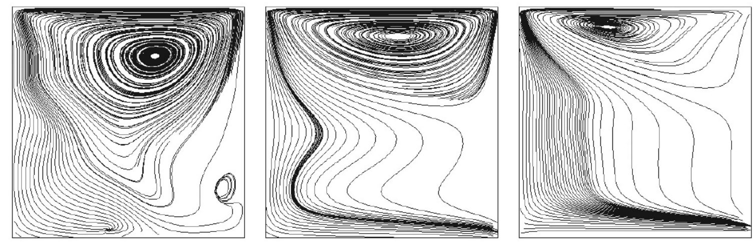
Mixed basis*:



$$\mathcal{A}_P = \begin{pmatrix} A & D^t & -Z^t & 0 \\ D & 0 & 0 & 0 \\ Z & 0 & F & B^t \\ 0 & 0 & B & 0 \end{pmatrix} \qquad \mathcal{P}_P = \begin{pmatrix} \hat{\mathcal{M}}_P & -\mathcal{Z}^t & 0 \\ 0 & \hat{X} & B^t \\ 0 & 0 & \hat{Y} \end{pmatrix}$$

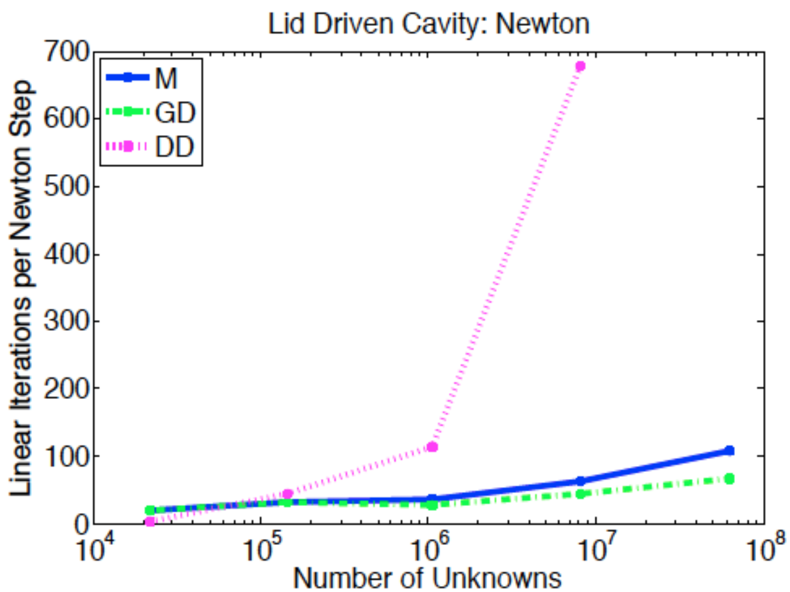
$$\hat{\mathcal{M}}_{P,2} = \begin{pmatrix} A + tD^t\bar{Q}_r^{-1}D & 0 \\ 0 & \frac{1}{t}\bar{Q}_r \end{pmatrix}$$

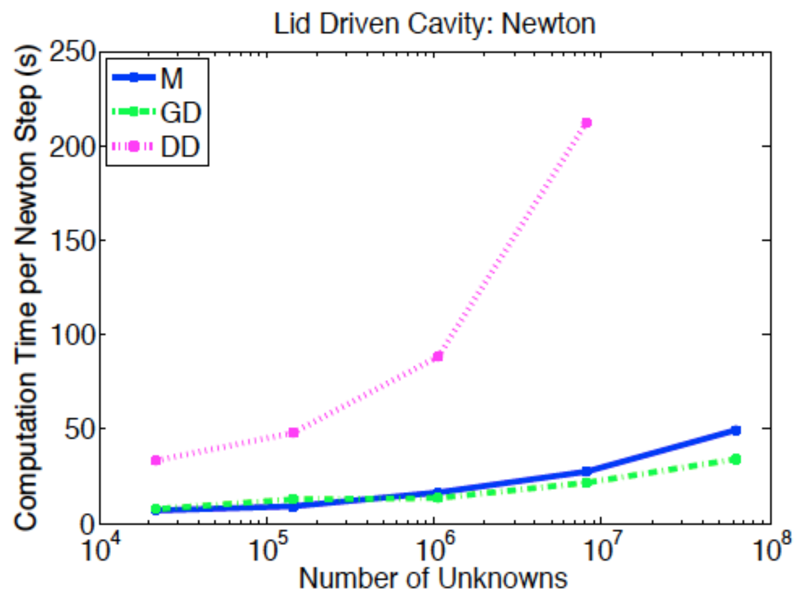
$$\hat{X} \approx F + \mathcal{Z}\hat{\mathcal{M}}_P^{-1}\mathcal{Z}^t, \quad Y = -B\hat{X}^{-1}B^t$$



$Re_m = 1$ $\quad$ $Re_m = 10$ $\quad$ $Re_m = 100$

Segregation into
- **H(grad) system AMG for velocity,**
- **Scalar H(grad) AMG for pressure,**
- **H(curl) AMG for magnetic field**





- Number of processors: 1, 8, 64, 512, 4096, $h = \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$
- $Re = 100$, $Re_m = 10$, $S = 1$

Drekar – Element types implemented with
***Intrepid (Bochev, Ridzal, Peterson)**

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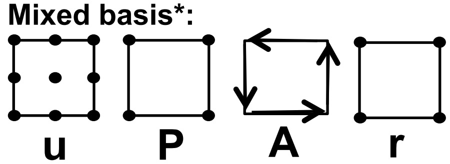
General Structure of Newton System: $\mathcal{J}\Delta\mathbf{x} = -\mathcal{F}$

$$\mathcal{J} = \begin{bmatrix} F & B_p^T & Z & \\ B_p & 0 & & \\ Y & & D & B_r^T \\ & & B_r & C_{\mathbf{A}} \end{bmatrix} \qquad \begin{aligned} \mathbf{x} &= [\mathbf{v}, P, \mathbf{A}, r]^T \\ \mathcal{F} &= [\mathbf{F_v}, F_P, \mathbf{F_A}, F_r]^T \end{aligned}$$

Magnetic Vector Potential Formulation: (Q2/Q1 Navier-Stokes, A -edge, Q1 Scalar Potential)

Mixed basis*:

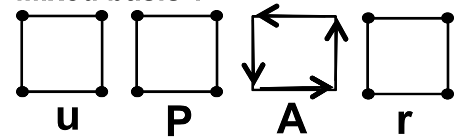


$\nabla \cdot \mathbf{B} = 0$ **to machine precision.**

Magentic Vector-Potential Formulation: (Q1 Navier-Stokes, A-edge, Q1 Lagrange Multiplier)

Structure of preconditioner and Maxwell ABF

Mixed basis*:

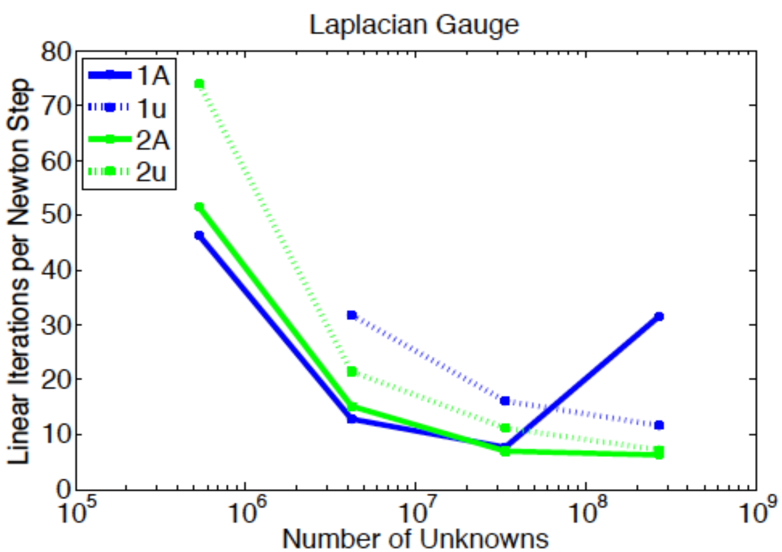


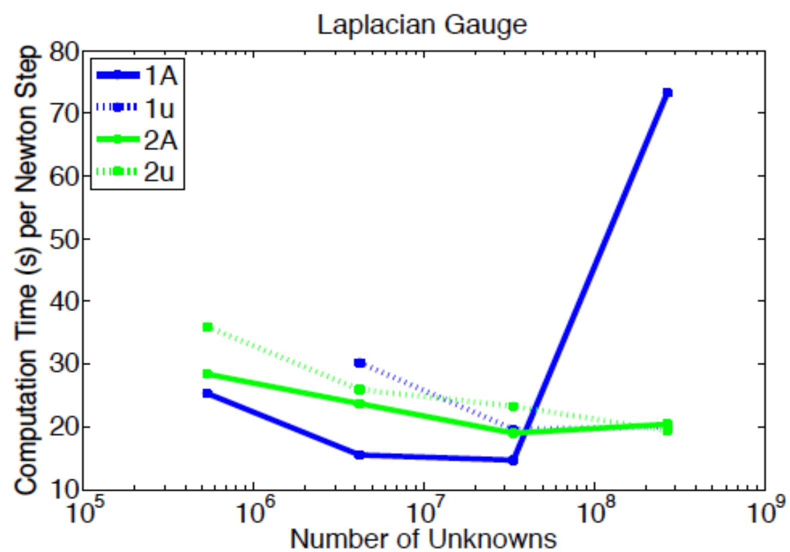
$$\begin{pmatrix} F & B^t & Z \\ B & C & 0 \\ Y & 0 & G \end{pmatrix} \approx \begin{pmatrix} F & 0 & Z \\ 0 & I & 0 \\ Y & 0 & G \end{pmatrix} \begin{pmatrix} F^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} F & B^t & 0 \\ B & C & 0 \\ 0 & 0 & I \end{pmatrix}$$

$$\text{or} \begin{pmatrix} F & B^t & 0 \\ B & C & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} F^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} F & 0 & Z \\ 0 & I & 0 \\ Y & 0 & G \end{pmatrix}$$

Segregation into

- **H(grad) system AMG for velocity,**
- **Scalar H(grad) AMG for pressure, Lagrange multiplier**
- **H(curl) AMG for magnetic vector potential**





Conclusions

- **Initial results for 3D Stabilized/VMS FE Lagrange multiplier and structure preserving discretizations for low-flow Mach number resistive MHD system is very encouraging.**

- **Robustness, efficiency and scalability of parallel Newton-Krylov solvers is very good.**

Preconditioning critical:

  - **FC-AMG (ML) scales well for stabilized / VMS 3D resistive MHD systems**
  - **ABF methods must have effective approximation of dominate off-diagonal coupling and time-scale represented.**
    - **ABF & AMG H(grad) results are encouraging for VMS Lagrange multiplier formulations.**
    - **Initial ABF & AMG [H(grad) + H(curl)] results for structure preserving formulations is promising**

- **Fully-implicit Newton-Krylov solvers have enabled initial adjoint based UQ capabilities**
  - **a posteriori error-estimation**
  - **Integrated sensitivity analysis**
  - **efficient surrogate model construction for UQ**

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