

**Gate-set tomography:  
calibration-free full characterization  
of quantum devices  
using error-amplifying circuits**

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# Part 1:

# Quantum Information

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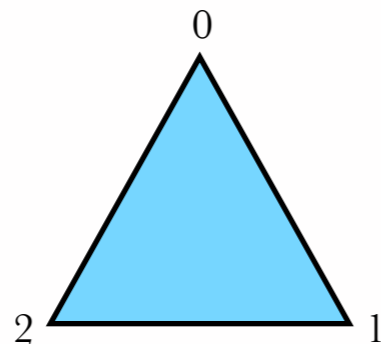
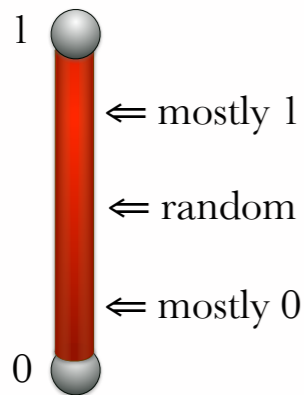
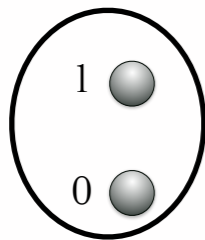
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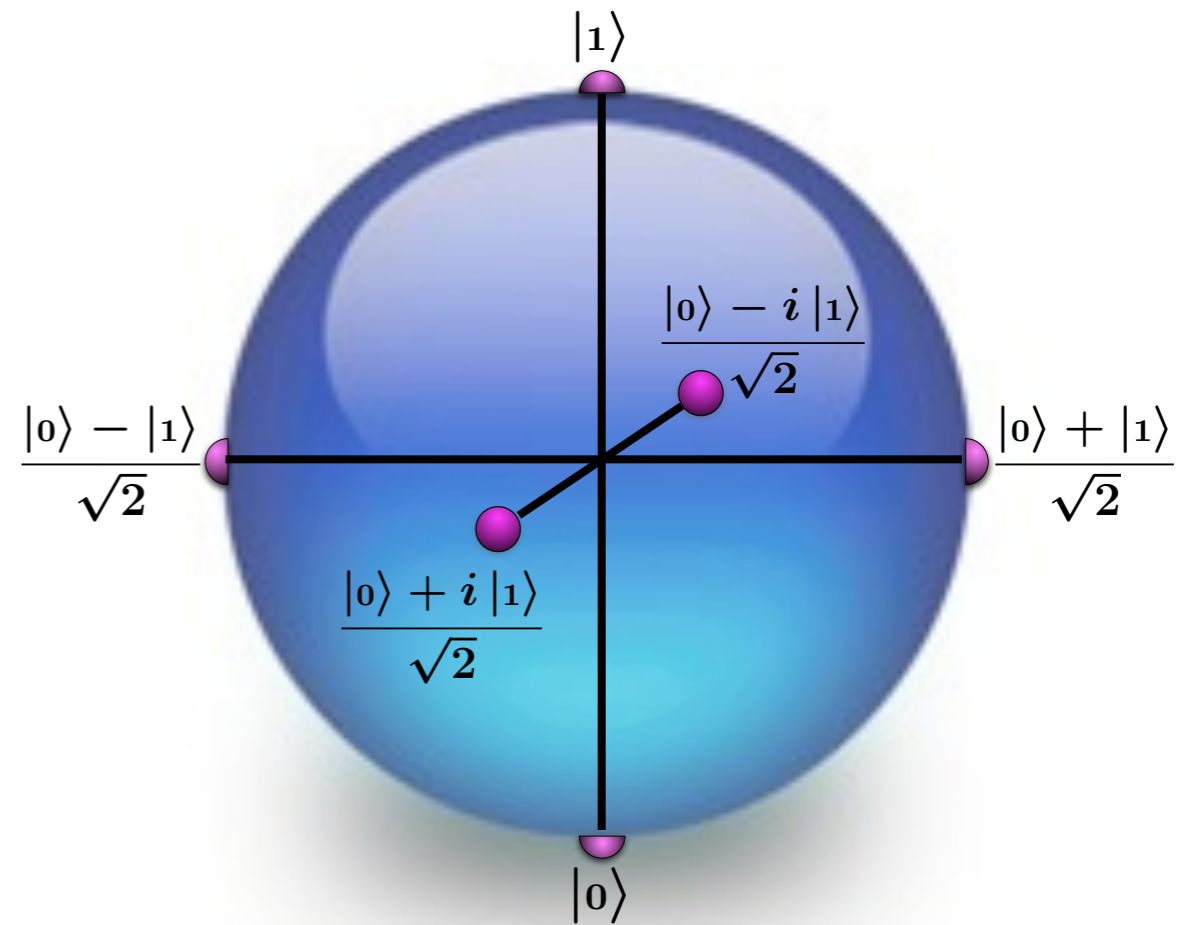
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# Qubits and Qudits



## Quantum Bit & the Bloch Sphere

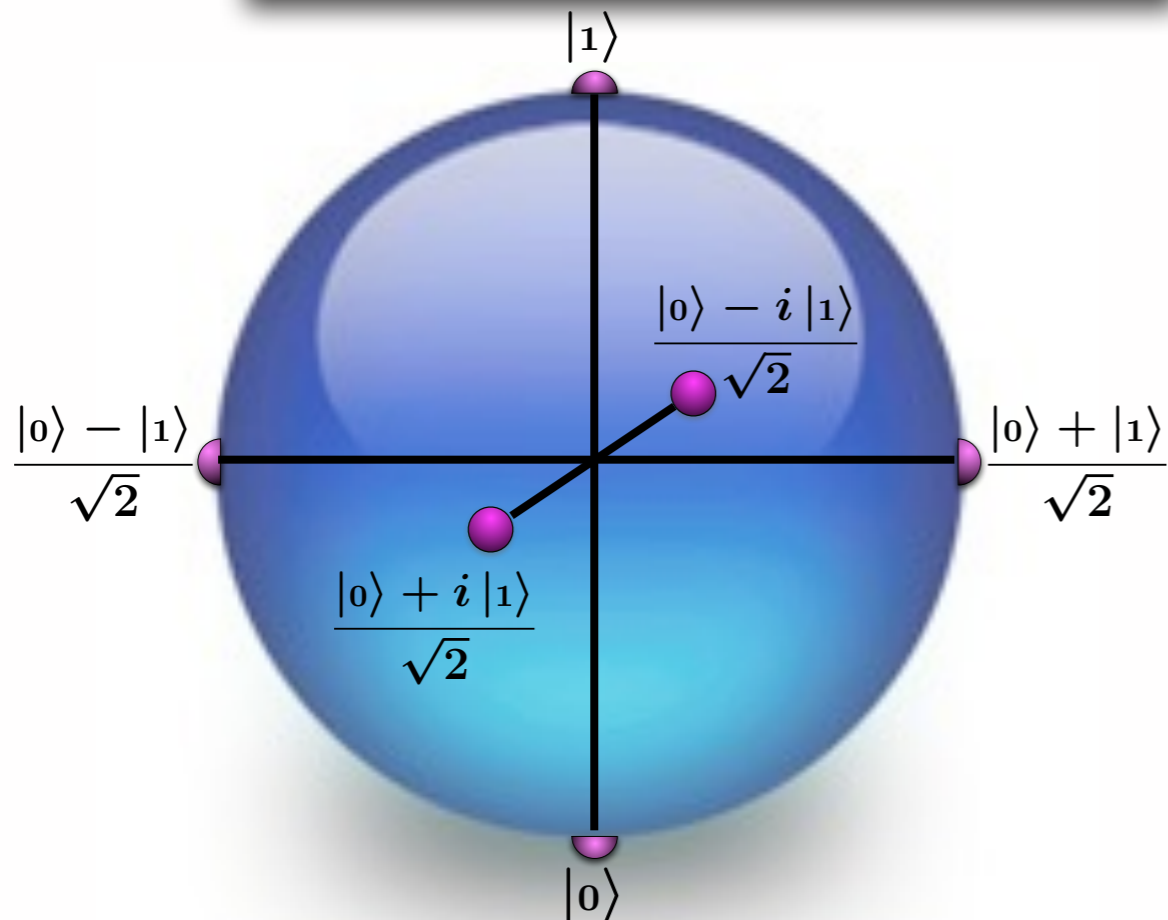


# The Bloch Sphere

Pure quantum states (state vectors) are like classical *definite* states -- e.g. “on” or “off” or “0” or “1”.

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

$$= \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$



Mixed quantum states (density matrices) are like classical probability distributions.

$$\rho = |\psi\rangle\langle\psi| = \vec{\psi}\vec{\psi}^* = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}$$

or

$$\rho = p_1 |\psi_1\rangle\langle\psi_1| + p_2 |\psi_2\rangle\langle\psi_2| \dots$$

so

$$\rho = \frac{1}{2} \mathbf{1} + x\mathbf{X} + y\mathbf{Y} + z\mathbf{Z}$$

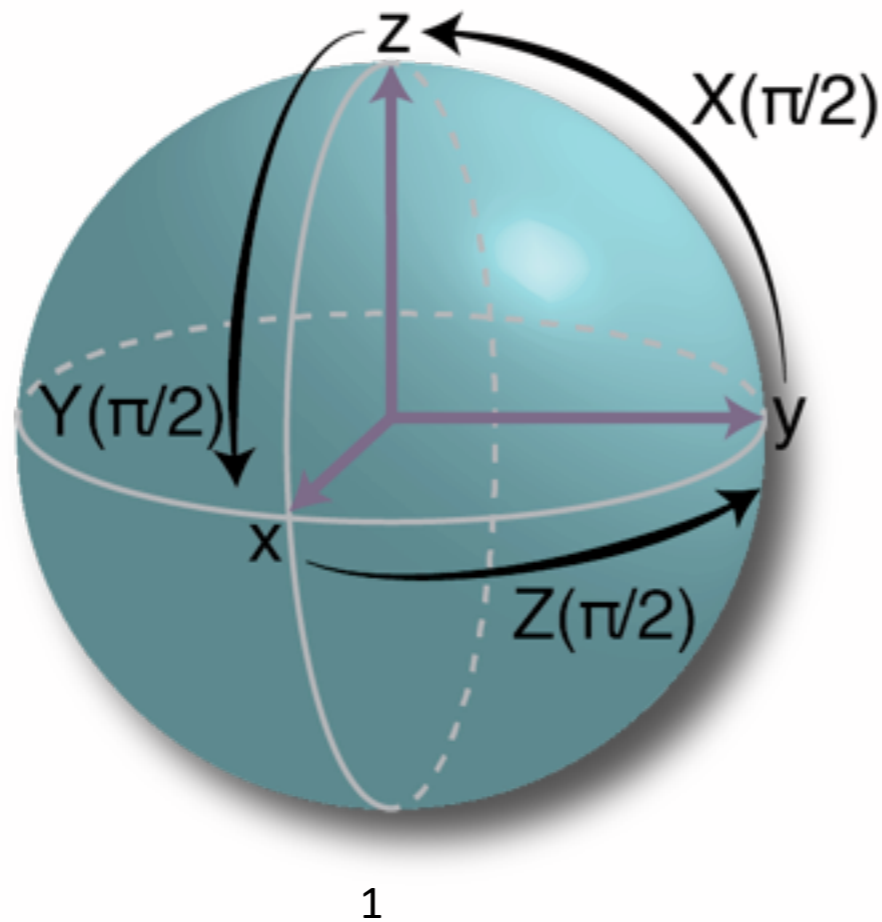
$$|\rho\rangle\rangle = \begin{pmatrix} 1/2 \\ x \\ y \\ z \end{pmatrix}$$

# Quantum Logic Gates

classical  
logic gates



quantum  
logic gates



Ideal logic gate  
=  
Unitary matrix

$$\mathbf{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

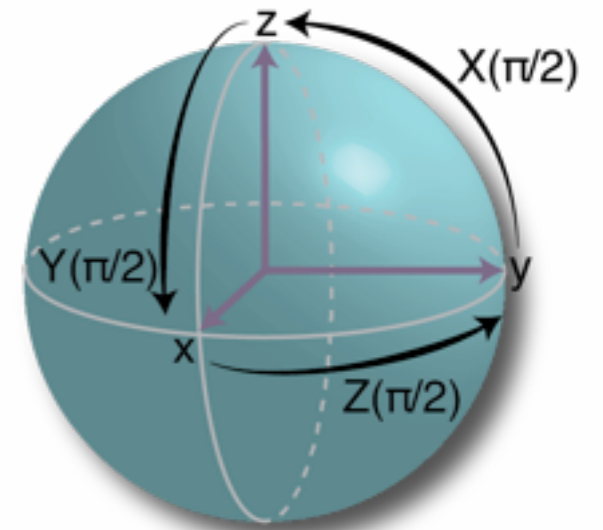
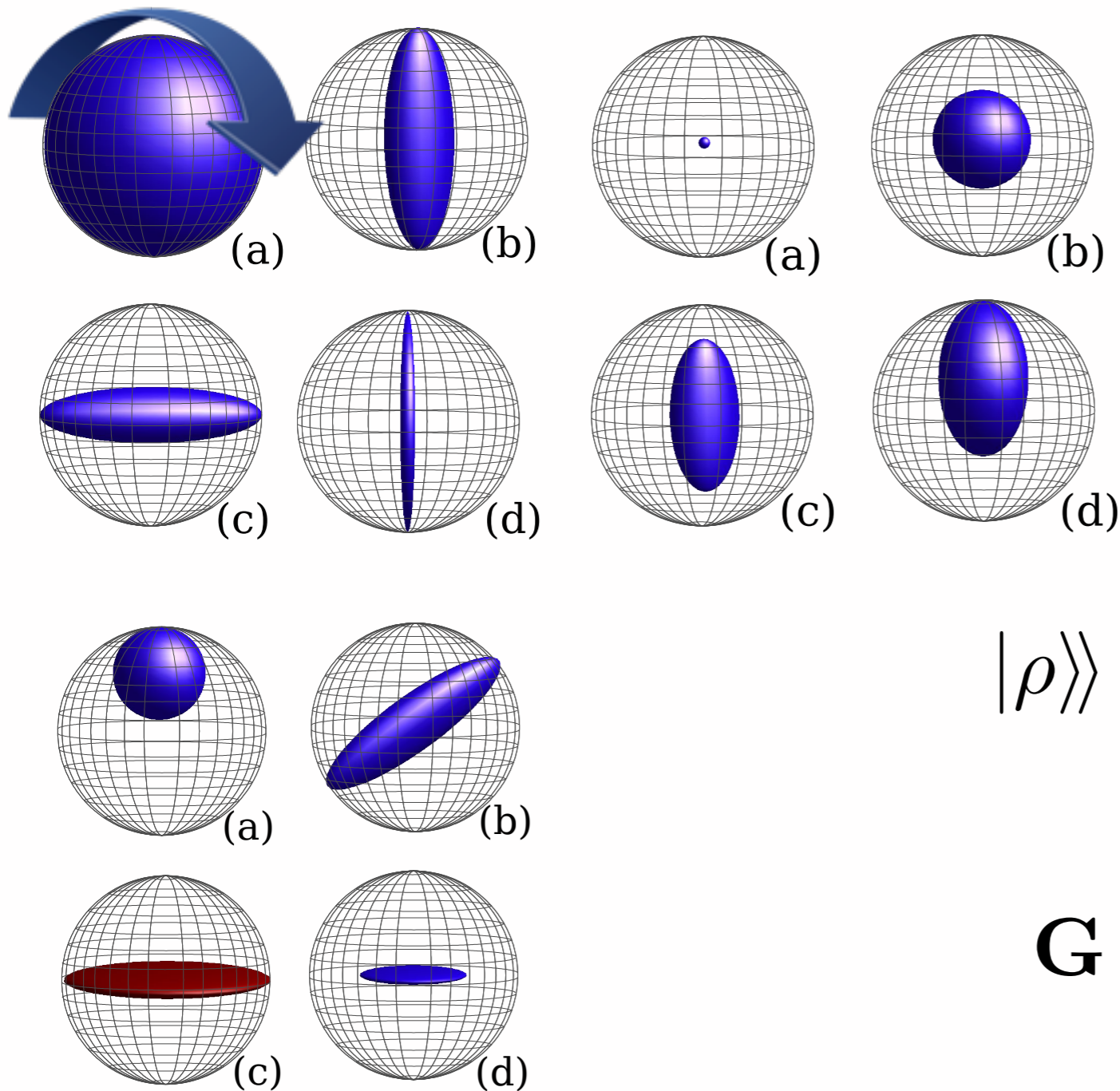
$$|\psi\rangle \rightarrow \mathbf{U} |\psi\rangle$$

$$\rho \rightarrow \mathbf{U} \rho \mathbf{U}^\dagger$$

$$|\rho\rangle\rangle \rightarrow \mathcal{U} |\rho\rangle\rangle$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{pmatrix} \begin{pmatrix} 1/2 \\ x \\ y \\ z \end{pmatrix}$$

# Noisy Quantum Operations



$$|\rho\rangle\rangle \rightarrow \mathbf{G} |\rho\rangle\rangle$$

Trace-changing block

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{pmatrix}$$

Non-unital (affine) block

Unital block

# Measurements & Born's Rule

- States exist solely to predict outcomes of observations.
- Measurements (observations) are sets of possible *events*.

$$\mathcal{M} = \{E_k : k = 1 \dots K\}$$

Simple (“pure” measurement)

$$\mathcal{M} = \{\langle\phi_1|, \langle\phi_2| \dots\}$$

$$\begin{aligned} Pr(k|\psi) &= |\langle\phi_k|\psi\rangle|^2 \\ &= \text{Tr}(|\phi_k\rangle\langle\phi_k||\psi\rangle\langle\psi|) \end{aligned}$$

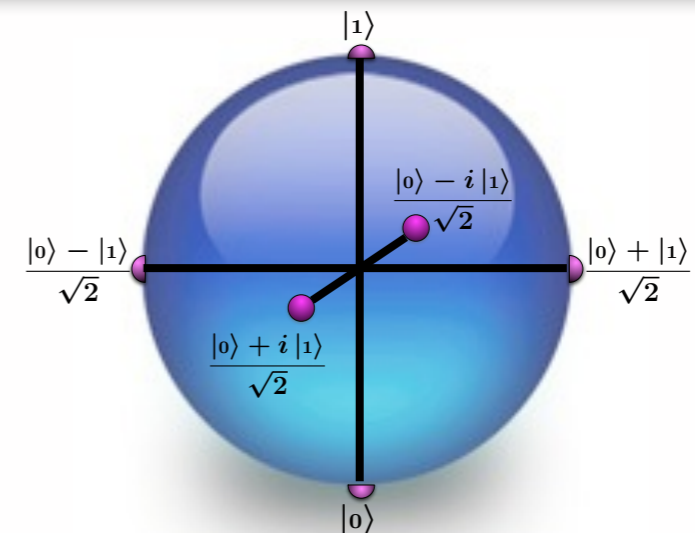
General (“POVM” measurement)

$$\mathcal{M} = \{E_1, E_2 \dots E_K\}$$

$$\begin{aligned} Pr(k|\rho) &= \text{Tr}(E_k\rho) \\ &= \langle\langle E_k|\rho\rangle\rangle \end{aligned}$$

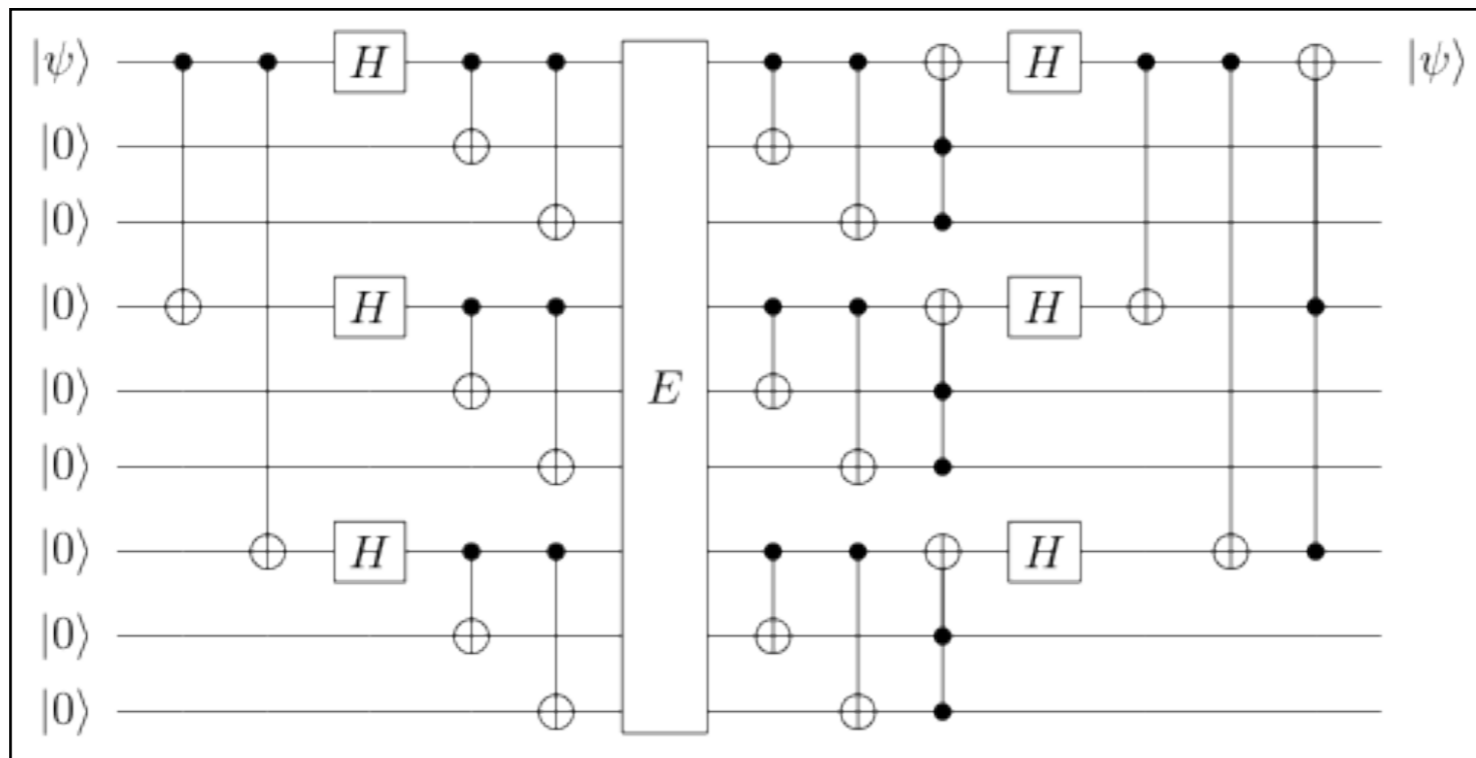
Probability for a *general* experiment:

$$Pr(\text{success}) = \langle\langle E_{\text{success}} | G_{\text{stuff}} | \rho \rangle\rangle$$



# Quantum Circuits

- Quantum logic experiment on qubits = *quantum circuit*.



- Circuit on 1 qubit = *gate sequence*



$$Pr(\bullet) = \langle\langle E | F_j G^{11} F_k | \rho \rangle\rangle$$

Part 2:  
Quantum Statistical Inference  
(okay, *tomography*)

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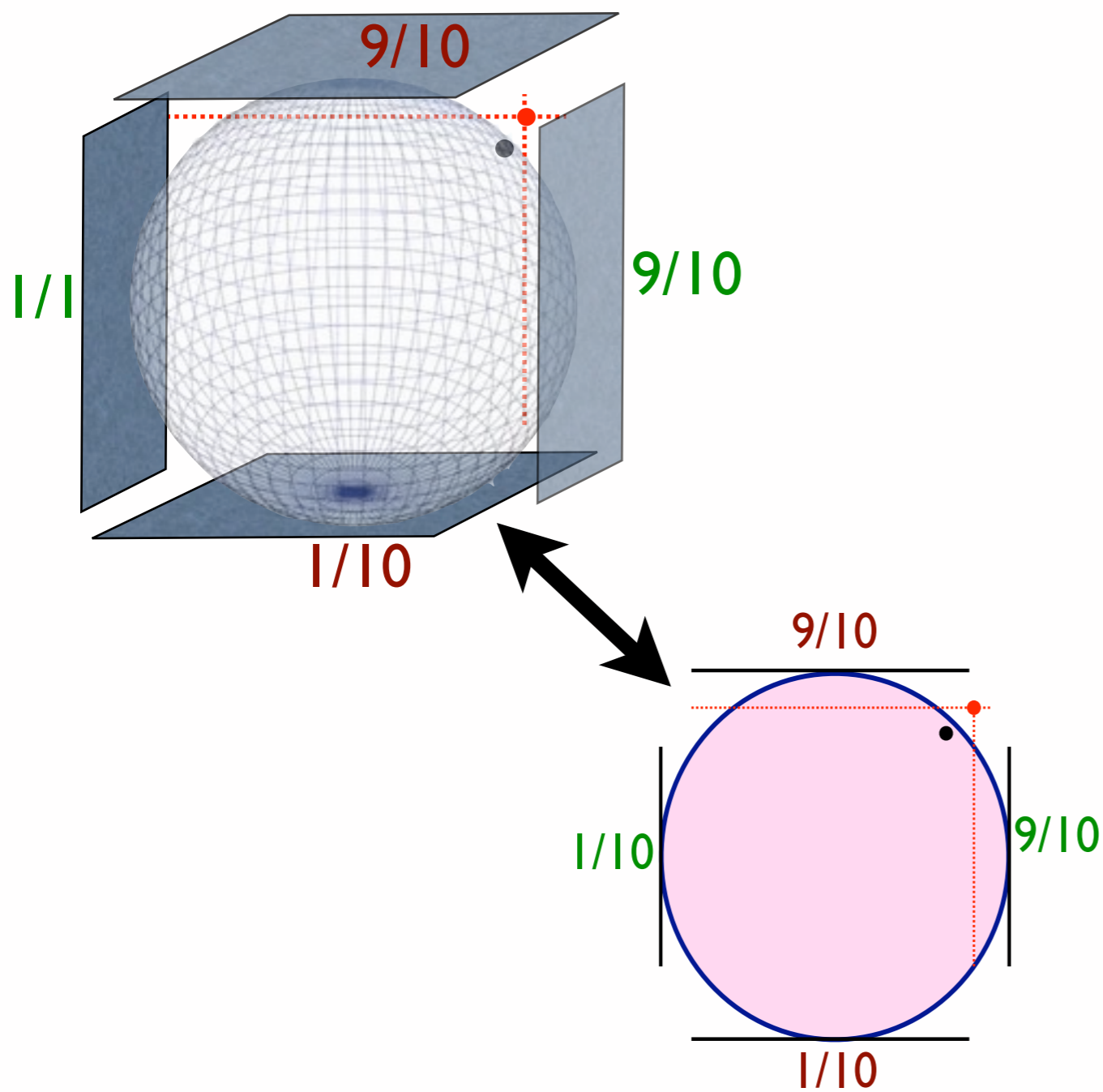


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# State Tomography



**Goal:** determine (estimate) a quantum state that can be parameterized like this:

$$\rho = \frac{1}{2} \mathbf{1} + x\mathbf{X} + y\mathbf{Y} + z\mathbf{Z}$$

$$|\rho\rangle\rangle = \frac{1}{2} |\mathbf{1}\rangle\rangle + x|\mathbf{X}\rangle\rangle + y|\mathbf{Y}\rangle\rangle + z|\mathbf{Z}\rangle\rangle$$

**Sol'n:** estimate each of the parameters by estimating observable probabilities:

$$z = \langle\langle \mathbf{Z} | \rho \rangle\rangle$$

$$= \langle\langle |0\rangle\langle 0| | \rho \rangle\rangle - \langle\langle |1\rangle\langle 1| | \rho \rangle\rangle$$

SO

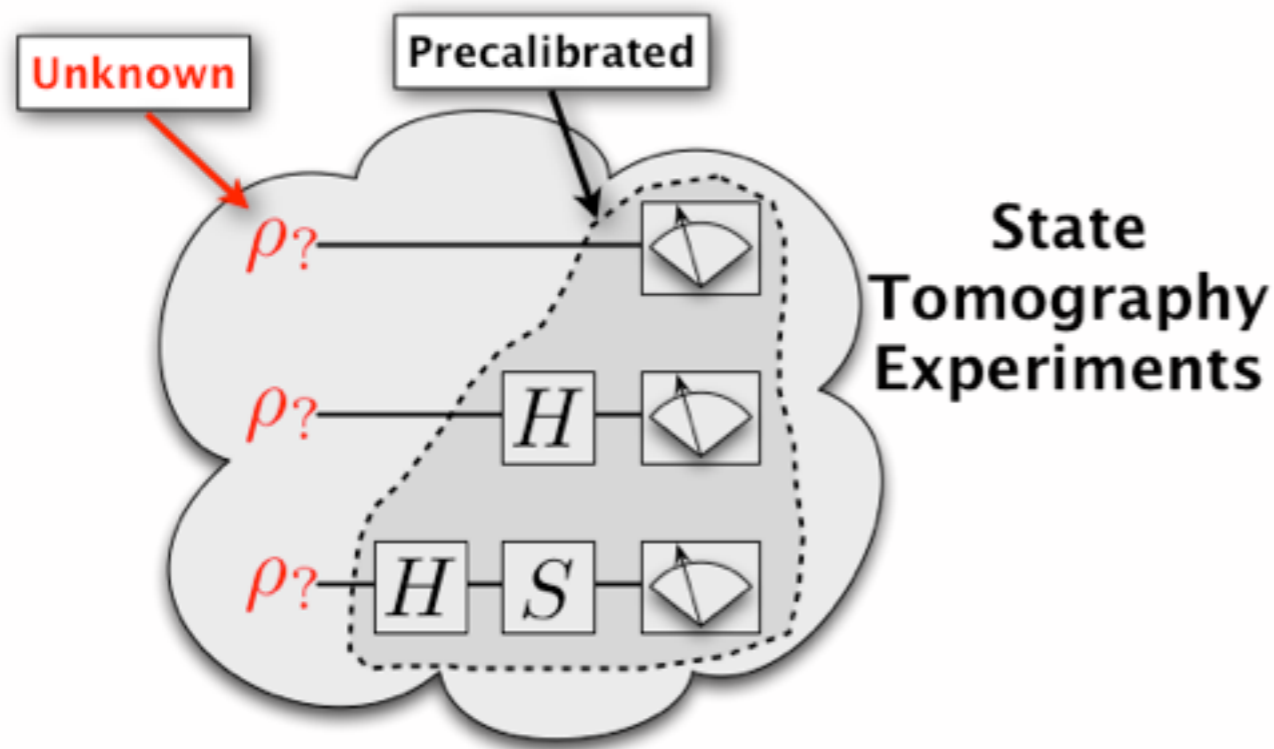
$$z = Pr(0|\rho) - Pr(1|\rho)$$

...etc, etc, for  $y$  and  $z$ .

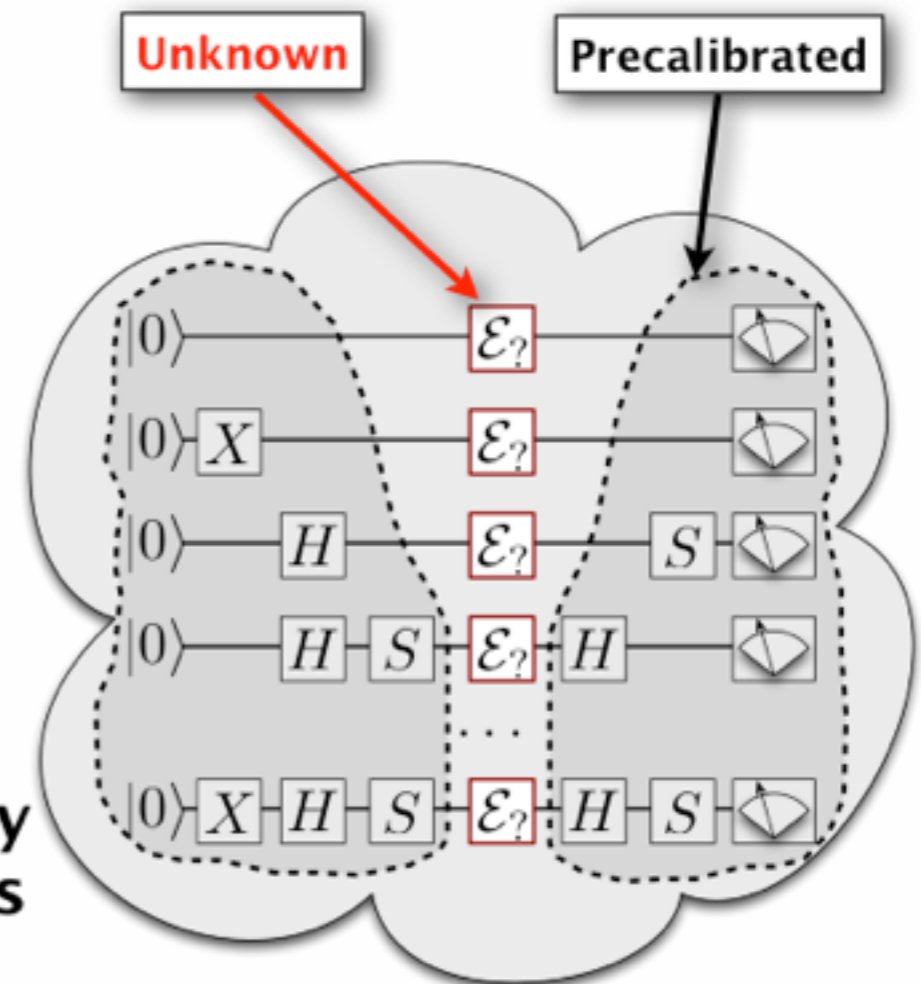
(and many more if the system is bigger than a single qubit...)

# Process Tomography

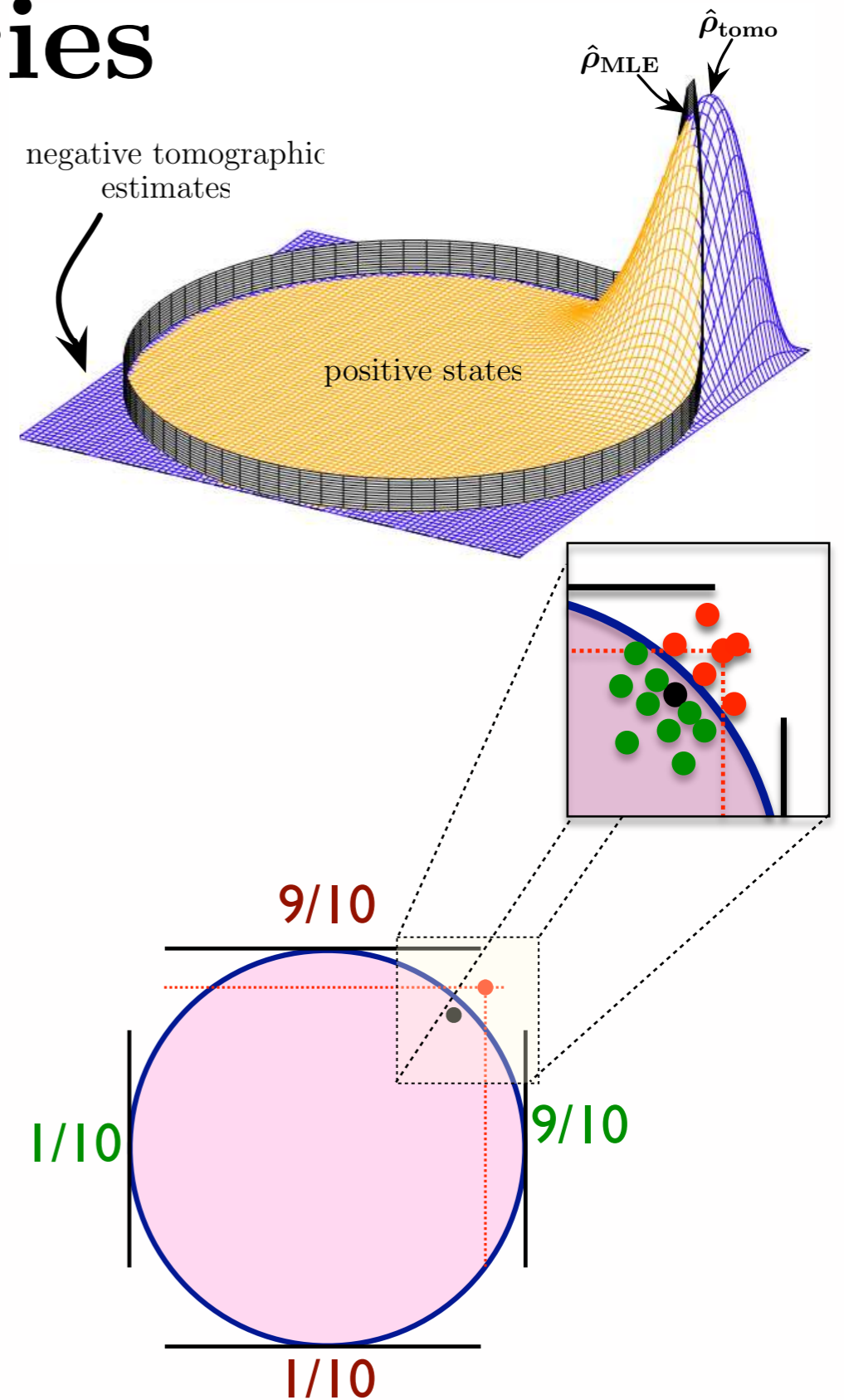
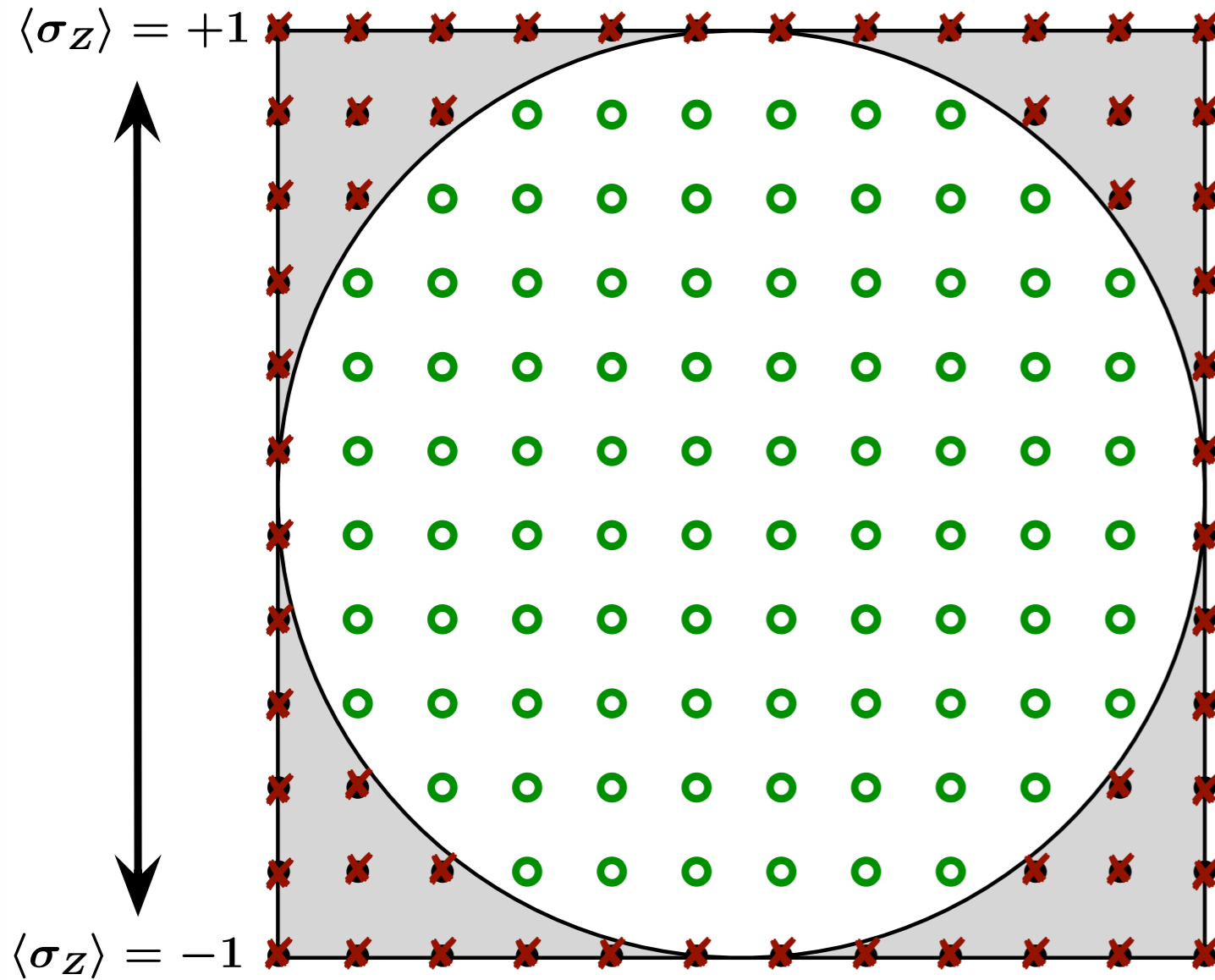
- Quantum processes (a.k.a. operations or logic gates) can be characterized in a very similar way to states.



**Process Tomography Experiments**

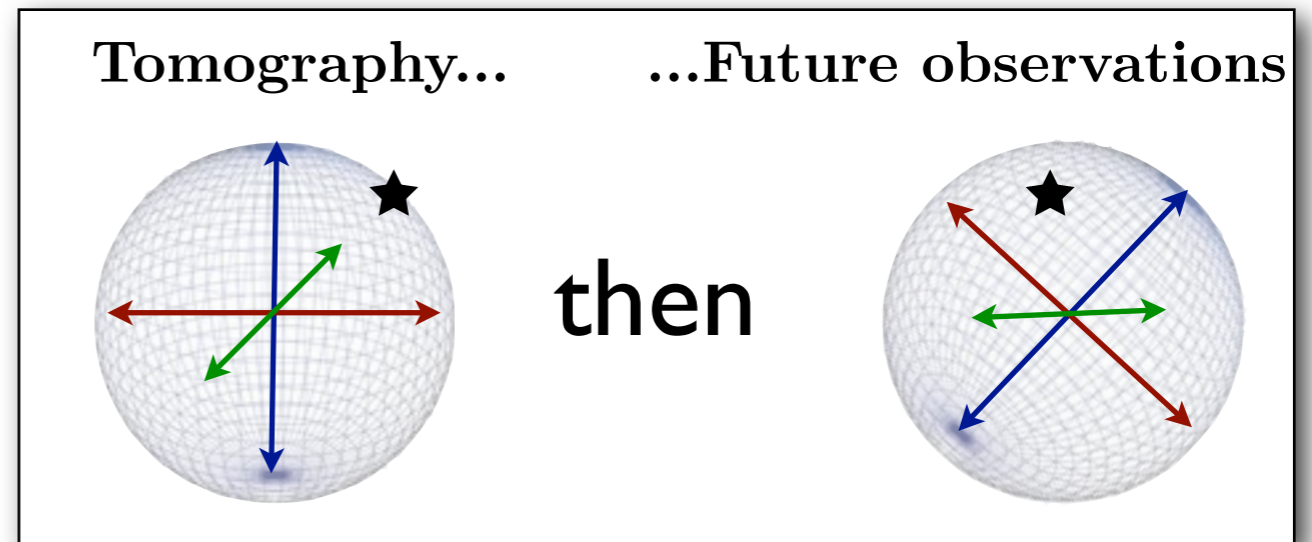
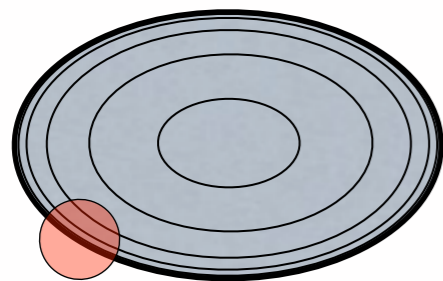


# Why quantum is different: Boundaries

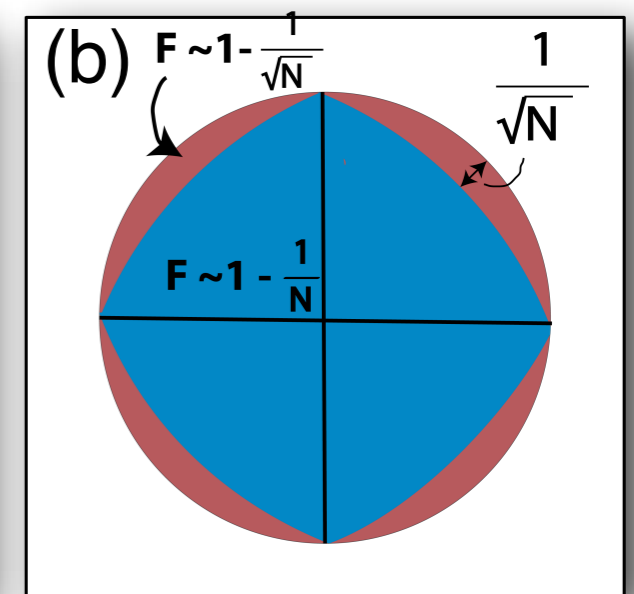
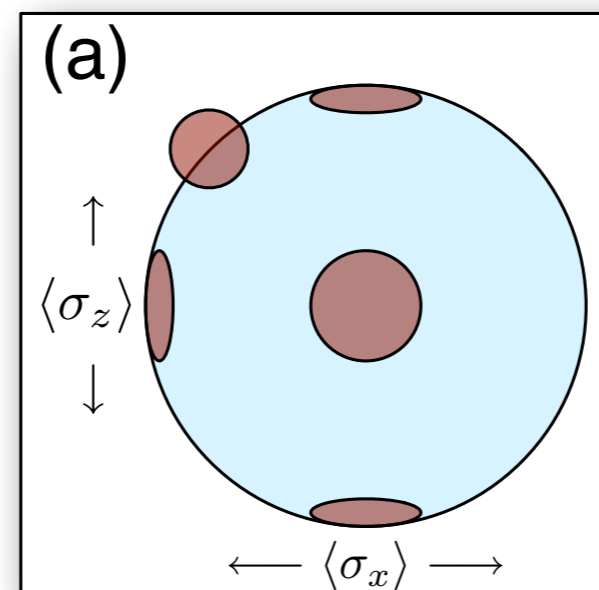
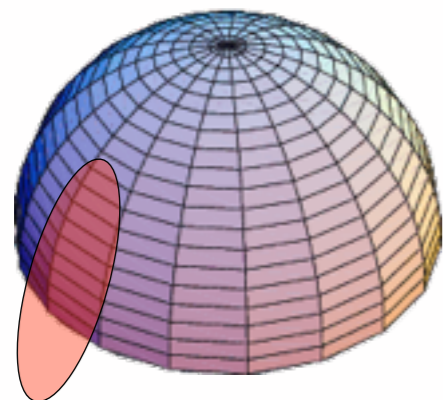


# Why quantum is different: Choice of measurements

## Bloch Sphere Geometry



## Fisher Metric Geometry



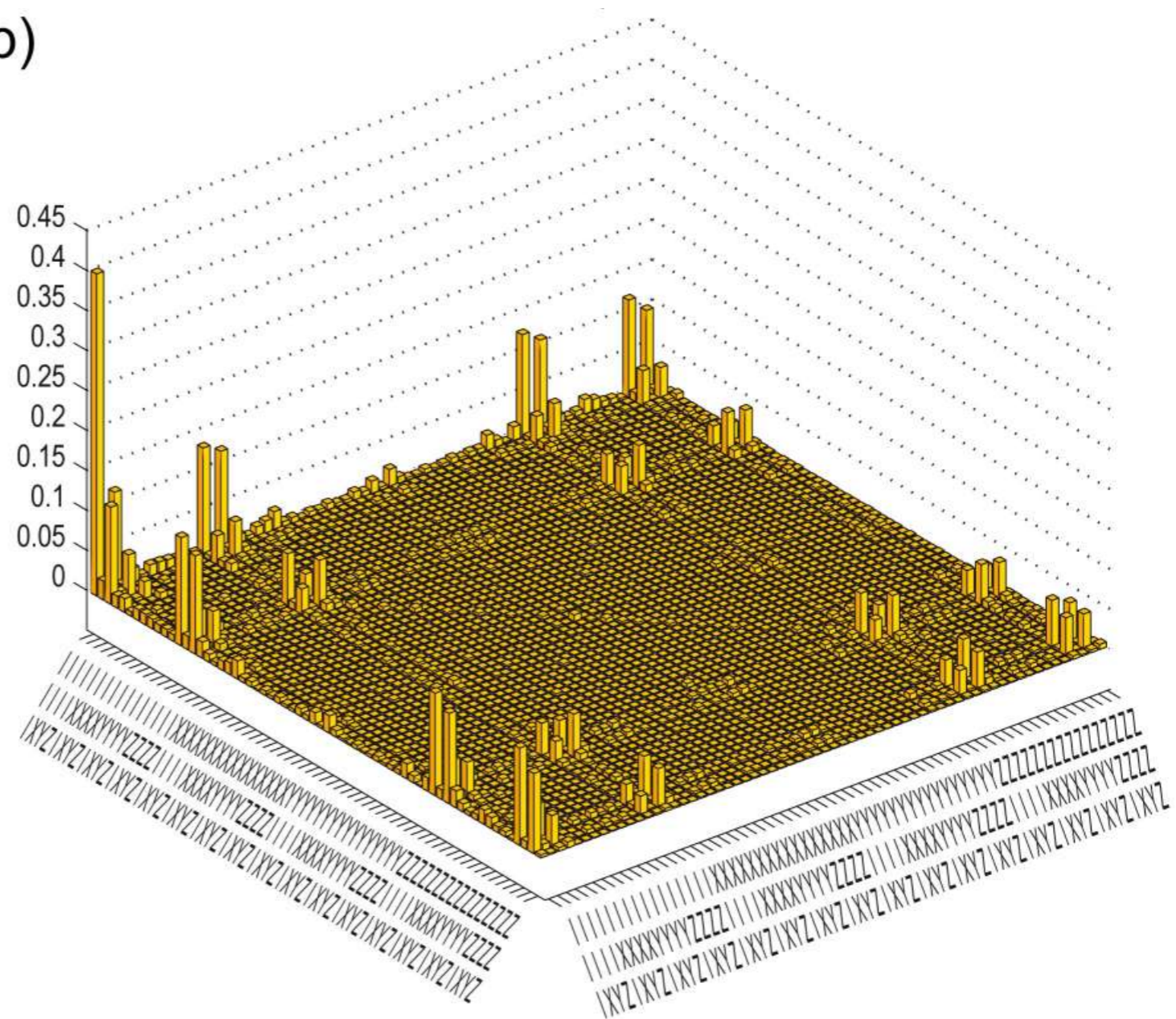
# Why quantum is different:

## Parameter overload!

Process tomography is the same for larger systems -- but requires  $O(d^4)$  different settings ( $16^N$  for  $N$  qubits!)

- The plot shows results of a 3-qubit (Toffoli gate) process tomography experiment (Innsbruck).
- 64 x 64 matrices have a lot of numbers in them! -- and each number corresponds to a different experimental configuration.
- Process tomography gets impractical *very* fast as  $N$  increases.

(b)



# Why gates are *not* like states

- There is a famous theorem in QI called the *Choi-Jamiolkowski isomorphism*:

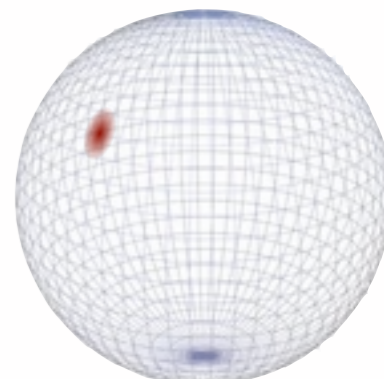
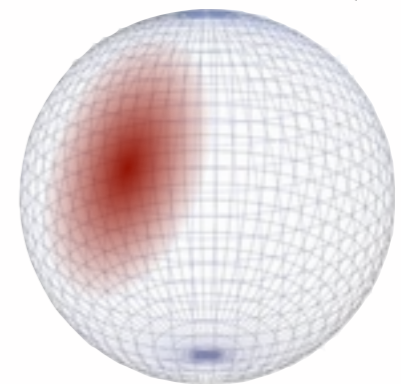
$$\rho_J \equiv (\mathbf{G}_S \otimes \mathbb{1}_A) \left[ |\Psi\rangle\langle\Psi| \right]$$

$$\text{where } |\Psi\rangle = d^{-1/2} \sum_k |kk\rangle$$

- If you think this means that *gates* are isomorphic to *states*, then you are missing something very important:

**Gates can be repeated sequentially.**

- If you realize this, then you can characterize a gate with much higher precision ( $1/N$ ) than the corresponding Jamiolkowski state ( $1/\sqrt{N}$ ).



# Part 3:

# Gate-set Tomography

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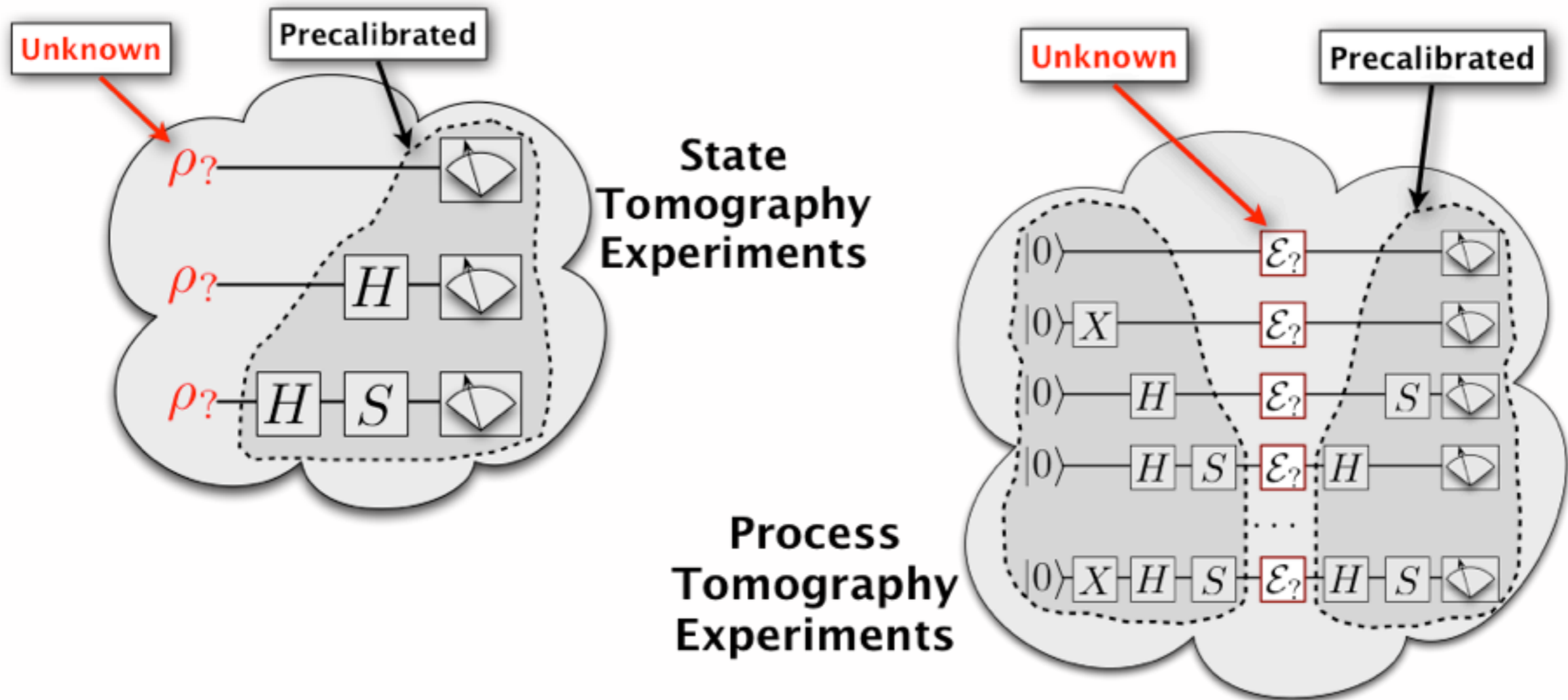
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# The problem with tomography



Standard tomography relies on precalibrated gates...

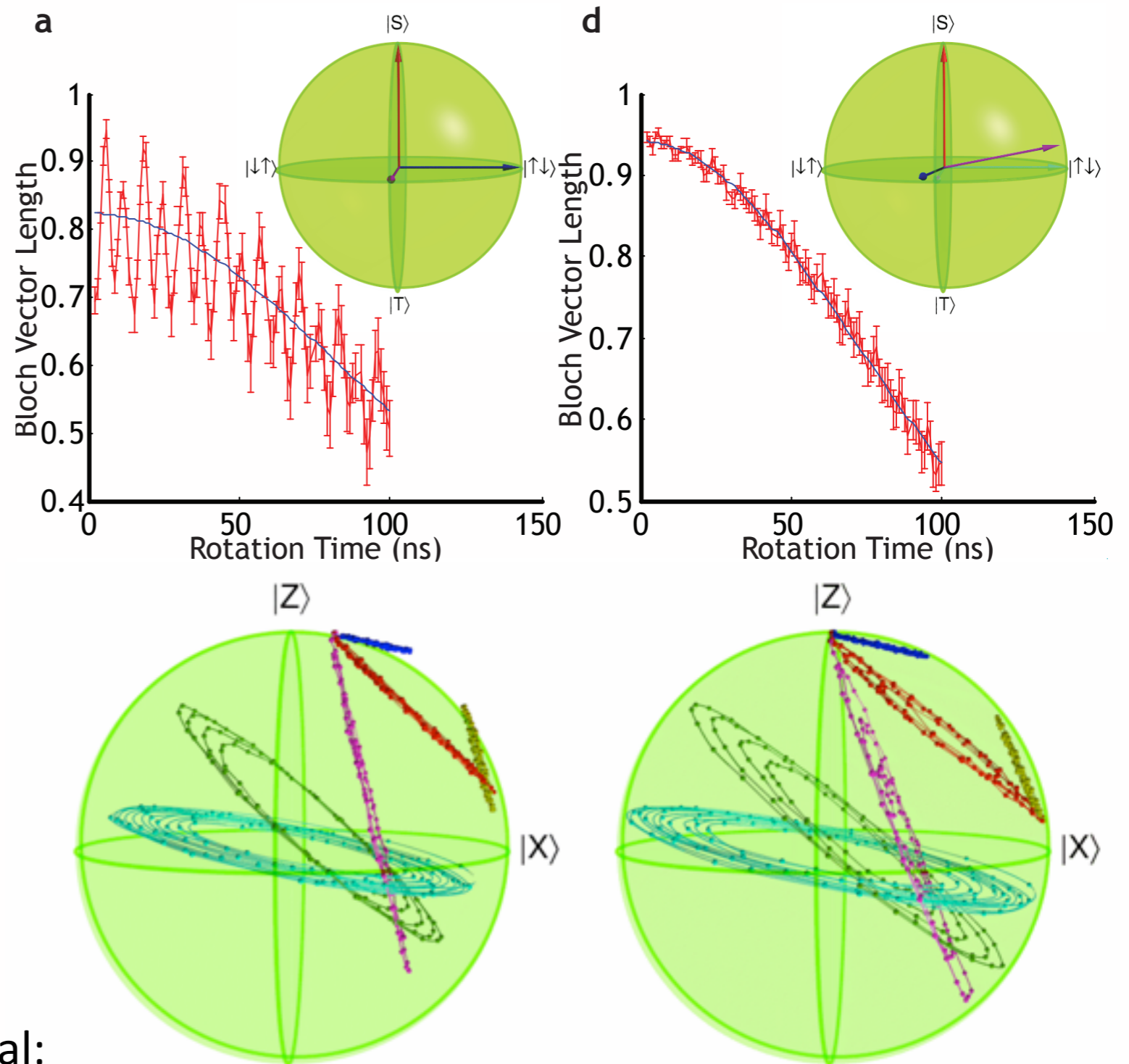
...but we don't have any of those in real hardware!

# Consequences of bad calibration

- 1) Estimated purity (length of Bloch vector) oscillates in a very non-Markovian way.
- 2) The tomographic estimate of the state is consistently outside of the Bloch sphere.

What went wrong? X and Y measurements are not native. They were implemented by *gates* that are not perfectly calibrated!

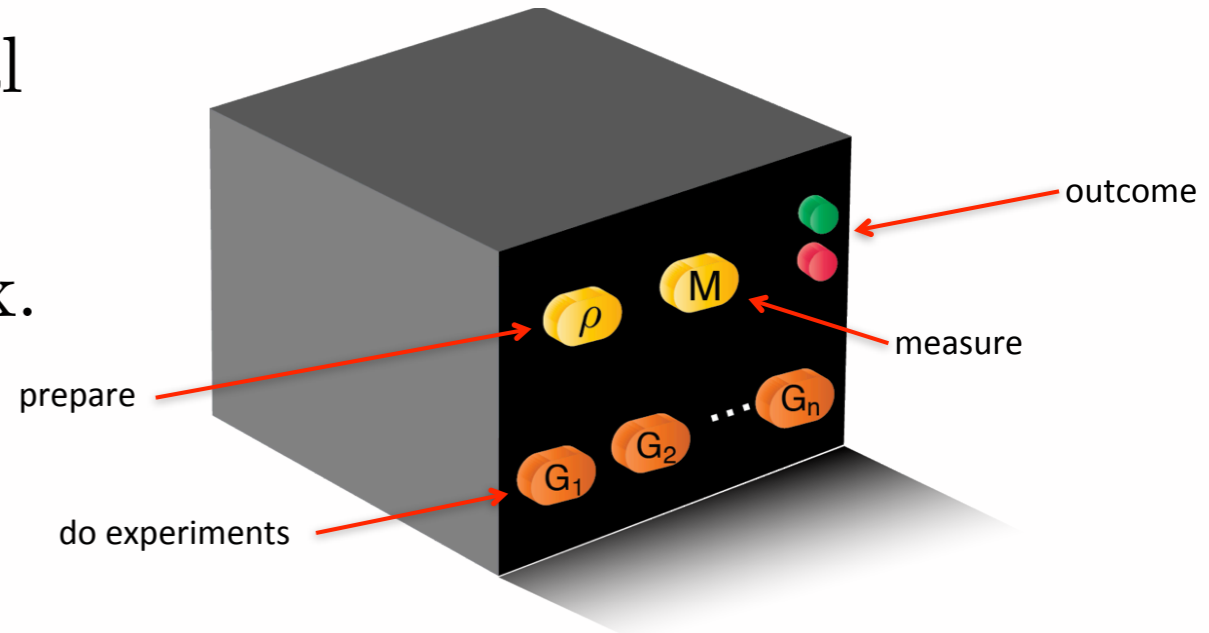
Figures and results courtesy of Oliver Dial:  
M.D. Shulman *et al*, **Science** 336, 202-205 (2012)



# Gate sets

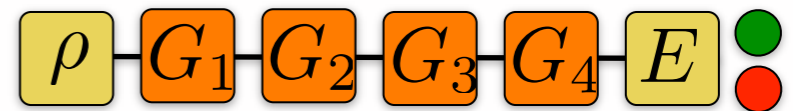
- A *gate set* is a complete statistical model for a quantum device (e.g. qubit) treated as a black box.

$$\left\{ \langle\langle E|, |\rho\rangle\rangle, G_1 \dots G_N \right\}$$



- The gate set framework explicitly rules out “external” reference frames -- the only way to interact with the quantum device is by pushing buttons and reading L.E.D.s.

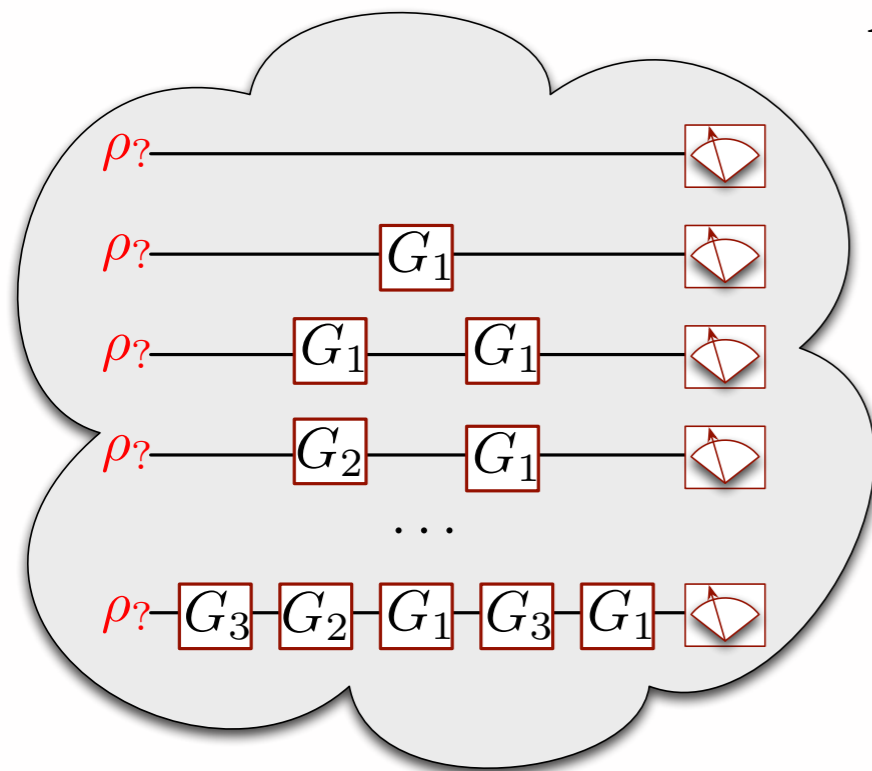
Everything we can “measure”  
is a probability of this form:



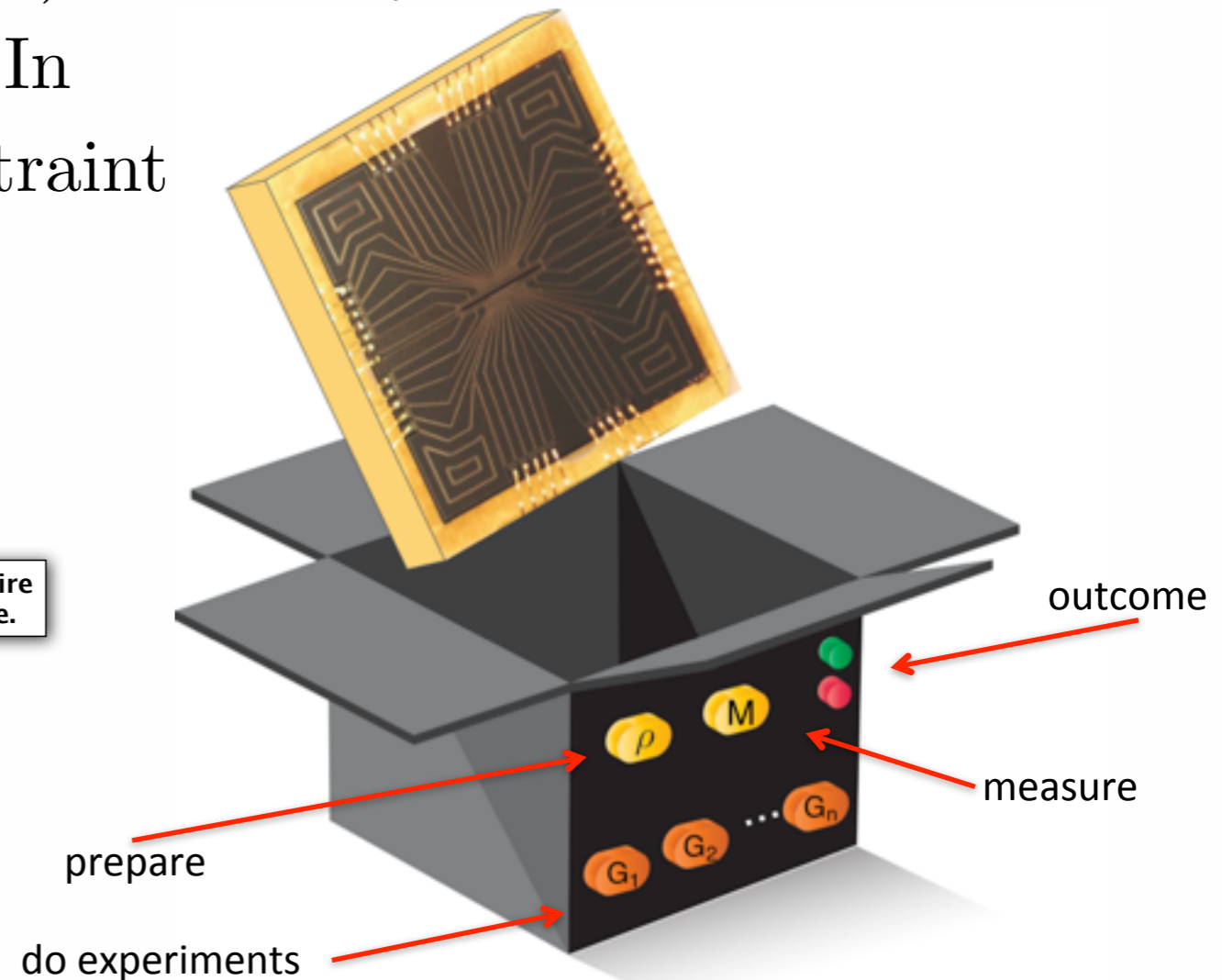
$$p(\bullet) = \langle\langle E | G_{s_1} \dots G_{s_L} | \rho \rangle\rangle$$

# Principles of GST

- A set of  $m$  gates has a few parameters (for 1 qubit,  $16m+8$ ).
- So we just measure (estimate) that many independent experimental probabilities. In *principle*, those should constraint the gate set.



Estimate the entire gate set at once.





# Linear GST

(1) Choose a set of *fiducial sequences*:

$$\{F_1, F_2, F_3, F_4\} \stackrel{\text{e.g.}}{=} \{G_1, G_2, G_3, G_2^2\}$$

(2) Do “tomography” by measuring:

$$(\tilde{G}_i)_{j,k} = \langle\langle E | F_j G_i F_k | \rho \rangle\rangle$$


$$\text{and } \tilde{\mathbb{1}}_{j,k} = \langle\langle E | F_j F_k | \rho \rangle\rangle$$


(3) Use linear algebra to get the estimate:

$$\hat{G}_i = \tilde{\mathbb{1}}^{-1} \tilde{G}_i, \text{ etc. } \iff \hat{G}_i = B^{-1} G_i B, \langle\langle \hat{E} | = \langle\langle E | B, |\hat{\rho}\rangle\rangle = B^{-1} |\rho\rangle\rangle$$

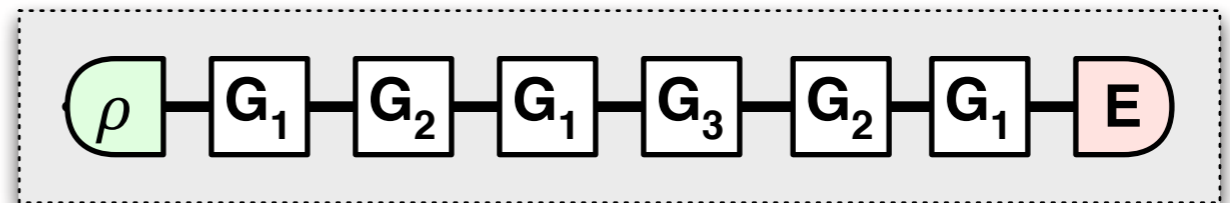
(4) Transform the estimated gateset by a *gauge transformation* to make it as close as possible to the desired target gateset:

$$\hat{G}_i \rightarrow S^{-1} \hat{G}_i S \text{ and } \langle\langle \hat{E} | \rightarrow \langle\langle \hat{E} | S \text{ and } |\hat{\rho}\rangle\rangle \rightarrow S^{-1} |\hat{\rho}\rangle\rangle$$

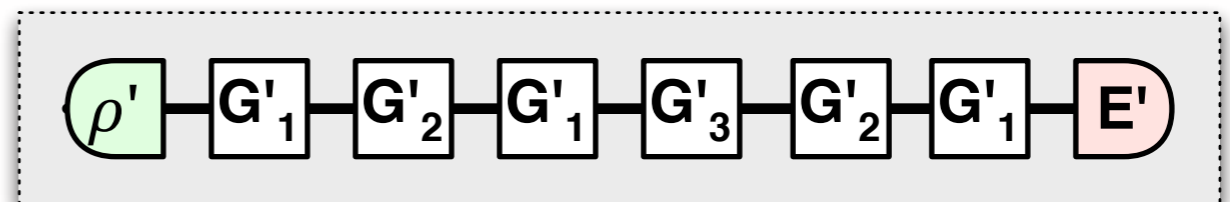
# The Gauge

- Let  $T$  be invertible.  
Transform the gateset:

$$\begin{aligned} \langle\langle E' | &= \langle\langle E | T \\ |\rho'\rangle\rangle &= T^{-1} |\rho\rangle\rangle \\ G'_k &= T^{-1} G_k T \end{aligned}$$



$$P(E) = \langle\langle E | G_1 \dots G_1 | \rho \rangle\rangle$$



$$\begin{aligned} P(E') &= \langle\langle E' | G'_1 \dots G'_1 | \rho' \rangle\rangle \\ &= \langle\langle E | T T^{-1} G_1 T T^{-1} \dots T T^{-1} G_1 T T^{-1} | \rho \rangle\rangle \\ &= P(E) \end{aligned}$$

- All observable probabilities are left unchanged!  
 $\Rightarrow \{G_k\}$  and  $\{G'_k\}$  are different *descriptions*  
of exactly the *same* physical device.

# Long circuit GST

- “Traditional” tomography is linear -- observable probabilities are *linear* functions of the parameters (e.g., elements of  $G_k$ ).
- This no longer holds for GST. Even in LGST, probabilities involve multiple  $G_k$ .
- So can we get an advantage from going whole-hog and using *really* nonlinear probabilities (many uses of  $G_k$  at once?)



- Yes! Long circuits can amplify small errors.
  - (1) What long circuit experiments should we perform?
  - (2) How do we fit a gateset to the resulting data?

**Part 4:**  
**Hyperaccuracy**  
**&**  
**Error-amplifying circuits**

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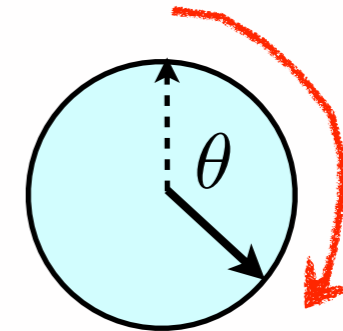
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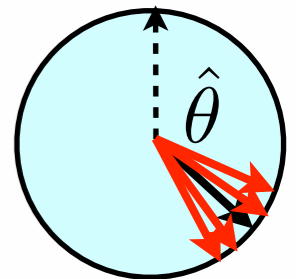
# Basics of Error Amplification

- Simple toy example:  
Goal: estimate  $\theta$ .

$$G = e^{i\theta}$$

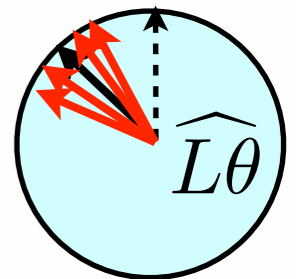


- $N$  observations of  $\langle\langle E | G | \rho \rangle\rangle$  yield  $\hat{\theta} = \theta \pm O(1/\sqrt{N})$ .



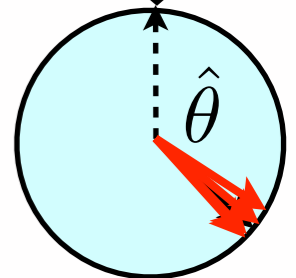
- But  $N$  observations of  $\langle\langle E | G^L | \rho \rangle\rangle$  yield

$$\widehat{L\theta} = L\theta \pm O(1/\sqrt{N}) \implies \hat{\theta} = \widehat{L\theta}/L = \theta \pm O(1/L\sqrt{N})$$



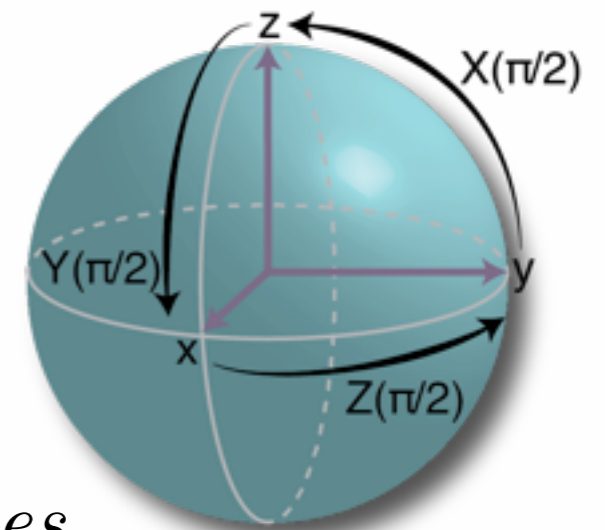
- **Problem:** *branch cuts*. We only observe rotations modulo  $2\pi$ , so if  $\delta\theta = \pi/2$ , is  $\theta$  equal to  $\pi/16$  or  $5\pi/16$ ?

- **Solution:** gather data at many scales:  $L=1,2,4,8,\dots$



# Germes and Germ-powers

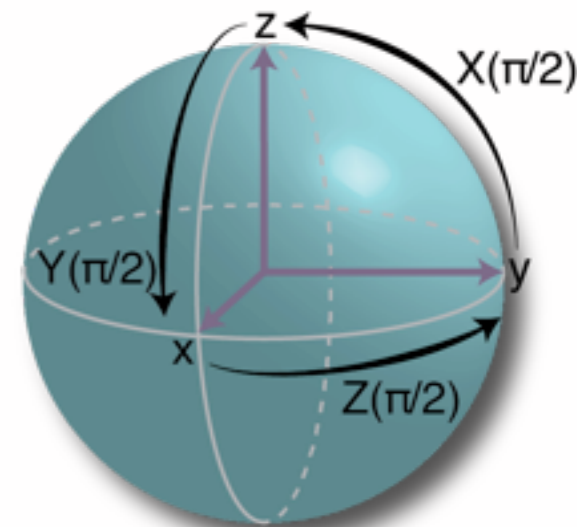
- If gates were scalars (e.g.  $G_k = e^{i\theta_k}$ ) we could just estimate sequences  $G_k^L$  for  $L=1,2,4,8\dots$  and infer the  $G_k$  from them.
- But the  $G_k$  are matrices... which make things more complex!
- Repeating  $G_k$  (i.e.,  $G_k^L$ ) only amplifies *eigenvalues* of  $G_k$ .
- In early experiments, we got great precision in rotation *angles*... but inaccurate rotation *axes*.
- **Solution:** we do many repetitions not just of single gates but of short *germs* -- e.g.  $G_1G_2$  -- that generate long sequences like  $(G_1G_2)^L$ . We call these *germ power sequences*.



# Completeness

- A set of experiments is *informationally complete* (IC) if its observable results constrain every parameter of the model (for gatesets, “every non-gauge parameter”).
- We want not just IC -- even LGST is IC. We want every parameter *amplified* proportional to  $L$  (sequence length).
- Designing IC germ sets is something of a black art right now (it’s a condition on the rank of a Jacobian), but we can do it for any given gate set. Example:

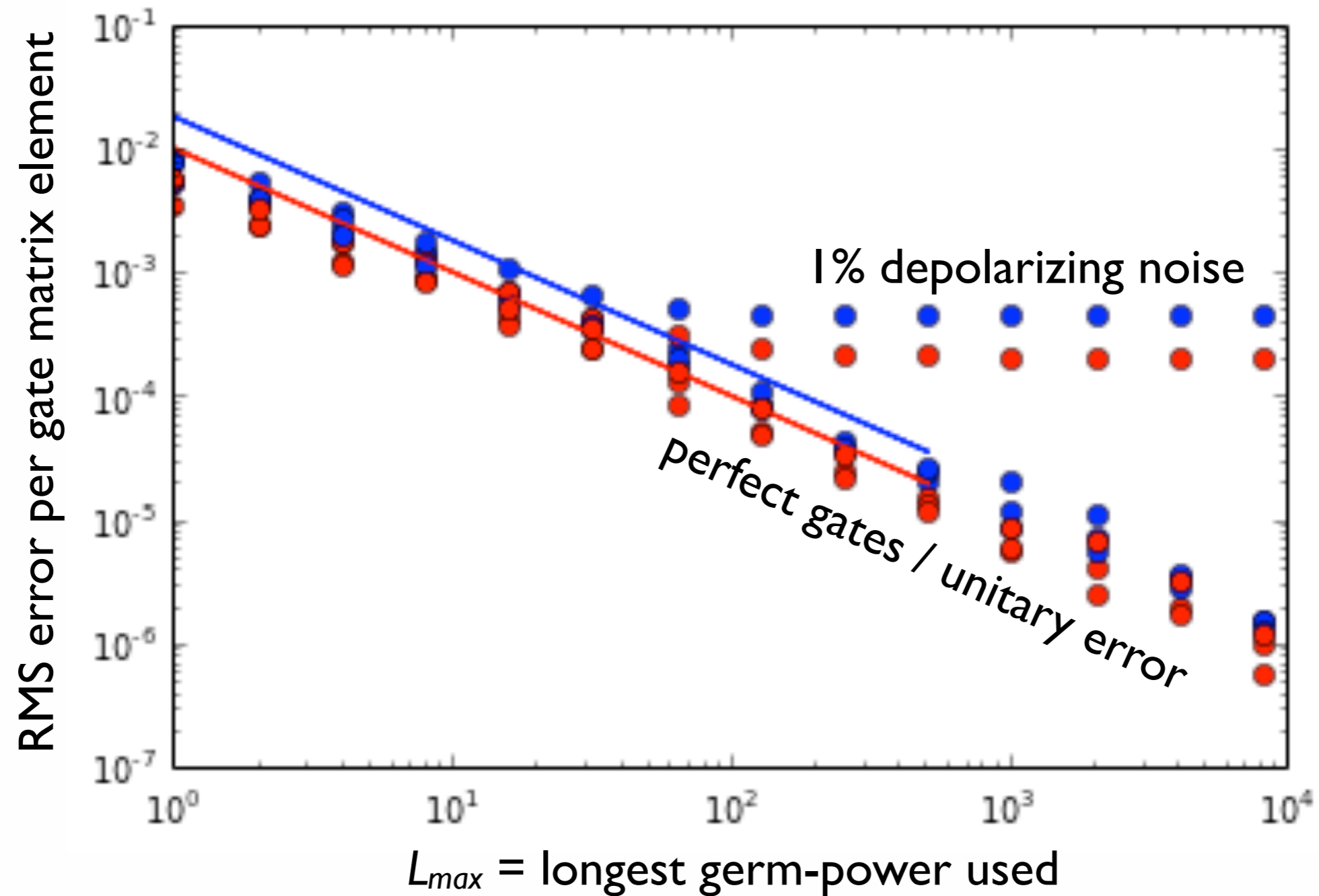
$$\{G_x, G_y, G_z, G_{\mathbb{1}}, G_x G_y, G_x G_z, G_y G_z, G_x G_x G_y, G_x G_x G_z, G_x G_y G_z, G_y G_y G_z\}$$



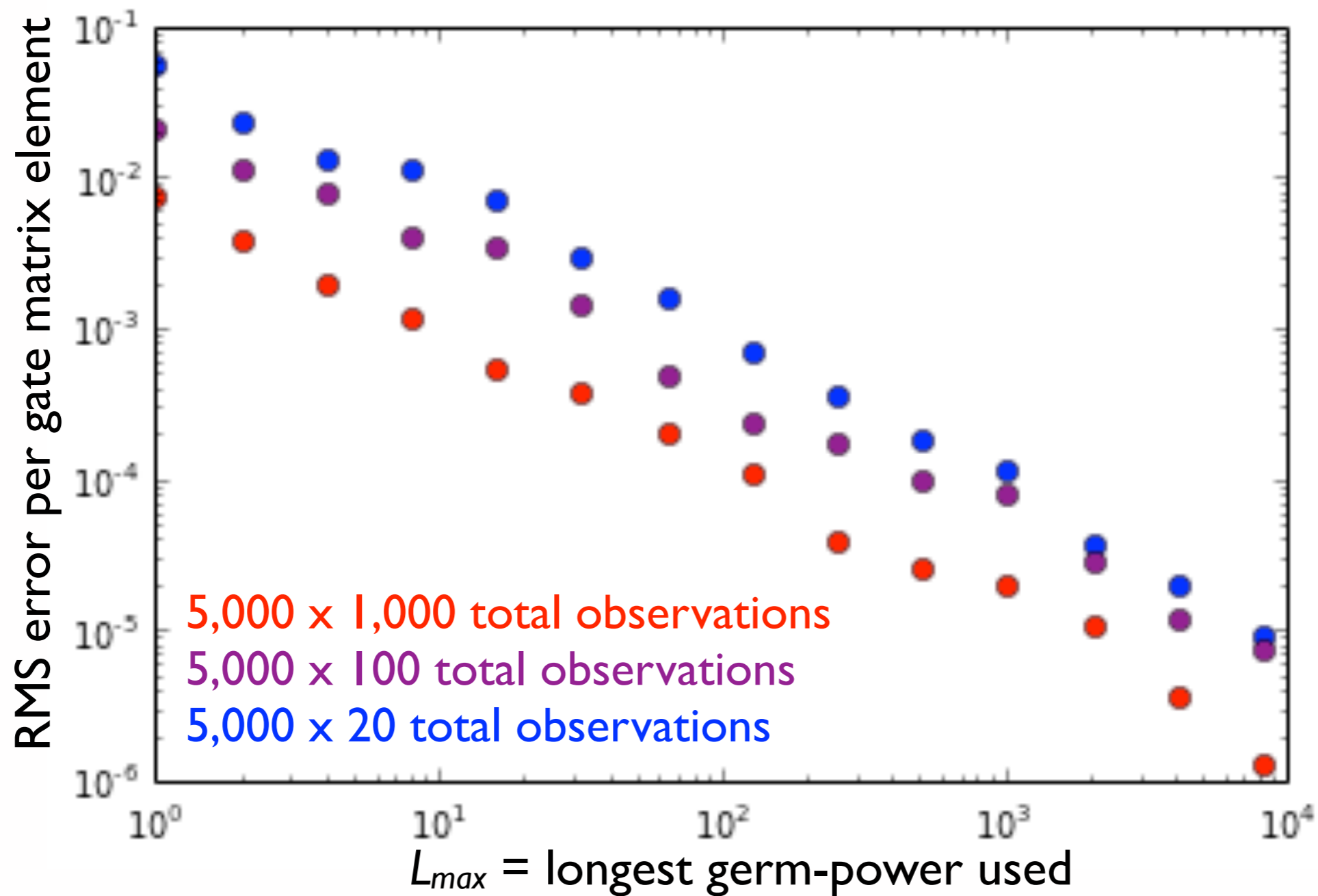
# eLGST and LSGST

- We need to fit a gateset  $\{\rho, E, G_k\}$  to long-sequence data.
- *Extended LGST*: **fast, simple, reasonably accurate**
  1. Do LGST using short sequences to get a rough estimate.
  2. Use long sequences of the form  $F_i(\mathbf{germ})^L F_j$  to get LGST estimates of the quantum operation for  $(\mathbf{germ})^L$ .
  3. Iteratively tweak the rough estimate to agree with estimated germ-powers of length  $L=2, 4, 8, \dots L_{\max}$ .
- *Least-squares GST*: **most accurate & statistically justified**
  1. Do LGST using short sequences to get a rough estimate.
  2. Iteratively improve the rough estimate by doing a weighted least-squares fit (minimize  $\chi^2$ ) to the same sequences used for eLGST.

# Hyperaccuracy: Simulations



# Efficiency: counting clicks



# Part 4: Experimental Results

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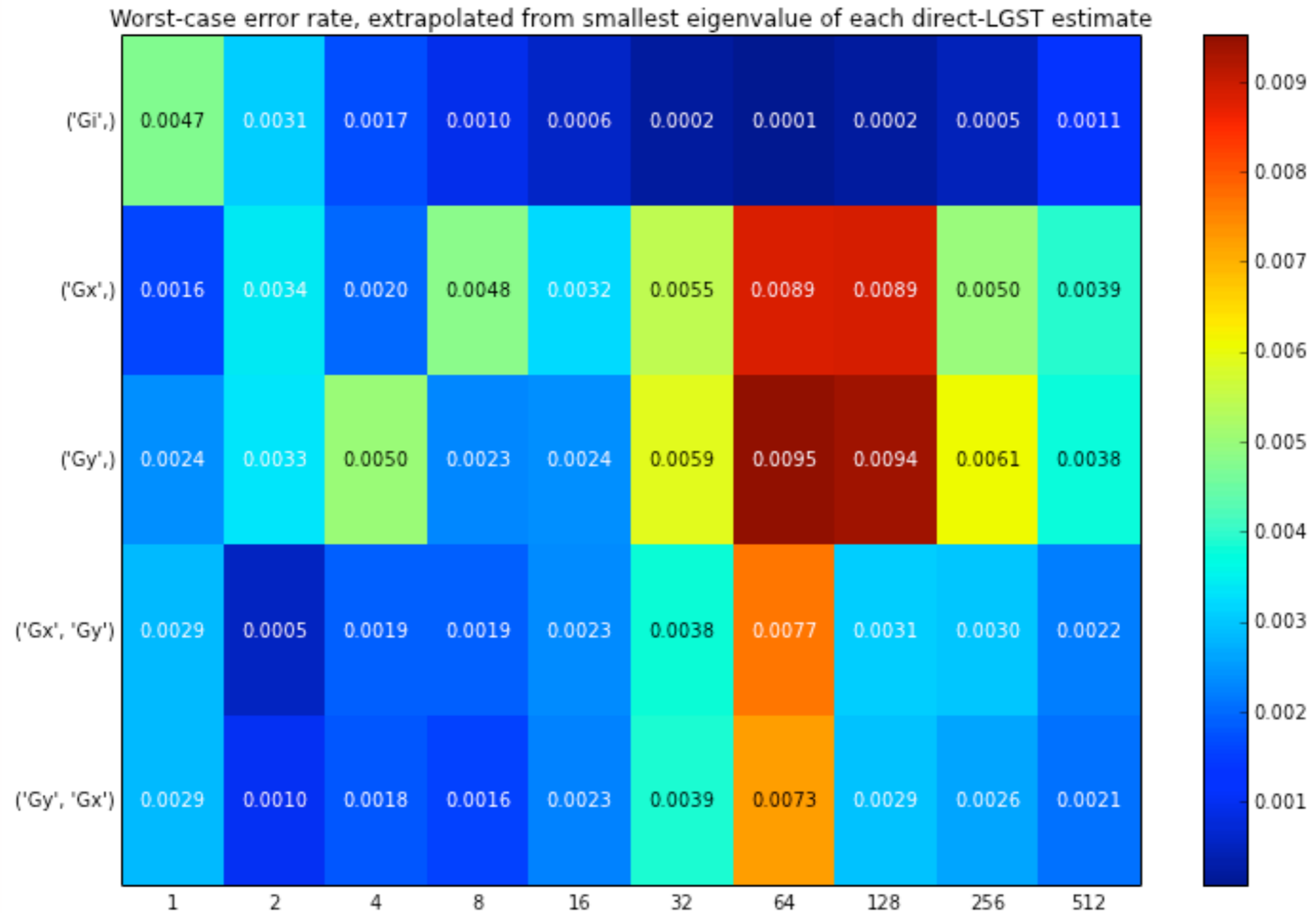
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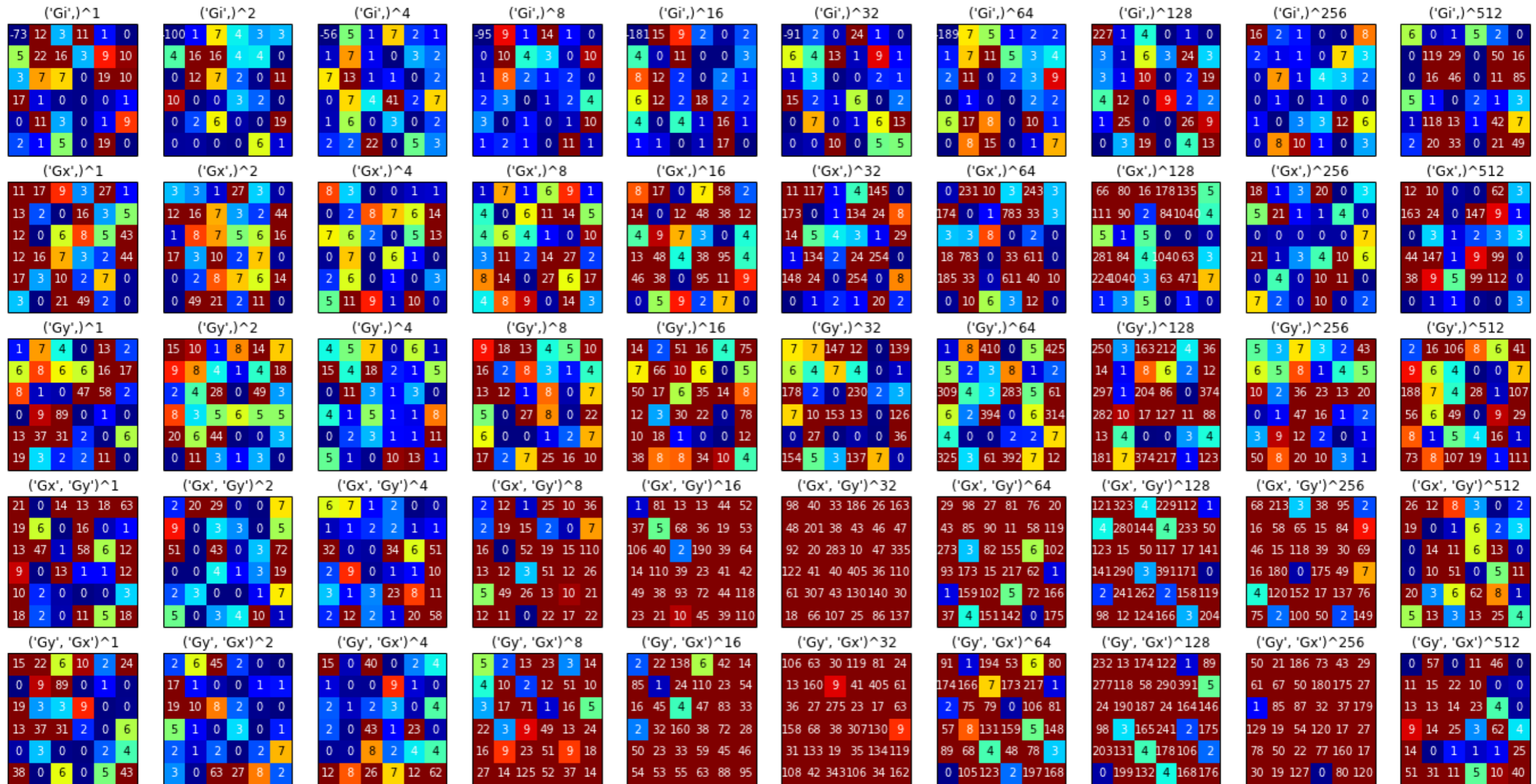
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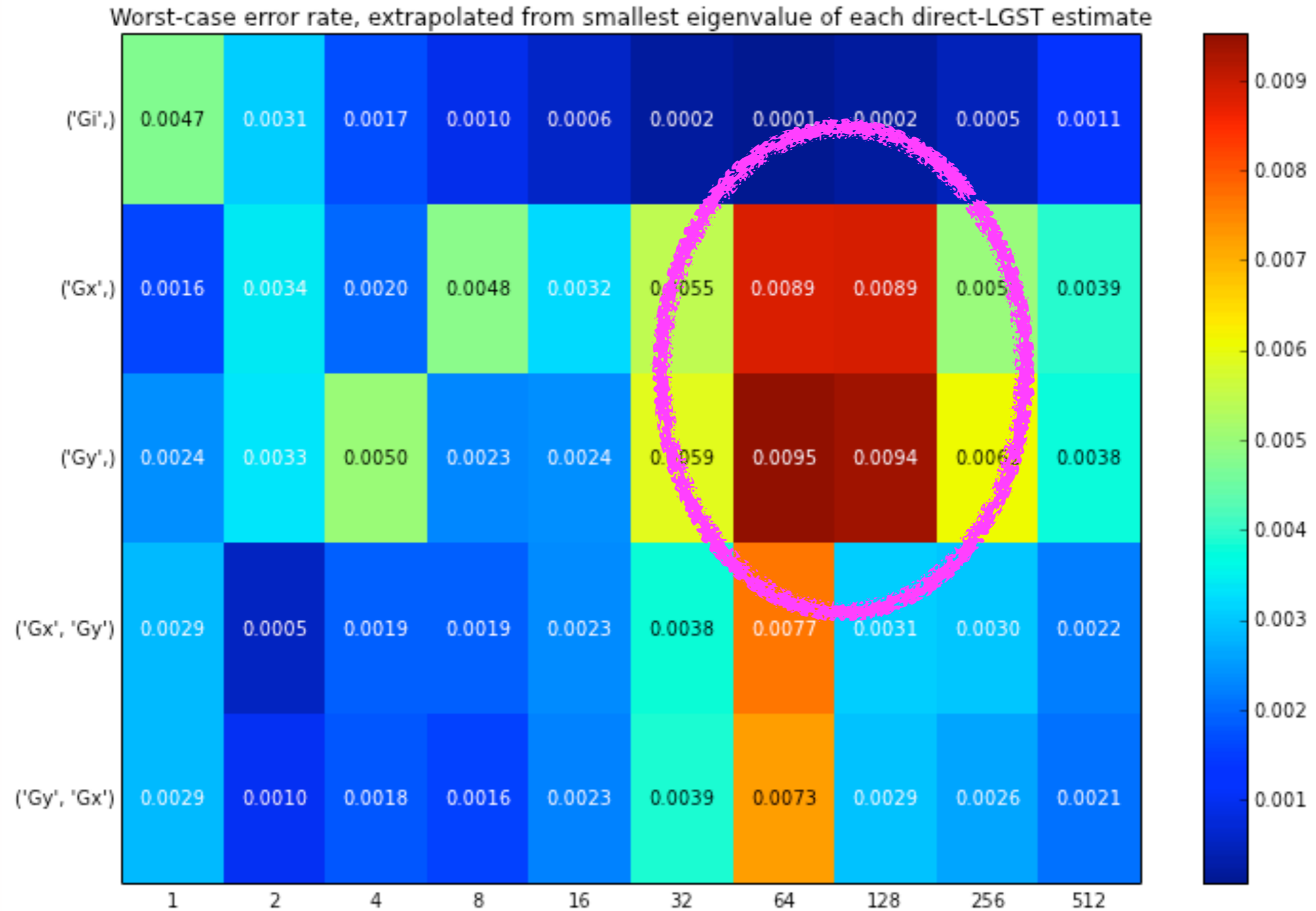
# Extracting Error Rates



# Self-consistency: $\chi^2$



# Non-Markovian noise

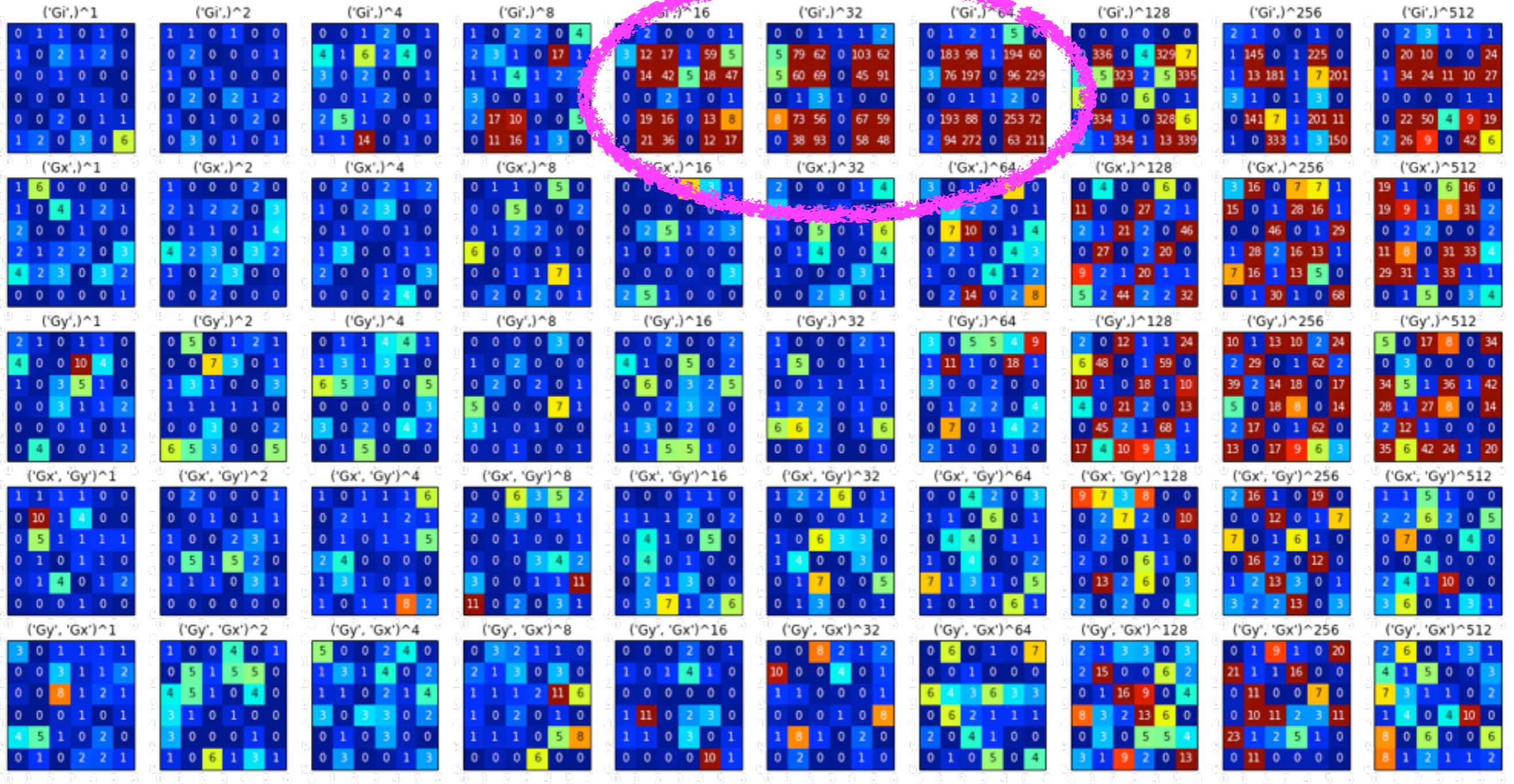


# Debugging: Improved Gates

Worst-case error rate, extrapolated from smallest eigenvalue, for each direct-LGST estimate



# Debugging: Improved $\chi^2$



# Conclusions & Future

- There's no excuse for doing the "old" tomography any more -- assuming precalibrated gates is unsafe and unnecessary.
- GST can achieve unprecedented accuracy & efficiency by leveraging long circuits that amplify gate errors.
- **Every (?) qubit has significant non-Markovian noise.**
- Serious statistical techniques are becoming more and more important for quantum device characterization. We rely critically on goodness-of-fit, and model selection is rapidly becoming an essential concept.
- **Quantum statistical inference is (after all) just another application of statistics... but with some near-unique caveats and challenges!**