SAND2015-1630C

Block Preconditioners for 3D Incompressible MHD with Mixed Finite Elements

Edward Phillips, Eric Cyr, John Shadid

Sandia National Laboratories

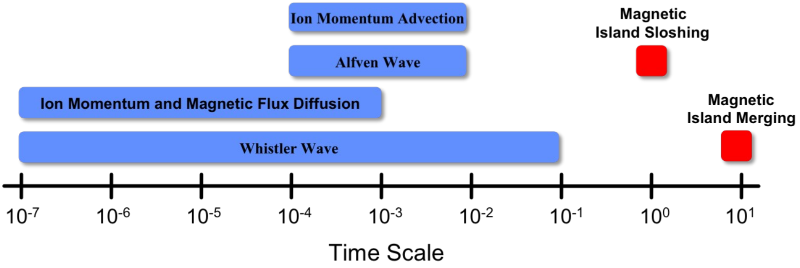
SIAM CSE
March 18, 2015

Sandia National Laboratories is a multi-program laboratory managed and operated by Sandia Corporation, a wholly owned subsidiary of Lockheed Martin Corporation, for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000.

Difficulties in Computational MHD

- Coupling between the Navier-Stokes equations and the Maxwell equations
- Prominent physics (e.g. the Alfvén wave) arise from this coupling
- Important to accurately capture the coupling (implicit fully coupled vs operator splitting schemes)
- Physical phenomena spanning over a wide range of length- and time-scales (restrictive time steps for explicit schemes)
- Conforming discretizations ($\vec{E}$ on edges/$\vec{B}$ on faces from physical perspective; some formulations require $\vec{B}$ on edges)

Fully Coupled Fully Implicit Schemes



- Implicit schemes allow stable time integration when following physical time scales of interest
- Fast time scales translate to stiff modes in discrete systems
- Need preconditioners that approximate coupled overlapping time scales that produce important modes
- Steady state is inifinite time step limit, extreme case

Robust Solvers

- We want preconditioners that capture the hydrodynamic-electromagnetic coupling

$$\begin{pmatrix} \mathcal{M} & \mathcal{Z} \\ \mathcal{Y} & \mathcal{N} \end{pmatrix} \begin{pmatrix} \mathbf{x}_{em} \\ \mathbf{x}_{fl} \end{pmatrix}$$

- Systems may have different discretizations for different DOFs (u and p nodal, Q_2-Q_1 for LBB stable, $\vec{E}$ on edges/$\vec{B}$ on faces, $\vec{B}$ on edges, $\vec{A}$ on edges)
- Motivates block preconditioners
  - Different physics separated $\rightarrow$ use preconditioning ideas developed for single physics
  - Different discretizations separated $\rightarrow$ use simple existing solvers for subsolves (vector convection-diffusion, scalar Laplacian, curl-curl operator on edges, etc.)

Goals of this Work

- Focus on a dual saddle point formulation ($\vec{B}$ on edges, magnetic Lagrange multiplier)
- Reflective of many difficulties in preconditioning MHD (strong coupling, different discretizations, unique operators)
- Develop block preconditioners that
  1. Account for coupling through outer structure
  2. Handle each saddle point subsystem well
  3. Use existing technologies for component solves
- Demonstrate robustness to nondimensional parameters and parallel scalability

A Lagrange Multiplier Fomulation

- D. Schötzau. Mixed finite element methods for stationary incompressible magnetohydrodynamics. Numer. Math., 96:771-800, 2004.
- $$\frac{\partial \vec{B}}{\partial t} + \frac{1}{Re_m}\nabla \times \nabla \times \vec{B} - \nabla \times (\vec{u} \times \vec{B}) + \nabla r = \vec{0}$$
  $$\nabla \cdot \vec{B} = 0$$
  $$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} - \frac{1}{Re}\Delta \vec{u} + \nabla p + S\vec{B} \times \nabla \times \vec{B} = \vec{0}$$
  $$\nabla \cdot \vec{u} = 0$$
- Integrate by parts so we can have $\vec{B} \in H(curl)$
- Q_2-Q_1 for $\vec{u}$-p, first order edge elements for $\vec{B}$, Q_1 for r results in a stable discretization

A Preconditioner Form

- The discrete system:

$$\begin{pmatrix} \mathcal{M} & \mathcal{Z} \\ \mathcal{Y} & \mathcal{N} \end{pmatrix} \begin{pmatrix} \mathbf{x}_{em} \\ \mathbf{x}_{fl} \end{pmatrix} = \begin{pmatrix} \mathcal{I} & 0 \\ \mathcal{Y}\mathcal{M}^{-1} & \mathcal{I} \end{pmatrix} \begin{pmatrix} \mathcal{M} & \mathcal{Z} \\ 0 & \mathcal{X} \end{pmatrix} \begin{pmatrix} \mathbf{x}_{em} \\ \mathbf{x}_{fl} \end{pmatrix}$$

$$\mathcal{X} = \mathcal{N} - \mathcal{Y}\mathcal{M}^{-1}\mathcal{Z}$$

- Use an approximation of the U factor as the preconditioner
- Requires approximations of the Maxwell system and the perturbed Navier-Stokes system
- If fluids are ordered first, a Schur complement is obtained on the electromagnetic DOFs (considering this for other formulations)
- We prefer here to put the Schur complement on the fluids as there are already several difficulties with the Maxwell system

The (Convective) Maxwell Saddle Point System

- Discrete system:

$$\begin{pmatrix} F_{\mathbf{B}} & D^t \\ D & 0 \end{pmatrix} \begin{pmatrix} \mathbf{B} \\ \mathbf{r} \end{pmatrix}$$

- Corresponding continuous system:

$$\begin{pmatrix} \frac{1}{\Delta t} I + \frac{1}{Re_m} \nabla \times \nabla \times - \nabla \times (\vec{a} \times \cdot) & \nabla \\ -\nabla \cdot & 0 \end{pmatrix} \begin{pmatrix} \vec{B} \\ r \end{pmatrix}$$

- The (1,1) block is singular at steady state and close to singular when Δt is large
- The curl-conforming convection-diffusion operator has a large null space (not all errors are well represented in the residual $\rightarrow$ traditional multigrid fails)
- May want to modify the (1,1) block so it is easier for component solvers (necessary for steady problems)

Grad Div Augmentation

- Based on augmented Lagrangian preconditioning (e.g. Benzi & Olshanskii)
- Augment the (1,1) block

$$\begin{pmatrix} F_{\mathbf{B}} & D^t \\ D & 0 \end{pmatrix} = \begin{pmatrix} I & -\frac{1}{R_m} D^t Q_{\mathbf{r}}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} F_{\mathbf{B}} + \frac{1}{Re_m} D^t Q_{\mathbf{r}} D & D^t \\ D & 0 \end{pmatrix}$$

- Approximately completes the Laplacian

$$G_{\mathbf{B}} := F_{\mathbf{B}} + \frac{1}{Re_m} D^t Q_{\mathbf{r}} D \sim \frac{1}{\Delta t} I - \frac{1}{Re_m} \Delta - \nabla \times (\vec{a} \times \cdot)$$

- Null space is now trivial, not close to singular if Δt is large
- Traditional multigrid can be used on this operator (errors are well represented in the residual)

The Grad Div Schur Complement

- Block LU decomposition:

$$\begin{pmatrix} G_{\mathbf{B}} & D^t \\ D & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ DG_{\mathbf{B}}^{-1} & 0 \end{pmatrix} \begin{pmatrix} G_{\mathbf{B}} & D^t \\ 0 & X_{\mathbf{r}} \end{pmatrix}$$

$$X_{\mathbf{r}} = -DG_{\mathbf{B}}^{-1}D^t$$

- Approximate the Schur complement with commutators, motivated by PCD preconditioner for Navier-Stokes
- Continuous commutator:

$$\nabla \cdot [\tfrac{1}{\Delta t} I - \tfrac{1}{Re_m}\Delta - \nabla \times (\vec{a} \times \cdot)] = [\tfrac{1}{\Delta t} I - \tfrac{1}{Re_m}\Delta_r]\nabla \cdot$$

- Discrete commutator:

$$DQ_{\mathbf{B}}^{-1}G_{\mathbf{B}} \approx L_{\mathbf{r}}Q_{\mathbf{r}}^{-1}D$$

- Schur complement:

$$X_{\mathbf{r}} \approx -Q_{\mathbf{r}}L_{\mathbf{r}}^{-1}A_{\mathbf{r}}$$

The Full Grad Div Approximation

- Two approximations

$$\begin{pmatrix} G_{\mathbf{B}} & D^t \\ 0 & -Q_{\mathbf{r}} L_{\mathbf{r}}^{-1} A_{\mathbf{r}} \end{pmatrix}, \quad \begin{pmatrix} G_{\mathbf{B}} & D^t L_{\mathbf{r}}^{-1}(L_{\mathbf{r}} + \frac{1}{Re_m} A_{\mathbf{r}}) \\ 0 & -Q_{\mathbf{r}} L_{\mathbf{r}}^{-1} A_{\mathbf{r}} \end{pmatrix}$$

- The second incorporates the upper triangular factor used for the original augmentation, requires an extra solve with $L_{\mathbf{r}}$
- Have proven bounds on the eigenvalues of both preconditioned system
- Steady versions

$$\begin{pmatrix} G_{\mathbf{B}} & D^t \\ 0 & -Q_{\mathbf{r}} \end{pmatrix}, \quad \begin{pmatrix} G_{\mathbf{B}} & 2D^t \\ 0 & -Q_{\mathbf{r}} \end{pmatrix}$$

- Steady state without convection: S.-L. Wu, T.-Z. Huang, and L. Li. Block triangular preconditioner for static Maxwell equations. Comput. Appl. Math., 30:589-612, 2011.

Mass Augmentation

- Through similar arguments, we obtain the mass augmentation preconditioner

$$\begin{pmatrix} F_{\mathbf{B}} + \frac{1}{Re_m} Q_{\mathbf{B}} & 0 \\ 0 & Re_m A_{\mathbf{r}} \end{pmatrix}$$

- Also has provable eigenvalue bounds
- Requires fewer solves and multiplies
- Makes the (1,1) block nonsingular for steady problems, further from singular for transients
- Traditional multigrid still can't be used on the (1,1) block, but there exist special edge based multigrid routines for operators of this type (with no convection)
- We use an eddy current Maxwell solver implemented in ML (Trilinos)
- J. Hu, R. Tuminaro, P. Bochev, C. Garasi, and A. Robinson. Toward an h-independent algebraic multigrid method for Maxwell's equations. SIAM J. Sci. Comput., 27:1669-1688, 2006.

The Perturbed Navier-Stokes System

- Schur complement $\mathcal{X} = \mathcal{N} - \mathcal{Y}\mathcal{M}^{-1}\mathcal{Z}$
- Working with the corresponding continuous operators, $-\mathcal{Y}\mathcal{M}^{-1}\mathcal{Z}$ can be approximated by a discretization of

$$\begin{pmatrix} \gamma S Re_m \vec{b} \times (\cdot \times \vec{b}) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{u} \\ p \end{pmatrix}$$

- The discrete approximation:

$$\mathcal{X}\mathbf{x}_{fl} \approx \begin{pmatrix} F_{\mathbf{B}} + K & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix}$$

- Saddle point system with zero order perturbation of the fluid convection-diffusion operator

Perturbed Navier-Stokes Schur Complement

- Upper triangular approximation

$$\begin{pmatrix} F_{\mathbf{B}} + K & B^t \\ 0 & X_{\mathbf{p}} \end{pmatrix}, \quad X_{\mathbf{p}} = -B(F_{\mathbf{B}} + K)^{-1} B^t$$

- The convection-diffusion operator commutes well with the divergence, but the coupled operator corresponding to K does not
- Motivated by our work on an exact penalty MHD formulation, we apply a modified version of LSC where the contribution of K is relaxed

$$X_{\mathbf{p}} \approx -(BQ_{\mathbf{u}}^{-1}B^t)[BQ_{\mathbf{u}}^{-1}(F_{\mathbf{B}} + \alpha K)Q_{\mathbf{u}}^{-1}B^t]^{-1}(BQ_{\mathbf{u}}^{-1}B^t)$$

- E. Phillips, H. Elman, E. Cyr, J. Shadid, and R. Pawlowski. A block preconditioner for an exact penalty formulation for stationary MHD. SIAM J. Sci. Comput., 36:B930-B951, 2014.

Component Solves

$$\mathcal{A} = \begin{pmatrix} \mathcal{M} & \mathcal{Z} \\ \mathcal{Y} & \mathcal{N} \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} \hat{\mathcal{M}} & \mathcal{Z} \\ 0 & \hat{\mathcal{X}} \end{pmatrix}$$

- $\hat{\mathcal{M}}$ is either the full grad div approximation incorporating the upper triangular term (GDf), the economy grad div approximation (GDe), or the mass augmentation approximation (M)
- $\hat{\mathcal{X}}$ is the upper tringular perturbed Navier-Stokes approximation

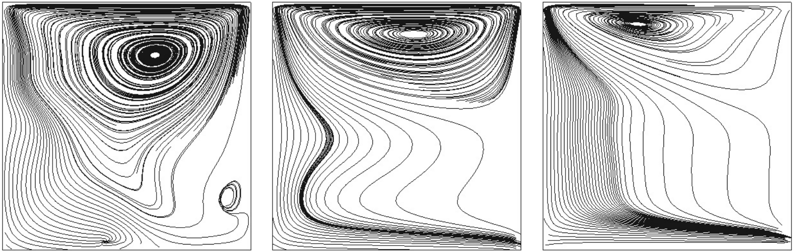
Operator	AMG Smoother	
$F_{\mathbf{u}} + K$	1 sweep ILU(0)	
$BQ_{\mathbf{u}}^{-1}B^t$	5 sweeps GS	
$A_{\mathbf{r}}, L_{\mathbf{r}}$	5 sweeps GS	
$F_{\mathbf{B}} + \frac{1}{Re_m} D^t Q_{\mathbf{r}}^{-1} D$	2 sweeps ILU(0)	wrapped in GMRES (10^{-3})
$F_{\mathbf{B}} + \frac{1}{Re_m} Q_{\mathbf{B}}$	Maxwell	wrapped in GMRES (10^{-3})

Implementation

- Implemented in the Trilinos framework
  - Drekar for finite element formulation
  - Teko for constructing block preconditioners
  - ML for multigrid
  - IFPACK for smoothers
- Backward Euler with fixed time step for transient problems
- Newton's method for nonlinear solves (10^{-4} residual reduction)
- GMRES for linear solves (10^{-3} residual reduction)

Test Problem

- 3D lid driven cavity with imposed magnetic field
- Domain $[0,1]^3$
- $\vec{u} = (1,0,0)$ on top, zero on all other walls
- $\vec{B} \times \vec{n} = (-1,0,0) \times \vec{n}$
- $Re = 100, S = 1$, steady solution $z = 0$ cross section



$Re_m = 1$ $Re_m = 10$ $Re_m = 100$

Transient Results

- $h = \frac{1}{16}, \Delta t = \frac{1}{16}, Re = 100$, integrated to $t = 2$
- $CFL_{Alfven} = \sqrt{S}\frac{\Delta t}{h}$ is a measure of how well the Alfvén wave is resolved by the time discretization
- $CFL_{Alfven} = 1$ if discrete time follows Alfvén speed
- Alfvén wave is stiff when $CFL_{Alfven} > 1$
- $Ha = \sqrt{SReRe_m}$ is a non-dimensional measure of coupling

	M			GD			E		
CFL_{Alfven} \ Ha	1	10	100	1	10	100	1	10	100
1	22	18	20	30	21	21	26	20	21
10	31	22	46	111	30	49	30	28	49
100	44	32	51	108	32	66	29	31	62

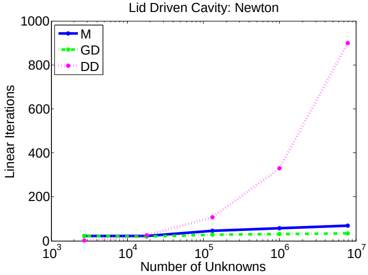
Steady Results

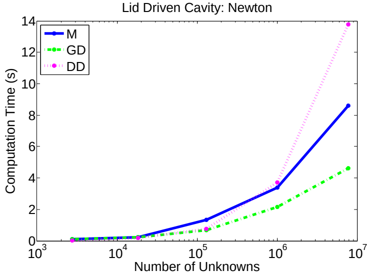
- $h = \frac{1}{16}, S = 1$
- GD and E give the same results for steady state

	M			GD/E		
Re \ Re_m	1	10	100	1	10	100
1	13	23	27	10	12	16
10	17	29	48	14	16	28
100	34	47	84	21	28	62

Steady Scaling Results

- Number of processors: 1, 8, 64, 512, 4096, $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$
- $Re = 100$, $Re_m = 10$, $S = 1$
- Compared to a domain decomposition preconditioner with one level of overlap and ILUTP on each subdomain





- Timings are dominated by the edge solves, growth largely due to more GMRES iterations being needed to reach 10^{-3} tolerance
- Need to improve these solves

Conclusions

- Developed block preconditioners for a dual saddle point MHD formulation that is robust to the non-dimensional parameters and scaled reasonably well
- Block structure and Schur complement approximations handle coupling
- Use augmentation to make the Maxwell subsystem solvable
- GD and E preconditioners use only traditional multigrid
- M preconditioner makes use of existing Maxwell multigrid algorithms
- Scaling can be improved by focusing on the $\vec{B}$ field component solves $\rightarrow$ improve smoothers to avoid using an inner GMRES solve
- The curl-conforming convection-diffusion operator arises in other MHD formulations
- We are currently using ideas developed for this formulation on vector potential and $\vec{E}$-$\vec{B}$ MHD formulations, extending to two-fluid MHD and full Maxwell