SAND2015-XXXX

Extended and Conformal Decomposition Finite Elements for 3D Compatible Discretizations

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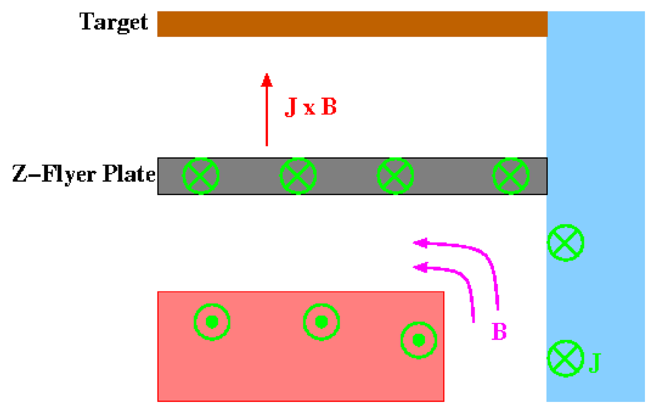
Motivating Application



Motivating application: Z-flyer plate
- Lorentz force accelerates plate towards target
- Goal: Shape current pulse to ensure plate
- is as *flat* and *solid* as possible on impact.

Current concentrates on material surfaces
- Material/void interfaces "count" for EM as voids must be meshed.
- Lagrangian only works for small deformation.
- Eulerian mixture models underdeveloped for EM.
- Solution: Interface tracking + local mesh or basis refinement.
- Basis refinement: eXtended Finite Element Method (XFEM).
- Mesh refinement: Conformal Decomposition Finite Element Method (CDFEM).

Governing PDEs

Nodes (2D/3D):
$$\tau\frac{\partial u}{\partial t}+\nabla\cdot\sigma\nabla u=0$$

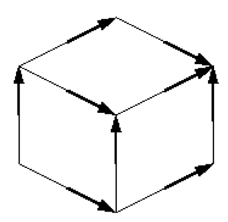
Edges (2D/3D) :
$$\sigma\frac{\partial \mathbf{E}}{\partial t}+\nabla\times\nu\nabla\times\mathbf{E}=0$$

Faces (3D):
$$\nu\frac{\partial \mathbf{F}}{\partial t}+\nabla\kappa\nabla\cdot\mathbf{F}=0$$

These PDEs capture electrostatics, magnetic diffusion and thermal diffusion (both nodal and flux-based).

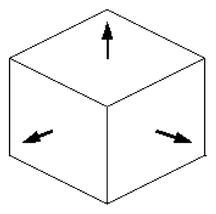
Approach

Edge and Face Element Discretizations
- Preserves **div curl = 0** and **curl grad = 0** discretely.
- Fact: Lowest order edge basis functions on simplicies (tris, tets) have a *constant tangential component* along any line.
- Fact: Lowest order face basis functions on simplicies (tris, tets) have a *constant normal component* along any plane.



Edge Basis

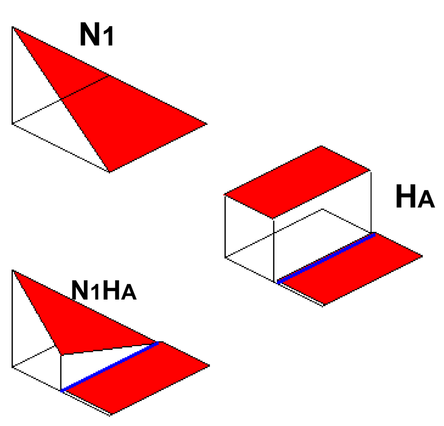
Face Basis



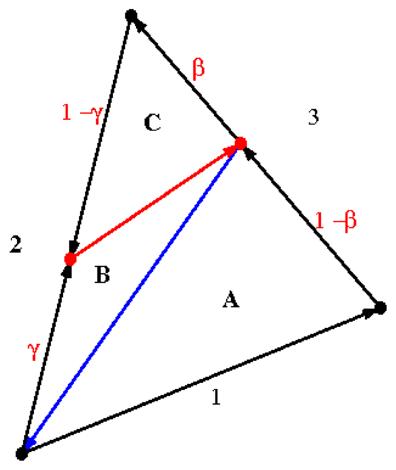
eXtended Finite Element Method (XFEM)
- Add *intra-element* discontinuities w/o changing mesh.
- Used here for **weak** (bonded materials) discontinuities.
- Uses Partition of Unity to preserve convergence:

$$u^h(x)=\sum_A\sum_I N_i(x)H_A(x)u_{I,A}$$

where the constant is in the span of the *F*'s
- Tie solution together w/ *virtual algebraic constraints*.
- Idea #1: Start w/ simplicies, decompose into simplicies.
- Idea #2: Apply just enough constraints to tie *everywhere*.



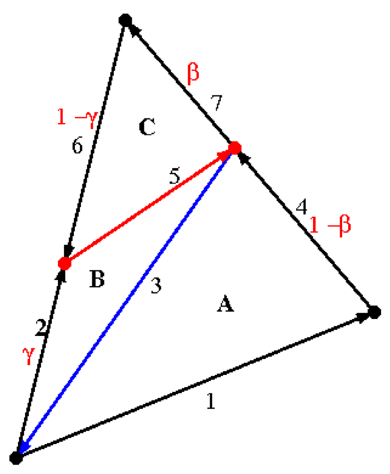
XFEM Basis
3 Edges x 3 Materials –
2 Curt * 1 Constraint
= 7 Effective DOFs



This allows us to prove **exact** equivalence to…

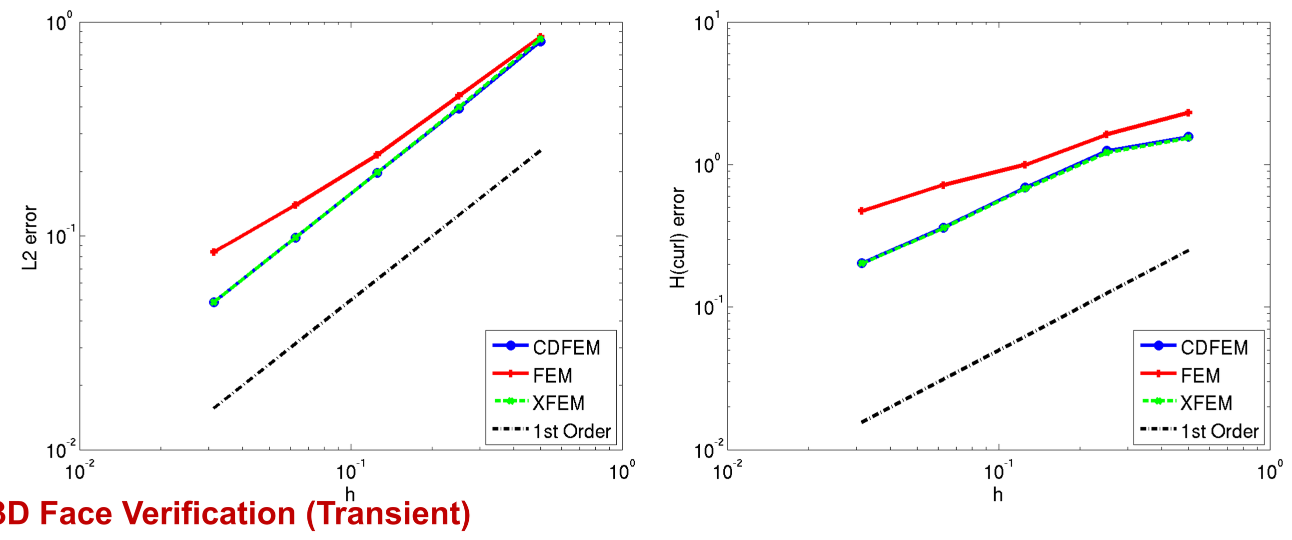
Conformal Decomposition Finite Element Method (CDFEM)

CDFEM Basis
7 Edges
= 7 Effective DOFs

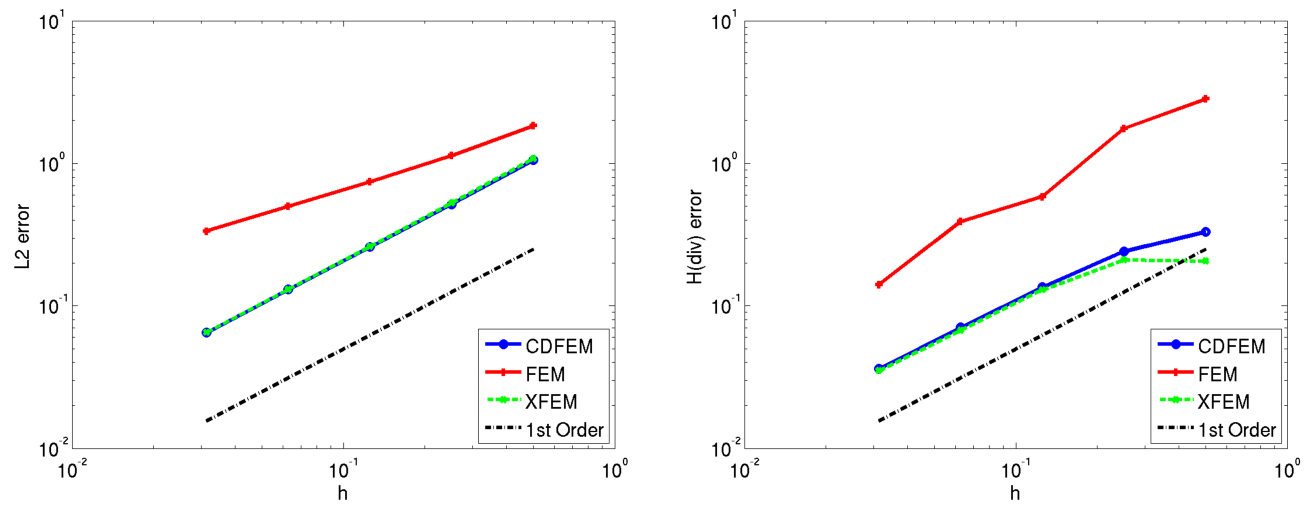


3D Verification Results

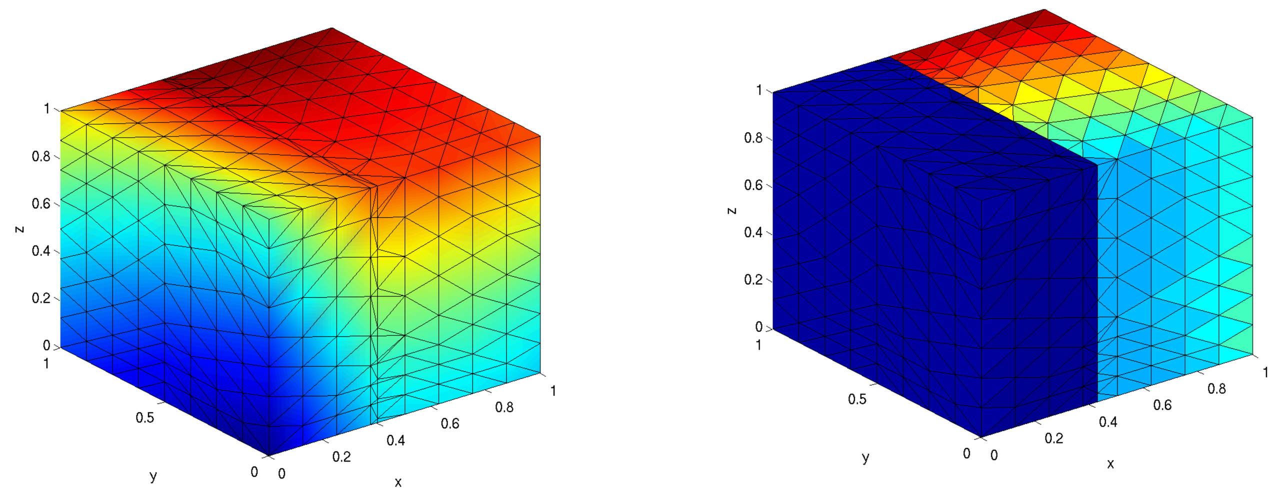
3D Edge Verification (Transient)



3D Face Verification (Transient)



3D Visualizations



Remap Algorithms

The starting point for our remap method is:
P. Bochev and M. Shashkov. Constrained interpolation (remap) of divergence-free fields. *Comput. Methods Appl. Mech. Engrg.*, 194:511–530, 2005.

The key: Think about remap as a method to transfer a representation of a field from one mesh to another.

Given the source field $\mathbf{B}^o$,

$$\lambda^e_{\mathrm{opt}}=\operatorname{argmin}\left|\|\mathbf{B}^o\|^2_e-\|\mathbf{B}^n(\lambda^e)\|^2_e\right|^2\quad\forall e\in T^n.$$

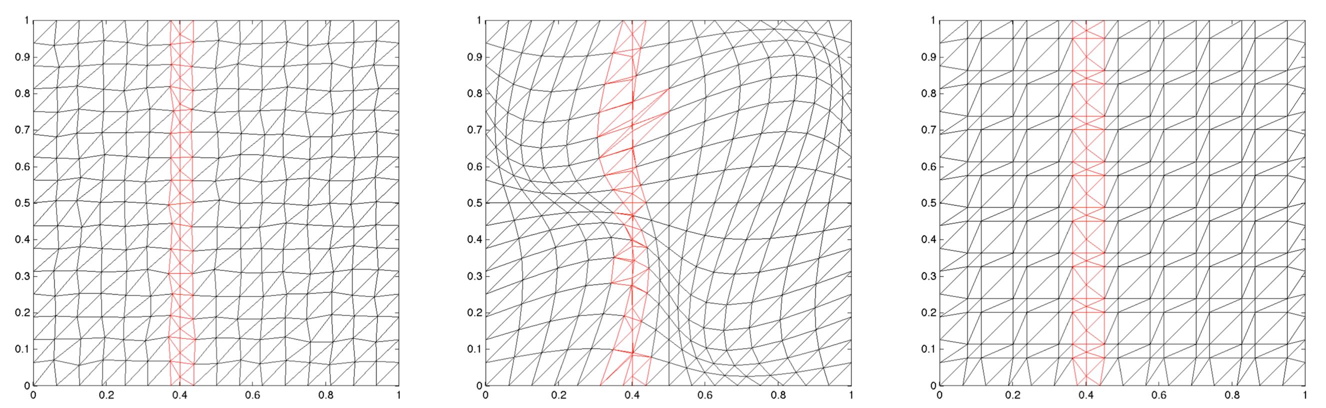
where (element-wise)

Low order (dissipative) field approximation $\mathbf{B}^n_{\mathrm{lo}}$	+	High order (accurate) field approximation $\mathbf{B}^n_{\mathrm{hi}}$	=	Accurate and conservative remapped field $\mathbf{B}^n=\lambda_{\mathrm{opt}}\mathbf{B}^n_{\mathrm{lo}}+(1-\lambda_{\mathrm{opt}})\mathbf{B}^n_{\mathrm{hi}}$

on the destination mesh T^n.

Remap Tests

Remap testing uses three different mesh motions, shown below.
A fixed material interface is located at *x* = 0.4; elements cut by the interface are enriched and shown in red. Each mesh increment is independently enriched.



RandomFixed — Random motion of interior nodes
Wave — Sinusoidal motion of interior nodes
Torture — Mesh compression and expansion

Remap Algorithm Overview

Construct 'low'- and 'high'-order interpolations of the source field on the destination mesh

↓

Form the desired reference norm from the source field

↓

(Locally) optimize the combination of low- and high-order interpolations such that the norm is preserved

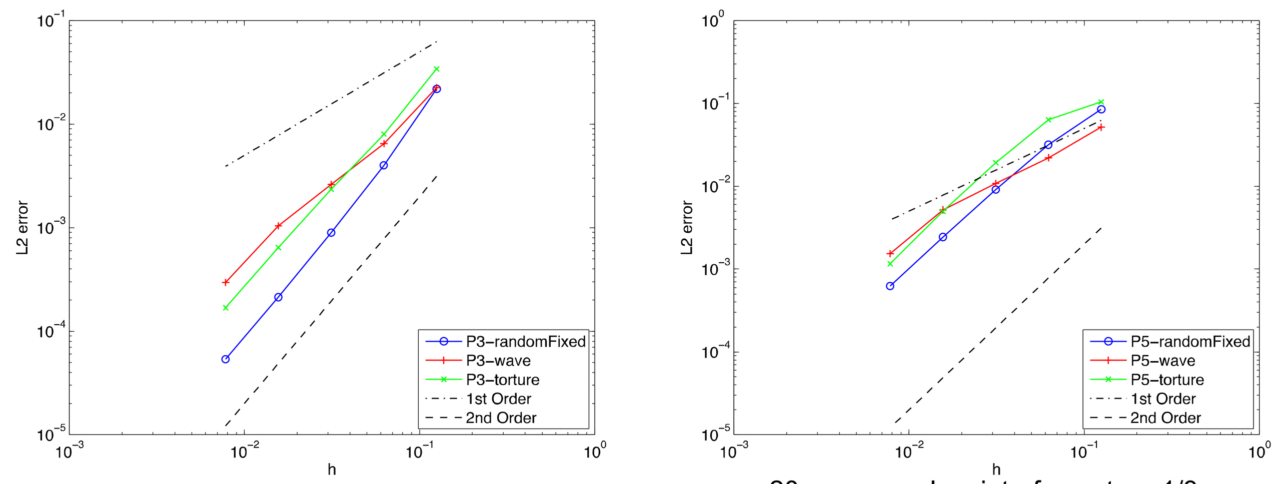
Patch recovery
Nodal: Standard procedure to extract element-wise quadratic approximations from linear FEM over element neighbors.
Edge: Extract coefficients for 2nd-order Nédélec edge shape function polynomials from the 1st order approximation.
Respects material interfaces.

Reference Norm
Perfect remap idealization: Integrate the known analytic solution over each destination element to define the reference norm.
Practical alternatives:
- Intersection remap of element quantities from source to destination elements.
- Approximate integration of source field on the dest mesh.

Optimization
Each element generates a simple squared quadratic objective function: roots at which local minima and maxima occur can be evaluated analytically (minimizes computational cost and improves accuracy).

2D Remap Results

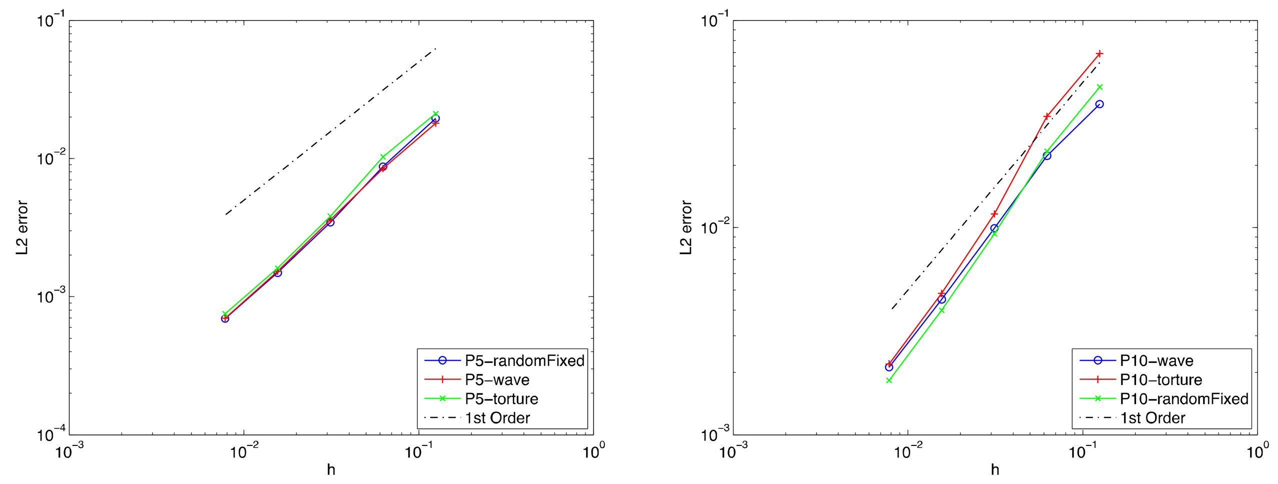
Nodal A-form (transverse magnetic):
Equation (1) solved assuming *A* = *A*(*x*, *y*), with a nodal compatible discretization, where *A* is a scalar potential of the **B** field such that **B** = grad(*A*).
Norm preserved is H^1 seminorm of the field, i.e., $|\mathrm{grad}(A)|^2 = \|\mathbf{B}\|^2$.



20 remap cycles, interface at *x* = 0.4
$$A=\cos(\pi y)(\sinh(\pi x)-\coth(\pi)\cosh(\pi x))$$

20 remap cycles, interface at *r* = 1/3
$$A=\begin{cases}2-r^2/4, & r<1/3,\\ 1-r^2, & r\geq 1/3\end{cases}$$

Compatible transverse electric:
Equation (1) solved assuming **A** = [A_x, A_y, 0], with an edge-based compatible discretization, where **A** is a vector potential of the B-field such that **B** = curl(**A**).
Norm preserved is H(curl) of the field, i.e., $|\mathrm{curl}(\mathbf{A})|^2 = |\mathbf{B}|^2$.



20 remap cycles, interface at *x* = 4/7
$$\mathbf{A}=\left[(1-8x+2y+8xy)/24,\left(-1+2x+4y-8x^2\right)/24\right]$$

20 remap cycles, interface at *x* = 4/7
$$\mathbf{A}=\begin{cases}\left[y\left(\frac{169}{266}-\frac{28}{19}x\right),0\right], & x<4/7\\ \left[y\left(\frac{1}{266}-\frac{98}{266}x\right),0\right], & x\geq 4/7\end{cases}$$

Current and Future Work

- Improved (piecewise linear) interface reconstruction.
- Mixed meshes (quad/tri and hex/pyramid/prism/tet).
- Extending remap algorithms to 3D.
- Improved patch recovery for edge element discretizatiobns.
- References:

  - R. Kramer, P. Bochev, C. Siefert and T. Voth. An extended finite element method with algebraic constraints (XFEM-AC) for problems with weak discontinuities. *Computer Methods in Applied Mechanics and Engineering*, Volume 266, pp. 70—80, 2013.
  - R. Kramer, P. Bochev, C. Siefert and T. Voth. Algebraically constrained extended edge element method (eXFEM-AC) for resolution of multi-material cells. *Journal of Computational Physics*, Volume 276, Pages 596—612, 2014.

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