LLNL-TR-678422

LAWRENCE
LIVERMORE
NATIONAL
LABORATORY

EMP-002a Phase Shift through the Ionosphere

R. Soltz, D. Simons, E. Fenimore, S. Wilks, T. Carey

October 20, 2015

Disclaimer

This work performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under Contract DE-AC52-07NA27344.

EMP-002a Phase Shift through the Ionosphere

R. Soltz, D. Simons, E. Fenimore, S. Wilks, T. Carey
Lawrence Livermore National Laboratory

In this note we review the derivation and use of the Ionospheric Transfer Function (ITF) in the DIORAMA model to calculate the propagation of a broad band ElectroMagnetic Pulse (EMP) through the Ionosphere in the limit of geometric optics. This note is intended to resolve a misunderstanding between the NDS VVA and EMP modeling teams regarding the appropriate use of the phase and group velocities in DIORAMA. The different approaches are documented in EMP-002 note, "Phase vs. Group" [1], generated by the LLNL DIORAMA VVA team, and the subsequent response from the DIORAMA EMP modeling team' [2].

We show that the form that we suggest for the EMP propagation through the Ionosphere is a direct solution to Maxwell's equations, using a Fourier Transform of the the electromagnetic wave equation in the WKB approximation. The result is expressed as an exponential of a path integral over the index of refraction, which is itself a solution the Appleton-Hartree Equation. The index of refraction may be written in terms of the phase velocity, but the main point is that the solution to Maxwells Equations depends upon the value of the index of refraction, and not its derivative, as would be the case for an explicit dependence on the group velocity.

We assume that the concepts of phase velocity, $v_p = \omega/k$, and group velocity, $v_g = \partial\omega/\partial k$, are well understood by both the EMP modeling and VVA teams, as described in the standard texts [3, 4]. Following the derivation in Sec. 1, we generate in Sec. 2 an analytic solution for a narrow band, Gaussian wave packet propagating through the ionosphere. According to our derivation this produces the expected transit time that is consistent with the group velocity. In Sec. 3 we present a numerical solution for a wide band pulse, to demonstrate that both a change in the formula and a sign change are required for DIORAMA to produce a result that conforms to Maxwells Equations and does not violate causality.

We provide concise derivations of the Appleton-Hartree Equation and the W.K.B. approximation in the Appendices. These are included in order to complete the connection between the proposed form of the Ionospheric Transfer Function (ITF) and Maxwells Equations, however, these derivations can be found in many standard texts and online.

1 Propagation of ElectroMagnetic Waves through the Ionosphere

We begin with well known Appleton-Hartree Equation for the index of refraction, $\eta = kc/\omega$, for a cold, collisionless plasma in the presence of an external magnetic field,

$$\eta^2 = 1 - \frac{2X(1-X)}{2(1-X) - (Y\sin\theta)^2 \pm \sqrt{(Y\sin\theta)^4 + 4(1-X)^2(Y\cos\theta)^2}}, \tag{1}$$

where $X = \omega_{pe}^2/\omega^2 \equiv ne^2/(m_e\epsilon_0\omega^2)$, $Y = \Omega_{ce}/\omega \equiv eB/(m_e\omega)$, and θ refers to the angle of Poynting vector, $\mathbf{E}\times\mathbf{B}$, with respect to the local magnetic field. The derivation for this equation is summarized in Appendix A. With the dispersive nature of the medium specified by the index of refraction, the spatial variation in the wave equation takes the form

$$\nabla^2\mathbf{E} + k^2(\mathbf{r})\mathbf{E} = 0, \tag{2}$$

$$\nabla^2\mathbf{E} + \frac{\omega^2}{c^2}\eta^2(\mathbf{r})\mathbf{E} = 0, \tag{3}$$

where the spatial dependence of the index of refraction has been made explicit. In one dimension, along the vertical direction, z, this simplifies to

$$\frac{d^2\mathbf{E}}{dz^2} + \frac{\omega^2}{c^2}\eta^2(\omega, z)\mathbf{E} = 0. \tag{4}$$

For the case in which the wavelength is much smaller than the length scale for variations in the index of refraction, the W.K.B. approximation can be employed. The first order solution is then given by

$$\mathbf{E}(z) = \frac{1}{\sqrt{\eta(\omega, z)}} \exp\left[\pm i\frac{\omega}{c}\int_0^z \eta(\omega, z')dz'\right]. \tag{5}$$

The W.K.B. approximation is explained in more detail in Appendix B. In what follows we will drop the $\frac{1}{\sqrt{\eta(\omega,z)}}$ term from our equations for two reasons: it has no effect on the calculation of the time delay, and it is canceled by the inverse term from the magnetic field when calculating the power spectrum with the Poynting vector.

The time-dependent solution for an electric field oscillation of a given frequency emanating upwards from the ground takes the form

$$\mathbf{E}(z, t) = \mathbf{A}\exp\left[-i\frac{\omega}{c}\int_0^z \eta(\omega, z)dz + i\omega t\right]. \tag{6}$$

To calculate the phase shift given by Eq. 6 it is simpler to work in the frequency domain, by applying a Fourier Transform to obtain the phase delay, and then inverting to recover the time dependence. This is the procedure used in DIORAMA. In the most recent DIORAMA release, version-11.4, the EMPPropagator Fourier Transform calls are wrapped in the EMPFFT.cpp methods. These methods call the FFTW library [5], which perform a discrete Fourier Transform between the time and frequency domains using the following convention expressed in the continuum limit,

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt, \tag{7}$$

$$h(t) = \int_{-\infty}^{\infty} H(\omega)e^{i\omega t}d\omega. \tag{8}$$

We note that this convention is common but not universal, .e.g see [6] for an example of Fourier Transforms with the opposite sign convention.

Using this convention, the amplitude for the electromagnetic wave of a given frequency is

$$\mathbf{E}(z, \omega) = \mathbf{A}_\omega \exp\left[-i\frac{\omega}{c}\int_0^z \eta(\omega, z')dz'\right], \tag{9}$$

and the time-dependence is then recovered by applying the inverse Fourier Transform to obtain

$$\mathbf{E}(z, t) = \int_{-\infty}^{\infty} \mathbf{A}_\omega \exp\left[-i\frac{\omega}{c}\int_0^z \eta(\omega, z')dz'\right] e^{i\omega t}d\omega, \tag{10}$$

$$= \int_{-\infty}^{\infty} \mathbf{A}_\omega \exp\left[i\omega t - i\frac{\omega}{c}\int_0^z \eta(\omega, z')dz'\right] d\omega. \tag{11}$$

We note that one could replace the term $\frac{\eta(\omega,z)}{c}$ with $\frac{1}{v_p(\omega,z)}$, the phase velocity at a specific height and for a specific frequency. However, the use of the index of refraction in Eq. 11 has the advantage that it makes the connection to Maxwells Equations more explicit.

2 Analytic Calculation for a Gaussian Wave Packet

Here we examine the simple case of a narrow, Gaussian wave packet propagating from the ground to a height, L, through a cold, collisionless plasma in the absence of a magnetic field.

$$A_\omega = A \exp\left[-\frac{(\omega-\omega_1)^2}{2\sigma^2}\right], \tag{12}$$

$$\eta(\omega, z) = \sqrt{1-\frac{\omega_p^2(z)}{\omega^2}} \tag{13}$$

$$\approx 1-\frac{\omega_p^2(z)}{2\omega^2}, \quad \text{for } \omega \gg \omega_p. \tag{14}$$

For a narrow wave packet, we can expand the quantity $\frac{\omega}{c}\eta(\omega, z)$ about ω_1.

$$\frac{\omega}{c}\eta(\omega, z) = \left(\frac{\omega_1}{c}-\frac{\omega_p^2(z)}{2\omega_1 c}\right)+\left(\frac{1}{c}+\frac{\omega_p^2(z)}{2\omega_1^2 c}\right)(\omega-\omega_1)+O\left((\omega-\omega_1)^2\right) \tag{15}$$

$$= \frac{\omega}{c}-\frac{\omega_p^2(z)}{\omega_1 c}+\frac{\omega_p^2(z)\omega}{2\omega_1^2 c}+O\left((\omega-\omega_1)^2\right) \tag{16}$$

$$= \frac{\omega}{c}-\frac{bn(z)}{\omega_1 c}+\frac{bn(z)\omega}{2\omega_1^2 c}+O\left((\omega-\omega_1)^2\right) \tag{17}$$

where we have isolated the dependence on the electron density, $n(z)$, with the substitution $b = \frac{e^2}{\epsilon_0 m_e}$, and the first order nature of this has been made explicit. The integral over the z-dependent index of refraction can then be evaluated in terms of the electron content, $N_e(L)$. To first order we have,

$$\frac{\omega}{c}\int_0^L \eta(\omega, z')dz' = \frac{\omega L}{c}-\frac{bN_e(L)}{\omega_1 c}+\frac{bN_e(L)\omega}{2\omega_1^2 c}. \tag{18}$$

The electric field at $z = L$ and t is then,

$$E(L,t) = A\int_{-\infty}^{\infty}\exp\left[-\frac{(\omega-\omega_1)^2}{2\sigma^2}+i\left(\omega t-\frac{\omega}{c}L+\frac{bN_e(L)}{\omega_1 c}-\frac{bN_e(L)\omega}{2\omega_1^2 c}\right)\right]d\omega, \tag{19}$$

$$= A\exp\left(i\frac{bN_e(L)}{\omega_1 c}\right)\int_{-\infty}^{\infty}\exp\left[-\frac{(\omega-\omega_1)^2}{2\sigma^2}+i\omega\tau\right]d\omega, \tag{20}$$

where we have defined, $\tau = t-\left(\frac{L}{c}+\frac{bN_e(L)}{2\omega_1^2 c}\right)$. Completing the square yields

$$E(L,t) = A\exp\left(i\frac{bN_e(L)}{\omega_1 c}\right)\int_{-\infty}^{\infty}\exp\left[-\left(\frac{\omega-\omega_1}{\sqrt{2}\sigma}-i\frac{\sigma}{\sqrt{2}}\tau\right)^2-\frac{\sigma^2\tau^2}{2}+i\omega_1\tau\right]d\omega, \tag{21}$$

$$= A\sqrt{2\pi}\sigma\exp\left[i\frac{bN_e(L)}{\omega_1 c}+i\omega_1\tau\right]\exp\left(\frac{-\sigma^2\tau^2}{2}\right). \tag{22}$$

Therefore the electric field will be Gaussian in time, having a width inversely proportional to the frequency and a peak at time $t_p = \frac{L}{c}+\frac{bN_e(L)}{2\omega_1^2 c}$. The effective velocity for this wave packet is therefore,

$$v_{gauss} = \frac{L}{t_p} = \frac{c}{1+\frac{bN_e(L)}{2\omega_1^2 L}}. \tag{23}$$

This is similar to the average group velocity, evaluated at ω_1 to first order,

$$v_{group} = \frac{1}{dk/d\omega} = \frac{c}{(d/d\omega)(\eta\omega)} = \frac{c}{[\eta(\omega) + \omega(d\eta/d\omega)]} = \frac{c}{\left[1 - \frac{bN_e(L)}{2\omega_1^2 L} + \omega_1 \frac{bN_e(L)}{\omega_1^3 L}\right]} \tag{24}$$

$$= \frac{c}{\left[1 + \frac{bN_e(L)}{2\omega_1^2 L}\right]}. \tag{25}$$

Thus we arrive at the expected result: the phase velocity enters through the index of refraction in the W.K.B. approximation, and the Fourier Transform converts this into a wave packet with a time delay that is consistent with the group velocity for a narrow band wave packet centered at $\omega = \omega_1$.

We note that in this first order calculation, the width of the packet is independent of N_e and does not spread out in time, a characteristic of a dispersive medium. This deficiency would be remedied by expanding to second order in $(\omega - \omega_1)$, which would introduce an imaginary term in the coefficient of ω^2 within the exponent. This effect will be taken into account in the next section, which provides a numerical example for a more realistic wave form that is closer to the calculation performed in DIORAMA.

3 Numerical Calculation

Here we repeat the exercise of calculating the ionospheric time delay, but for a double-exponential electric pulse of the form

$$V(t) = \frac{V_0}{e^{-(t-t_0)/a} + e^{-(t-t_0)/\tau_b}}, \tag{26}$$

with V_0=100 Volts/meter (at 1 meter), a=1 nsec, and b=15 nsec. The propagation of this wave form is solved numerically using the full Appleton-Hartree equation and a Fast Fourier Transform routine, $\mathcal{F}$,

$$E(L,t) = \mathcal{F}\left[A_\omega \exp\left(-\frac{\omega L}{v_p}\right)\right], \tag{27}$$

with phase velocity defined by the index of refraction from Eq 1, and

$$\omega_{pe} = 8.98 \cdot 10^{-12}\left(\frac{\text{TEC}}{L}\right), \tag{28}$$

$$\Omega_{ce} = 2.8 \cdot 0.45 \text{ Gauss}, \tag{29}$$

$$\theta = \pi/4. \tag{30}$$

For this example, the total electron content, TEC=$60 \cdot 10^{16}$ e/m^2, and $L = 5 \cdot 10^5$ m. For comparison, we also perform a similar calculation substituting the group velocity for the phase velocity in Eq. 27, where the group velocity is evaluated numerically from the derivative of the index of refraction,

$$v_g = \frac{c}{\eta(\omega) + \omega(dn/d\omega)} \tag{31}$$

The results are shown in Fig. 1. The red band shows the results using the phase velocity in Eq. 27, with the signal arriving after the 1667 microsecond time of flight for the vacuum transit time. Substituting the group velocity produces the green band, which precedes the vacuum transit time, violating causality. The blue and purple bands show the impact of including a narrow band receiver filter at 25 $\pm$ 2 MHz. Note that the two bands are not symmetric about the vacuum transit time, indicating that reversing the

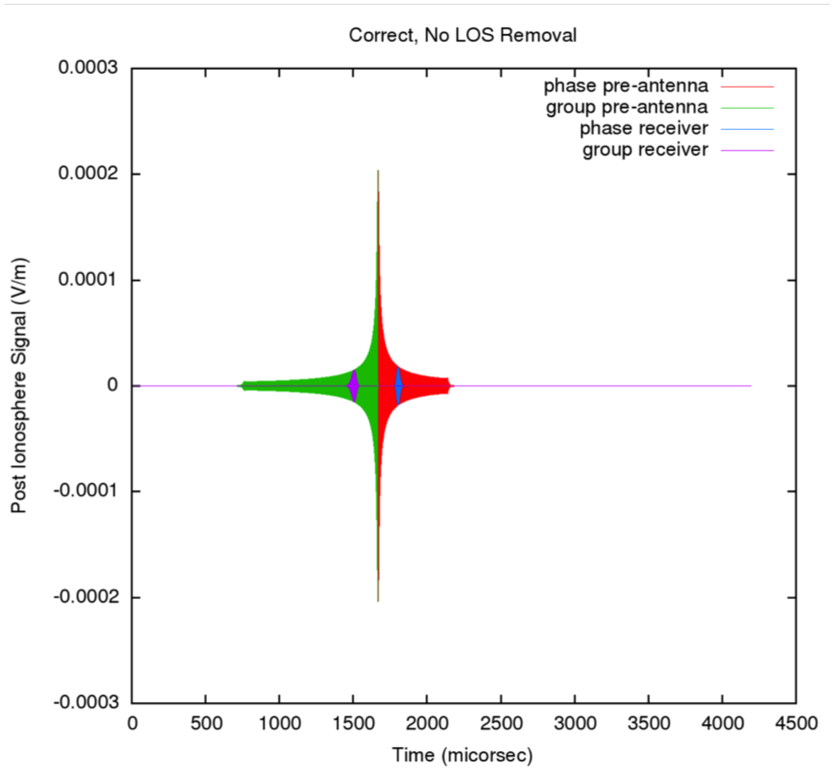

Figure 1: Phase delay for Gaussian wave packet using both phase (red) and group (green) velocities.

sign in the Fourier Transform combined with the use of group velocity in place of the phase velocity are not fully compensating.

These two errors are evident in the most recent version of the DIORAMA EMP package, lanl_emp-1.0.0.alpha.2plus. Line 2190 of EMPPropagator.cpp reads,

```
delay = (s1 + s3) / emp::constant::kc0 + s2 / group_vel;
```

We therefore conclude that for the DIORAMA EMP package to be valid, these lines should be replaced with

```
delay = (s1 + s3) / emp::constant::kc0 - s2 * eta_over_c;
```

where `eta_over_c` is the index of refraction divided by the speed of light. We also request that a similar correction be made in any other instances in which group velocity is incorrectly used to generate a phase delay in the frequency domain.

The Appendices which follow are only provided for completeness, to illustrate that the Ionospheric Transfer Function of Eq. 11 follows directly from Maxwells Equations.

A Derivation of Appleton-Hartree Equation

We begin with the time dependent expressions of Maxwell's Equations and the Langevin equation of motion for the electrons in the ionosphere:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{32}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(-en\mathbf{u} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right), \tag{33}$$

$$m\frac{d\mathbf{u}}{dt} = -e\left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right) - m\nu\mathbf{u}, \tag{34}$$

where e, m, n, $\mathbf{u}$, are the electron charge, mass, density, and four-velocity. We will neglect the electron collision frequency, ν, as well as cross terms in $\mathbf{u}$ and the time dependent components of $\mathbf{B}$. The frequency of the ion motion, $\omega_{\mathrm{pi}} \equiv \sqrt{\frac{nZ^2e^2}{m_i\epsilon_0}}$, is also assumed to be negligible. If we furthermore assume plane wave solutions of the form $\exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t)$, these equations reduce to the following set of coupled equations between the electric field and the electron current:

$$\mathbf{k} \times \mathbf{k} \times \mathbf{E} + \frac{\omega^2}{c^2} = \frac{i\omega en}{\epsilon_0 c^2}\mathbf{u}, \tag{35}$$

$$\mathbf{u} + \frac{ie}{m\omega}(\mathbf{u} \times \mathbf{B_0}) = \frac{-ie}{m\omega}\mathbf{E}, \tag{36}$$

where $\mathbf{B_0}$ is the local magnetic field. Solving these equations leads to the familiar Appleton-Hartree equation for the index of refraction, $\eta = kc/\omega$,

$$\eta^2 = 1 - \frac{2X(1-X)}{2(1-X) - (Y\sin\theta)^2 \pm \sqrt{(Y\sin\theta)^4 + 4(1-X)^2(Y\cos\theta)^2}}, \tag{37}$$

where $X = \omega_{pe}^2/\omega^2 \equiv ne^2/(m_e\epsilon_0\omega^2)$, $Y = \Omega_{ce}/\omega \equiv eB/(m_e\omega)$, and θ refers to the angle of Poynting vector with respect to the local magnetic field.

B The W.K.B. Approximation

The W.K.B. approximation is used to generate solutions to second-order differential equations of the form

$$\frac{d^2y(z)}{dz^2} \pm p(z)^2 y(z) = 0. \tag{38}$$

for regions where p(x) is large relative to the variation in $y(x)$. This is the case for Eq. 4 where ω/c is large and η varies slowly, i.e. the wave-length is much smaller than the length-scale for the plasma-density gradient:

$$\frac{d^2\mathbf{E}}{dz^2} + \frac{\eta^2(z)}{\epsilon^2}\mathbf{E} = 0, \tag{39}$$

where we have set $\frac{\omega}{c} = \frac{1}{\epsilon}$ to emphasize the scale of the pre-factor, $\frac{\eta}{\epsilon}$. The derivation of the W.K.B. approximation follows a common practice in solving differential equations, by guessing an initial solution, and then modifying it in an iterative fashion. Eq. 39 suggests a solution of the form,

$$E(z) = e^{i\phi(z)}, \tag{40}$$

which yields the following differential equation for $\phi(z)$:

$$i\frac{d^2\phi}{dz^2} - \left(\frac{d\phi}{dz}\right)^2 + \frac{\eta^2}{\epsilon^2} = 0. \tag{41}$$

If we assume the second derivative to be much less than the other two terms, we have

$$\left(\frac{d\phi}{dz}\right)^2 = \left(\frac{\eta}{\epsilon}\right), \tag{42}$$

$$\phi(z) = \pm\frac{1}{\epsilon}\int_0^z \eta(z')dz'. \tag{43}$$

This is the zero'th order W.K.B. approximation to Eq. 39, which takes the form

$$E(z) = \exp\left[\pm\frac{i}{\epsilon}\int \eta(z)dz\right]. \tag{44}$$

This is a reasonable approximation if we have $\frac{d^2\phi}{dz^2} = \frac{d\eta}{dz} \ll \frac{\eta^2}{\epsilon}$, which is more commonly written as,

$$\left|\frac{d}{dz}\frac{1}{\eta(z)}\right| \ll \frac{\omega}{c}. \tag{45}$$

Higher order terms of the W.K.B. approximation are obtained by iterating upon this solution. This process can be expressed more compactly as

$$E(z) = \left(\exp\left[\pm\frac{i}{\epsilon}\int \eta(z)dz\right]\right)\left(\phi_0(z) + \epsilon\phi_1(z) + \epsilon^2\phi_2(z) + \ldots\right). \tag{46}$$

Inserting Eq. 46 to all orders into Eq. 39 yields,

$$\left(\phi_0(z) + \epsilon\phi_1(z) + \epsilon^2\phi_2(z) + \ldots\right)' + \frac{\eta'}{2\eta}\left(\phi_0(z) + \epsilon\phi_1(z) + \epsilon^2\phi_2(z) + \ldots\right) \mp \frac{i\epsilon}{2\eta}\left(\phi_0(z) + \epsilon\phi_1(z) + \epsilon^2\phi_2(z) + \ldots\right)''.$$

Setting ϵ to zero leads to,

$$\phi_0'(z) + \frac{\eta'}{2\eta}\phi_0(z) = 0. \tag{47}$$

This yields the following pre-factor of the first-order W.K.B. approximation,

$$\phi_0(z) = \frac{1}{\sqrt{\eta(z)}}, \tag{48}$$

which matches the first-order W.K.B. solution given in Eq. 5.

References

[1] DIORAMA VVA Team, EMP-002, *Phase vs. Group*, January 8, 2015.

[2] M. Light, *Response to Phase vs. Group*, April 13, 2015.

[3] M. Born and E. Wolf, *Principles of Optics*, Cambridge University Press, 1959, pp. 18–20.

[4] J.D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, *Second Ed.*, 1975, pp 299–302.

[5] Documentation for the discrete, 1-D, real FFT function developed within the *Fastest Fourier Transform in the West* can be found here, http://www.fftw.org/doc/The-1d-Real_002ddata-DFT.html .

[6] W.H. Press, S.A. Teukolsky, W.T. Vetterling, B.P. Flannery, *Numerical Recipes in C: The Art of Scientific Computing*, Cambridge University Press, *Second Ed.*, 1992, p 496.