SANDIA REPORT

SAND2016-0509
Unlimited Release
Printed January 2016

Notes on a New Coherence Estimator

Douglas L. Bickel

Prepared by
Sandia National Laboratories
Albuquerque, New Mexico 87185 and Livermore, California 94550

Sandia National Laboratories is a multi-program laboratory managed and operated by Sandia Corporation, a wholly owned subsidiary of Lockheed Martin Corporation, for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000.

Issued by Sandia National Laboratories, operated for the United States Department of Energy by Sandia Corporation.

Printed in the United States of America. This report has been reproduced directly from the best available copy.

Available to DOE and DOE contractors from
U.S. Department of Energy
Office of Scientific and Technical Information
P.O. Box 62
Oak Ridge, TN 37831

Telephone: (865) 576-8401
Facsimile: (865) 576-5728
E-Mail: reports@adonis.osti.gov
Online ordering: http://www.osti.gov/bridge

Available to the public from
U.S. Department of Commerce
National Technical Information Service
5285 Port Royal Rd.
Springfield, VA 22161

Telephone: (800) 553-6847
Facsimile: (703) 605-6900
E-Mail: orders@ntis.fedworld.gov
Online order: http://www.ntis.gov/help/ordermethods.asp?loc=7-4-0#online

SAND2016-0509
Unclassified Unlimited Release
Printed January 2016

Notes on a New Coherence Estimator

Douglas L. Bickel
ISR Analysis and Application
Sandia National Laboratories
PO Box 5800
Albuquerque, NM 87185

ABSTRACT

This document discusses some interesting features of the new coherence estimator in [1]. The estimator is derived from a slightly different viewpoint. We discuss a few properties of the estimator, including presenting the probability density function of the denominator of the new estimator, which is a new feature of this estimator. Finally, we present an approximate equation for analysis of the sensitivity of the estimator to the knowledge of the noise value.

ACKNOWLEDGEMENTS

The preparation of this report is the result of an unfunded research and development activity.

Sandia National Laboratories is a multi-program laboratory managed and operated by Sandia Corporation, a wholly owned subsidiary of Lockheed Martin Corporation, for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000.

CONTENTS

This page left intentionally blank.

1. Executive Summary

This document discuss some characteristics and an interpretation of the new “change estimator” from [1]. The probability density function of the denominator is derived, which is the new feature of this estimator. The new estimator is sensitive to the relative knowledge of the value of the noise used in the estimator and we present an analysis of this sensitivity.

.

This page left intentionally blank.

2. Introduction

This document derives the estimator in [1] from a slightly different perspective. In particular, we discuss the interpretation of the new "change estimator" in [1] and the properties of the denominator. We focus on the denominator because mathematically it is the new feature in the estimator in [1]. Finally, we present the sensitivity of the new estimator to the relative knowledge of the value of the noise used in the estimator.

Coherent change detection (CCD) focuses on estimation of coherence between complex images as an indicator of change; however coherent change between images has many potential sources [2]. Therefore, as is implied in [1], it is not change in coherence that we wish to detect, but a specific type of change. Namely, as called out in [1], we wish to detect specifically "anthropogenic and zoogenic" changes.

Under the conditions where we can assume that the only significant sources of change between the images are due to thermal noise and/or "anthropogenic and zoogenic" change between the time the first image and second image are taken, then from [2] we can write the total coherence as:

$$\mu_{tot} = \mu_{az} \cdot \mu_{snr} \tag{1}$$

where in this case μ_{az} is the desired "anthropogenic and zoogenic" temporal change that we wish to detect.

We emphasize that in the applications where we wish to find change, it is not the coherence in and of itself that is important, but the use of coherence to detect the desired change.

We can generate the result in [1], by recognizing that we wish to estimate and detect the desired change μ_{az} by solving equation (1) for:

$$\mu_{az} = \frac{\mu_{tot}}{\mu_{snr}} \tag{2}$$

Since this is the case, we need to estimate μ_{snr}.

From [2] (and other references therein):

$$\mu_{snr} = \frac{\sqrt{snr_1 snr_2}}{\sqrt{(snr_1 + 1)(snr_2 + 1)}} \tag{3}$$

where snr_1 and snr_2 are the true expected values of the signal-to-noise ratios for the clutter cells in each of the two images that we correlate. We can rewrite equation (3) as:

$$\mu_{snr} = \frac{\sqrt{\left(\frac{\sigma_{s1}^2}{\sigma_{n1}^2}\right)\left(\frac{\sigma_{s2}^2}{\sigma_{n2}^2}\right)}}{\sqrt{\left(\frac{\sigma_{s1}^2+\sigma_{n1}^2}{\sigma_{n1}^2}\right)\left(\frac{\sigma_{s2}^2+\sigma_{n2}^2}{\sigma_{n2}^2}\right)}} \tag{4}$$

where σ_{si}^2 is the signal power of the i^{th} image, and σ_{ni}^2 is the corresponding noise power for that image.

If we let the signal-plus-noise ratio for the i^{th} image be:

$$spnr_i = \left(\frac{\sigma_{si}^2+\sigma_{ni}^2}{\sigma_{ni}^2}\right) = snr_i + 1 \tag{5}$$

then we can rewrite equation (3) as:

$$\mu_{snr} = \frac{\sqrt{(spnr_1 - 1)(spnr_2 - 1)}}{\sqrt{(spnr_1)(spnr_2)}} \tag{6}$$

At this point we will note that we are only able to deal with estimates of the coherence and coherence quantities. The maximum likelihood estimate of the total coherence is given by (see [1,2]):

$$|\hat{\mu}_{tot}| \approx \frac{\left|\sum_{n=0}^{L-1} x_{1,n} x_{2,n}^*\right|}{\sqrt{\sum_{n=0}^{L-1} |x_{1,n}|^2 \sum_{n=0}^{L-1} |x_{2,n}|^2}} \tag{7}$$

where $x_{i,n}$ is the "voltage" for the i^{th} image and the n^{th} pixel, and L (assumed to be) independent looks.

It can be shown, given L independent samples under the assumption that $x_{i,n}$ is radiometrically calibrated, then the maximum likelihood estimate of $spnr_i$ is:

$$\hat{s}pnr_i \approx \frac{\left(\frac{1}{L}\right)\sum_{n=0}^{L-1} |x_{i,n}|^2}{\sigma_{ni}^2} \tag{8}$$

where σ_{ni}^2 is the variance of the noise for the i^{th} image.

Similarly the maximum likelihood estimator for snr_i can be shown to be:

$$\hat{s}nr_i \approx \hat{s}pnr_i - 1 \tag{9}$$

If we can assume that we can substitute the maximum likelihood values into equation (3), and use the various equations above, we get an approximation for equation (2) of:

$$\begin{aligned}
\left|\hat{\mu}_{az}\right| \approx \frac{\left|\hat{\mu}_{tot}\right|}{\hat{\mu}_{snr}} &\approx \frac{\left|\sum_{n=0}^{L-1} x_{1,n} x_{2,n}^{*}\right|}{\sqrt{\sum_{n=0}^{L-1}\left|x_{1,n}\right|^2 \sum_{n=0}^{L-1}\left|x_{2,n}\right|^2}} \left(\frac{\sqrt{\left(\hat{s}pnr_1\right)\left(\hat{s}pnr_2\right)}}{\sqrt{\left(\hat{s}pnr_1 - 1\right)\left(\hat{s}pnr_2 - 1\right)}} \right) \\
&\approx \frac{\left|\sum_{n=0}^{L-1} x_{1,n} x_{2,n}^{*}\right|}{\sqrt{\sum_{n=0}^{L-1}\left(\left|x_{1,n}\right|^2 - \sigma_{n1}^2\right)\sum_{n=0}^{L-1}\left(\left|x_{2,n}\right|^2 - \sigma_{n2}^2\right)}}
\end{aligned} \tag{10}$$

For now, let $u = \sum_{n=0}^{L-1}\left(\left|x_{1,n}\right|^2 - \sigma_{n1}^2\right)$ and $v = \sum_{n=0}^{L-1}\left(\left|x_{2,n}\right|^2 - \sigma_{n2}^2\right)$, then we want to know more about $\sqrt{uv}$. We know from the inequality of the arithmetic and geometric means that:

$$\frac{u+v}{2} \ge \sqrt{uv} \tag{11}$$

where the equality only occurs for $u = v$. Therefore, as effectively in [1], the approximation is made as:

$$\sqrt{uv} \approx \frac{u+v}{2} \tag{12}$$

In other words, for the application at hand:

$$\begin{aligned}
\sqrt{\sum_{n=0}^{L-1}\left(\left|x_{1,n}\right|^2 - \sigma_{n1}^2\right)\sum_{n=0}^{L-1}\left(\left|x_{2,n}\right|^2 - \sigma_{n2}^2\right)} &\approx \frac{\sum_{n=0}^{L-1}\left(\left|x_{1,n}\right|^2 - \sigma_{n1}^2\right) + \sum_{n=0}^{L-1}\left(\left|x_{2,n}\right|^2 - \sigma_{n2}^2\right)}{2} \\
&\approx \frac{\sum_{n=0}^{L-1}\left|x_{1,n}\right|^2 + \sum_{n=0}^{L-1}\left|x_{2,n}\right|^2 - L\sigma_{n1}^2 - L\sigma_{n2}^2}{2}
\end{aligned} \tag{13}$$

Again, in making the above approximation of the geometric mean as the arithmetic mean, we have assumed that:

$$\sum_{n=0}^{L-1}\left(\left|x_{1,n}\right|^2-\sigma_{n1}^2\right)\approx\sum_{n=0}^{L-1}\left(\left|x_{2,n}\right|^2-\sigma_{n2}^2\right) \tag{14}$$

In the general case as equation (11) shows, the approximation of the denominator in [1], and in equation (13), will be larger than the true value[1].

Making these assumptions and pulling together the equations above, we get the key equation from [1]:

$$\left|\hat{\mu}_{az}\right|\approx\frac{2\left|\sum_{n=0}^{L-1}x_{1,n}x_{2,n}^*\right|}{\sum_{n=0}^{L-1}\left(\left|x_{1,n}\right|^2-\sigma_{n1}^2\right)+\sum_{n=0}^{L-1}\left(\left|x_{2,n}\right|^2-\sigma_{n2}^2\right)}=\frac{2\left|\sum_{n=0}^{L-1}x_{1,n}x_{2,n}^*\right|}{\sum_{n=0}^{L-1}\left|x_{1,n}\right|^2+\sum_{n=0}^{L-1}\left|x_{2,n}\right|^2-L\sigma_{n1}^2-L\sigma_{n2}^2} \tag{15}$$

We do not have to, but to simplify the interpretation of the results in this document from here on we will assume that the signal and noise powers are equivalent between each image for the rest of this document, i.e. $\sigma_{s1}^2=\sigma_{s2}^2$, and $\sigma_{n1}^2=\sigma_{n2}^2$.

[1] A quick thought experiment on the approximation shows that if we had no noise and the one image were say 1/10th the amplitude of the other image, then coherence from equation (10) would be 1 as it should be; whereas, the coherence using equation (15) would be nearly 0.2. Also, note that [1] recommends some scaling to try to limit the effect of this problem.

3. Statistical behavior of the denominator

We now focus on the denominator of the estimator from [1], which is the last line in equation (13). More details on the derivation are given in Appendix A and Appendix B. Again, although we do not have to, we will assume that $\sigma_{s1}^2 = \sigma_{s2}^2 = \sigma_s^2$ and $\sigma_{n1}^2 = \sigma_{n2}^2 = \sigma_n^2$.

We need a couple of preliminaries before we describe the statistics of the denominator. First, let's represent the denominator as:

$$z = \frac{\sum_{n=0}^{L-1} \left| x_{1,n} \right|^2 + \sum_{n=0}^{L-1} \left| x_{2,n} \right|^2 - 2L\sigma_n^2}{2} \tag{16}$$

Let us make a change in random variable, by adding a constant, and then scaling by a constant, such that we have the new random variable, y, given by:

$$y = \frac{2\left(z + L\sigma_n^2\right)}{\sigma_n^2} = \frac{\sum_{n=0}^{L-1} \left| x_{1,n} \right|^2 + \sum_{n=0}^{L-1} \left| x_{2,n} \right|^2}{\sigma_n^2} \tag{17}$$

The random variable, y, has the form which permits the use of equation (25) in the appendix[2]. For this statistic, we let $\theta_1 = \theta_2 = \theta = \frac{\sigma_s^2 + \sigma_n^2}{\sigma_n^2}$, L is of course the number of independent looks, and μ_{tot} is, as before, the true coherence. Based upon this, it can be shown that the probability density function (pdf) of y is as follows (see appendices):

$$f_Y\left(y\right) = \frac{\sqrt{\pi}\, y^{L-1/2}}{2^{L-1/2} \left|\mu_{tot}\right|^{L-1/2} \sqrt{1-\left|\mu_{tot}\right|^2}\,\Gamma\left(L\right)\theta^{L+1/2}} \exp\left\{-\frac{y}{\theta\left(1-\left|\mu_{tot}\right|^2\right)}\right\} I_{L-1/2}\left[\frac{\left|\mu_{tot}\right| y}{\theta\left(1-\left|\mu_{tot}\right|^2\right)}\right]$$

$$y \ge 0 \Rightarrow z \ge -L\sigma_n^2 \tag{18}$$

An important case of equation (18) occurs when $\left|\mu_{tot}\right| = 0$. For this case, equation (18) becomes a gamma pdf of (see Appendix A):

[2] Note that it is not required to normalize the random variable by the scalar, σ_n^2 , but the author chose to mainly for plotting convenience.

$$f_Y(y) = \frac{y^{2L-1}}{\theta^{2L}\Gamma(2L)} \exp\left\{-\frac{y}{\theta}\right\} \qquad y \geq 0 \tag{19}$$

The mean of the variable z from equation (16) is given by:

$$\langle z \rangle = L\sigma_s^2 \tag{20}$$

and the variance of the variable z from equation (16) is given by:

$$\operatorname{var}(z) = \frac{L\left(\sigma_s^2 + \sigma_n^2\right)\left(1 + |\mu_{tot}|^2\right)}{2} \tag{21}$$

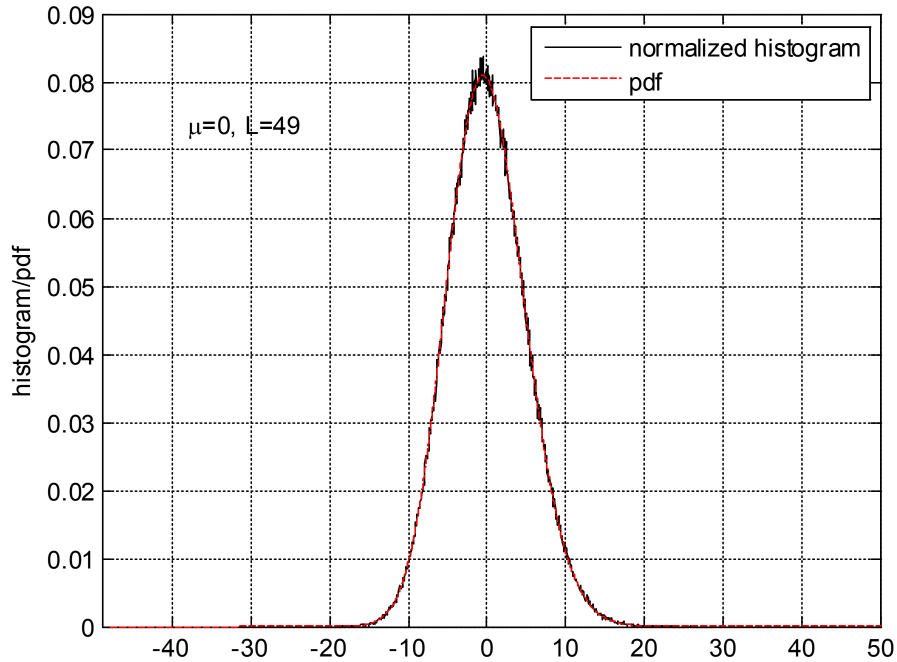
Figure 1 shows the simulated histograms versus pdf for a couple different cases. Again in these plots we are showing the pdfs of the variable z from equation (16) to help show what is really happening with the denominator variable. For simplification we assume that $\sigma_{n1}^2 = \sigma_{n2}^2 = 1$. Figure 1 shows the simulated histograms versus pdf for a couple different cases. Again in these plots we are showing the pdfs of the variable z from equation (16) to help show what is really happening with the denominator variable[3].

An interesting note is that there is always a non-zero probability that the denominator can be less than zero which means that $|\mu_{az}|$ in equation (15) can be negative. Typically this will only occur for low signal, and hence $|\mu_{tot}|$ should be close to zero. Nevertheless, we should probably consider either thresholding or using an absolute value on the denominator. Figure 2 shows the probability of the denominator being negative versus total coherence, $|\mu_{tot}|$, assuming $|\mu_{az}| = 1$ (i.e., no "change")[4].

In addition, values of the denominator can be close to zero with non-zero probability. Ideally these would only occur where the numerator from [1] is zero, but this is not always the case. This means we can get very large values of the estimate of $|\mu_{az}|$ with some probability. Of course, higher coherence and more looks help this situation. As mentioned in [1], a threshold for values of $|\mu_{az}|$ can be set at one.

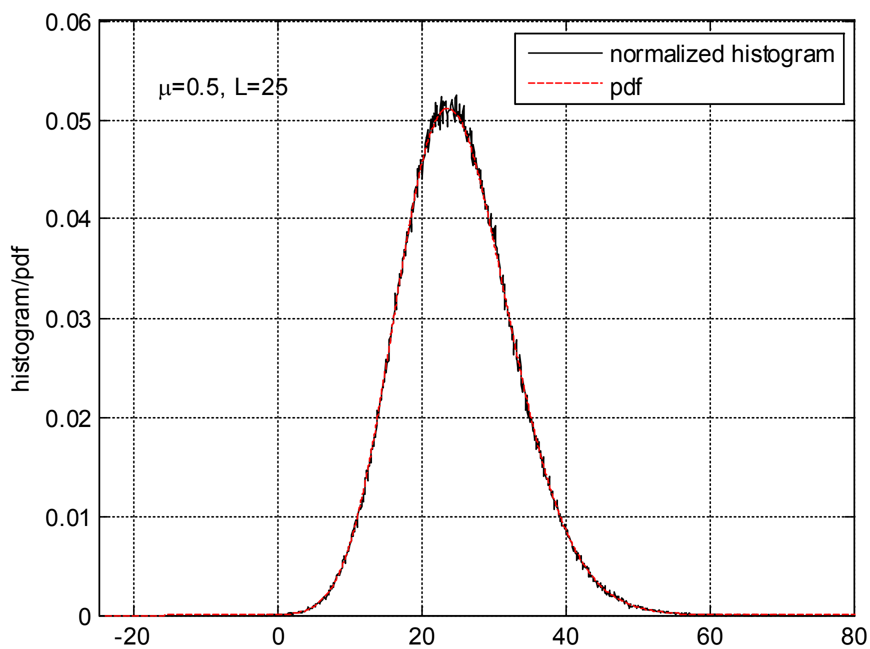
[3] Note we really have the random variable z normalized by σ_n^2.

[4] Unfortunately, in general the cumulative distribution function has to be determined numerically.



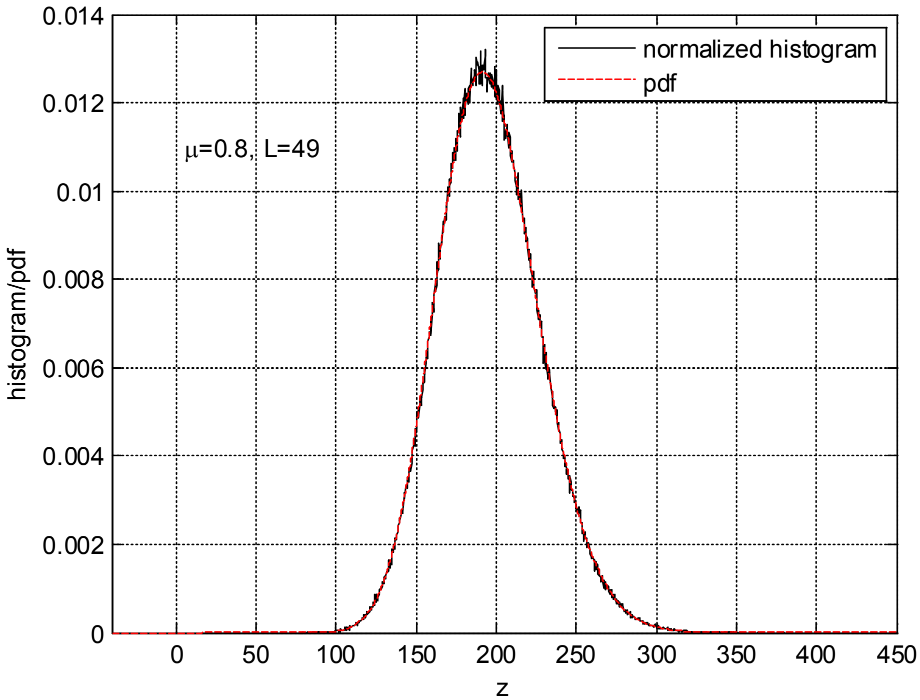
normalized histogram
pdf
μ=0, L=49
histogram/pdf

(a)



normalized histogram
pdf
μ=0.5, L=25
histogram/pdf

(b)



(c)

Figure 1: Comparison of simulated histograms to pdf in equation (18): a) $|\mu| = 0, L = 49$, b) $|\mu| = 0.5, L = 25$, c) $|\mu| = 0.8, L = 49$

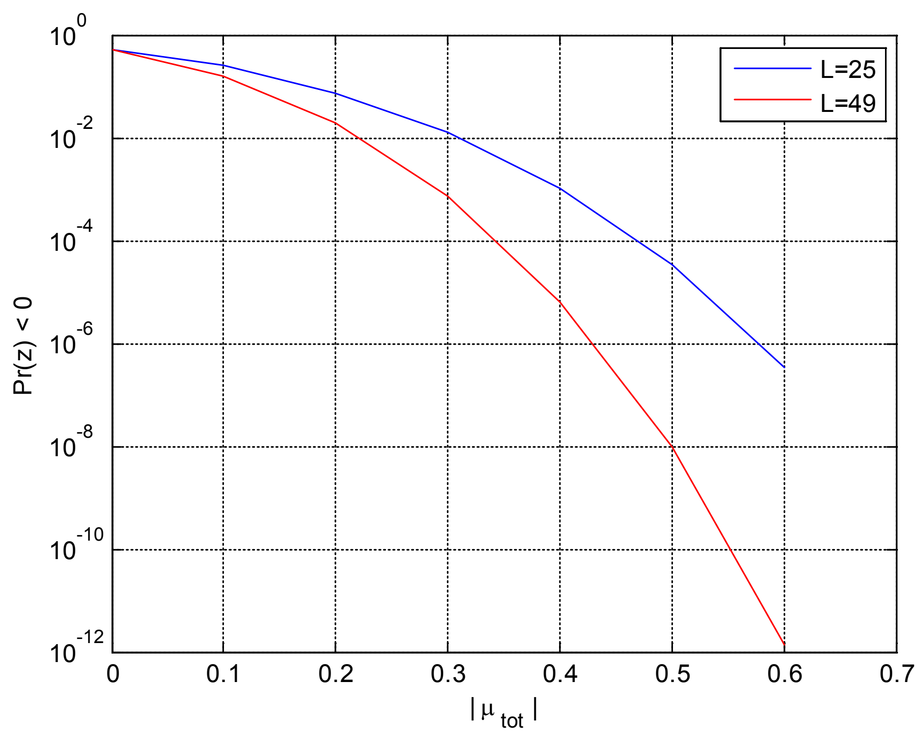


Figure 2: Probability of the denominator being less than zero versus coherence

4. Sensitivity to knowledge of noise power

We now look at the sensitivity of the estimator to our knowledge (or estimate) of the noise power. From equation (15), the first-order error can be found to be:

$$\varepsilon_{|\hat{\mu}_{az}|} \approx \left[\frac{2\left|\sum_{n=0}^{L-1} x_{1,n} x_{2,n}^{*}\right|}{\sum_{n=0}^{L-1}\left|x_{1,n}\right|^{2} + \sum_{n=0}^{L-1}\left|x_{2,n}\right|^{2} - 2L\sigma_{n}^{2}} \right] \left[\frac{\varepsilon_{\sigma_n^2}}{\left(\sum_{n=0}^{L-1}\left|x_{1,n}\right|^{2} + \sum_{n=0}^{L-1}\left|x_{2,n}\right|^{2} - 2L\sigma_{n}^{2}\right) \Big/ (2L)} \right] \tag{22}$$

where $\varepsilon_{\sigma_n^2}$ is the error in the noise.

We can rewrite this equation as approximately:

$$\varepsilon_{|\hat{\mu}_{az}|} \approx |\hat{\mu}_{az}| \left(\frac{1}{s\hat{n}r} \right) \left(\frac{\varepsilon_{\sigma_n^2}}{\sigma_n^2} \right) \tag{23}$$

where $\varepsilon_{|\hat{\mu}_{az}|}$ is the error in the estimated change coherence due to misknowledge of the noise, $\varepsilon_{\sigma_n^2} \big/ \sigma_n^2$ is the error in our noise value used as a fraction of the true noise value, $|\hat{\mu}_{az}|$ is the estimated change coherence which ideally is one when there is no change, and we let $s\hat{n}r$ be the average estimated SNR for both images.

Note from equation (23) that this error is a scaling error, i.e., it is proportional to the estimated change coherence (first term on lhs). It is scaled by the reciprocal of the estimated signal to noise ratio (second term on lhs), and is relative to a fractional error in noise (third term on lhs).

Using $|\hat{\mu}_{az}| \approx 1$ (expected mean for no change case) and $s\hat{n}r \approx snr$ so we can approximate as:

$$\varepsilon_{|\hat{\mu}_{az}|} \approx \left(\frac{1}{snr} \right) \left(\frac{\varepsilon_{\sigma_n^2}}{\sigma_n^2} \right) \tag{24}$$

Figure 3 shows a plot of equation the error using equation (24) versus the average error from simulation results. In all of these we are assuming that there is no change. We assume that $\varepsilon_{\sigma_n^2} \big/ \sigma_n^2 = 0.1$ but we plot the scale factor for this term, i.e., the approximately "1/snr" term. The figure shows that the approximation in equation (24) is a bit optimistic but reasonable. What it does not show is that as the coherence goes lower than 0.6 we start to have issues in the simulation with getting close to the divide by zero denominator effects discussed above.

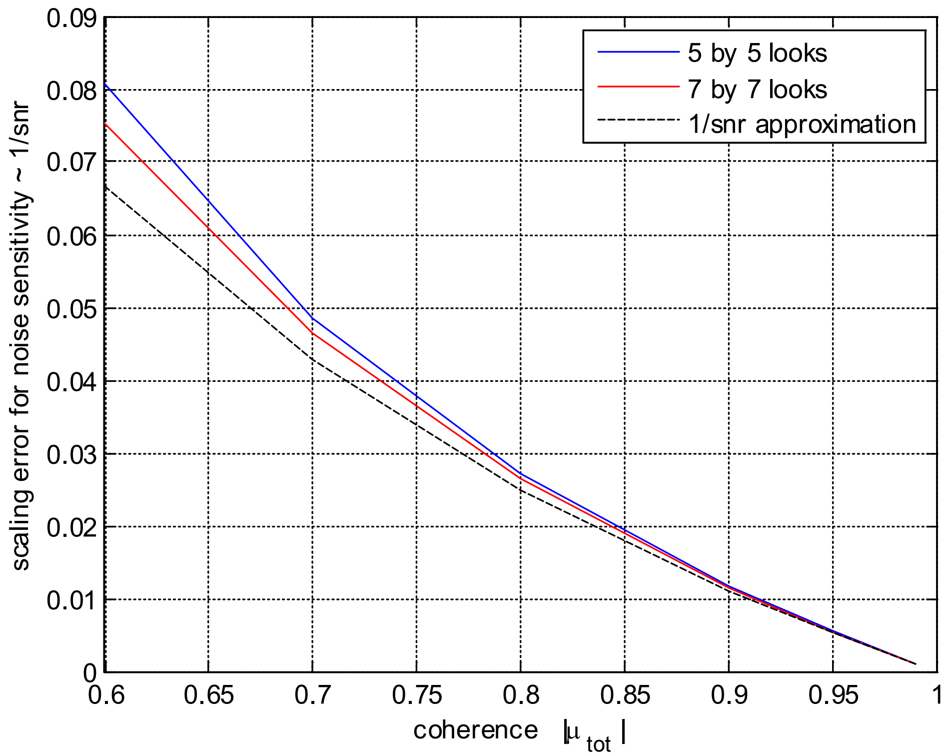


Figure 3: Scale factor for noise error - approximately equal to 1/snr in equation (24)

5. Conclusion

We have discussed some attributes of the interesting new estimator presented in [1]. We re-derived this estimator from a slightly different viewpoint and showed that it attempts to reveal the desired change by removing the change due to thermal noise. We showed some statistical properties of the denominator of this estimator. Finally we presented the sensitivity of the estimator to errors in the noise estimate.

This page left intentionally blank.

6. References

[1] D. E. Wahl, D. A. Yocky, C. V. Jakowatz, Jr., “A New Maximum-Likelihood Change Estimator for Two-Pass SAR Coherent Change Detection”, *SAND2014-18179*, Sept 2014.

[2] D. L. Bickel, “SAR Image Effects on Coherence and Coherence Estimation”, *SAND2014-0369*, January, 2014.

[3] H. Holm, M. Alouini, “Sum and difference of two squared correlated Nakagami variates in connection with the McKay distribution”, *IEEE Transactions on Communications*, Vol. 52, Iss 8, August 2004.

[4] “Scattering Scattering and Inverse Scattering in Pure and Applied Science”, ed., R. Pike, P. Sabatier, Academic Press, 2002.

This page left intentionally blank.

7. Appendix A: Derivation of the denominator of the estimator

This appendix derives the probability density function (pdf) for the denominator for the estimator discussed in this document.

We want the distribution of the sum of two correlated gamma distributed random variables. Given, a random variable, $X = X_1 + X_2$, where X_1 and X_2 are correlated gamma distributed random variables with shape parameter k and scale parameters θ_1 and θ_2, respectively. Given this from [3], X is distributed as a McKay type 1 distribution for the sum of squared-Nakagami (Gamma) distributed random variables:

$$f_X(x) = \frac{\sqrt{\pi}\left(c^2-1\right)^{a+1/2} x^a}{2^a b^{a+1}\Gamma\left(a+\frac{1}{2}\right)} \exp\left(-\frac{c}{b}x\right) I_a\left(\frac{x}{b}\right) H(x) \tag{25}$$

where:

$\Gamma(w)$ - is the gamma function of the argument w

$I_n(w)$ - is the modified Bessel function of the first kind of argument w and order n

$H(w)$ - Heaviside step function of argument w

$$a = k - 1/2 \tag{26}$$

$$b = \frac{2\theta_1\theta_2(1-\rho)}{\sqrt{(\theta_1+\theta_2)^2 - 4\theta_1\theta_2(1-\rho)}} \tag{27}$$

$$c = \frac{\theta_1+\theta_2}{\sqrt{(\theta_1+\theta_2)^2 - 4\theta_1\theta_2(1-\rho)}} \tag{28}$$

ρ - is the correlation coefficient between X_1 and X_2

Note that:

$$\rho = \frac{\langle X_1 X_2\rangle}{\sqrt{\langle |X_1|^2\rangle\langle |X_2|^2\rangle}} \tag{29}$$

where $\langle\phi\rangle$ is the expected value of ϕ.

We assume that the underlying process for generating the random variables X_1 and X_2 is:

$$X_1 = \sum_{i=0}^{k-1} \left|\tau_{1,i}\right|^2$$
$$X_2 = \sum_{i=0}^{k-1} \left|\tau_{2,i}\right|^2 \tag{30}$$

where $\tau_{1,i}$ and $\tau_{2,i}$ are zero mean bivariate Gaussian distributed given by:

$$f\left(\tau_{1,i},\tau_{2,i}\right) \sim \frac{1}{2\sigma_{\tau 1}\sigma_{\tau 2}\sqrt{1-|\mu|^2}} \exp\left[-\frac{1}{2\left(1-|\mu|^2\right)}\left(\frac{\tau_{1,i}^2}{\sigma_{\tau 1}^2}+\frac{\tau_{2,i}^2}{\sigma_{\tau 2}^2}-2\mu\frac{\tau_{1,i}\tau_{2,i}}{\sigma_{\tau 1}\sigma_{\tau 2}}\right)\right] \tag{31}$$

Note that are these variables are independent for differing values of *i*, i.e., we are dealing with a multivariate random variable in a somewhat simplified manner. We further assume that $\tau_{1,i}$ same variance, $\sigma_{\tau 1}^2$, for all *i*. Likewise, $\tau_{2,i}$ same variance, $\sigma_{\tau 2}^2$, for all *i*[5].

We now use the results from Appendix B along with the assumption from that appendix that $\theta = \theta_1 = \theta_2$ to evaluate b , and c above.

$$b = \frac{2\theta_1\theta_2\left(1-\rho\right)}{\sqrt{\left(\theta_1+\theta_2\right)^2-4\theta_1\theta_2\left(1-\rho\right)}} = \frac{2\theta^2\left(1-|\mu|^2\right)}{\sqrt{\left(2\theta\right)^2-4\theta^2\left(1-|\mu|^2\right)}} = \frac{\theta\left(1-|\mu|^2\right)}{|\mu|} \tag{32}$$

$$c = \frac{\theta_1+\theta_2}{\sqrt{\left(\theta_1+\theta_2\right)^2-4\theta_1\theta_2\left(1-\rho\right)}} = \frac{2\theta}{\sqrt{\left(2\theta\right)^2-4\theta^2\left(1-|\mu|^2\right)}} = \frac{1}{|\mu|} \tag{33}$$

Plugging into equation (25) yields :

[5] Although we do not have to, we will soon simplify even further and assume that $\sigma_\tau^2 = \sigma_{\tau 1}^2 = \sigma_{\tau 2}^2$.

$$f_X(x) = \frac{\sqrt{\pi}\left(\frac{1-|\mu|^2}{|\mu|^2}\right)^k x^{k-1/2}}{2^{k-1/2}\left[\frac{\theta\left(1-|\mu|^2\right)}{|\mu|}\right]^{k+1/2}\Gamma(k)} \exp\left[-\frac{x}{\theta\left(1-|\mu|^2\right)}\right] I_{k-1/2}\left[\frac{|\mu|x}{\theta\left(1-|\mu|^2\right)}\right] H(x)$$

$$f_X(x) = \frac{\sqrt{\pi}x^{L-1/2}}{2^{L-1/2}\Gamma(L)\theta^{L+1/2}|\mu|^{L-1/2}\sqrt{1-|\mu|^2}} \exp\left[-\frac{x}{\theta\left(1-|\mu|^2\right)}\right] I_{L-1/2}\left[\frac{|\mu|x}{\theta\left(1-|\mu|^2\right)}\right] H(x) \tag{34}$$

where we have let the number of looks be $L = k$, in the last equation.

We note an interesting case is when $|\mu| = 0$ then this simplifies to a gamma distribution since we assumed the same statistics for both X_1 and X_2:

$$f_Y(y) = \frac{y^{2L-1}}{\theta^{2L}\Gamma(2L)} \exp\left\{-\frac{y}{\theta}\right\} \qquad y \geq 0 \tag{35}$$

The above equation requires repeated use of L'Hôpital's rule and the duplication formula for gamma distributions.

This page left intentionally blank.

8. Appendix B: Important moments

This appendix derives some moments that are important for this document. One of the main equations we want is to rewrite the correlation coefficient, ρ, between X_1 and X_2 in terms of the correlation coefficient, μ, between $\tau_{1,i}$ and $\tau_{2,i}$.

Where:

$$\mu = \frac{\left\langle \tau_1 \tau_2^* \right\rangle}{\sqrt{\left\langle |\tau_1|^2 \right\rangle \left\langle |\tau_2|^2 \right\rangle}} \tag{36}$$

From the previous equation we can see that:

$$\left\langle \tau_1 \tau_2^* \right\rangle \left\langle \tau_2 \tau_1^* \right\rangle = \left\langle \tau_1 \tau_2^* \right\rangle \left\langle \tau_1 \tau_2^* \right\rangle^* = \left\langle |\tau_1|^2 \right\rangle \left\langle |\tau_2|^2 \right\rangle |\mu|^2 \tag{37}$$

We will need the moment theorem for circular complex Gaussian random processes [4]:

$$\left\langle \tau_1 \tau_1^* \tau_2 \tau_2^* \right\rangle = \left\langle \tau_1 \tau_1^* \right\rangle \left\langle \tau_2 \tau_2^* \right\rangle + \left\langle \tau_1 \tau_2^* \right\rangle \left\langle \tau_1^* \tau_2 \right\rangle \tag{38}$$

Combining equations (37) and (38):

$$\left\langle \tau_1 \tau_1^* \tau_2 \tau_2^* \right\rangle = \left\langle |\tau_1|^2 |\tau_2|^2 \right\rangle = \left\langle \tau_1 \tau_1^* \right\rangle \left\langle \tau_2 \tau_2^* \right\rangle + \left\langle |\tau_1|^2 \right\rangle \left\langle |\tau_2|^2 \right\rangle |\mu|^2 = \left\langle |\tau_1|^2 \right\rangle \left\langle |\tau_2|^2 \right\rangle \left(1 + |\mu|^2\right) \tag{39}$$

Since $\tau_{1,i}$ and $\tau_{2,i}$ are Gaussian with the same statistics, then $\tau_{1,i}^2$ and $\tau_{2,i}^2$ are exponentially distributed. Again, even though it is not required we will assume the mean of both of these exponential distributions is θ.

Therefore equation (39) becomes:

$$\left\langle \tau_1 \tau_1^* \tau_2 \tau_2^* \right\rangle = \left\langle |\tau_1|^2 |\tau_2|^2 \right\rangle = \theta^2 \left(1 + |\mu|^2\right) \tag{40}$$

We want, ρ, from equation (29):

$$\rho = \frac{\left\langle \sum_{i=0}^{k-1} |\tau_{1,i}|^2 \sum_{i=0}^{k-1} |\tau_{2,i}|^2 \right\rangle - \left(\left\langle \sum_{i=0}^{k1} |\tau_1|^2 \right\rangle \left\langle \sum_{i=0}^{k-1} |\tau_2|^2 \right\rangle \right)}{\sqrt{\left[\left\langle \left(\sum_{i=0}^{k-1} |\tau_1|^2 \right)^2 \right\rangle - \left\langle \left(\sum_{i=0}^{k-1} |\tau_1|^2 \right) \right\rangle^2 \right] \left[\left\langle \left(\sum_{i=0}^{k-1} |\tau_2|^2 \right)^2 \right\rangle - \left\langle \left(\sum_{i=0}^{k-1} |\tau_2|^2 \right) \right\rangle^2 \right]}} \tag{41}$$

We start with:

$$\left\langle X_1 X_2 \right\rangle = \left\langle \sum_{i=1}^{k} \left| \tau_{1,i} \right|^2 \sum_{i=1}^{k} \left| \tau_{2,i} \right|^2 \right\rangle \tag{42}$$

We note that:

$$\left\langle \left| \tau_{1,i} \right|^2 \left| \tau_{2,j} \right|^2 \right\rangle = \left\langle \left| \tau_{1,i} \right|^2 \right\rangle \left\langle \left| \tau_{2,j} \right|^2 \right\rangle \text{ for } i \neq j \tag{43}$$

because of the independence of these terms.

Then:

$$\left\langle X_1 X_2 \right\rangle = k \left\langle \left| \tau_1 \right|^2 \left| \tau_2 \right|^2 \right\rangle + k\left(k-1\right) \left\langle \left| \tau_1 \right|^2 \right\rangle \left\langle \left| \tau_2 \right|^2 \right\rangle \tag{44}$$

which is:

$$\left\langle X_1 X_2 \right\rangle = k\theta^2 \left(1 + \left|\mu\right|^2\right) + k\left(k-1\right)\theta^2 = \left(k\theta\right)^2 \left(1 + \frac{\left|\mu\right|^2}{k}\right) \tag{45}$$

and:

$$\left\langle \sum_{i=1}^{k} \left| \tau_1 \right|^2 \right\rangle \left\langle \sum_{i=1}^{k} \left| \tau_2 \right|^2 \right\rangle = \left(k\theta\right)^2 \tag{46}$$

So the numerator in equation (41) is:

$$\left\langle \sum_{i=0}^{k-1} \left| \tau_{1,i} \right|^2 \sum_{i=0}^{k-1} \left| \tau_{2,i} \right|^2 \right\rangle - \left(\left\langle \sum_{i=0}^{k-1} \left| \tau_1 \right|^2 \right\rangle \left\langle \sum_{i=0}^{k-1} \left| \tau_2 \right|^2 \right\rangle \right) = \left(k\theta\right)^2 \left(1 + \frac{\left|\mu\right|^2}{k}\right) - \left(k\theta\right)^2 = \left|\mu\right|^2 k\theta^2 \tag{47}$$

The denominator of equation (41) is the geometric mean of the variances of the two gamma distributions. Since we assume that the variances are the same for both variables, then the denominator is:

$$\sqrt{\left[\left\langle \left(\sum_{i=0}^{k-1} \left| \tau_1 \right|^2 \right)^2 \right\rangle - \left\langle \left(\sum_{i=0}^{k-1} \left| \tau_1 \right|^2 \right) \right\rangle^2 \right] \left[\left\langle \left(\sum_{i=0}^{k-1} \left| \tau_2 \right|^2 \right)^2 \right\rangle - \left\langle \left(\sum_{i=0}^{k-1} \left| \tau_2 \right|^2 \right) \right\rangle^2 \right]} = \sqrt{k\theta^2}\sqrt{k\theta^2} = k\theta^2 \tag{48}$$

So putting equations (47) and (48) into equation (41) leads to:

$$\rho = |\mu|^2 \tag{49}$$

From [3] the mean of the distribution in equation (25) is:

$$\langle X \rangle = \frac{(2a+1)bc}{c^2 - 1} \tag{50}$$

and the variance of the distribution is:

$$\left\langle \left(X - \langle X \rangle \right)^2 \right\rangle = \frac{(2a+1)b^2\left(c^2+1\right)}{\left(c^2-1\right)^2} \tag{51}$$

Substituting in $\theta = \theta_1 = \theta_2$ and equation (49) into equations (27) and (28):

$$b = \frac{\theta\left(1 - |\mu|^2\right)}{|\mu|} \tag{52}$$

$$c = \frac{1}{|\mu|} \tag{53}$$

Substituting in the above equations and equation (26):

$$\langle X \rangle = 2k\theta \tag{54}$$

$$\left\langle \left(X - \langle X \rangle \right)^2 \right\rangle = \left(2k\theta^2\right)\left(1 + |\mu|^2\right) \tag{55}$$

Now we want to look at the specific case of

$$z = \frac{\sum_{n=0}^{L-1} \left|x_{1,n}\right|^2 + \sum_{n=0}^{L-1} \left|x_{2,n}\right|^2 - 2L\sigma_n^2}{2} \tag{56}$$

For the random variable, z, if $\theta = \sigma_{x1}^2 = \sigma_{x2}^2 = \sigma_s^2 + \sigma_n^2$ then $\theta = \sigma_s^2 + \sigma_n^2$ and the mean of the distribution of z is:

$$\langle z \rangle = \frac{2L\left(\sigma_s^2 + \sigma_n^2\right) - 2L\sigma_n^2}{2} = L\sigma_s^2 \tag{57}$$

Next we look at the variance of z. The addition of a constant does not affect the variance, but of course, the scaling does scale the variance. Therefore the variance is:

$$\left\langle \left(z - \langle z \rangle \right)^2 \right\rangle = \frac{2L\left(\sigma_s^2 + \sigma_n^2\right)^2 \left(1 + |\mu|^2\right)}{4} = \frac{L\left(\sigma_s^2 + \sigma_n^2\right)^2 \left(1 + |\mu|^2\right)}{2} \tag{58}$$

9. Appendix C: Symbols and terminology

Variable definitions

$H(w)$ - is the Heaviside step function of argument w

$I_n(w)$ - is the modified Bessel function of the first kind of argument w and order n

L - independent looks averaged to estimate the coherence between two images

snr_i, snr_1, snr_2, snr - true signal-to-noise ratio of image #*i*, image #1, image #2, and in general, respectively, (unitless)

$\hat{s}nr_i, \hat{s}nr_1, \hat{s}nr_2, \hat{s}nr$ - estimate of the signal-to-noise ratio of image #*i*, image #1, image #2, and in general, respectively, (unitless)

$spnr_i, spnr_1, spnr_2, spnr$ - true signal-*plus*-noise ratio of image #*i*, image #1, image #2, and in general, respectively, (unitless)

$\hat{s}pnr_i, \hat{s}pnr_1, \hat{s}pnr_2, \hat{s}pnr$ - estimate of the signal-*plus*-noise ratio of image #*i*, image #1, image #2, and in general, respectively, (unitless)

$x_{1,n}$ - "voltage" of the n^{th} complex image pixel from image #1

$x_{2,n}$ - "voltage" of the n^{th} complex image pixel from image #2

$\varepsilon_{\sigma_n^2}$ - is the bias error in the assumed noise power (note this is not a random variable)

$\varepsilon_{\mu_{az}}$ - is the error in the estimated value for $\hat{\mu}_{az}$

$\Gamma(w)$ - is the gamma function of the argument w

μ_{az} - population coherence of "anthropogenic and zoogenic" changes between two complex images

$\hat{\mu}_{az}$ - estimate of the coherence due to "anthropogenic and zoogenic" changes (i.e., sample coherence) between two complex images

μ_{snr} - true coherence due to thermal noise between two complex images

μ_{tot} - total population coherence between two complex images

$\hat{\mu}_{tot}$ - estimate of the total coherence (i.e., sample coherence) between two complex images

ρ - correlation coefficient between the square (of the absolute values) between the corresponding samples of two images

$\sigma_{s1}^2, \sigma_{s2}^2, \sigma_s^2$ - signal power in image #1, image #2, and in general, respectively

$\sigma_{n1}^2, \sigma_{n2}^2, \sigma_n^2$ - noise power in image #1, image #2, and in general, respectively

This page left intentionally blank.

10. Distribution

Unlimited Release

Unlimited Release

1	MS 0509	A. J. Medina	5300	(electronic copy)
1	MS 0519	D. L. Bickel	5344	(electronic copy)
1	MS 0519	A. Martinez	5344	(electronic copy)
1	MS 0519	R. M. Naething	5344	(electronic copy)
1	MS 0519	A. M. Raynal	5344	(electronic copy)
1	MS 0519	K. R. Czuchlewski	5346	(electronic copy)
1	MS 0519	M. G. Chen	5346	(electronic copy)
1	MS 0519	R. D. West	5346	(electronic copy)
1	MS 0519	J. A. Ruffner	5349	(electronic copy)
1	MS 0519	A. W. Doerry	5349	(electronic copy)
1	MS 0519	L. M. Klein	5349	(electronic copy)
1	MS 0519	R. C. Ormesher	5332	(electronic copy)
1	MS 0532	S. P. Castillo	5340	(electronic copy)
1	MS 0532	B. L. Burns	5340	(electronic copy)
1	MS 0532	W. H. Hensley	5344	(electronic copy)
1	MS 0532	E. D. Gentry	5344	(electronic copy)
1	MS 0532	C. Musgrove	5344	(electronic copy)
1	MS 0532	M. C. Chang	5348	(electronic copy)
1	MS 0532	D. M. Small	5348	(electronic copy)
1	MS 0532	M. E. Thompson	5348	(electronic copy)
1	MS 0533	S. Gardner	5342	(electronic copy)
1	MS 0533	T. P. Bielek	5342	(electronic copy)
1	MS 0533	D. Harmony	5342	(electronic copy)
1	MS 0533	J. A. Hollowell	5342	(electronic copy)
1	MS 0533	R. Riley	5342	(electronic copy)
1	MS 0533	B. G. Rush	5342	(electronic copy)
1	MS 0533	D. G. Thompson	5342	(electronic copy)
1	MS 0533	K. W. Sorensen	5345	(electronic copy)
1	MS 0533	D. F. Dubbert	5345	(electronic copy)
1	MS 1202	J. G. Chow	5349	(electronic copy)
1	MS 0899	Technical Library	9536	(electronic copy)

Sandia National Laboratories