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Brittle fracture phase-field modeling of a short-rod specimen

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Brittle fracture phase-field modeling of a short-rod specimen

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Abstract

Predictive simulation capabilities for modeling fracture evolution provide further insight into quantities of interest in comparison to experimental testing. Based on the variational approach to fracture, the advent of phase-field modeling achieves the goal to robustly model fracture for brittle materials and captures complex crack topologies in three dimensions.

As part of the 2015 summer internship of the first author, this report presents a comparison between cohesive zone, phase-field, and gradient damage models for simulating the short-rod fracture experiment. Crack growth is captured in both phase-field and gradient damage models through a diffuse fracture model without the need to know the location of failure *a priori*. Numerical formulations for the phase-field and gradient damage models are elaborated on to discuss differences and similarities between the two. These fracture methods are demonstrated by simulation of the short-rod fracture test [14] and the Brazilian splitting test for Indiana Limestone. Previously published results by Rinehart et al. [14] use a cohesive zone model for simulating the experiments. The phase-field and gradient damage model solutions are compared and contrasted to these results. In addition, several verification activities are performed such as demonstrating convergence with mesh refinement. Intricacies associated with running explicit time integration for the phase-field and gradient damage computational models are commented on throughout this study.

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This work would not have been completed without contribution and inspiration from many individuals at Sandia National Labs. My growth as an analytical thinker is credited to my mentor Joe Bishop. He took time to foster enough curiosity on the subject of fracture mechanics and allowed me to find excitement in the work presented to me as the hurdle to overcome for my summer internship program. Thank you for letting me borrow a section of your library, in particular Jean Lemaitre's *Short Course on Damage Mechanics*, which became the best way to familiarize myself with intellectual boundaries on this topic. Your patience and willingness to allow me to pioneer the direction of my summer work created opportunities for me to understand my natural direction of development.

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Nomenclature

SR Short-rod

CZM Cohesive-zone model

PF Phase-field

GDM Gradient damage model

R1 First mesh refinement

R2 Second mesh refinement

FE Finite element

Chapter 1

Introduction

Computational fracture models permit detailed investigation of quantities of interest when compared to experimental methods. The goal of predictive modeling has driven development of several fracture theories and computational models. Of the first, Griffith's theory of brittle fracture assigns an equivalence between the energy needed to create a unit area of fracture surface and a known critical value (Bazant and Planas [1]). Some developments of computational methods include the cohesive zone (CZ) method (Camacho and Ortiz [4]) and the extended finite element method (Dolbow and Belytschko [6]). Although these computational methods provide significant advances, they struggle to robustly model three-dimensional problems containing complex crack topologies including branching and coalescence. To circumvent these challenges, modifications of the Ginzburg-Landau evolution theory have been proposed as phase-field and general gradient damage formulations. Based on the variational energy balance (Francfort and Marigo [7]), the phase-field formulation uses a thermodynamically consistent framework to represent damage (Miehe et al. [12]). A similar method is the gradient damage model introduced by Lorentz and Godard [8]. A major advantage of these models is the exclusion of discontinuities from the formulation. Fracture evolution is approximated by a scalar field through a diffusive crack approach, which results in a spatially smooth continuum formulation (Miehe et al. [11]). It can be shown that the Γ -convergence of the phase-field formulation recovers linear elastic fracture mechanics as the length scale parameter associated with the crack bandwidth approaches zero (May et al. [10]). The gradient damage model converges to a cohesive zone model as the length scale approaches zero (Lorentz and Godard [8]).

This report uses the newly introduced phase-field and gradient damage model capabilities in the Sierra Solid Mechanics code, to model full scale three-dimensional cases of fracture growth for the short-rod (SR) experiments (Rinehart et al. [14]). Both computational fracture models are investigated for convergence with mesh refinement. In addition, a cylinder splitting test, called a Brazilian test, is modeled using phase-field, and the predicted peak load is compared to experimental results. The phase-field formulation may be extended to multi-physics applications based on the current framework available in Sierra Solid Mechanics. As these computational methods are undergoing development in the Sierra Solid Mechanics code, this work provides supportive information for determining future development activity.

Chapter 2

Formulation

Phase-field model of fracture

For applications in brittle fracture, a type of gradient damage model, termed phase-field, was recently developed from the variational approach to fracture (Bourdin et al. [3]). Given below is a summary of the phase-field formulation as found in [11, 12]. Consider an arbitrary bounded domain, Ω , of up to three-dimensions with an external boundary, $\partial\Omega = \partial\Omega_g \cup \partial\Omega_h$, where $\partial\Omega_g$ and $\partial\Omega_h$ represent the Dirichlet and Neumann boundaries respectively. An internal discontinuity surface, Γ , represents the crack (Figure 2.1). Enforcing irreversibility, the internal boundary must have, $\Gamma(t) \subseteq \Gamma(t + \Delta t)$ for any $\Delta t > 0$, as the crack path evolves in time.

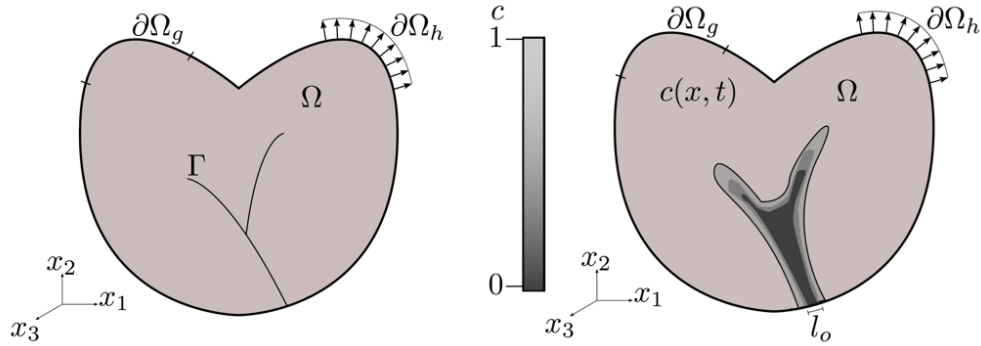


Figure 2.1: Schematic of a solid body, Ω , approximated by using a phase-field variable (left), where l_o is the width of the smeared crack, and a representation of the crack as an internal discontinuity, Γ (right). The phase-field parameter, c , is zero if the material is fully damaged and equals one if fully intact.

For an isotropic linear elastic model, the elastic strain energy density, ψ_e is given by

$$\psi_e(\boldsymbol{\epsilon}) = \frac{1}{2} \lambda \epsilon_{ii} \epsilon_{jj} + \mu \epsilon_{ij} \epsilon_{ij} \quad (2.1)$$

where λ and μ are the Lamé constants and $\boldsymbol{\epsilon}$ is the infinitesimal strain tensor. Within Griffith's theory of brittle fracture, the energy required to create a unit area of fracture

surface is equal to the critical fracture energy density, $\mathcal{G}_c$ (Borden et al. [2]). The total potential energy, Ψ , is the sum of elastic strain energy and fracture energy. Fracture energy is determined as the critical energy density, $\mathcal{G}_c$, integrated over the fracture surface.

$$\Psi(\epsilon, \Gamma) = \int_{\Omega} \psi_e(\epsilon) d\Omega + \int_{\Gamma} \mathcal{G}_c d\Gamma. \quad (2.2)$$

To minimize the variational form of the total potential energy, it is beneficial to approximate the fracture energy by a volumetric integral as opposed to a surface integral (Bourdin et al. [3]). This change requires the introduction of a scalar-valued variable dubbed as the phase-field, c , which determines the presence of a crack. The phase-field variable has a value of 0 where the material is fully broken and a value of 1 where the material is fully intact (Figure 2.1). The fracture surface now is approximated by a functional dependent on the phase-field, c , and a length scale parameter, l_o , which will determine the bandwidth of diffusivity in the crack topology (for further definition of the functional refer to [12]). In this work, a second-order Taylor expansion of the crack functional is used

$$\Gamma_{l_o}(c) = \int_{\Omega} \underbrace{\frac{1}{4l_o} [(c-1)^2 + 4l_o |\nabla c|^2]}_{\gamma_{c,2}} d\Omega. \quad (2.3)$$

The internal discontinuity, Γ , is approximated by a smeared surface functional, Γ_{l_o} , where $\gamma_{c,2}$ is the second-order Taylor expansion of the crack density. The fracture energy can now be stated as follows,

$$\Psi_{fracture} = \int_{\Gamma} \mathcal{G}_c d\Gamma \approx \int_{\Omega} \mathcal{G}_c \gamma_{c,2} d\Omega. \quad (2.4)$$

From minimizing the crack functional with the following boundary conditions, $c(0) = 0$ and $\frac{dc}{dx}(x) \rightarrow 0$ as $|x| \rightarrow \infty$, the Euler-Lagrange equation is given by (Borden et al. [2]),

$$c - 1 - 4l_o^2 \Delta c = 0. \quad (2.5)$$

In the one-dimensional case, the phase-field scalar field parameter is determined by an exponential function, which is a solution to the minimization of the above equation (Figure 2.2),

$$c(x) = 1 - e^{-\frac{|x|}{2l_o}}. \quad (2.6)$$

In practice, the fracture energy is approximated using eq. (2.4). The strain energy density must also be reformulated to include the degradation of stress in tension, which restricts crack propagation under compression (Miehe et al. [11]),

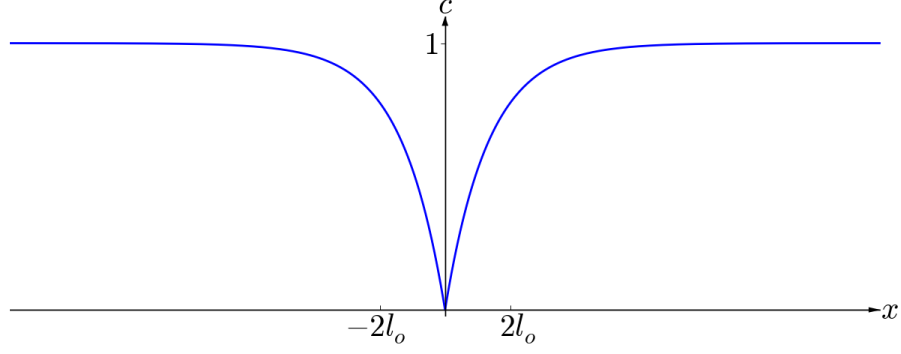


Figure 2.2: Phase-field approximation of a crack surface in one dimension. The crack is centered about $x = 0$. The phase-field parameter c is fully damaged in the center and has an effective bandwidth of $4l_o$, where l_o is assigned as an input to the simulation (Table 4.1).

$$\psi_e(\boldsymbol{\epsilon}, c) = g(c)\psi_e^+(\boldsymbol{\epsilon}) + \psi_e^-(\boldsymbol{\epsilon}) \quad (2.7)$$

where $g(c)$ is a monotonically increasing degradation function that must meet the following conditions: $g(0) = 0$, $g(1) = 1$, and $\frac{dg}{dc}(0) = 0$. The simplest function that satisfies all given conditions is

$$g(c) = c^2. \quad (2.8)$$

The total phase-field approximation of potential energy becomes

$$\Psi(\boldsymbol{\epsilon}, c) = \int_{\Omega} g(c)\psi_e^+(\boldsymbol{\epsilon}) + \psi_e^-(\boldsymbol{\epsilon}) + \frac{1}{4l_o}\mathcal{G}_c[(1-c)^2 + 4l_o|\nabla c|^2]d\Omega. \quad (2.9)$$

Assembling the Lagrange energy functional, with the inclusion of external work, $W_{ext}(\dot{\mathbf{u}})$, and kinetic energy, $K(\dot{\mathbf{u}})$, leads to the energy balance where $\dot{\mathbf{u}} = \frac{d\mathbf{u}}{dt}$ is the time derivative of the position vector or the velocity,

$$\frac{d}{dt}K + \frac{d}{dt}\Psi - W_{ext} = 0. \quad (2.10)$$

Borden et al. [2] gives details for obtaining the strong form for the second-order initial value problem. To enforce the irreversibility condition of crack growth, a strain-history field is introduced, $\mathcal{H}$. This field satisfies the Kuhn-Tucker conditions for loading and unloading. Now the tensile contribution of strain energy density in the equation of motion associated with solving the phase-field parameter is replaced by the strain-history field (Miehe et al. [11]).

$$(S) \begin{cases} \nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \rho \ddot{\mathbf{u}} & \text{on } \Omega \times]0, T[, \\ \frac{4l_o c \mathcal{H}}{\mathcal{G}_c} + c - 4l_o^2 \Delta c = 1 & \text{on } \Omega \times]0, T[, \\ \mathbf{u} = \mathbf{g} & \text{on } \partial\Omega_{\mathbf{g}} \times]0, T[, \\ \boldsymbol{\sigma} \mathbf{n} = \mathbf{h} & \text{on } \partial\Omega_{\mathbf{h}} \times]0, T[, \\ \nabla c \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times]0, T[, \\ \mathbf{u} = \mathbf{u}_o & \text{on } \Omega \times 0, \\ \dot{\mathbf{u}} = \mathbf{v}_o & \text{on } \Omega \times 0, \\ c = c_o & \text{on } \Omega \times 0, \end{cases} \quad (2.11)$$

where $\boldsymbol{\sigma} = g(c) \frac{\partial \psi_{\epsilon}^+}{\partial \epsilon} + \frac{\partial \psi_{\epsilon}^-}{\partial \epsilon}$, $\mathbf{g}$ and $\mathbf{h}$ are imposed boundary conditions on the position vector in $\partial\Omega_{\mathbf{g}}$ and $\partial\Omega_{\mathbf{h}}$ respectively (Figure 2.1), $\mathbf{n}$ is the outward pointing unit normal of the boundary, and the initial conditions are stated as $\mathbf{u}_o$ for displacements, $\mathbf{v}_o$ for velocities, and c_o for the phase-field.

A critical stress, σ_c , is defined as the value of stress at which both the stress and the phase-field parameters begin to decrease. This crack nucleation stress can be found from $\mathcal{G}_c$ and l_o ,

$$\sigma_c = \frac{9}{16} \sqrt{\frac{E \mathcal{G}_c}{6l_o}}. \quad (2.12)$$

With this formulation, the weak form of the problem can be expressed after defining appropriate trial and weighting spaces and applying integration by parts. $\boldsymbol{\mathcal{S}}_t$ and $\tilde{\boldsymbol{\mathcal{S}}}_t$ are the trial spaces and $\boldsymbol{\mathcal{V}}$ and $\tilde{\boldsymbol{\mathcal{V}}}$ are the weighting spaces.

$$\begin{aligned} \boldsymbol{\mathcal{S}}_t &= \{\mathbf{u}(t) \in (H^1(\Omega))^d \mid \mathbf{u}(t) = \mathbf{g} \text{ on } \partial\Omega_{\mathbf{g}}\}, \\ \tilde{\boldsymbol{\mathcal{S}}}_t &= \{c(t) \in H^1(\Omega)\}, \\ \boldsymbol{\mathcal{V}} &= \{\mathbf{w} \in (H^1(\Omega))^d \mid \mathbf{w} = 0 \text{ on } \partial\Omega_{\mathbf{g}}\}, \\ \tilde{\boldsymbol{\mathcal{V}}} &= \{q \in H^1(\Omega)\}. \end{aligned} \quad (2.13)$$

A Galerkin semi-discretization of the weak form is obtained by discretizing the trial solution spaces and test function spaces (Borden et al. [2]). The resulting variational problem is given by

$$(G) \left\{ \begin{array}{l} \text{Given } \mathbf{g}, \mathbf{h}, \mathbf{b}, \mathbf{u}_o, \dot{\mathbf{u}}_o, \text{ and } c_o, \text{ find } \mathbf{u}^h(t) \in \mathcal{S}_t^h \text{ and } c^h(t) \in \tilde{\mathcal{S}}_t^h, t \in [0, T], \\ \text{such that } \forall \mathbf{w}^h \in \mathcal{V}^h \text{ and } \forall q^h \in \tilde{\mathcal{V}}^h, \\ (\rho \ddot{\mathbf{u}}^h, \mathbf{w}^h)_\Omega + (\boldsymbol{\sigma}, \nabla \mathbf{w}^h)_\Omega = (\mathbf{h}, \mathbf{w}^h)_{\partial\Omega_h} + (\mathbf{b}, \mathbf{w}^h)_\Omega, \\ (\frac{4l_o c^h \mathcal{H}}{\mathcal{G}_c} + c^h, q^h)_\Omega + (4l_o^2 \nabla c^h, \nabla q^h)_\Omega = (1, q^h)_\Omega, \\ (\rho \mathbf{u}^h(0), \mathbf{w}^h)_\Omega = (\rho \mathbf{u}_o, \mathbf{w}^h)_\Omega, \\ (\rho \dot{\mathbf{u}}^h(0), \mathbf{w}^h)_\Omega = (\rho \dot{\mathbf{u}}_o, \mathbf{w}^h)_\Omega \end{array} \right. \quad (2.14)$$

Both the trial solution and test function spaces are defined using the same set of basis functions.

Gradient damage model for fracture

A gradient damage model has been proposed by (Lorentz et al. [9]), and is summarized below. It can be shown that this computational model converges to a cohesive zone model as the length-scale, l_D , approaches zero. This model holds to a classic damage mechanics convention of the scalar parameter, c , where $c = 0$ corresponds to a fully intact material and $c = 1$ represents ultimate damage (opposite of the phase-field formulation).

This gradient damage model isolates the length-scale parameter, l_D , from characteristic material parameters, which differs from the phase-field formulation (eq. 2.12). For an isotropic linear elastic model, the elastic strain energy density is given by eq. 2.1. Damage progressively weakens the stiffness through a degradation function applied on the tensile components of the elastic strain energy density.

$$\psi_e(\boldsymbol{\epsilon}, c) = g(c)\psi_e^+(\boldsymbol{\epsilon}) + \psi_e^-(\boldsymbol{\epsilon}) \quad (2.15)$$

where $g(c) \in [0, 1]$, is a stiffness function that degrades the linear stress-strain relationship by monotonically decreasing as c increases, and is given by

$$g(c) = \frac{(1 - c)^2}{1 + (m - 2)c + (1 + pm)c^2}. \quad (2.16)$$

The internal parameters, p and m are subject to the constraints $p \geq 1$ and $m \geq p + 2$ based on physical considerations. For comparison to phase-field, computational models reported use $m = 3$ and $p = 1$ in all simulations (Lorentz and Godard [8]).

Similar to phase-field, the fracture energy is integrated over the domain of the bulk material as

$$\Psi_{fracture}(c) = \int_\Omega \frac{b}{2} (\nabla c)^2 d\Omega, \quad (2.17)$$

where b is a positive constant that controls the localization bandwidth of damage. For a one-dimensional problem, the current top damage value shown in Figure 2.3 describes crack topology is visualized at the ultimate damage stage of $c = 1$ of crack nucleation.

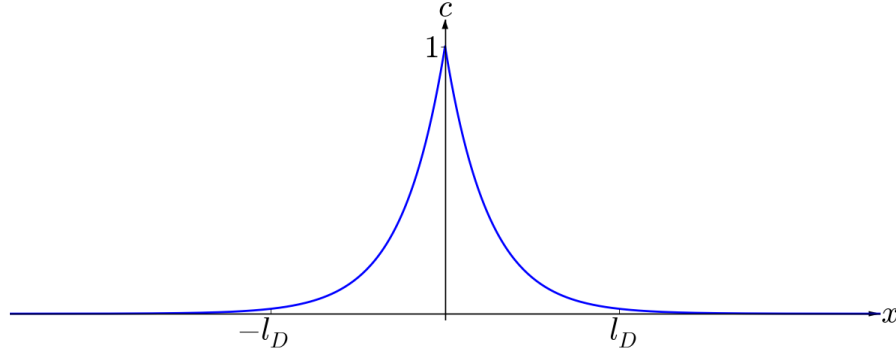


Figure 2.3: Gradient damage approximation of the crack in one dimension. Assuming the crack is centered about $x = 0$, the damage scalar, c , will express full damage in the center with a value of one and diffuse to a fully intact value of zero within an effective bandwidth of $2l_D$.

The total gradient damage approximation of potential energy proposed by Lorentz and Godard [8] becomes

$$\Psi(\epsilon, c) = \int_{\Omega} [g(c)\psi_e(\epsilon, c) + \frac{b}{2}(\nabla c)^2] d\Omega, \quad (2.18)$$

The dissipation potential is given by

$$\mathcal{D}(\dot{c}) = \int_{\Omega} [k\dot{c} + \mathcal{I}_{\mathbb{R}_+}(\dot{c})] d\Omega, \quad (2.19)$$

where $k \geq 0$. The indicator function, $\mathcal{I}_{\mathbb{R}_+}(\dot{c})$, enforces the time derivative of the scalar value of damage, $\dot{c} \geq 0$ (Lorentz and Godard [8]), which retains irreversibility.

The motivation of the gradient damage implementation is to isolate the length scale, l_D , from the characteristic parameters that describe the material, E , the Young's modulus, $\mathcal{G}_c$, the critical fracture energy density, σ_c , the peak stress, and p , the shape parameter confined to being $p = 1$. The length scale represents half the damage bandwidth. This differs from the phase field formulation (Figures 2.2 and 2.3), and can be set independent of the macroscopic parameters. Given the constraints on the shape parameter and the internal parameter, m , a restriction is imposed on the length scale,

$$l_D \leq \frac{3}{2(p+2)} \frac{E\mathcal{G}_c}{\sigma_c^2}. \quad (2.20)$$

This formulation is discretized using the same techniques as for the phase-field model. The key difference between the models is the independence of the respective length scale

parameter from the material properties. Also, the Γ -convergence of this gradient damage model has been proven to give a cohesive zone model as the length parameter approaches zero, where phase-field has a Γ -convergence to Griffith linear elastic fracture mechanics (May et al. [10]).

Chapter 3

Fracture experimentation of Indiana Limestone

Short-rod experimental method

Rinehart et al. [14] have performed a series of short-rod (SR) fracture experimentation on Indiana Limestone. Below is a summary of the experimental procedure, and its comparison to a linear softening cohesive fracture model for the short-rod test configuration (Rinehart et al. [14]).



Figure 3.1: Images of short-rod fracture samples, (a) outside view, (b) inside fracture surface has propagated completely, and (c) inside fracture surface with an angled illumination to capture detailed crack features Rinehart et al. [14]

Samples of Indiana Limestone that were weakly 1-2 cm planar-bedded calcarenite with medium- to coarse-grain size were machined to have a SR test configuration (see Figure 3.1). The SR specimen measured 25.76 ± 0.25 mm in diameter and 38.64 ± 0.39 mm in height. Four specimens were prepared by making a 0.90 mm wide slot cut through the center of the top leaving a 58° chevron tip, a schematic of the geometry and test set-up are shown in Figure 3.2.

For the fracture test, 5 mm wide aluminum connectors were epoxied to the top surface and loaded as shown in Figure 3.3. The SR samples were pulled apart by the connectors at a constant displacement rate of $0.001 \frac{\text{mm}}{\text{s}}$ until complete separation of the damaged specimen. The force of the actuator in the short-rod was recorded as the aluminum connectors displaced at a constant rate.

The measured peak loads captured a wide variation with as much as 25% difference

between test samples (Figure 3.4).

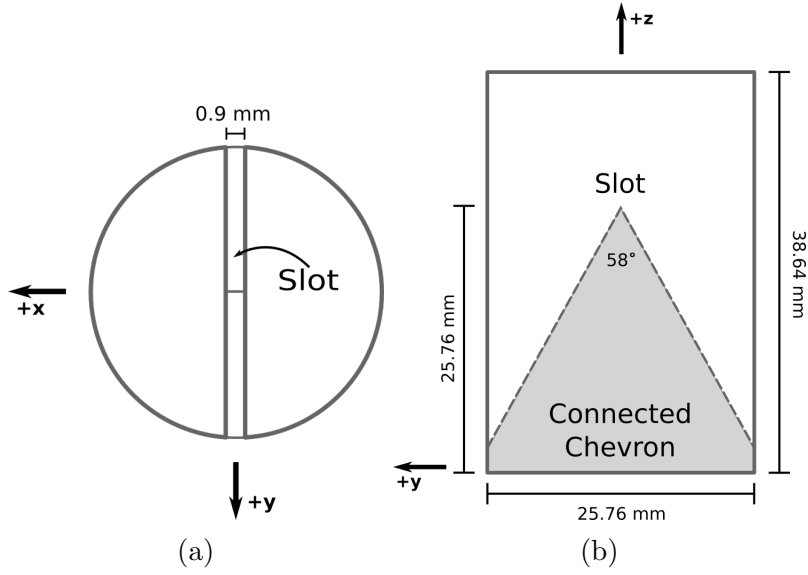


Figure 3.2: Schematic of the short-rod specimen including the coordinate axis used in model, (a) top view and (b) side view.

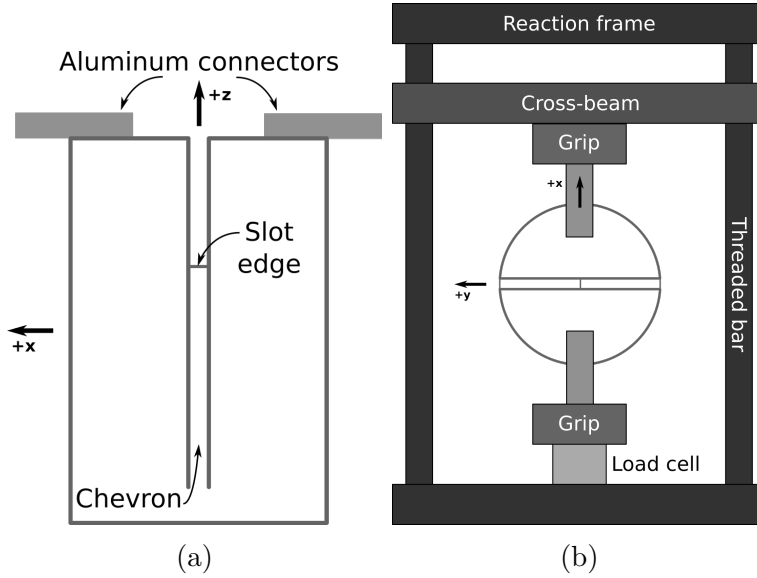
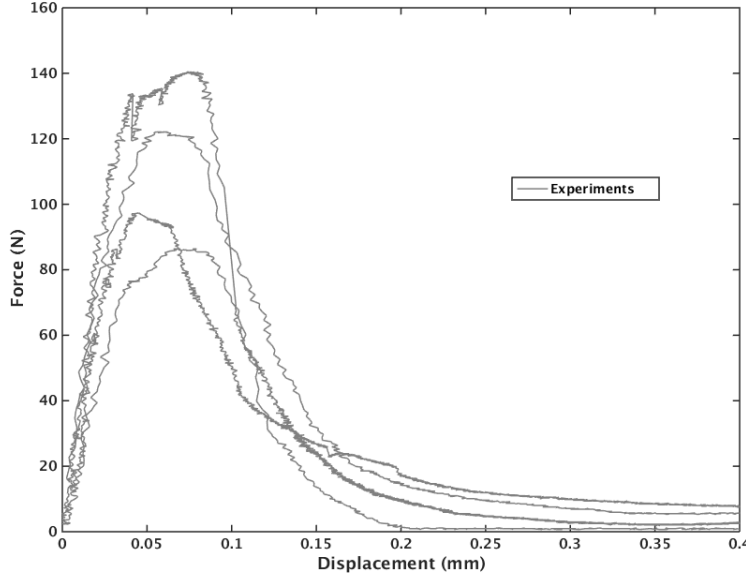


Figure 3.3: Schematic of the short-rod fracture test including the coordinate axis used in model, (a) aluminum connectors attached to the test sample and (b) short-rod specimen. The aluminum connectors pull apart the short-rod test specimen using a constant displacement rate of $0.001 \frac{mm}{s}$ until the crack has propagated through the chevron to the bottom most point of the domain.



Experiment	Peak Force (N)	Displacement (mm)
1	86	0.065
2	97	0.045
3	122	0.055
4	140	0.074

Figure 3.4: Four short-rod samples demonstrate variation in the peak force and the displacement at which the peak force is reached.

Cohesive zone model

The loading rate was later before reaching failure is used for calibration of the cohesive zone model parameters (Table 3.1).

A cohesive zone model uses a traction separation relationship to model fracture. The cohesive zone model used in the work Rinehart et al. [14] was a bilinear relationship with linear elastic loading to a peak stress and linear softening once the crack width has reached a critical value (Bazant and Planas [1]). This linear softening model is the traction law of separation between cohesive surfaces (Figure 3.5). The simulation was generated using the finite-element software Abaqus/Standard 6.13 to model the SR experiment quasi-statically. Parameters for the SR fracture test model were determined from previously outlined experiments (Table 3.1).

This model assumed a cohesive process zone of 20% of the chevron thickness, so 0.18 mm thick centered about where the single-trace crack is known to form *a priori*. This zone contains a check for when the stress has reached a critical value for a Mode I type deformation, and applies the traction separation law as deformation continues to occur until a given distance, w_1 , has been reached where no tensile stress should be present. Results of the quasi-static simulation are compared to the experimental results in Figure 3.6. These results will be used for comparison to the results of the phase-field modeling given in Chapter 4.

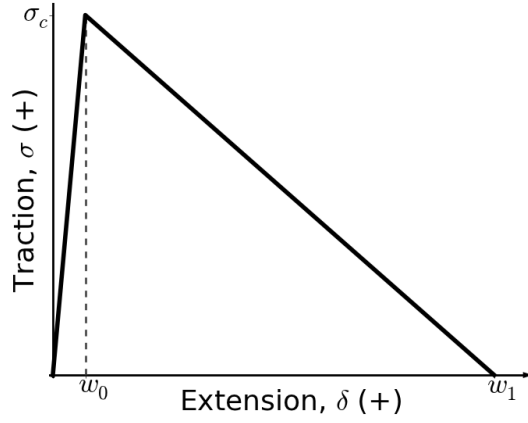


Figure 3.5: Cohesive crack traction as a function of crack width. Linear elastic loading leads to a critical value in tension at an opening width, w_0 , and begins a linear decline until a completely cohesionless crack opening is reached, w_1 .

Table 3.1: Parameters for cohesive zone model of short-rod specimen

Parameter	Value	Units
Young's modulus, E	7.0	GPa
Poisson's ratio, ν	0.155	
Density, ρ	2250	kg/m ³
Critical stress, σ_c	5.9	MPa
Mesh size (mean), h	0.33	mm
Cohesionless separation, w_1	0.0115	mm

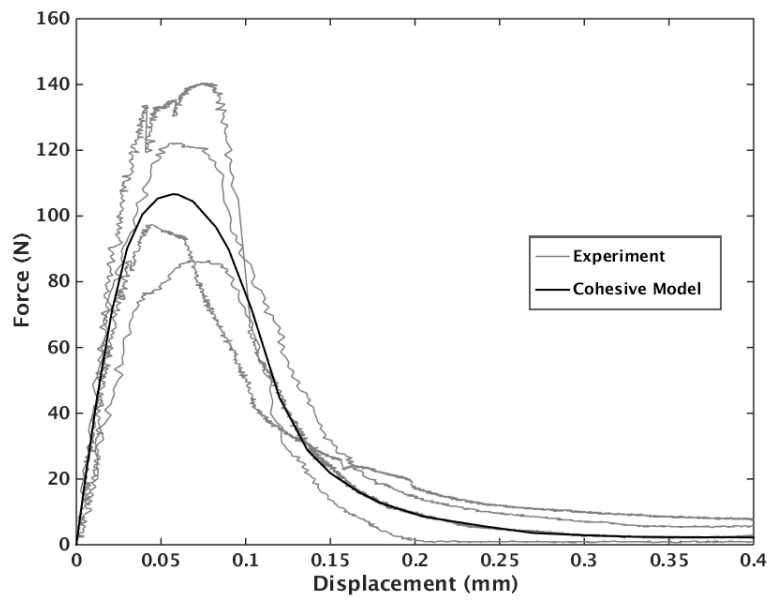


Figure 3.6: Load-deflection curve of the cohesive zone model (CZM) overlaid with experimental results. The CZM predicts a peak force of 107 N at a displacement of 0.0570 mm which falls within experimental data.

Chapter 4

Brittle fracture phase-field modeling

This chapter presents results from the phase-field fracture modeling of the SR geometry. The phase-field formulation summarized in Chapter 2 is solved via an explicit time integration scheme staggered with an implicit solve for the phase-field parameter, c . Representative numerical examples of regularized brittle fracture models are presented in [2], [12], and [11] using the framework of a staggered solve for the displacement field and phase-field approximations.

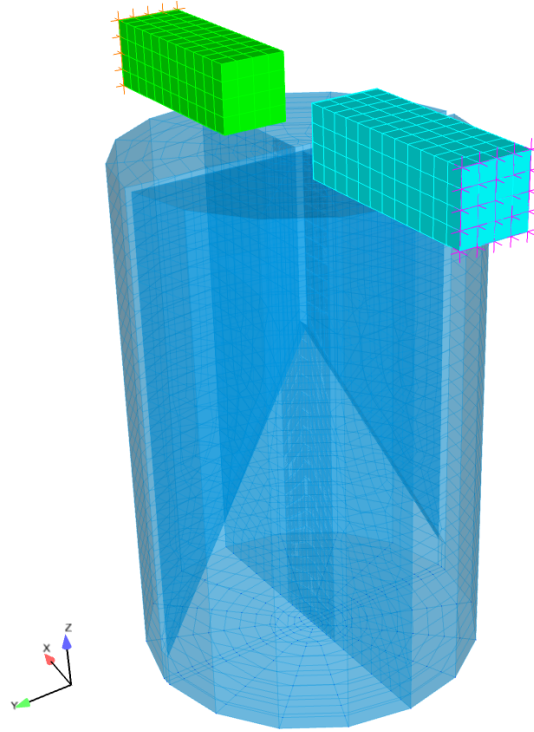


Figure 4.1: Finite element mesh of the short-rod specimen, with two node-sets identified by + symbols. All node-sets were fixed in both y and z-directions. The aluminum connectors had a fixed displacement imposed perpendicular to the crack propagation plane (in the $\pm x$ -direction) of $0.01 \frac{m}{s}$.

Finite-element model

The short-rod finite-element model was built at full-scale without exploiting symmetry for simplicity in the simulation. The model includes the aluminum connectors as shown in Figure 3.3 where the displacement boundary conditions are applied. The bottom end of the model is set to be fixed in the y - and z -directions (Figure 4.1). Details of input data can be found in Appendix A.1, for use in running simulations in the SIERRA Solid Mechanics code (Version 4.37.8).

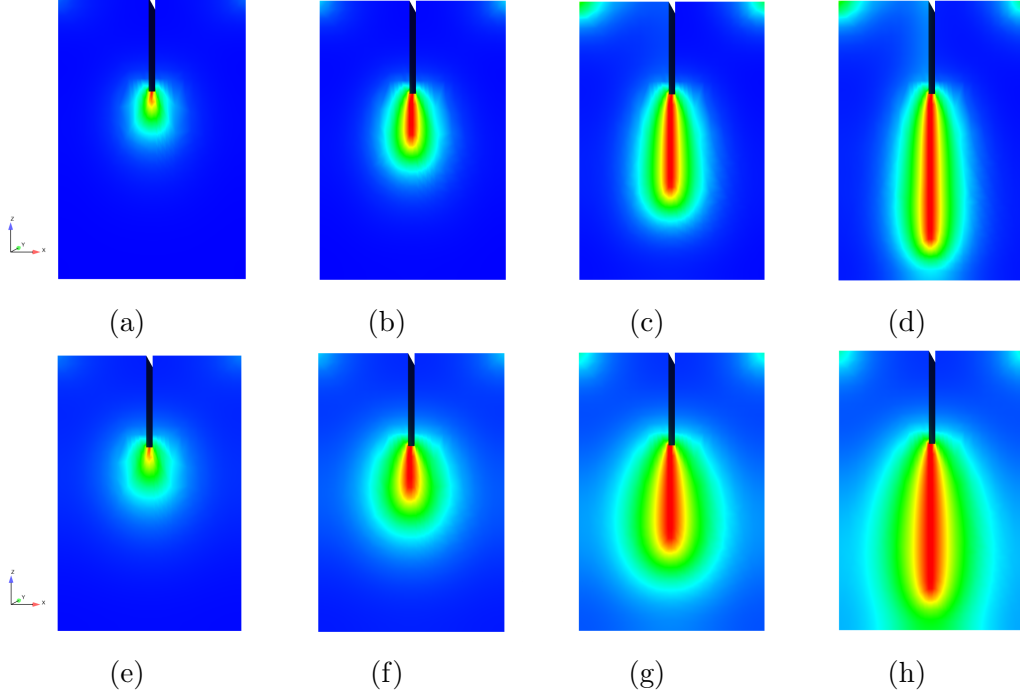


Figure 4.2: Length scale comparison over crack topology evolution, (a) through (d) represent a smaller parameter by a factor of 2 than (e) through (h) with a constant mesh from Figure 4.1, $l_o \text{ (a-d)} < l_o \text{ (e-h)}$.

Equation 2.12 gives a relation between the crack smearing length, l_o , the nucleation stress, σ_c , and critical fracture energy density, $\mathcal{G}_c$. From experimentation (Rinehart et al. [14]), the critical stress and fracture energy density were determined to be 5.9 MPa and $34 \frac{\text{J}}{\text{m}^2}$, respectively. By using these parameters, a length scale of $l_o = 0.359 \text{ mm}$ is recommended. To best capture fracturing characteristics, it is understood that the length scale parameter should be much larger than the mesh size where the crack will form ($l_o \gg h$ where h is the mesh size). The length scale parameter, l_o , represents the extent of crack smearing within the domain, Ω , as the phase-field parameter approaches a fully damaged value of 0. As the length scale parameter decreases, $l_o \rightarrow 0$, the crack evolution will more closely represent linear elastic fracture mechanics with a sharper crack surface (see Figure 4.2).

Miehe et al. [11] suggests to have the mesh length be less than half the length scale

parameter. This results in a mesh size of $h = 0.179$ mm for this material, resulting in a simulation with over six million elements. Considering dimensions of the SR specimen (Figure 3.2), a coarser mesh was implemented to capture results with a cost effective approach. In doing so, the value of the length scale was changed. A base model was generated using similar parameters as the cohesive zone model (Table 4.1), with the modification of the length scale parameter. The increase in the length scale parameter causes the critical stress to decrease from what was observed experimentally to a value of $\sigma_c = 2.254$ MPa, since the critical fracture energy density remained unchanged. This change accommodates the ability to implement a coarser mesh.

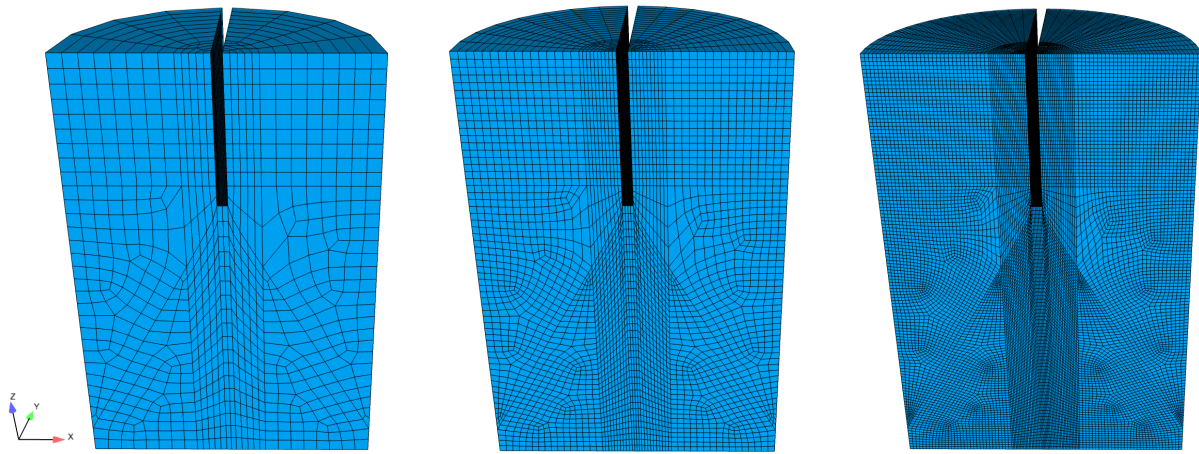
Table 4.1: Parameters for phase-field model of short-rod specimen

Parameter	Value	Units
Young's modulus, E	7.0	GPa
Poisson's ratio, ν	0.155	
Density, ρ	2250	kg/m ³
Fracture energy density, $\mathcal{G}_c$	34	J/m ²
Length scale, l_o	2.473	mm

The base model mesh is refined globally by halving the mesh size in all three dimensions. This approach is applied twice to get three overall meshes for comparison (Table 4.2). The actual mesh refinement is captured in Figure 4.3 where the mesh size varies over the entire domain, so the mesh size reported is the value along the chevron slice. The length scale was held constant for the refinement study meaning the ratio, $\frac{l_o}{h}$, varied with each case.

Table 4.2: Mesh refinement details for the short-rod test configuration

Case	No. of elements	Mesh size, h (mm)
Base	6,344	1.431
R1	50,752	0.716
R2	406,016	0.358



(a) Case: Base

(b) Case: R1

(c) Case: R2

Figure 4.3: Short-rod mesh refinement where the mesh size is halved globally for each case. All meshes were generated using linear 8-node hexahedron elements. The mesh size reported is the measure of mesh in the center of the chervon in the SR geometry. The base case (a) contains 6344 elements, the first refinement (b) has 50752 elements, and the second refinement (c) has 406016 elements.

Results

To approximate a quasi-static analysis, a slow displacement rate was imposed on the aluminum connectors. A rate study showed small variations in the peak load with different displacement rates applied. Slowing down this rate reduces inertial effects throughout the simulation (Figure 4.4).

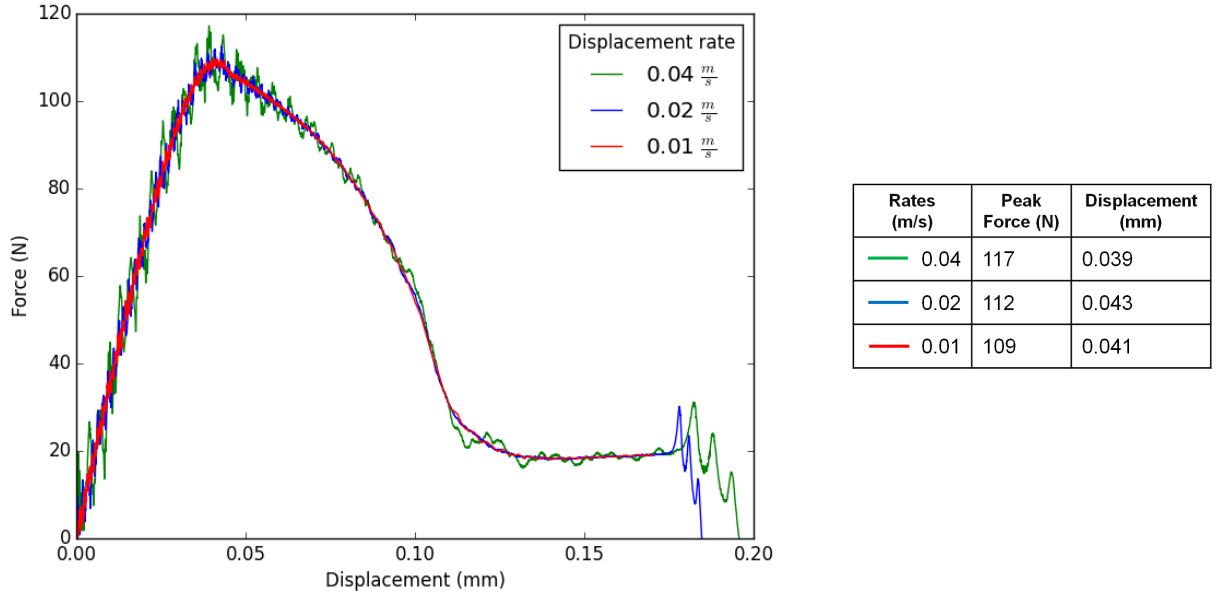


Figure 4.4: Load-deflection curves for various applied displacement rates of the aluminum connectors on the short-rod specimen. Although the difference between peak loads is small, oscillations due to dynamic effects are undesirable. The rate chosen for analysis is $0.01 \frac{m}{s}$ since it is the rate that captures minimal amplitudes of oscillations. After a displacement of 0.18 mm, the crack has fully propagated through the bottom surface of the model.

The phase-field model produced a peak load similar to the cohesive zone model, and within the ranges of the experimental data (Figure 4.5). Since phase-field modeling does not explicitly represent the fracture surface, the unloading curve does not relax to a fully unloaded state. Oscillations in the linear elastic loading are due to inertial effects in the problem. The SR specimen is expected to damage parallel to the bottom surface, which is the lowest surface on the z axis, and should propagate mainly in the chevron slice (Figure 3.3). As a qualitative check, the phase-field scalar evolution in Figure 4.6 shows a crack topology as anticipated with red representing fully damaged bulk material and blue representing fully intact bulk.

In the interest of reducing computing cost, a reduction in the amount of implicit solves for the phase-field parameter, c , was implemented. Instead of solving for the phase-field parameter at every time step, the solve instead occurs after a given input of time step skips.

The difference between load-deflection curves for the lower interval skips (4 and 16) is small (Figure 4.7). Therefore, the implicit solver was set to find the phase-field parameter once every sixteen explicit time steps.

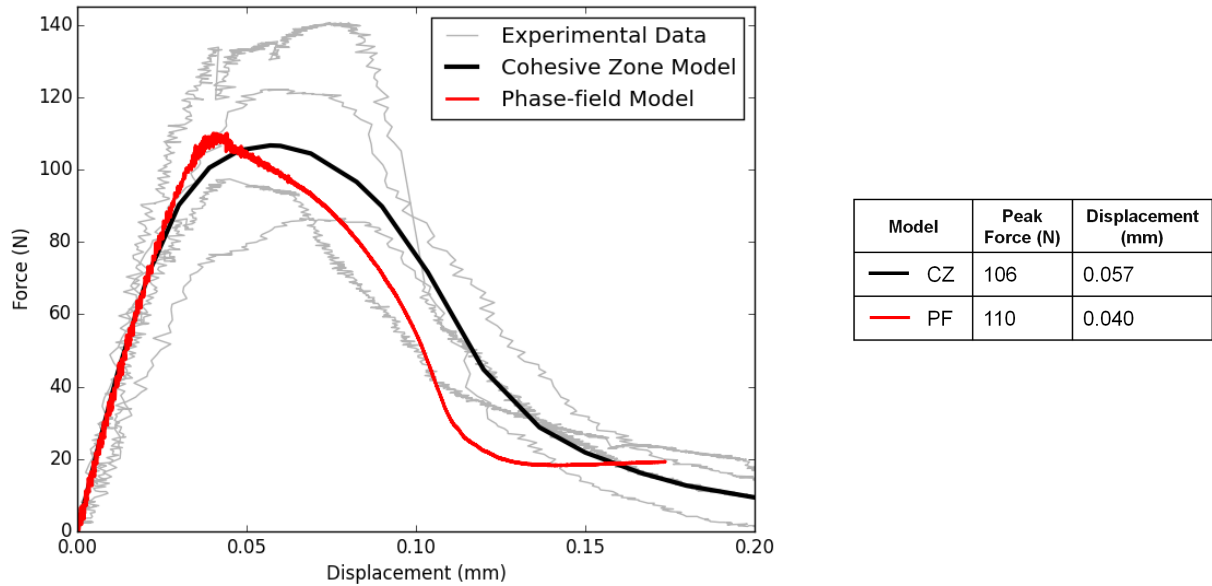


Figure 4.5: Load-deflection curves of the cohesive zone and phase-field models with the experimental results of the short-rod specimen. The peak load falls within the range of experimental results, and is 3 percent different than the CZM solution.

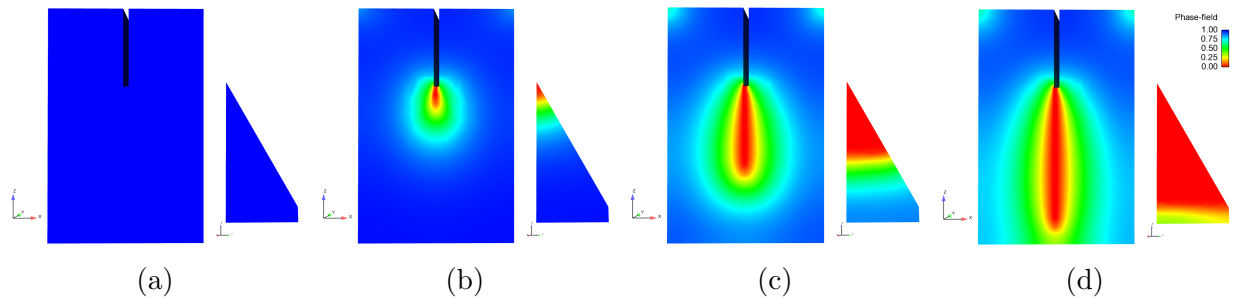


Figure 4.6: Qualitative confirmation of the planar propagation of crack topology as the short-rod specimen approaches a fully damaged state in the chevron region. Crack propagation occurs from (a) $t = 0.0000$ s, (b) $t = 0.0045$ s, (c) $t = 0.0093$ s, and (d) $t = 0.0132$ s.

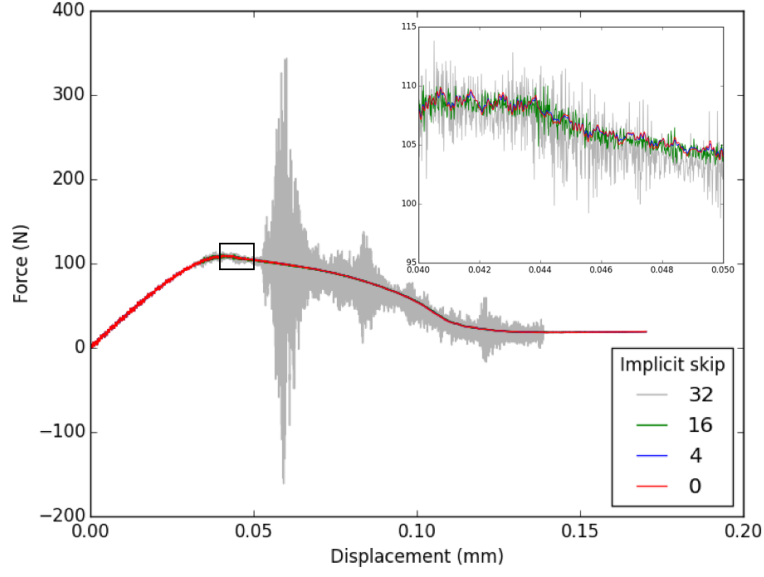
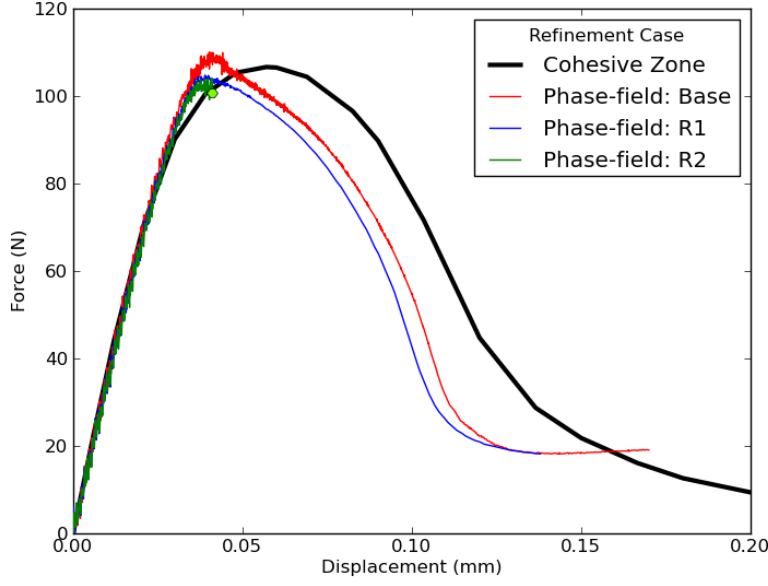


Figure 4.7: Load-deflection curves for an application of various time-step skip intervals for the implicit phase-field solve. The implicit skip chosen was to solve for the phase-field variable every 16 time steps. The difference in the solution between 0, 4, and 16 time step skips is negligible, where 32 skips causes the solution to go unstable after a displacement of 0.05 mm.

As evidenced by the load deflection curves, the phase-field results were insensitive to mesh refinement. The most refined case, R2, introduced instabilities at the beginning of the unloading process, but was included to compare the peak load to the other cases (Figure 4.8).



Model	Peak Force (N)	Displacement (mm)
Base	110	0.040
R1	104	0.039
R2	104	0.040

Figure 4.8: Load-deflection curves for mesh refinement study. Due to the solution running unstable, R2, was truncated at initiation of the unloading process. All refinement cases capture a peak load at a displacement comparable to the CZM, which falls within the experimental results.

To further investigate convergence with mesh refinement the strain energy, dissipated energy, and crack velocity were considered. The internal energy is a sum of the strain and dissipated energies (Figure 4.9).

$$\Psi = \mathcal{E}_e + \mathcal{E}_d, \quad (4.1)$$

where,

$$\mathcal{E}_e = \int_{\Omega} g(c)\psi_e^+ + \psi_e^- d\Omega, \quad (4.2)$$

and

$$\mathcal{E}_d = \int_{\Omega} \frac{1}{4l_o} \mathcal{G}_c [(1-c)^2 + 4l_o |\nabla c|^2] d\Omega. \quad (4.3)$$

Velocity of the crack tip was measured at a post-processing stage. The leading location of the crack tip was taken to be at a value of $c = 0.25$, as extracted by defining an iso-curve. The velocity, $\mathbf{v}$, is determined by central difference method on the position values, $\mathbf{x}$ (Figure 4.10).

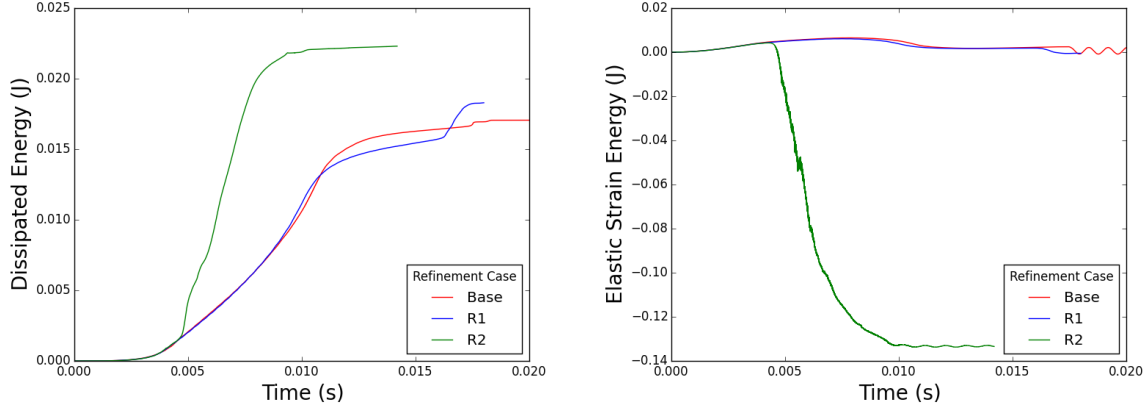


Figure 4.9: Investigation of convergence with mesh refinement of the dissipated and elastic strain energies convergence. The finest mesh case, R2, became unstable at the onset of the unloading process, but was observed to follow the first two cases during loading of the short-rod specimen.

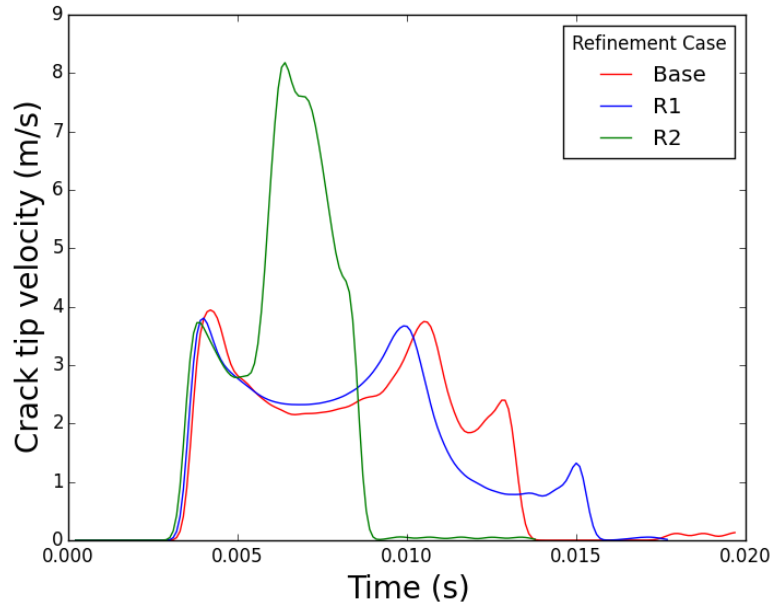


Figure 4.10: Crack tip velocity over time of the short-rod specimen using phase-field approximation. The initial peak occurs when the crack tip first appears in the domain and the second peak generates when the domain begins to widen at the location of crack initiation. After 0.02 seconds, the simulation has developed a crack through the bottom most point of the domain. The simulation of the finest mesh, R2, became unstable at $t \approx 0.005$ s.

Phase-field simulation of Brazilian splitting test

The Brazilian splitting test is a popular method of characterizing the tensile strength of brittle materials. The bulk material, usually a cylindrical geometry, is subjected to compressive loading along the length of the body (Figure 4.11). A constant displacement is applied onto connecting strips on top and bottom of the test configuration, and terminates once complete fracture occurs. A peak load is recorded, and used to determine the tensile splitting strength, σ_c (Carmona et al. [5]). Rinehart et al. [14] ran sixteen tests to obtain a critical stress used in the cohesive zone models generated.

Here, a phase-field model was created to reproduce the peak load found from experimentation. The splitting strength of the material was held fixed and both the length scale and critical fracture energy density were adjusted accordingly (Table 4.3). Details of input data can be found in A.2, which is used for running models in the SIERRA Solid Mechanics code (Version 4.37.8).

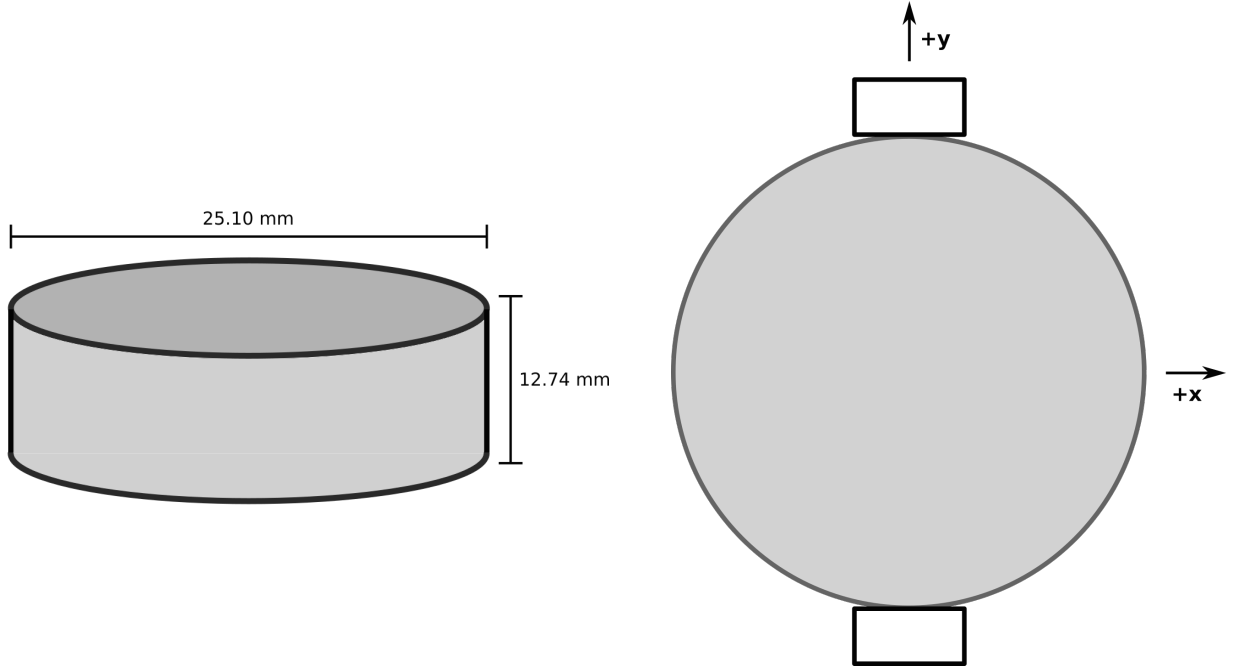


Figure 4.11: Schematic of Brazilian splitting test geometry and test set up. A constant displacement is applied in a compressive direction until fracture occurs.

The actual Brazilian experiment was loaded axially at $0.0025 \frac{\text{mm}}{\text{s}}$ (Rinehart et al. [14]). In the interest of running the model in a realistic amount of wall time, the displacement rate of the connecting strips was held at $0.01 \frac{\text{m}}{\text{s}}$ (Figure 4.12). Like the SR model, the Brazilian model uses a coarser mesh than anticipated to conserve CPU time with a mesh size of $h = 0.251 \text{ mm}$ (Figure 4.12). Using a length scale ratio of $\frac{l_0}{h} = 4$, the fracture energy density was modified to $98 \frac{\text{J}}{\text{m}^2}$, while the critical stress was held fixed to match experimental results at 5.9 MPa.

Table 4.3: Parameters for phase-field model of Brazilian splitting test

Parameter	Value	Units
Young's modulus, E	7.0	GPa
Poisson's ratio, ν	0.155	
Density, ρ	2250	kg/m ³
Fracture energy density, $\mathcal{G}_c$	98	J/m ²
Length scale, l_o	1.034	mm

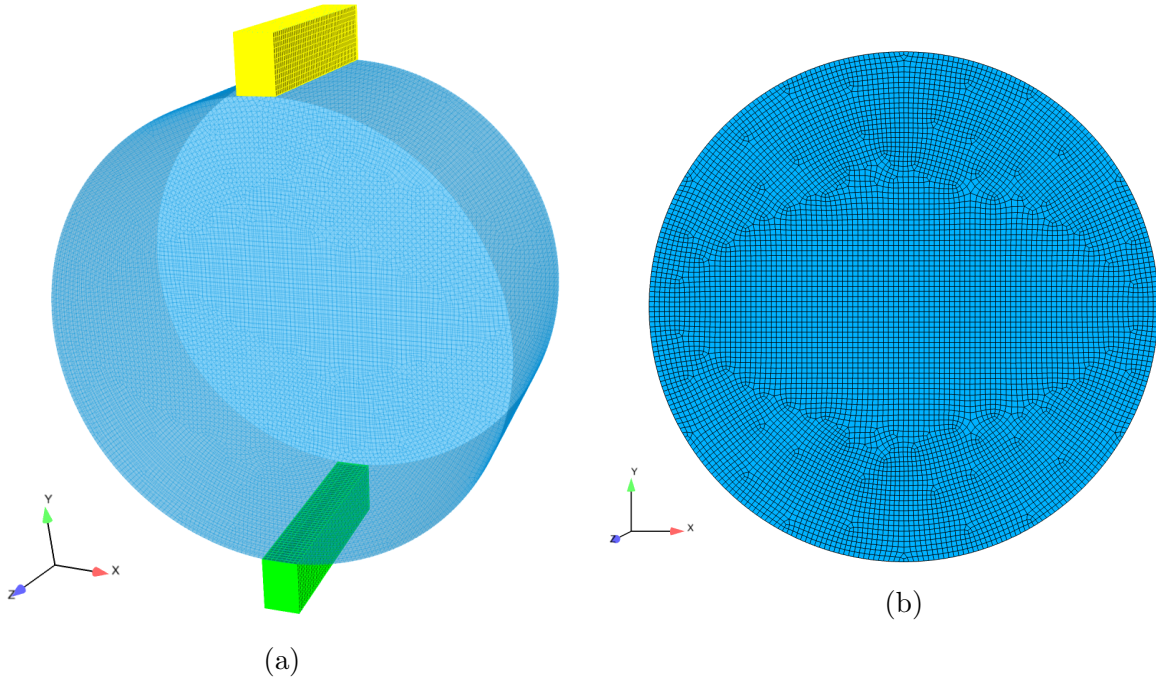


Figure 4.12: Finite-element model of the Brazilian test. (a) The loading strips were assigned a constant displacement rate in $\pm y$ -direction causing compression of the bulk material. The displacement rate of the model was fixed at $0.01 \frac{m}{s}$ on both connectors. (b) Brazilian splitting test mesh generated using linear hexahedron elements with 427788 elements in the entire domain.

A peak load was determined by selecting a geometry matching one of the experimental cases. The resultant peak load of that experiment is plotted with the phase-field solution in Figure 4.13. The interesting attribute of the Brazilian test schematic is its typical behavior of failure, which includes crack branching and coalescence. Three-dimensional crack topology is shown in Figure 4.14 using a phase-field value of $c = 0.001$ as a damage threshold. The crack propagation in the structure initiates quickly and transitions to a V-shape around the connecting strips are applying constant displacement in compression.

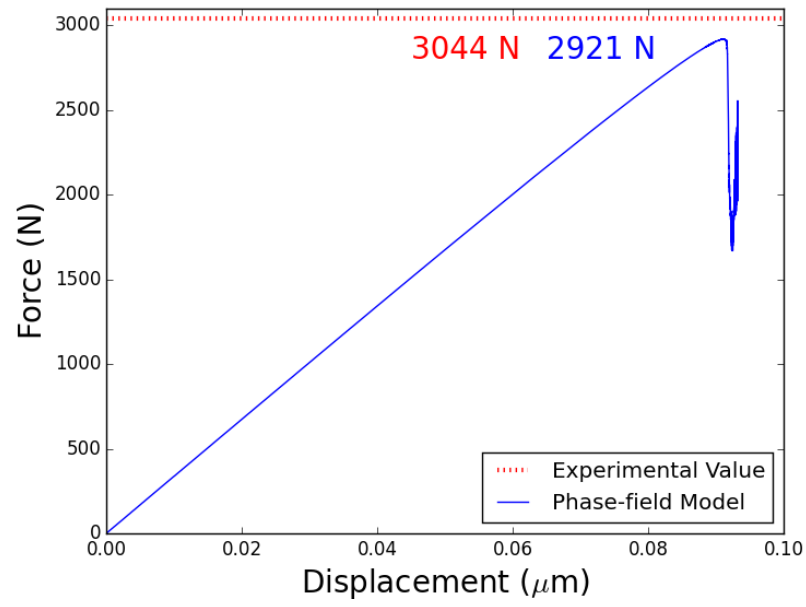


Figure 4.13: Load-deflection curve of Brazilian splitting test model compared to experimental results. The model resulted in a 4 percent difference from the experimental result.

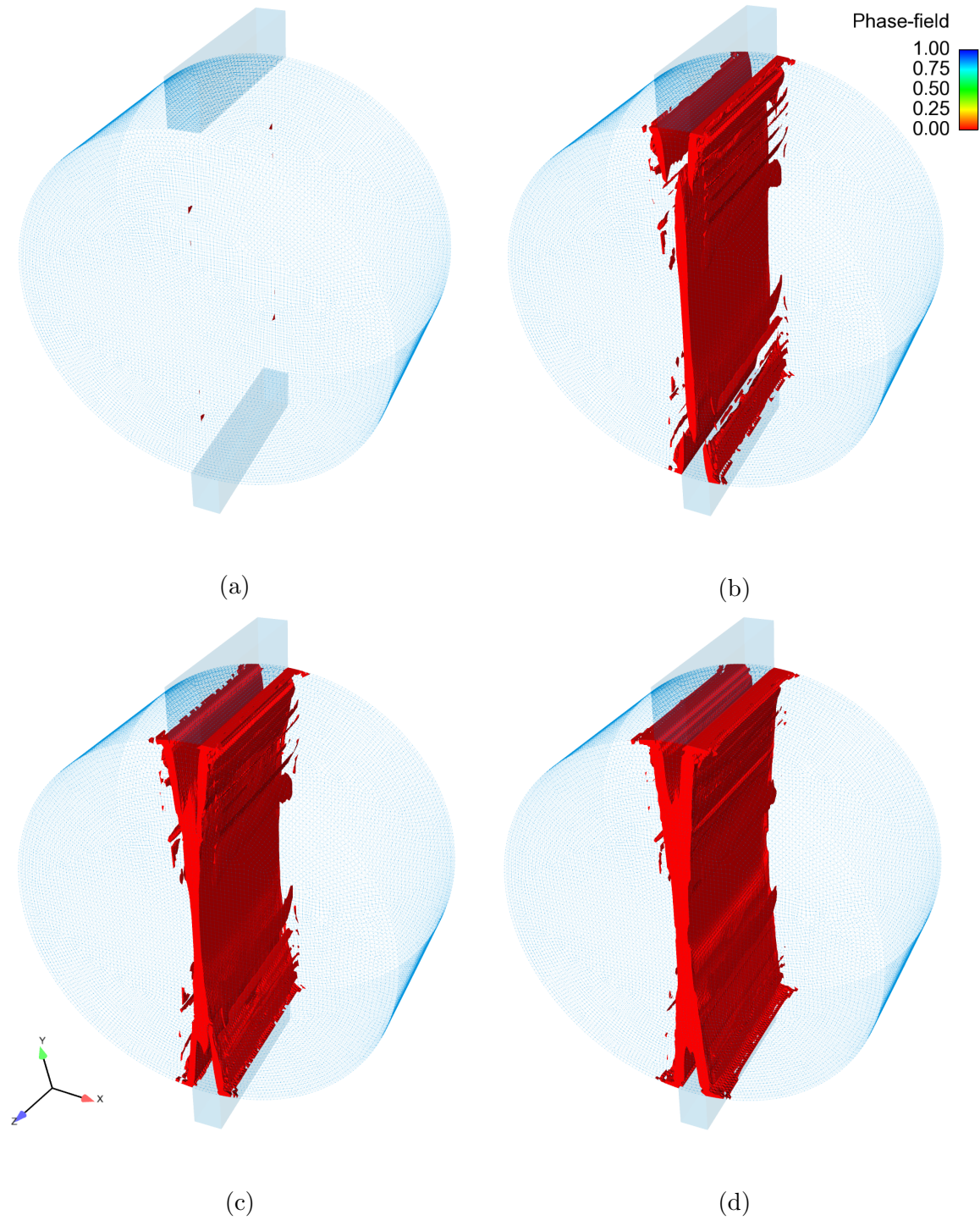


Figure 4.14: Phase-field simulation of the Brazilian test, with chronology going from (a) through (d), threshold crack topology was determined by, $c \leq 0.001$. Time was captured at (a) $t = 0.00690$ s, (b) $t = 0.00699$ s, (c) $t = 0.0700$ s, and (d) $t = 0.00703$ s. Crack bifurcation develops between (b) and (c) close to the contacting loading strips.

Chapter 5

Brittle fracture gradient damage modeling

Results from phase-field modeling are compared to an explicit gradient damage model based on the formulation summarized from (Lorentz et al. [9]). This model solves for the damage parameter, c , via explicit time integration using an unstructured three-dimensional mesh. Details of input data can be found in Appendix A.3, for the SIERRA Solid Mechanics simulation (Version 4.37.8).

Comparison between phase-field and gradient damage modeling of the short-rod fracture experiment

The gradient damage model was examined and compared to the results obtained from the phase-field model of a SR test specimen. The domain constructed from previous models was reused for consistency in analysis with the same displacement boundary conditions (Figure 4.1).

The material parameters are in Table 5.1, with the shape parameter, p , being a unique parameter to this formulation. Since the material characteristics have been defined as a separate set from the length scale, both the critical stress and the fracture energy density need to be considered as input for the simulation. As mentioned previously, these models hold fixed parametric values of $p = 1$ and $m = 3$.

Table 5.1: Parameters for gradient damage model of short-rod test specimen

Parameter	Value	Units
Young's modulus, E	7.0	GPa
Poisson's ratio, ν	0.155	
Density, ρ	2250	kg/m ³
Fracture energy density, $\mathcal{G}_c$	34	J/m ²
Critical stress, σ_c	5.9	MPa
Length scale, l_D	3.40	mm

The material parameters reflect values from experimentation with the exception of the length scale, l_D , which was chosen solely to satisfy the inequality in eq. 2.20. The displacement rate used in the phase-field analysis is also used in this model as $0.01 \frac{m}{s}$. Comparing the same mesh results in a load-deflection curve that represents different peak loads (Figure 5.1).

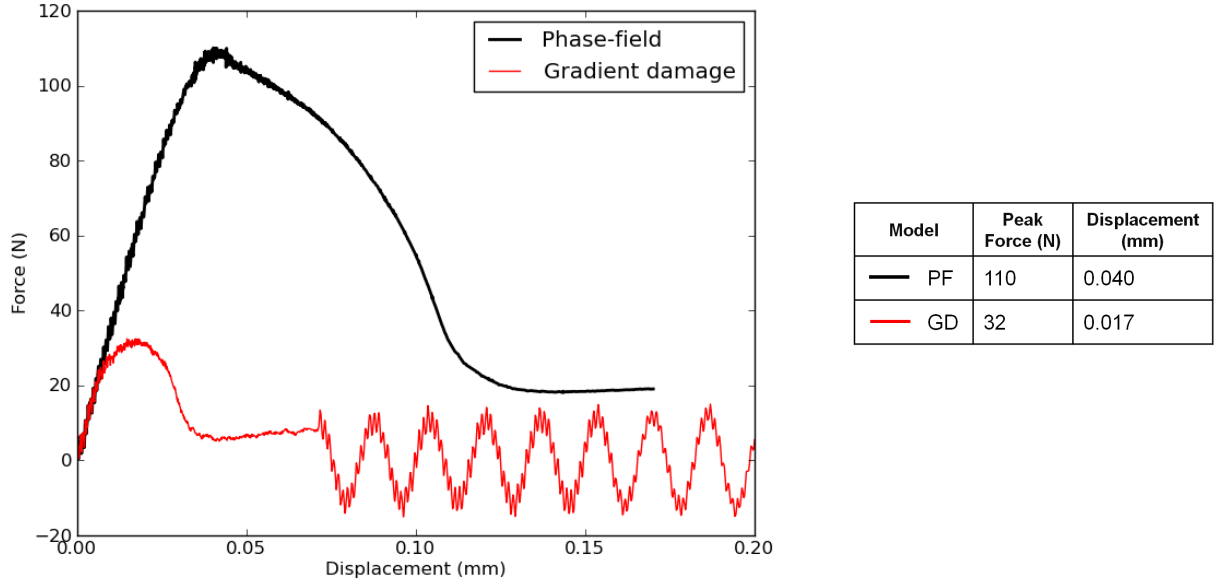


Figure 5.1: Load-deflection curves of the phase-field model and gradient damage model for the coarsest mesh configuration. Note the peak load of the gradient damage model significantly falls below that of the phase-field model which infers further consideration to parameters not properly calibrated to the material.

When considering computing cost, the gradient damage model runs significantly faster than the phase-field model even when including an implicit solve skip once every sixteen time steps. The gradient damage model requires half the wall time using the same amount of processors. The base model was refined globally in previous work, and reused here for a study on the refinement of the gradient damage model (Figure 4.3) with details given in Table 5.1. To emulate the phase-field refinement, the length scale l_D was held constant for each mesh case. Results for three mesh refinements are shown. The results are relatively insensitive to mesh refinement. Due to the inability of the damage model to represent the fracture surface, the simulation was truncated once the crack tip reached the lowest point of the SR specimen (Figure 5.2). The displacement rate of $0.01 \frac{m}{s}$ could be slowed down further to observe fewer oscillations in the solution.

As known from experiments and the phase-field model, the crack evolution should develop symmetrically over time. The coarsest mesh manages to represent this phenomena (Figure 5.3), whereas further refinements do not (Figure 5.4).

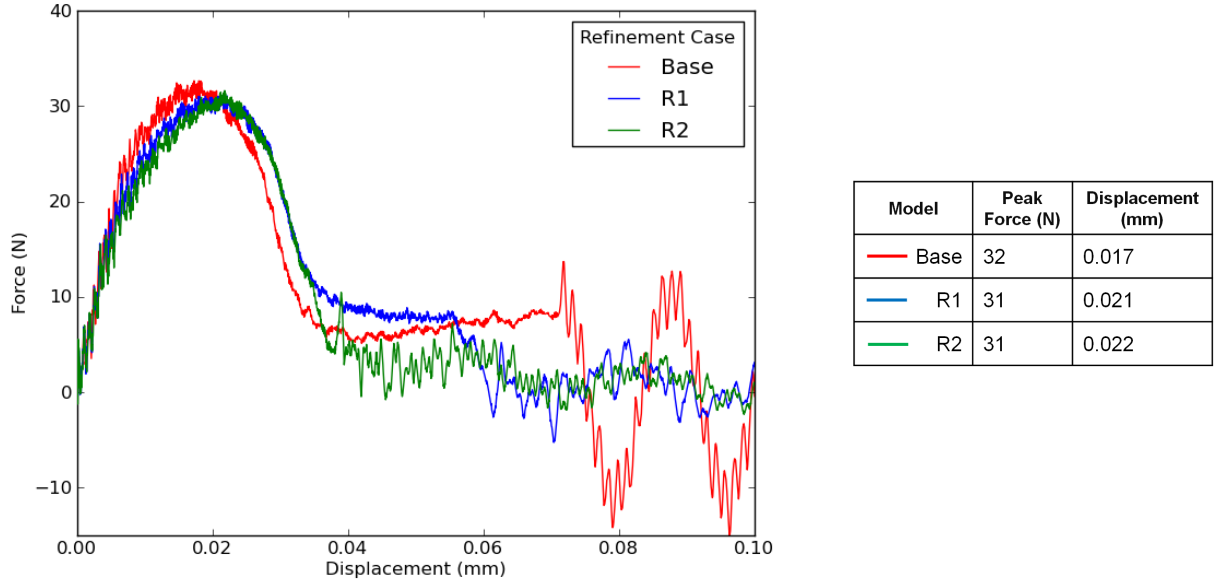


Figure 5.2: Load-deflection curves of the gradient damage model for assessing convergence with mesh refinement. The peak loads fall within a range of 1 N between the coarsest mesh to the finest mesh, and there is a stronger similarity between the two finer refinement cases.

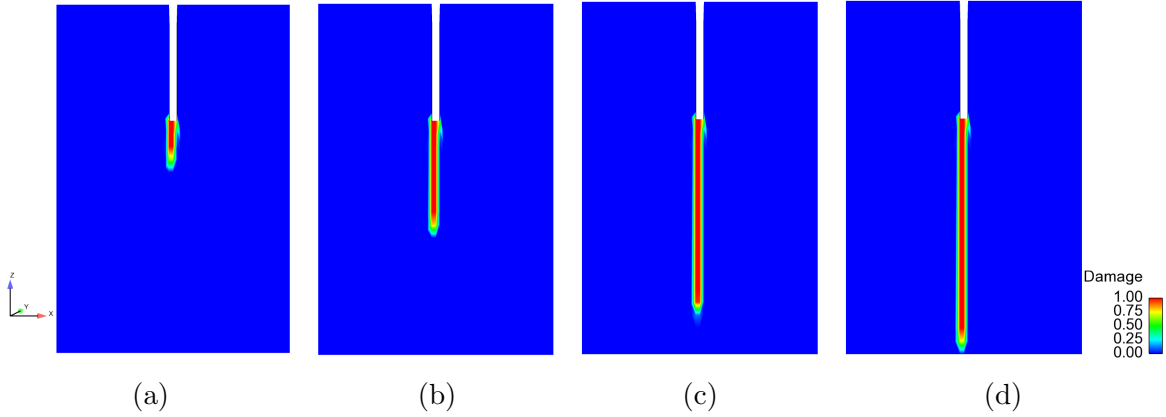


Figure 5.3: Symmetric development of crack topology for the base case of gradient damage modeling. Crack propagation occurs from (a) $t = 0.0010$ s, (b) $t = 0.0020$ s, (c) $t = 0.0030$ s, and (d) $t = 0.0040$ s. Note the sharper crack bandwidth in comparison to the phase-field smearing, where this formulation captures a tighter decay of the scalar damage parameter.

Further investigation of global parameters is executed by considering the elastic strain and dissipated energy over time (Figure 5.5). Divergence of these energies is apparent, and resulting from numerical instabilities associated with the formulation.

The crack tip velocity is obtained from post-processing, by recording the position of an iso-curve of the damage parameter value of $c = 0.75$, which is assumed as the leading crack

edge. The velocity, $\mathbf{v}$, is determined by central difference method on the position values, $\mathbf{x}$ (Figure 5.6).

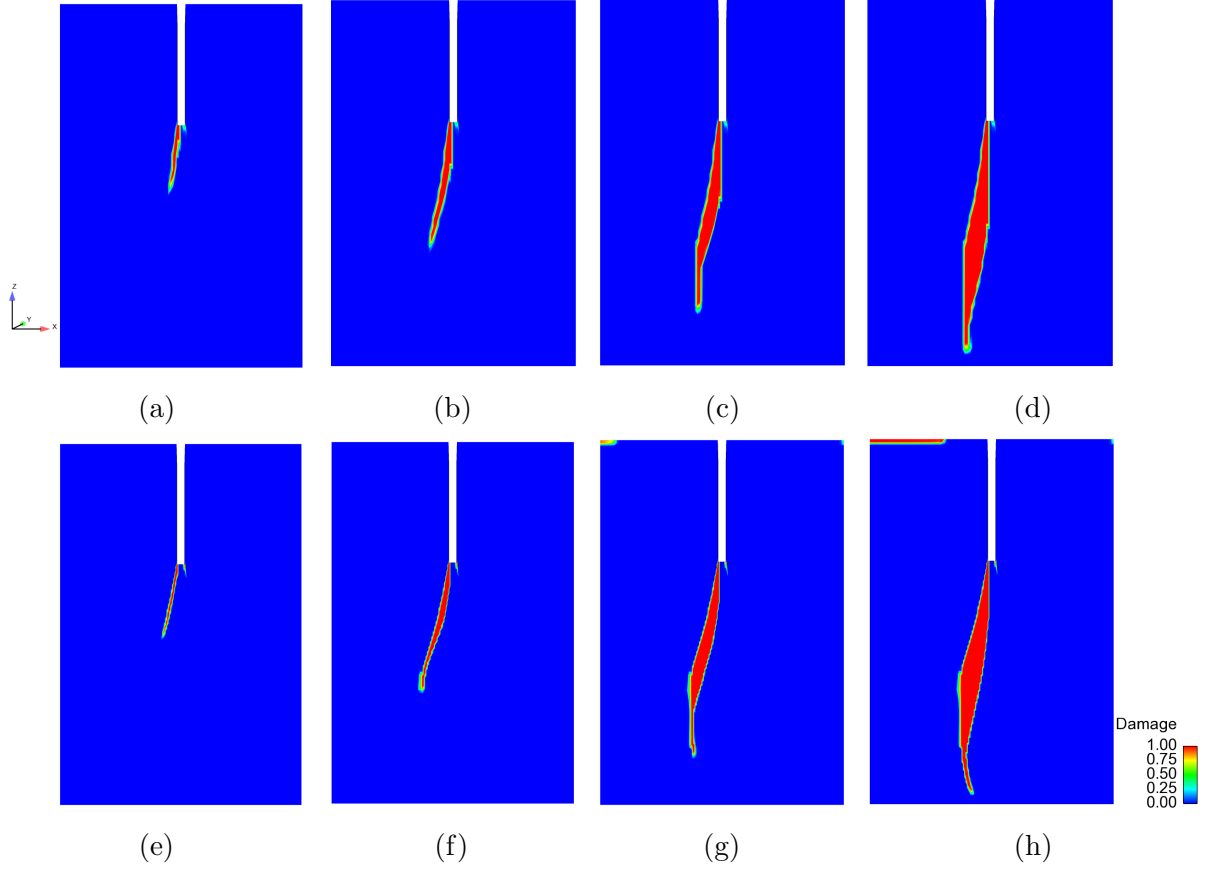


Figure 5.4: Asymmetric development of crack topology for both refinement cases of gradient damage modeling. (a) through (d) represent the first refinement, R1, and (e) through (h) are the second refinement, R2. Crack propagation occurs from (a) $t = 0.0010$ s, (b) $t = 0.0020$ s, (c) $t = 0.0030$ s, and (d) $t = 0.0040$ s. Due to numerical instabilities not yet resolved in the numerical formulation, the crack topology does not display symmetric propagation.

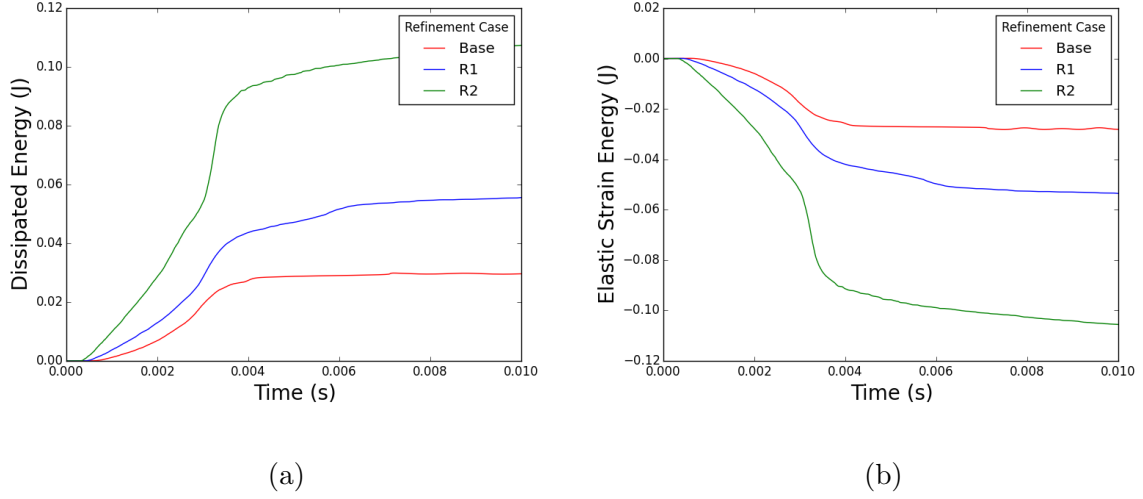


Figure 5.5: Short-rod gradient damage model expressing (a) elastic strain energy stated by eq. 4.2 and (b) dissipated energy stated by eq. 4.3 over time. After 0.007 seconds, the simulation crack topology has reached the bottom of the domain, and has fully damaged. Both plots display non-convergent solutions due to numeric instabilities involved in solving the gradient damage parameter.

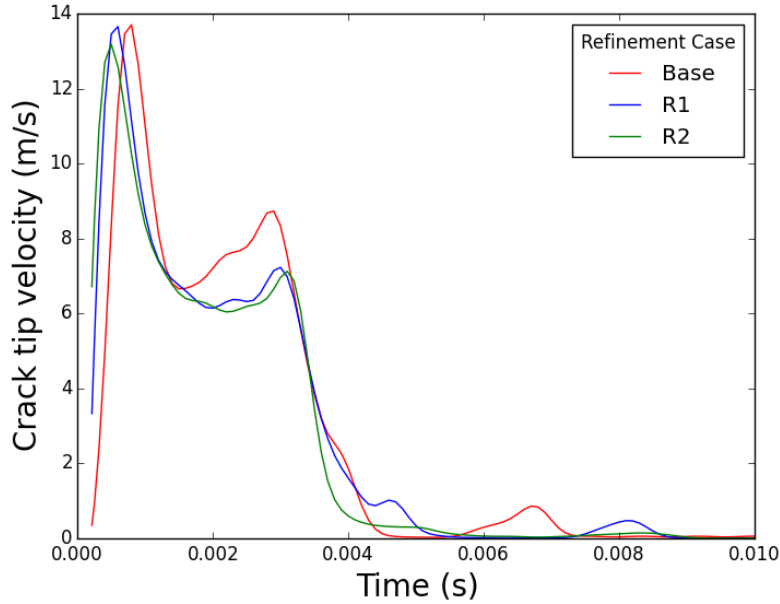


Figure 5.6: Crack tip velocity over time of the short-rod specimen using gradient damage approximation of the crack evolution. The initial peak occurs when the crack tip first appears in the domain and the second peak generates when the domain begins to widen at the location of crack initiation. After 0.007 seconds, the simulation has developed a crack through the bottom most point of the domain.

Chapter 6

Conclusions

Using a rate-independent dynamic phase-field model for brittle fracture provides a thermodynamically consistent damage evolution to represent failure of the bulk material. Benefits of phase-field and gradient damage modeling in general are the freedom from having to numerically track discontinuities via the displacement field and the ability to create crack nucleation, branching, and coalescence in three dimensions seamlessly. The phase-field formulation has limiting convergence to a Griffith's type linear elastic fracture mechanics model. It was compared to a general gradient damage formulation for the short-rod test specimen.

Phase-field did not display the same time step sensitivity, since the scalar parameter was solved for by Newton's method, but when global refinement of the mesh was applied a second time, the solution became unstable at the initiation of unloading. As the ratio of length scale to mesh size increased past a certain value convergence ceased to be met. This awareness can be implemented when considering new developments for predictive fracture modeling. The Brazilian tensile splitting simulation exhibited crack coalescence in three-dimensions and reproduced the peak load obtained from experimentation.

The gradient damage model converges to a cohesive type model, and allows for the length-scale parameter to be an independent value from the nucleation stress and the critical fracture energy density. Although this model did not capture comparable representative quantitative results, it did show convergence of the crack tip velocity and the load-deflection curves with mesh refinement. Numerical instabilities caused sensitivity to the solutions making the dissipated energy and elastic strain energy divergent.

With an appropriate length scale to mesh size ratio, phase-field modeling for brittle fracture can robustly predict crack topology in three dimensions. This improves upon current commonly adopted methods, such as cohesive zone modeling, which require *a priori* knowledge of the crack path.

Future Work

Since this study was carried out during constantly upgrading code, the capabilities of phase-field modeling expanded with each new feature. At this point in time, the linear elastic phase-field models can be run in both quasi-static and dynamic applications. There are

additional considerations to the formulation not presented in this paper when working in a quasi-static implementation. A possible venue to further understand capabilities available would be to develop a quasi-static model of the short-rod specimen to compare to experimental results. To further understand convergence with mesh refinement, the length scale parameter must be allowed to change in accordance to a prescribed ratio to mesh size while holding either the nucleation stress or the critical fracture energy density fixed.

Other common test specimen could be considered for modeling, such as notched three-point bending tests or the double torsion fracture tests. The double torsion test is used to measure subcritical fracture. Since the fracture pattern is unimportant for the creation of a phase-field simulation, exploration into unique geometries can also be considered.

Another set of opportunities of phase-field modeling comes with expanding the functional associated with fracture energy, $\Gamma_{c,n}$, to higher orders. This has been demonstrated by Borden et al. [2] to have higher accuracy. This requires higher-order basis functions, which can be handled in an isogeometric analysis (IGA) or reproducing kernel particle method (RKPM) framework. Since phase-field and gradient damage models fail to resolve fracture surfaces, coupling these methods with element death or extended finite element method would be useful for large deformation fracture. A caveat with these couplings would be returning to a mesh dependent visualization of crack evolution.

Phase-field modeling can also be extended to coupled multi-physics problems. This is the motivation of geomechanics applications for modeling reservoir-scale effects of coupled flow, chemical, and mechanical damage processes (Mikeli et al. [13]). Taking advantage of the thermodynamically consistent framework has been shown to be advantageous for coupled problems.

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Appendix A

SIERRA Input Data

A.1 Short-rod using phase-field

Base Case Model:

- Piecewise linear displacement rate: $0.01 \frac{m}{s}$
- Material model: `phase_field_linear_elastic`
- Limestone short-rod specimen
 - Density: $2250 \frac{kg}{m^3}$
 - Young’s modulus: 7.0 GPa
 - Poisson’s ratio: 0.155
 - Length scale: 2.473 mm
 - Conditioning coefficient: 0.0
 - Fracture energy density: $34.034 \frac{J}{m^2}$
- Aluminum connectors
 - Density: $2700 \frac{kg}{m^3}$
 - Young’s modulus: 68.9 GPa
 - Poisson’s ratio: 0.33
 - Length scale: 0.1 mm
 - Conditioning coefficient: 0.0
 - Fracture energy density: $1000.0 \frac{J}{m^2}$
- FE section: Fully integrated formulation
- FE procedure: Presto
- Termination time: 0.020 s

- Fixed displacement: bottom surface of SR in y and z-directions
- Prescribed displacements: on aluminum connectors in $\pm x$ -directions
- Tied contact between connectors and SR
- Phase-field parameters
 - Solve increment: 16
 - Feti residual norm tolerance: 1E-10

First Refinement Case Model:

- Piecewise linear displacement rate: $0.01 \frac{m}{s}$
- Material model: phase_field_linear_elastic
- Limestone short-rod specimen
 - Density: $2250 \frac{kg}{m^3}$
 - Young's modulus: 7.0 GPa
 - Poisson's ratio: 0.155
 - Length scale: 2.473 mm
 - Conditioning coefficient: 0.0
 - Fracture energy density: $34.034 \frac{J}{m^2}$
- Aluminum connectors
 - Density: $2700 \frac{kg}{m^3}$
 - Young's modulus: 68.9 GPa
 - Poisson's ratio: 0.33
 - Length scale: 0.1 mm
 - Conditioning coefficient: 0.0
 - Fracture energy density: $1000.0 \frac{J}{m^2}$
- FE section: Fully integrated formulation
- FE procedure: Presto
- Termination time: 0.018 s
- Fixed displacement: bottom surface of SR in y and z-directions
- Prescribed displacements: on aluminum connectors in $\pm x$ -directions

- Tied contact between connectors and SR
- Phase-field parameters
 - Solve increment: 16
 - Feti residual norm tolerance: 1E-10

Second Refinement Case Model:

- Piecewise linear displacement rate: $0.01 \frac{m}{s}$
- Material model: phase_field_linear_elastic
- Limestone short-rod specimen
 - Density: $2250 \frac{kg}{m^3}$
 - Young's modulus: 7.0 GPa
 - Poisson's ratio: 0.155
 - Length scale: 2.473 mm
 - Conditioning coefficient: 0.0
 - Fracture energy density: $34.034 \frac{J}{m^2}$
- Aluminum connectors
 - Density: $2700 \frac{kg}{m^3}$
 - Young's modulus: 68.9 GPa
 - Poisson's ratio: 0.33
 - Length scale: 0.1 mm
 - Conditioning coefficient: 0.0
 - Fracture energy density: $1000.0 \frac{J}{m^2}$
- FE section: Fully integrated formulation
- FE procedure: Presto
- Termination time: 0.020 s
- Fixed displacement: bottom surface of SR in y and z-directions
- Prescribed displacements: on aluminum connectors in $\pm x$ -directions
- Tied contact between connectors and SR
- Phase-field parameters
 - Solve increment: 16
 - Feti residual norm tolerance: 1E-10

A.2 Brazilian test using phase-field

Base Case Model:

- Piecewise linear displacement rate: $0.01 \frac{m}{s}$
- Material model: `phase_field_linear_elastic`
- Limestone Brazilian test specimen:
 - Density: $2250 \frac{kg}{m^3}$
 - Young’s modulus: 7.0 GPa
 - Poisson’s ratio: 0.155
 - Length scale: 1.034 mm
 - Conditioning coefficient: 0.0
 - Fracture energy density: $98.135 \frac{J}{m^2}$
- Connecting structures
 - Density: $615 \frac{kg}{m^3}$
 - Young’s modulus: 9.346 GPa
 - Poisson’s ratio: 0.31
 - Length scale: 0.5 mm
 - Conditioning coefficient: 0.0
 - Fracture energy density: $2000.0 \frac{J}{m^2}$
- FE section: Fully integrated formulation
- FE procedure: Presto
- Termination time: 0.010 s
- Fixed displacement: Connectors in x and z-directions
- Prescribed displacements: Connectors in $\pm y$ -directions
- Tied contact between connectors and Brazilian test specimen
- Phase-field parameters
 - Solve increment: 16
 - Feti residual norm tolerance: 1E-10

A.3 Short-rod using gradient damage model

Base Case Model:

- Piecewise linear displacement rate: $0.01 \frac{m}{s}$
- Material model: `gradient_damage_explicit`
- Limestone short-rod specimen
 - Density: $2250 \frac{kg}{m^3}$
 - Young's modulus: 7.0 GPa
 - Poisson's ratio: 0.155
 - Length scale: 3.4 mm
 - Conditioning coefficient: 0.0
 - Fracture energy density: $34.034 \frac{J}{m^2}$
 - Critical stress: 5.919 MPa
 - Cohesive shape: 1.0
 - Phase viscosity: 0.1
- Aluminum connectors
 - Density: $2700 \frac{kg}{m^3}$
 - Young's modulus: 68.9 GPa
 - Poisson's ratio: 0.33
 - Length scale: 0.1 mm
 - Conditioning coefficient: 0.0
 - Fracture energy density: $1000.0 \frac{J}{m^2}$
 - Critical stress: 580 MPa
 - Cohesive shape: 1.0
 - Phase viscosity: 0.1
- FE section: Fully integrated formulation
- FE procedure: Presto
- Termination time: 0.020 s
- Fixed displacement: bottom surface of SR in y and z-directions
- Prescribed displacements: on aluminum connectors in $\pm x$ -directions

- Tied contact between connectors and SR
- Phase-field parameters
 - Initial value = 0.0
 - Solve explicit: on

First Refinement Case Model:

- Piecewise linear displacement rate: $0.01 \frac{m}{s}$
- Material model: gradient_damage_explicit
- Limestone short-rod specimen
 - Density: $2250 \frac{kg}{m^3}$
 - Young's modulus: 7.0 GPa
 - Poisson's ratio: 0.155
 - Length scale: 3.4 mm
 - Conditioning coefficient: 0.0
 - Fracture energy density: $34.034 \frac{J}{m^2}$
 - Critical stress: 5.919 MPa
 - Cohesive shape: 1.0
 - Phase viscosity: 0.1
- Aluminum connectors
 - Density: $2700 \frac{kg}{m^3}$
 - Young's modulus: 68.9 GPa
 - Poisson's ratio: 0.33
 - Length scale: 0.1 mm
 - Conditioning coefficient: 0.0
 - Fracture energy density: $1000.0 \frac{J}{m^2}$
 - Critical stress: 580 MPa
 - Cohesive shape: 1.0
 - Phase viscosity: 0.1
- FE section: Fully integrated formulation
- FE procedure: Presto
- Termination time: 0.020 s

- Fixed displacement: bottom surface of SR in y and z-directions
- Prescribed displacements: on aluminum connectors in $\pm x$ -directions
- Tied contact between connectors and SR
- Phase-field parameters
 - Initial value = 0.0
 - Solve explicit: on

Second Refinement Case Model:

- Piecewise linear displacement rate: $0.01 \frac{m}{s}$
- Material model: gradient_damage_explicit
- Limestone short-rod specimen
 - Density: $2250 \frac{kg}{m^3}$
 - Young's modulus: 7.0 GPa
 - Poisson's ratio: 0.155
 - Length scale: 3.4 mm
 - Conditioning coefficient: 0.0
 - Fracture energy density: $34.034 \frac{J}{m^2}$
 - Critical stress: 5.919 MPa
 - Cohesive shape: 1.0
 - Phase viscosity: 0.1
- Aluminum connectors
 - Density: $2700 \frac{kg}{m^3}$
 - Young's modulus: 68.9 GPa
 - Poisson's ratio: 0.33
 - Length scale: 0.1 mm
 - Conditioning coefficient: 0.0
 - Fracture energy density: $1000.0 \frac{J}{m^2}$
 - Critical stress: 580 MPa
 - Cohesive shape: 1.0
 - Phase viscosity: 0.1
- FE section: Fully integrated formulation

- FE procedure: Presto
- Termination time: 0.020 s
- Fixed displacement: bottom surface of SR in y and z-directions
- Prescribed displacements: on aluminum connectors in $\pm x$ -directions
- Tied contact between connectors and SR
- Phase-field parameters
 - Initial value = 0.0
 - Solve explicit: on

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