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Title: Solitary waves in nonlinear Dirac equation: from field theory to Dirac materials

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Solitary Waves in Nonlinear Dirac Equation

FROM FIELD THEORY TO DIRAC MATERIALS

Avadh Saxena (Los Alamos National Lab)

1. SOLITONS: in nature and physics.

2. Balance between nonlinearity and dispersion.

3. Soliton bearing systems and equations: NLS, KdV, sine-Gordon.

4. Nonlinear Dirac solitons: scalar & vector interaction.

5. Stability criteria and simulations.

6. Conclusions (and comments on quantum elasticity).

F. Cooper, A. Khare, F.G. Mertens, B. Mihaila, N.R. Quintero, S. Shao

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SOLVAY CONFERENCE 1927

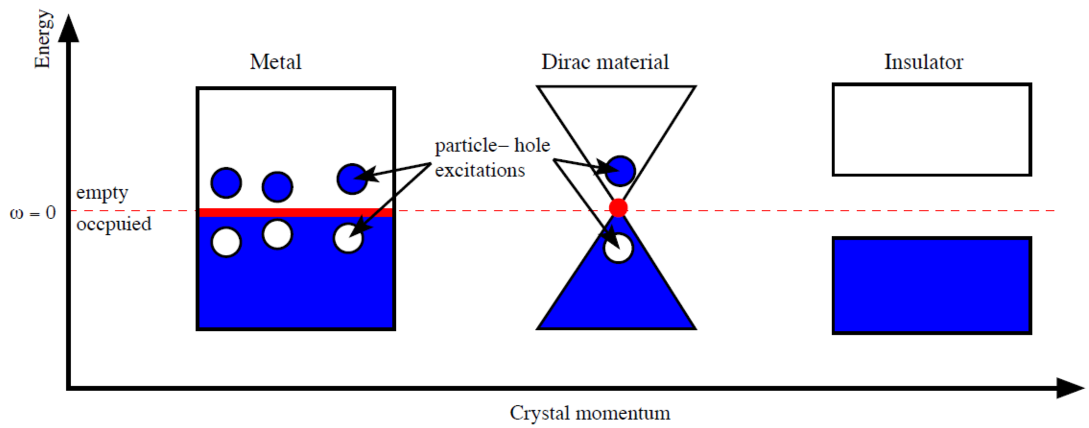
A. PICARD E. HENRIOT P. EHRENFEST Ed. HERSEN Th. DE DONDER E. SCHRÖDINGER E. VERSCHAFFELT W. PAULI W. HEISENBERG R.H FOWLER L. BRILLOUIN

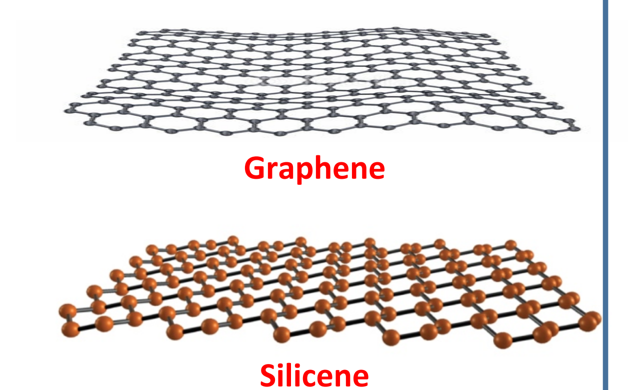
P. DEBYE M. KNUDSEN W.L. BRAGG H.A. KRAMERS P.A.M. DIRAC A.H. COMPTON L. de BROGLIE M. BORN N. BOHR

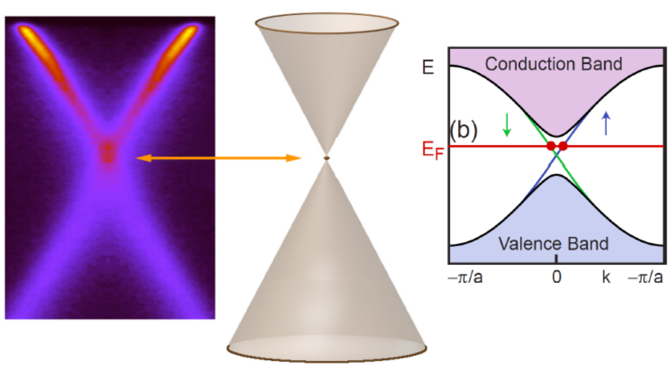
I. LANGMUIR M. PLANCK Mme CURIE H.A. LORENTZ A. EINSTEIN P. LANGEVIN Ch.E. GUYE C.T.R. WILSON O.W. RICHARDSON

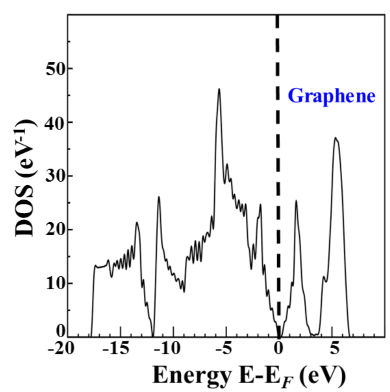
Absents : Sir W.H. BRAGG, H. DESLANDRES et E. VAN AUBEL

DIRAC MATERIALS



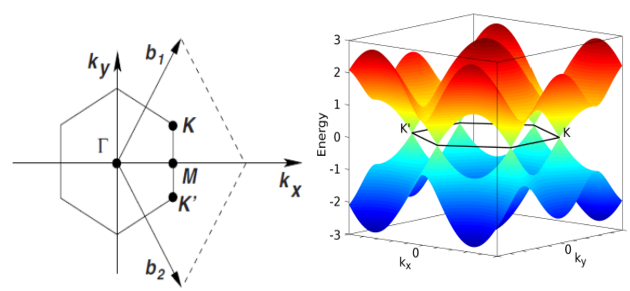






ARPES on Graphene:
Advanced Light Source at LBNL

DFT Calculation of DOS:
T-4 Division, LANL



Lattice Structure and Dirac points
In Hexagonal Dirac Materials

Possible Relevance

1. BE condensate in a honeycomb lattice in the long wavelength limit

Ref: Haddad and Car, Physics D238, Eurp Phys. Lett. 94

2. Multicomponent BEC order parameter has an exact spinor structure

3. Binary optical wave guide arrays

Ref. Ann. Phys. **340**, 179, J. Optics Soc. of Ame. B31, 1132

4. Nonlinear Dynamics in Honeycomb lattices

PRA **84**, 021802(R)

5. Nonlinear Diffraction in Photonic Graphine

Opt. Lett. **36**, 3762

JOHN SCOTT RUSSELL (1808-1882)

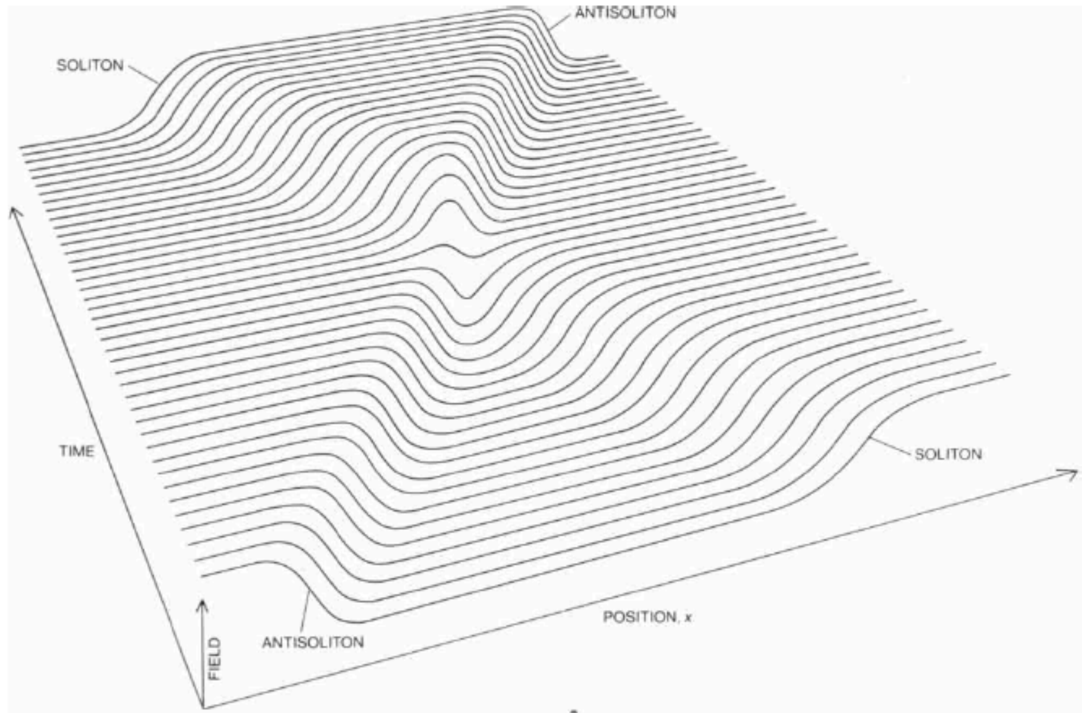
1834: First observation of solitons (Edinburgh).

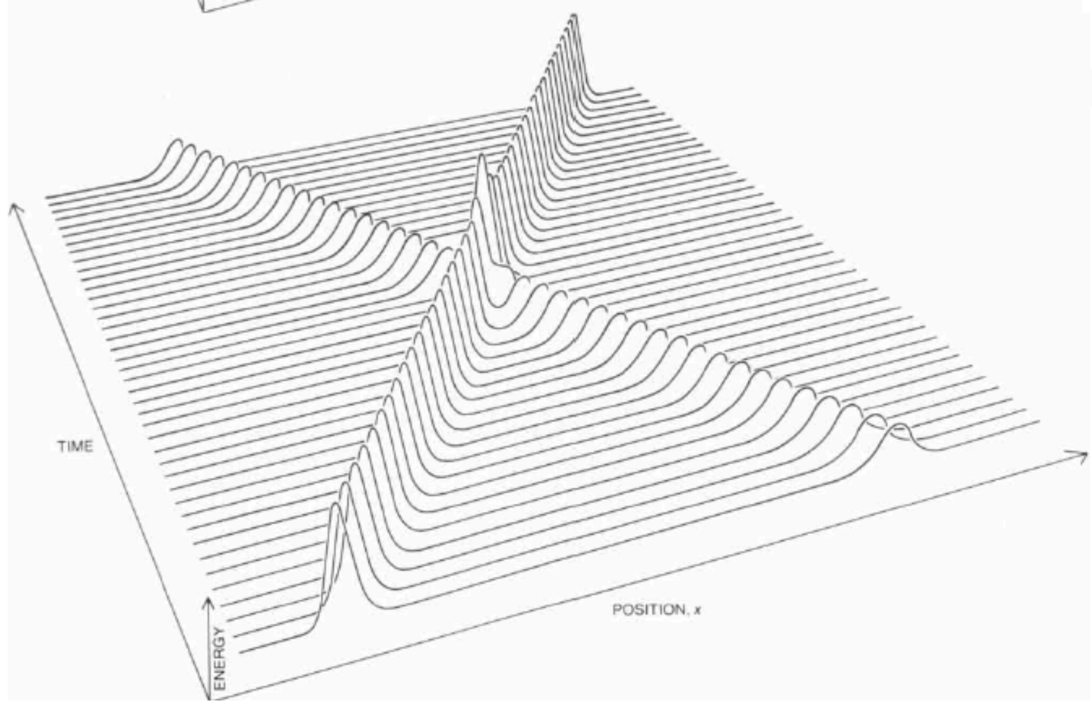
"Report on Waves": 1844

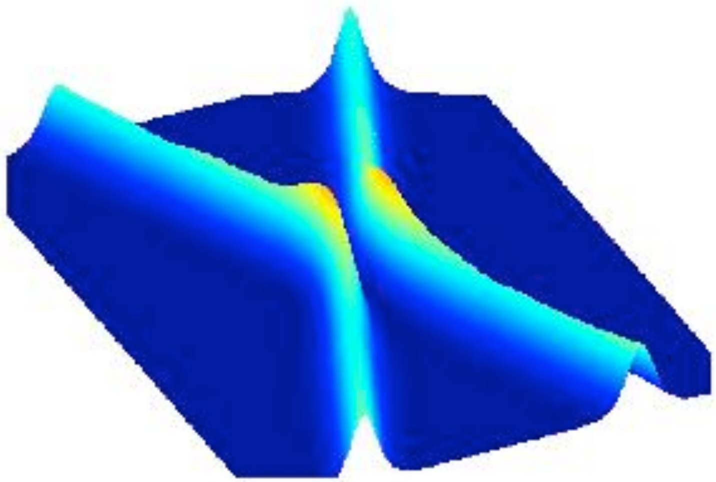
1895: Korteweg-de Vries (KdV)

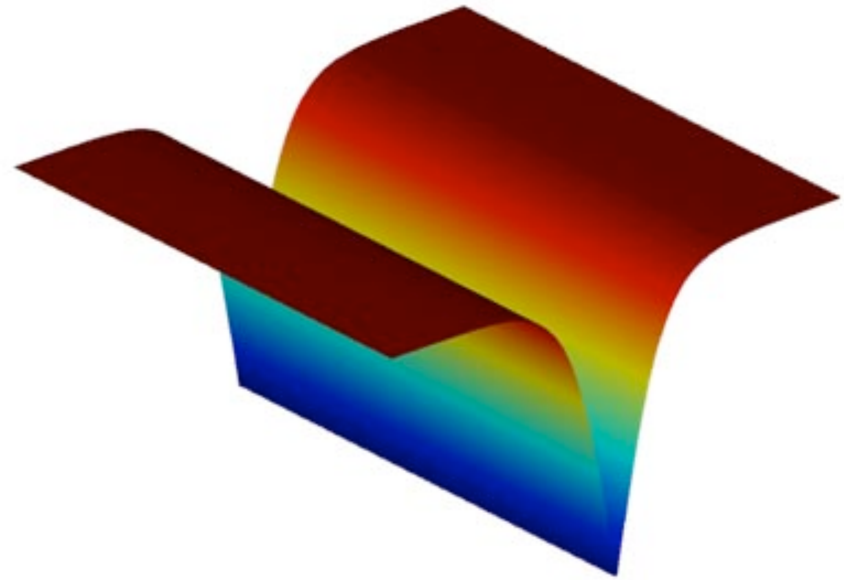
1965: Zabusky-Kruskal (numerics, FPU)

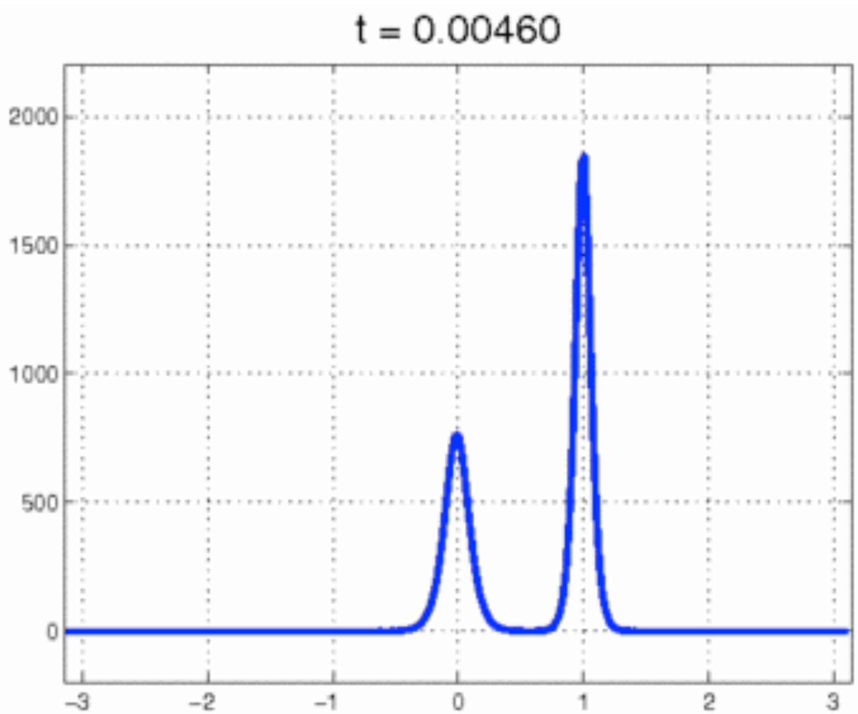
Great Red Spot (storm) on Jupiter


ANTISOLITON
SOLITON
TIME
SOLITON
FIELD
ANTISOLITON
POSITION, x


TIME
ENERGY
POSITION, x



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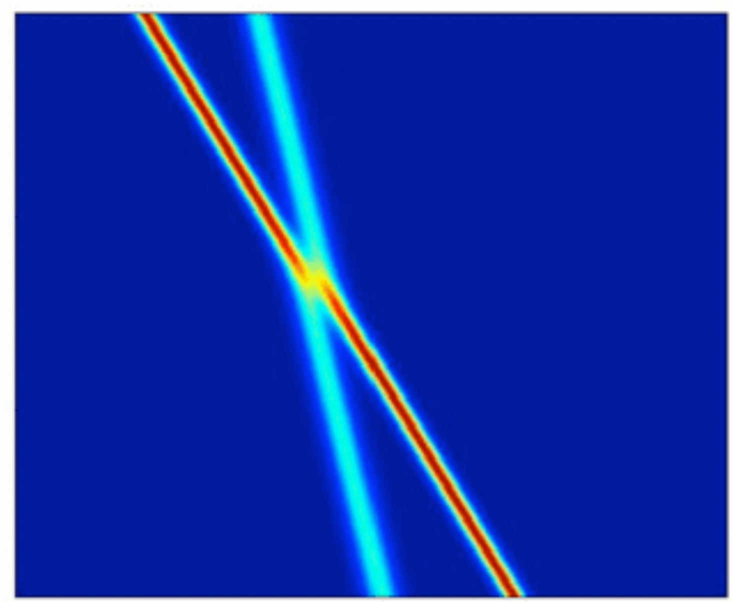
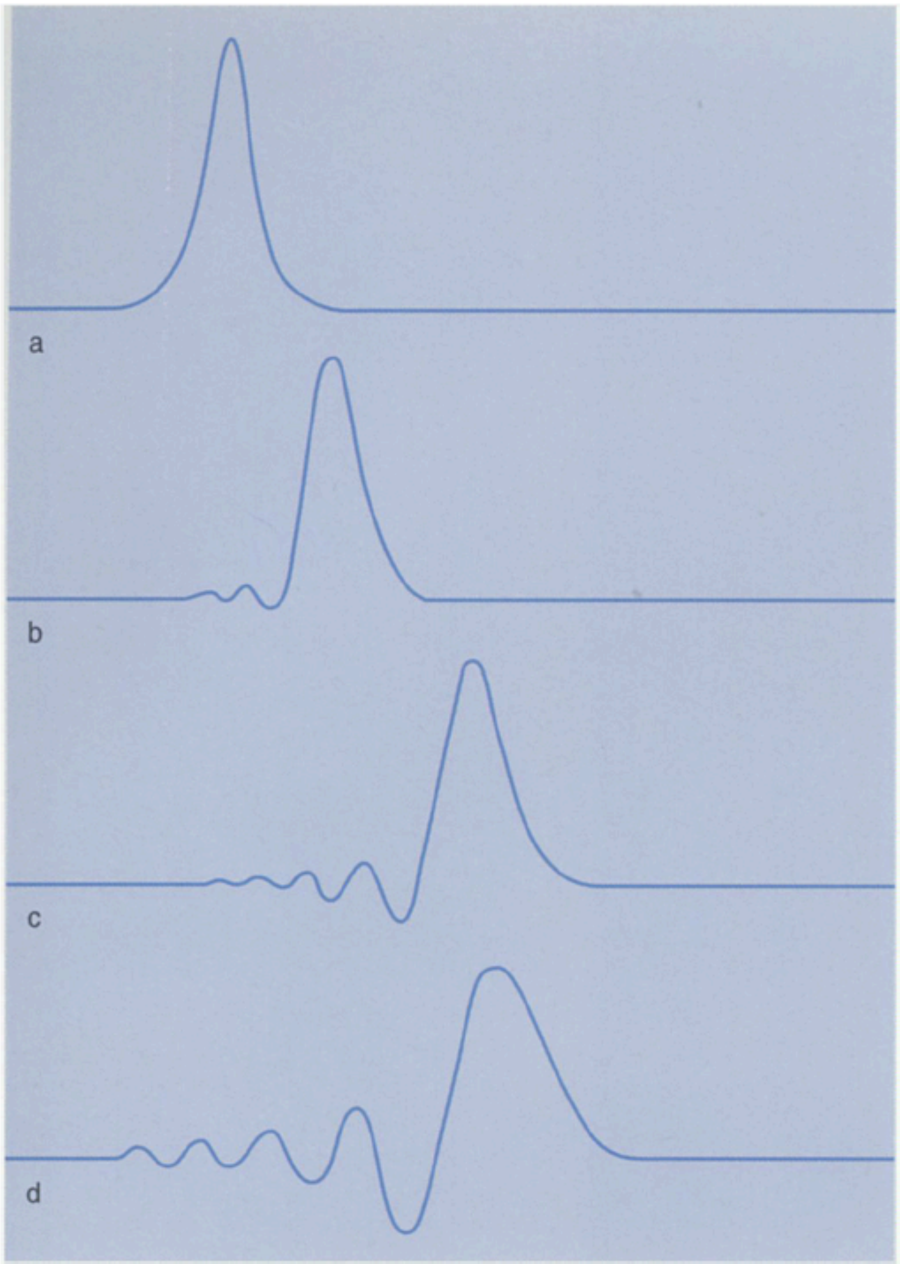
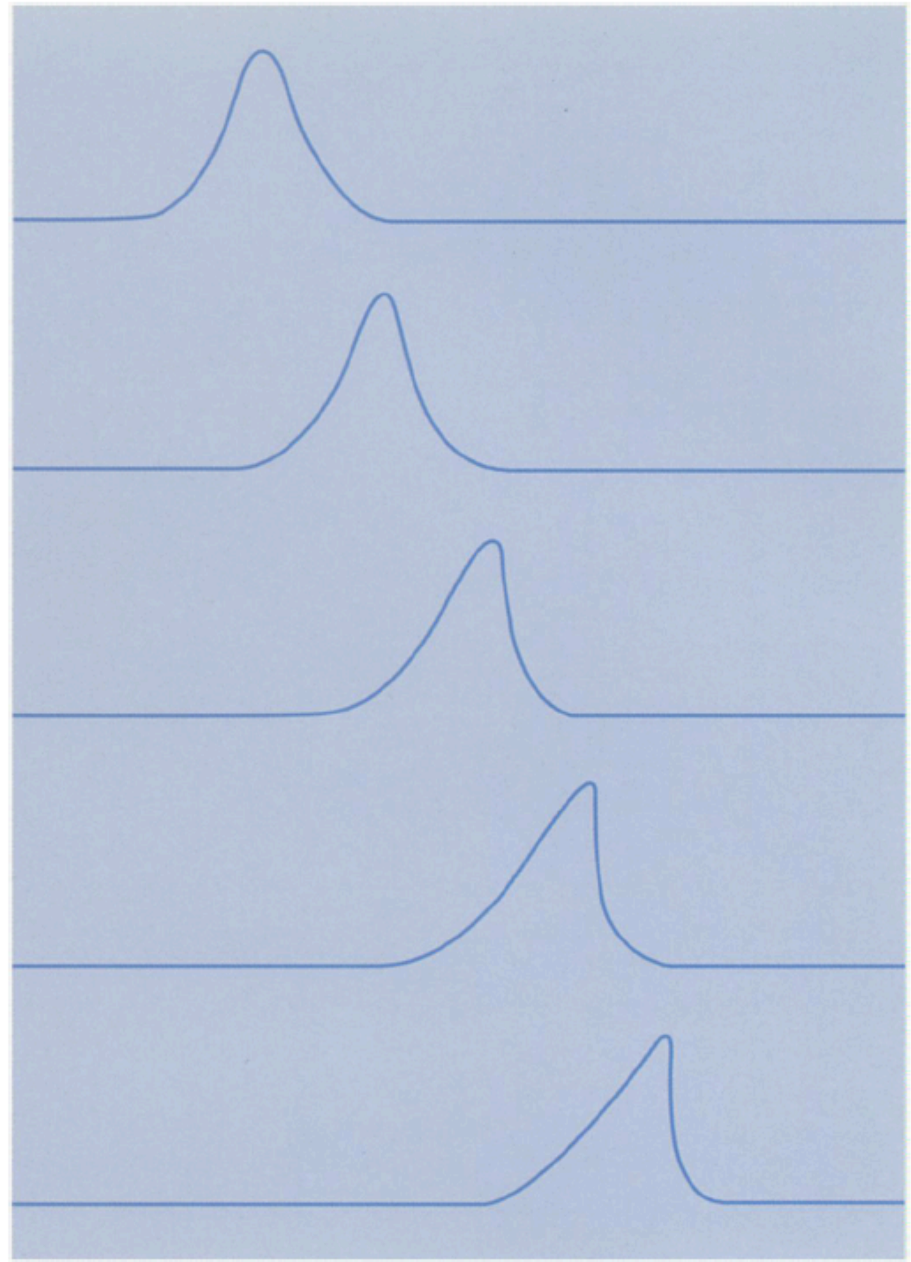
2000
1500
1000
500
0
−3
−2
−1
0
1
2
3



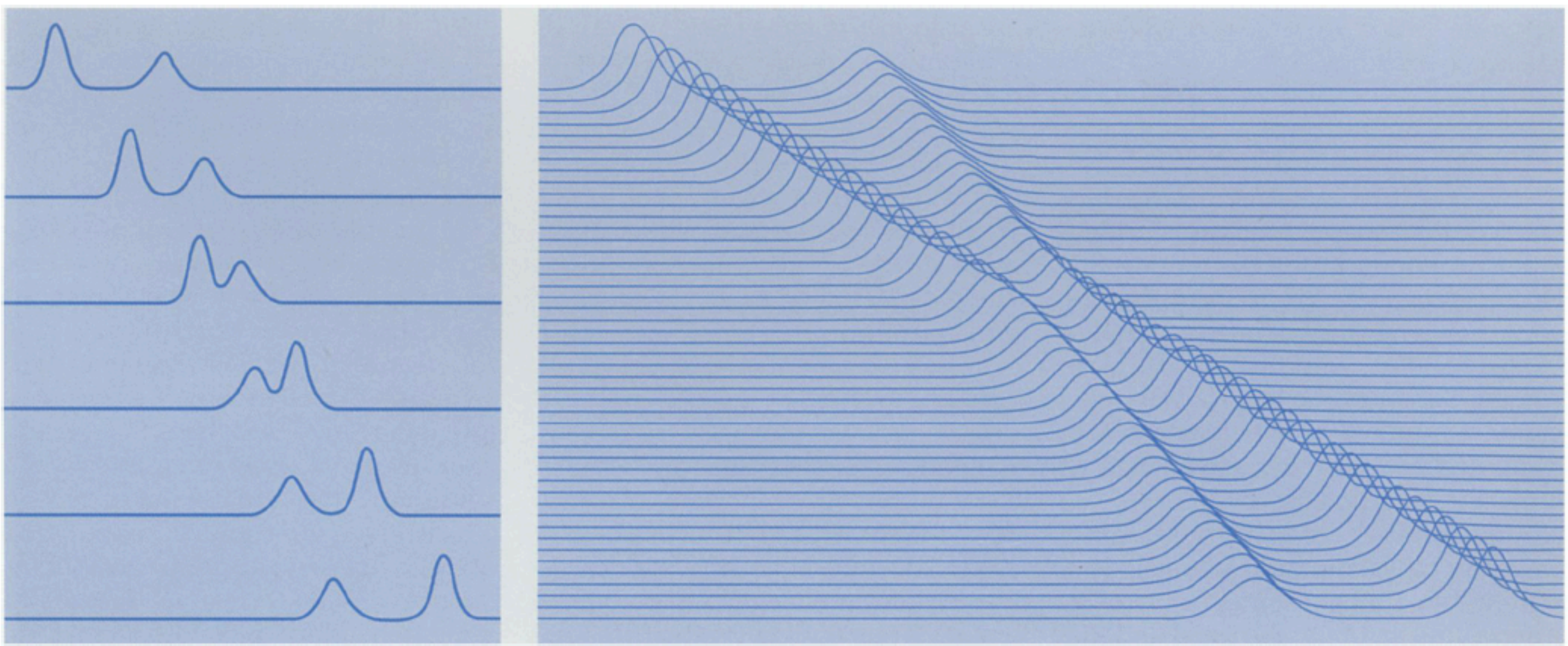
DISPERSION



NONLINEARITY



SOLITONS GO THROUGH EACH OTHER WITHOUT SHAPE CHANGE



I. SOLITON BEARING EQUATIONS

EXACT KINK SOLUTIONS

Consider the Ginzburg-Landau free energy

$$F_{GL}(\phi, \phi') = V(\phi) + \frac{g}{2}\phi'^2,$$

After two integrations we get

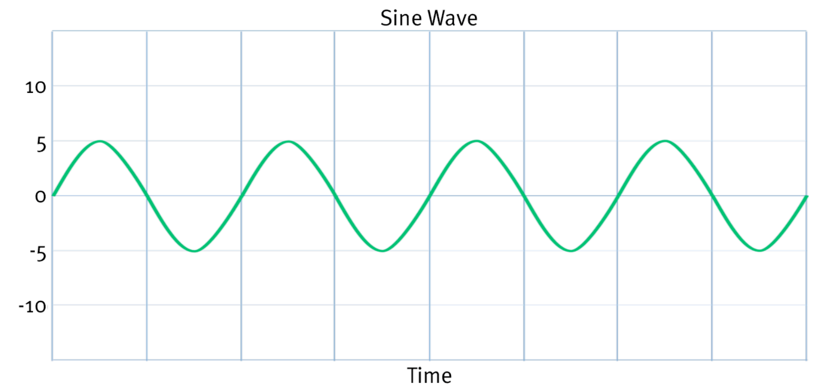
$$\pm\sqrt{\frac{2}{g}}(x - x_0) = \int_{\phi(x_0)}^{\phi(x)} \frac{d\phi}{\sqrt{V(\phi) - V_0}}.$$

QUADRATURE

Sine-Gordon Equation

$$\phi_{tt} - \phi_{xx} + \sin\phi = 0$$

$$V(\phi) = 1 - \cos\phi$$

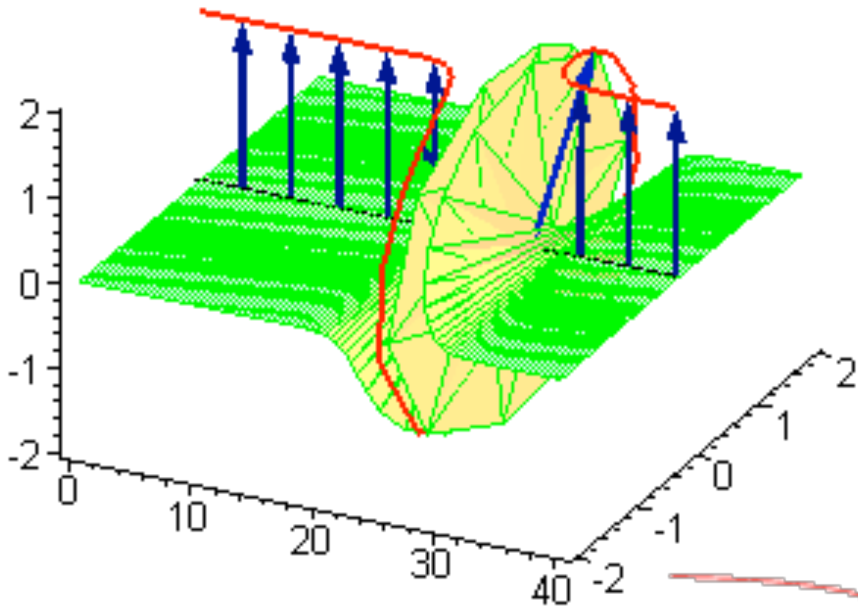


Soliton solution:

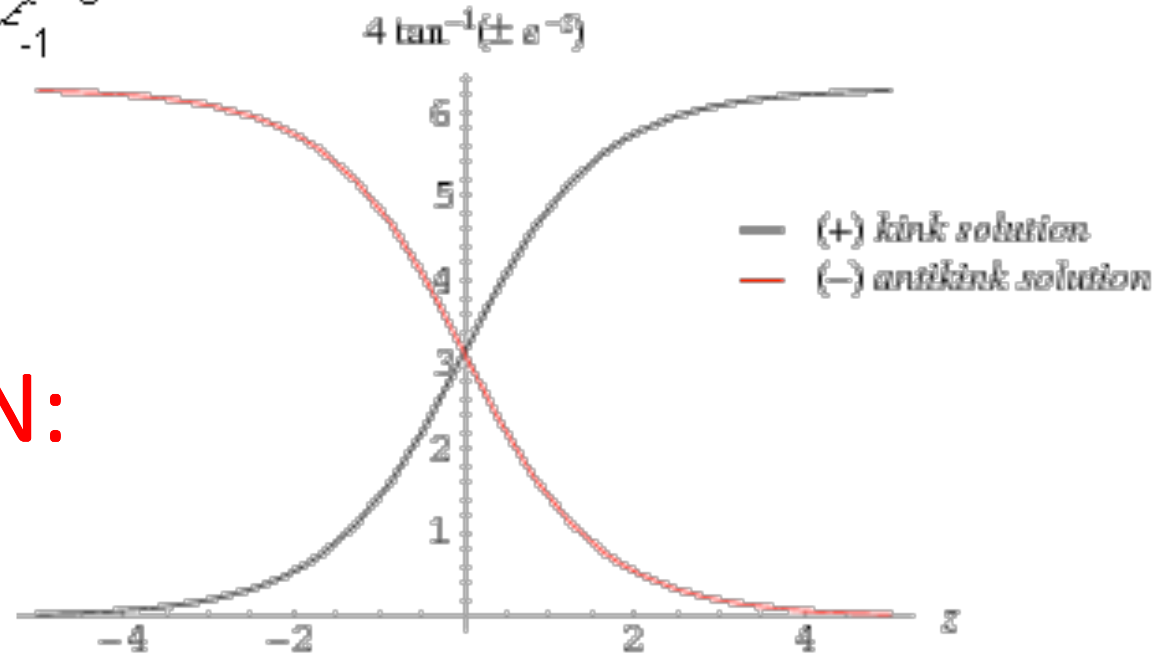
$$\phi(x,t) = 4\,\mathrm{arctan}\,\exp[\gamma(x - vt)]$$

$$\gamma^2 = 1/(1 - v^2)$$

NONLINEAR PENDULUM



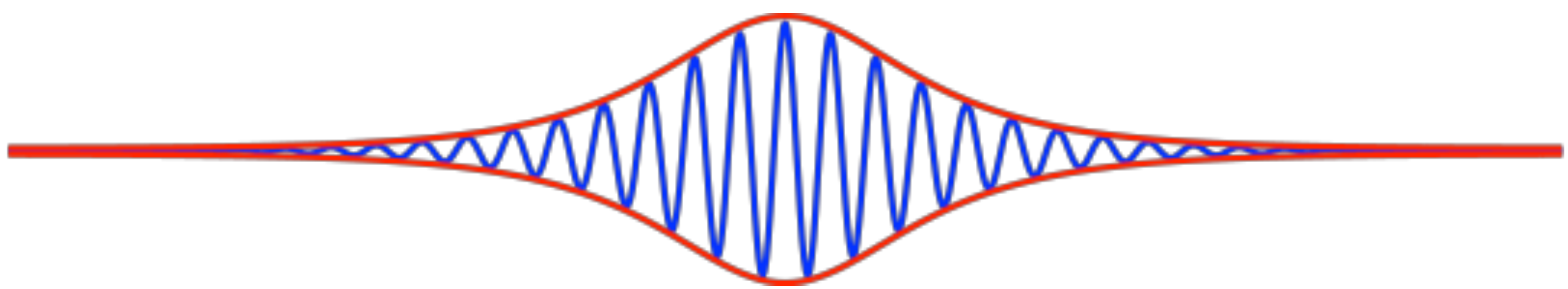
KINK SOLUTION:



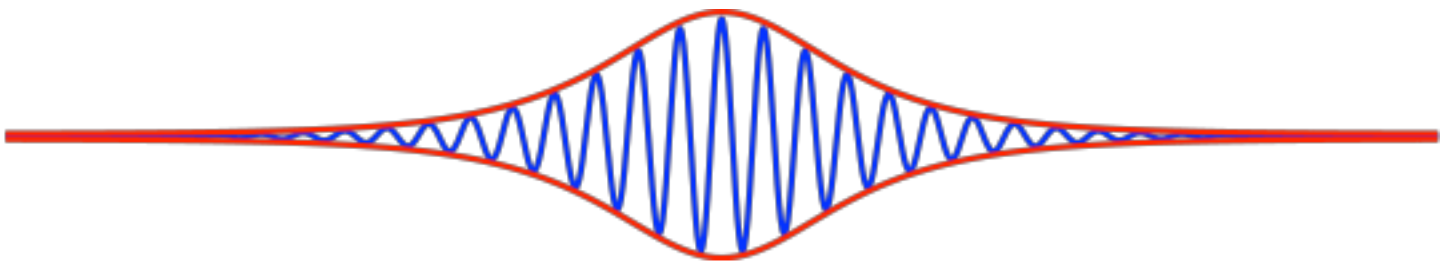
Sine-Gordon breather

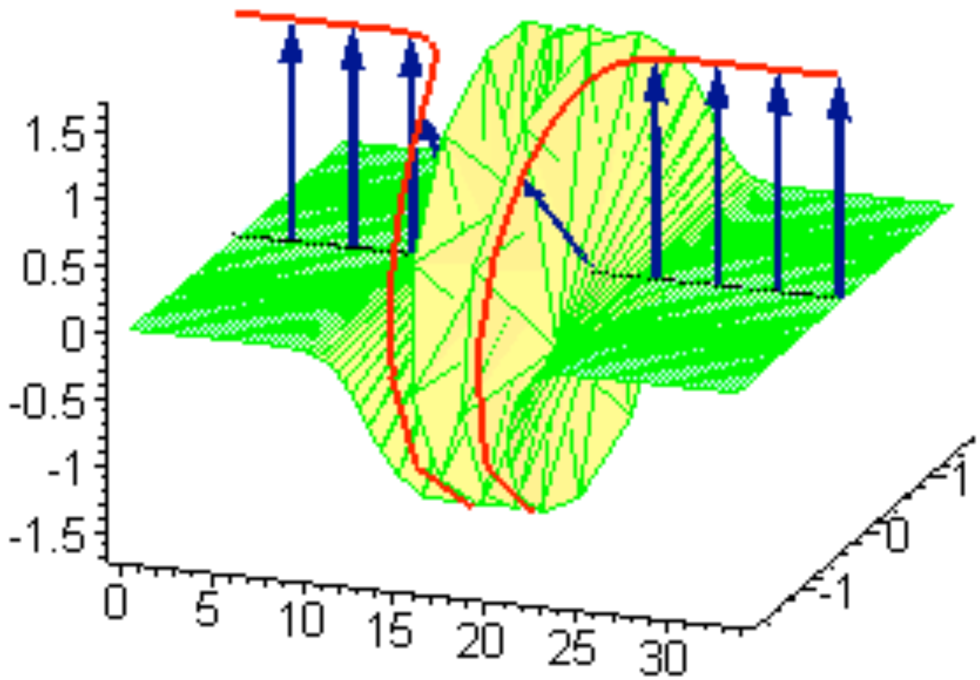
$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin u = 0,$$

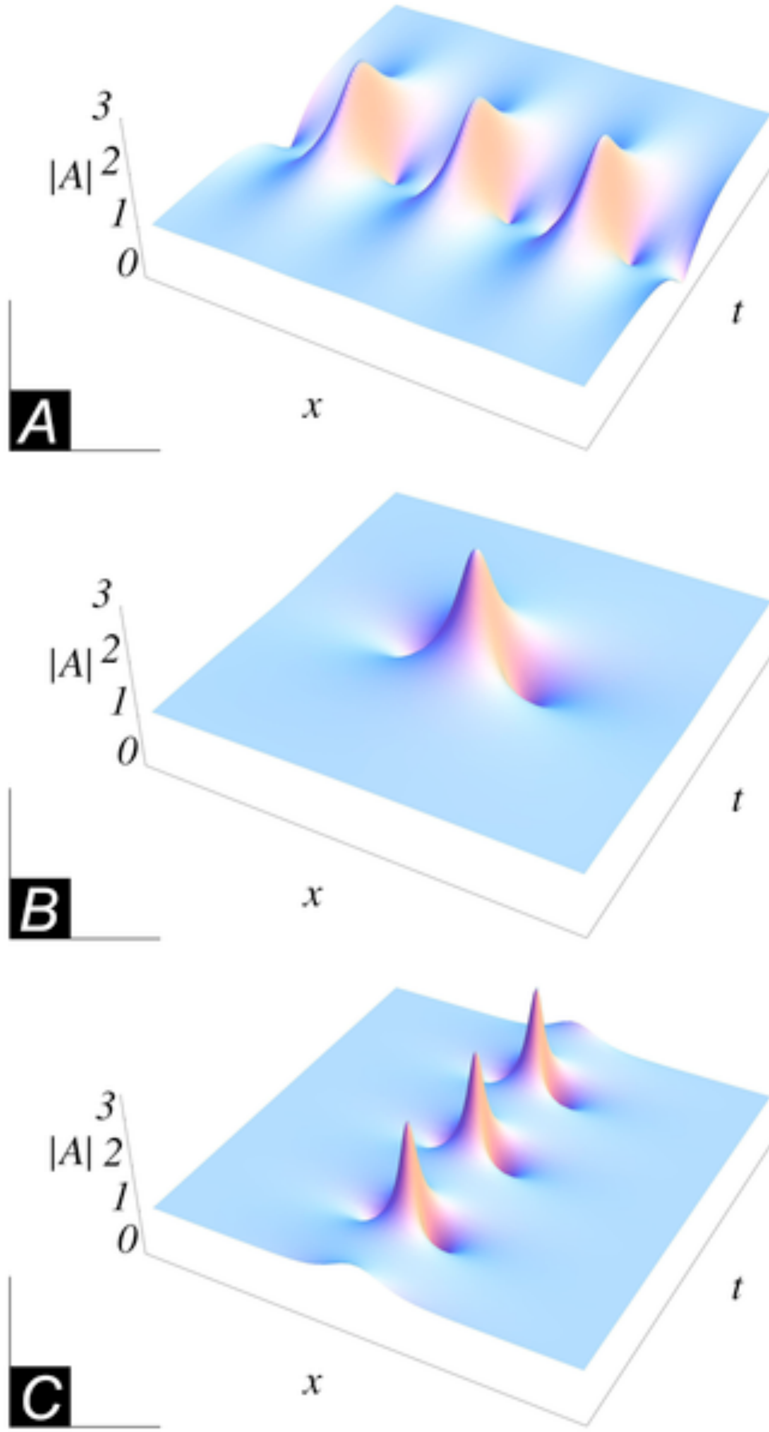
$$u = 4 \arctan\left(\frac{\sqrt{1-\omega^2}\;\cos(\omega t)}{\omega\;\cosh(\sqrt{1-\omega^2}\;x)}\right),$$



BREATHERS, ILMs

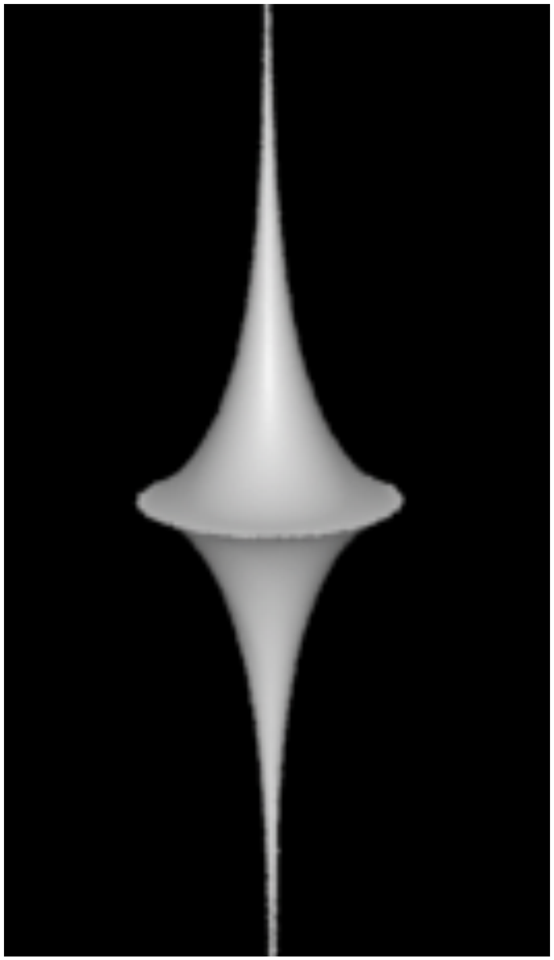
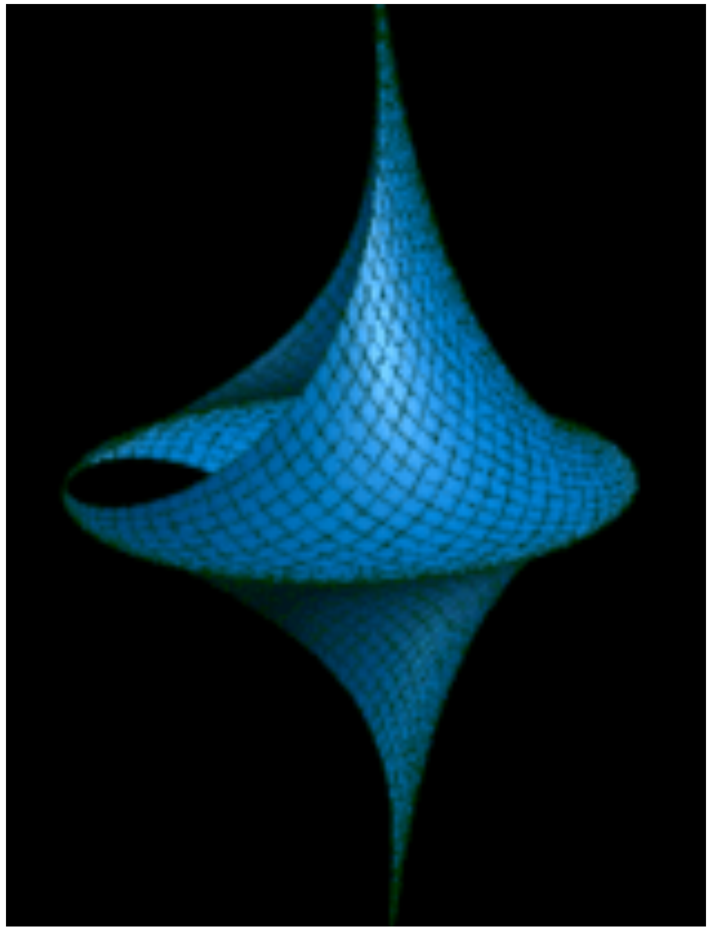
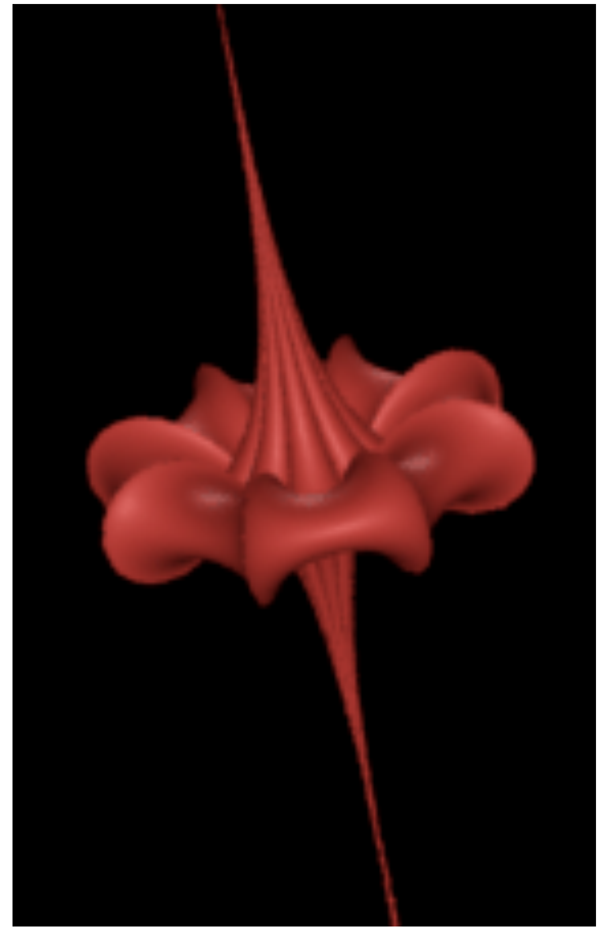






3
|A| 2
1
0
t
x
A
3
|A| 2
1
0
t
x
B
3
|A| 2
1
0
t
x
C

$$\phi_{uv} = \sin\phi$$ -ve Gaussian curvature

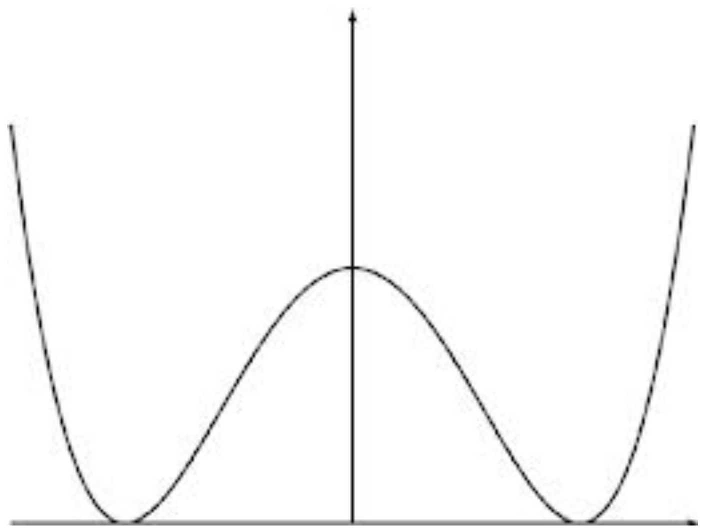




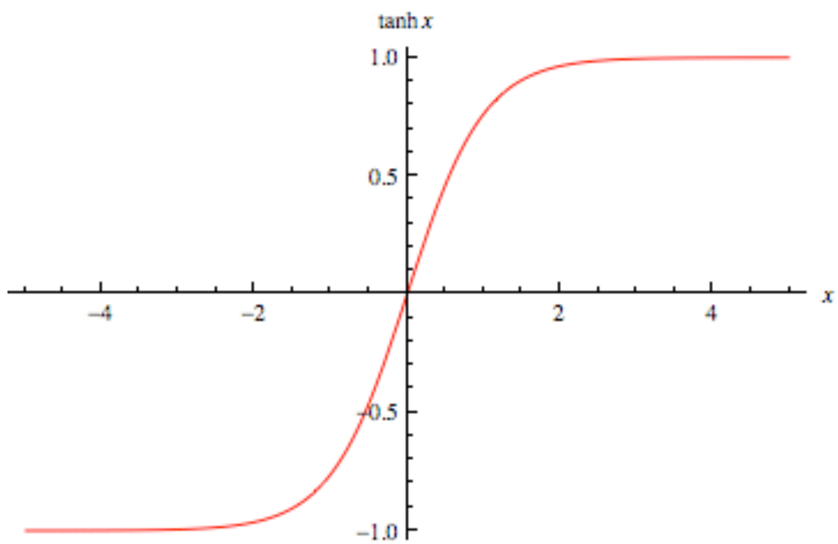
DOUBLE WELL (ϕ^4) POTENTIAL

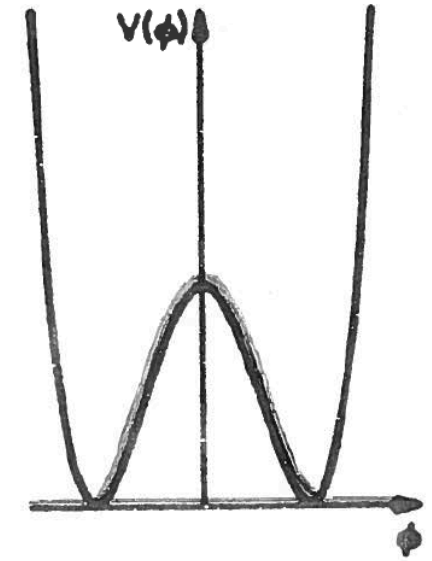
$$V(\phi) = \phi^2(\phi^2 - a^2)$$

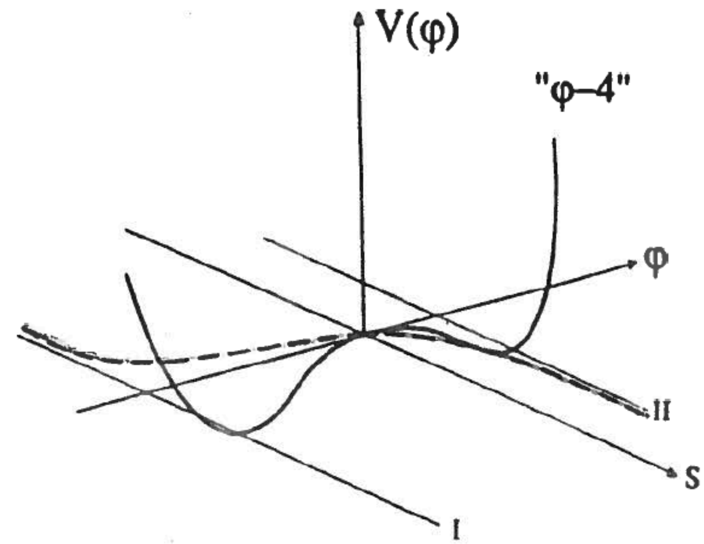
SOLITON (second order transition):

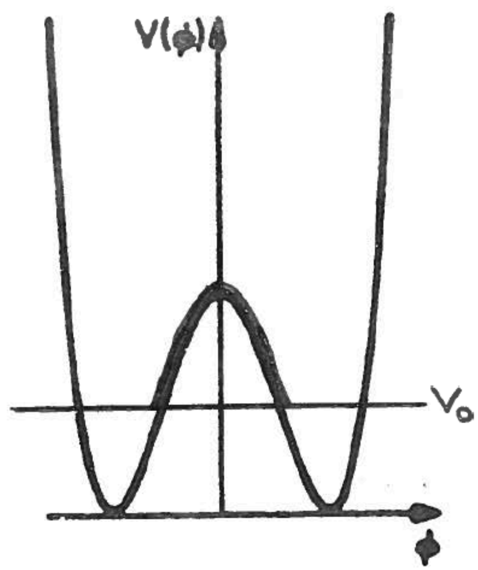
$$\phi(x) = \phi_0 \tanh(x/\xi)$$

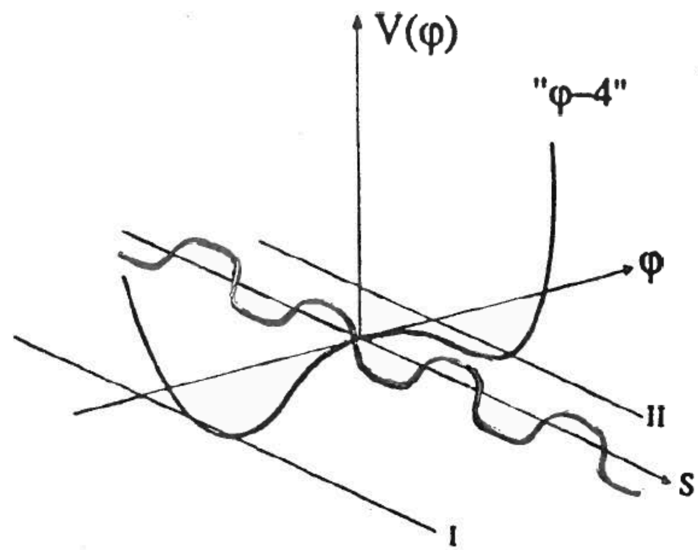





V(ϕ)
ϕ

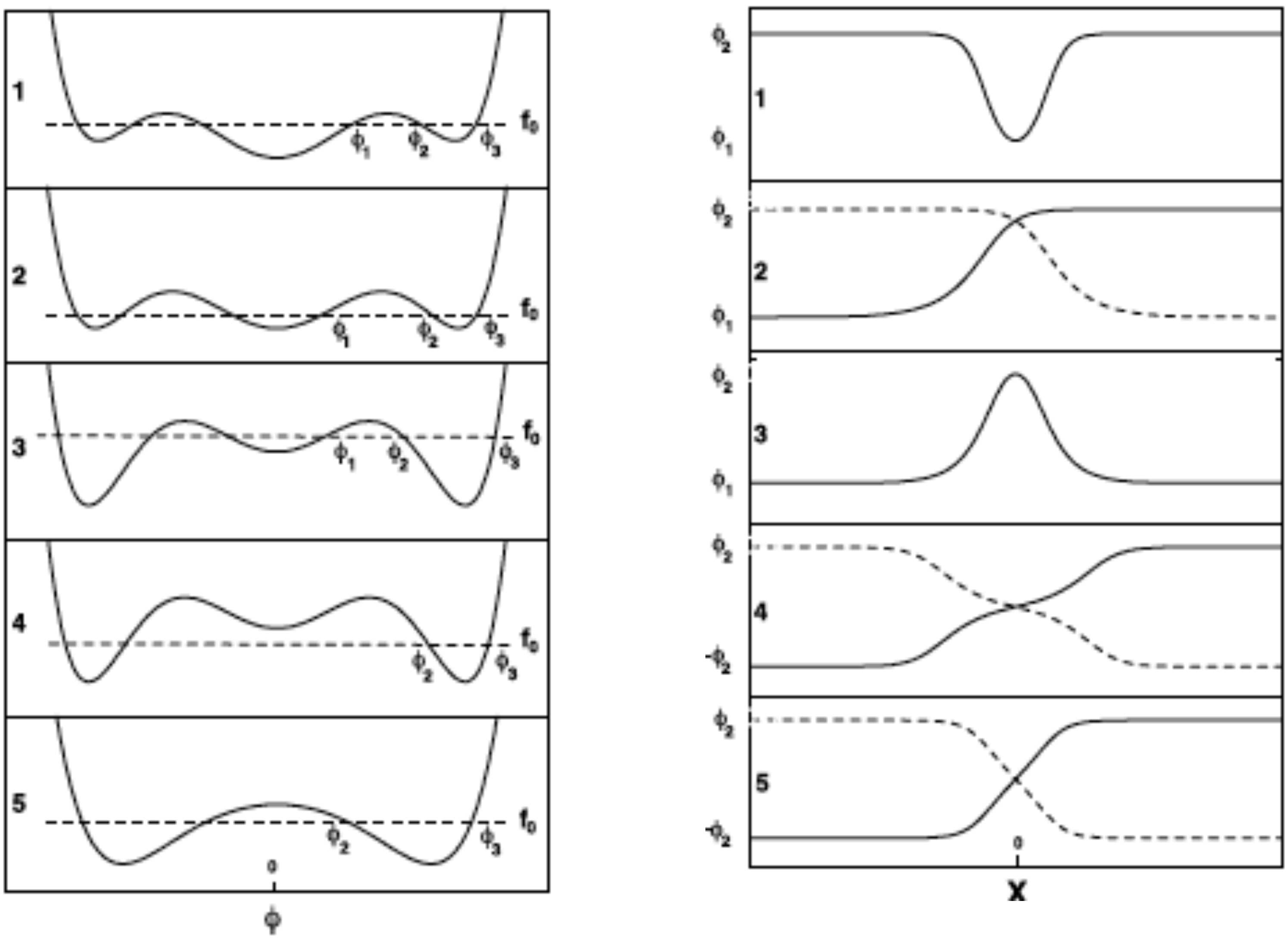

V(φ)
"φ–4"
φ
II
S
I


V(ϕ)
V₀
ϕ


V(φ)
"φ–4"
φ
II
S
I

TRIPLE WELL (ϕ^6) POTENTIAL

$$V(\phi) = a\phi^2 + b\phi^4 + c\phi^6$$



ϕ^6 Model Soliton Solutions

First order transition

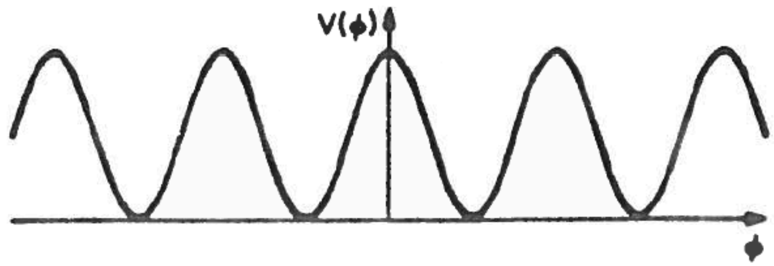
Pulse: $$\phi(x) = \frac{\pm\phi_1}{\sqrt{1-\alpha^2\tanh^2(\frac{x-x_0}{\xi})}}$$

Kink: $$\phi(x) = \frac{\pm\phi_1\gamma\tanh\frac{x-x_0}{\xi}}{\sqrt{1-\gamma^2\tanh^2\frac{x-x_0}{\xi}}}.$$

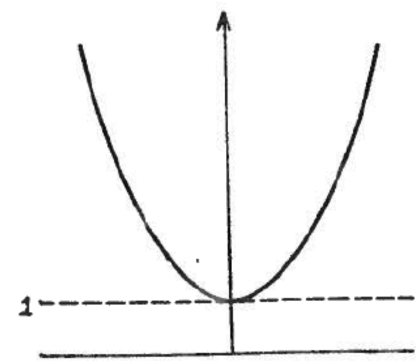
HALF-KINK: $$\phi(x) = \sqrt{3}\left[4+\exp\left(\mp\frac{x-x_0}{2\sqrt{g/3}}\right)\right]^{-1/2}.$$

Pulse: $$\phi(x) = \frac{\pm\phi_2\operatorname{sech}\frac{x-x_0}{\xi}}{\sqrt{1-\beta^2\tanh^2\frac{x-x_0}{\xi}}}$$

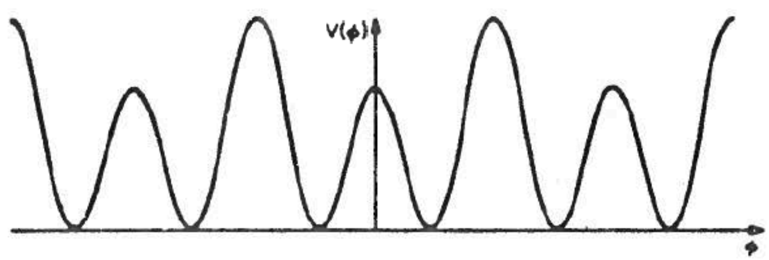
$$V_{SG}(\phi) = \cos\phi.$$



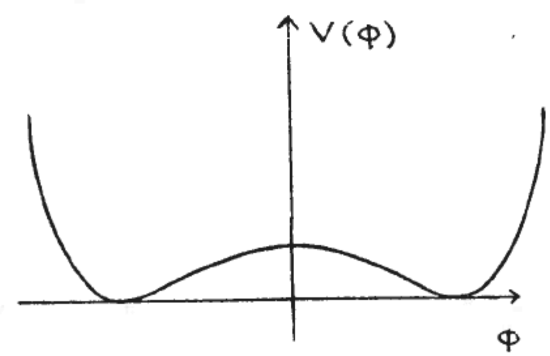
$$V_{SHG}(\phi) = \cosh\phi.$$



$$V_{DSG}(\phi) = \alpha\sin\phi + \beta\sin\frac{\phi}{2}.$$

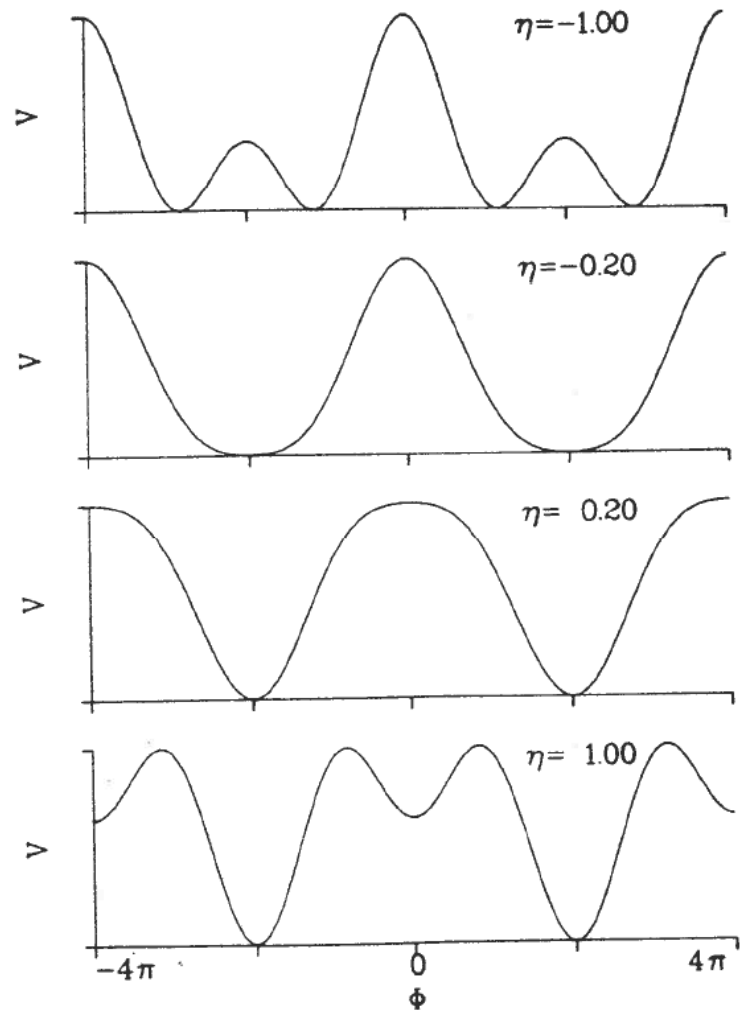


$$V_{DSHG}(\phi) = (\zeta\cosh 2\phi - n)^2$$



DOUBLE SINE-GORDON

$$V_{DSG} = -\frac{4}{1+|4\eta|}\left[-\cos\frac{\phi}{2} + \eta\cos\phi\right].$$



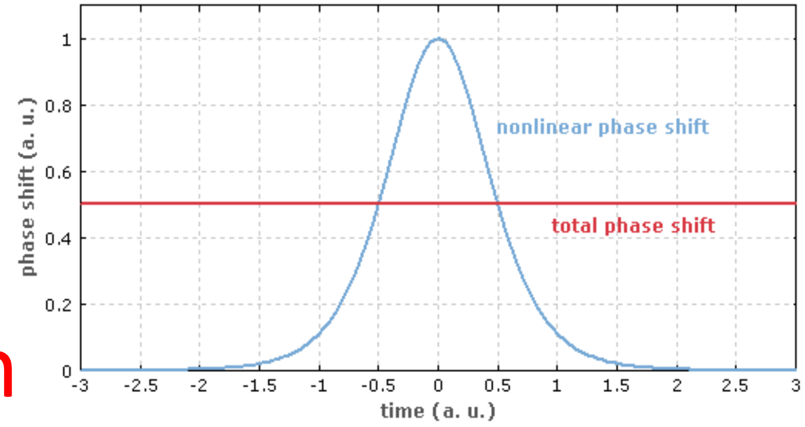
Nonlinear Schrodinger Equation (NLS)

$$i\phi_t + \phi_{xx} - \kappa|\phi|^2\phi = 0$$

Envelope Soliton:

$$\phi(x,t) = \phi_0 sech((x - vt)/\xi)$$

Nontopological soliton (pulse)



Nonlinear Dirac Equation

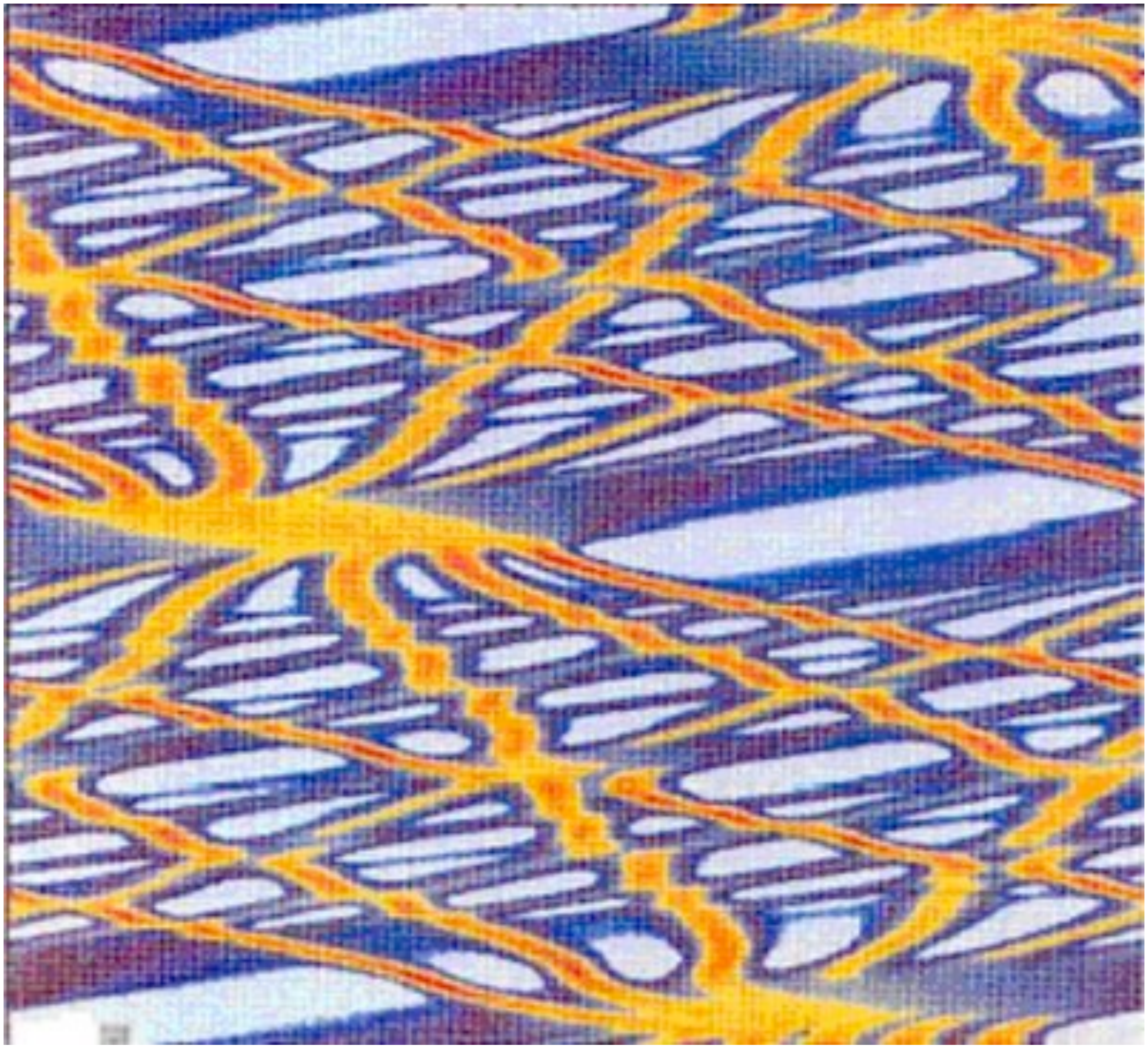
Korteweg-de Vries (KdV) Equation

$$\phi_t + \phi_x^3 + 6\phi\phi_x = 0$$

Soliton:

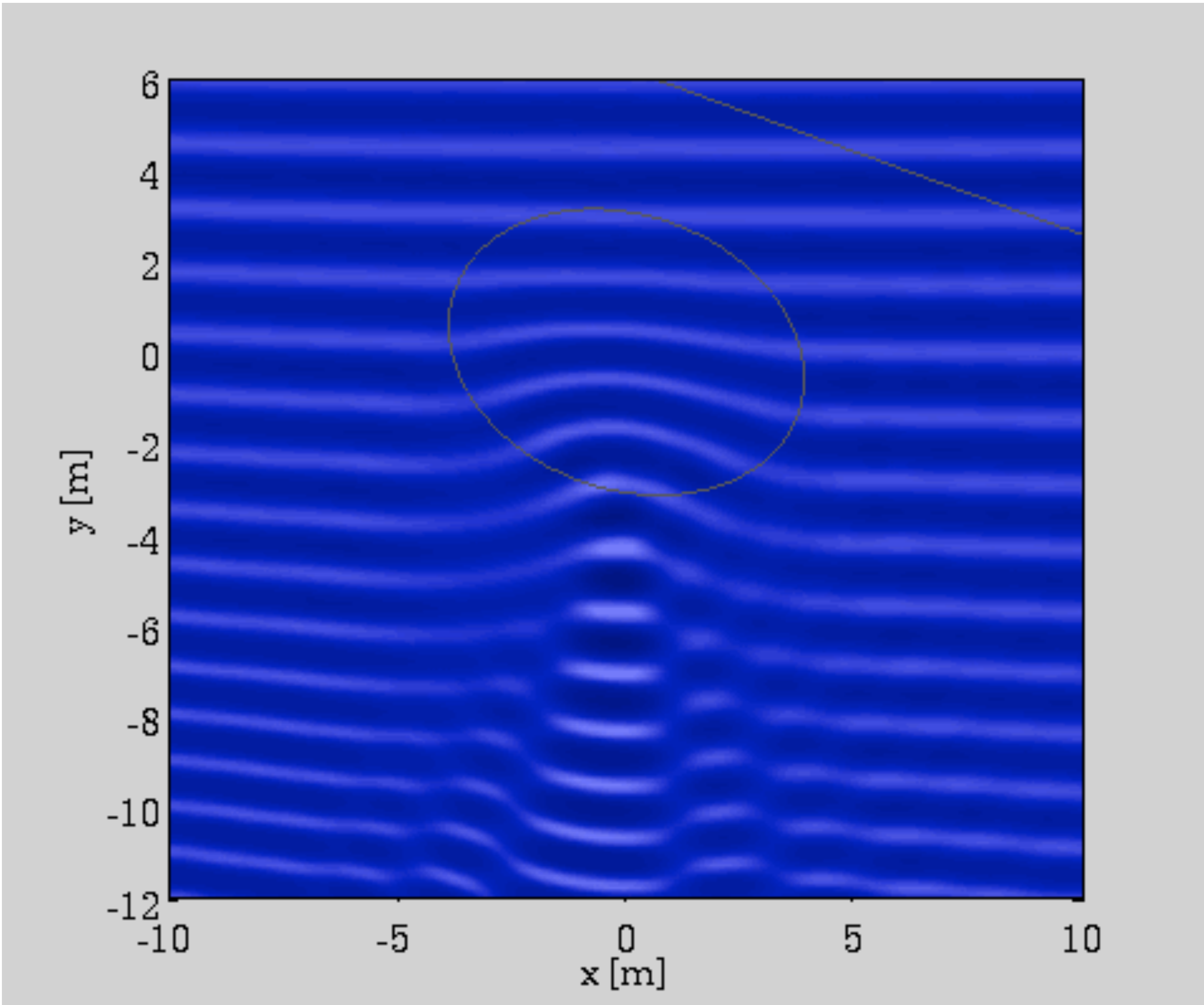
$$\phi(x,t) = \frac{c}{2} sech^2 \left[\frac{\sqrt{c}}{2}(x - ct - a) \right]$$

KdV solitons: near-recurrence



BOUSSINESQ EQUATION

$$\phi_{tt} - \phi_{xx} - \phi_{xxxx} + 3(\phi^2)_{xx} = 0$$

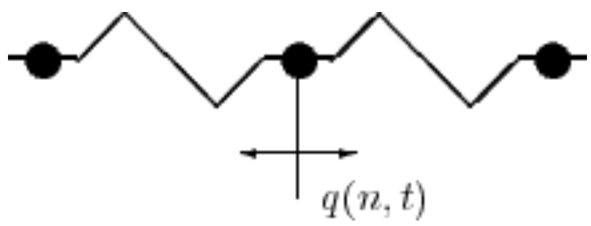


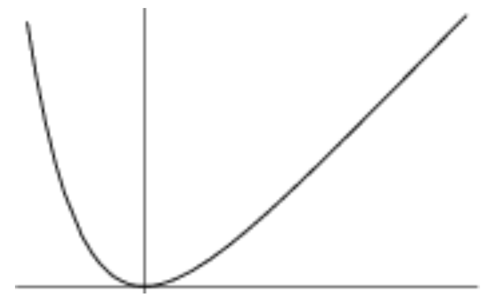
DISCRETE SOLITON EQUATIONS

- Discrete sine-Gordon (Frenkel-Kontorova)
- Discrete nonlinear Schrodinger (DNLS)
- Discrete ϕ^4, ϕ^6 models.

- Peierls-Nabarro barrier
- Exceptional discretization
- Conserved quantities; ILMs.

The Toda Lattice: Discrete Solitons





$$p_t(n,t) = \exp -[q(n,t) - q(n-1,t)] - \exp -[q(n+1,t) - q(n,t)]$$

$$q_t(n,t) = p(n,t)$$

SOLITON:

$$q_1(n,t) = q_+ + \log\left(\frac{1 + \frac{\gamma}{1-\mathrm{e}^{-2\kappa}}\exp(-2\kappa n + 2\sigma\sinh(\kappa)t)}{1 + \frac{\gamma}{1-\mathrm{e}^{-2\kappa}}\exp(-2\kappa(n+1) + 2\sigma\sinh(\kappa)t)}\right),$$

II. NONLINEAR DIRAC EQUATION (NLD)

NONLINEAR DIRAC EQUATION: SOLITONS:

Introduction

Not much work on Soliton solutions in fermionic theories

Exceptions

Gross-Neuve and Massive Thirring models (1+1)

$$\mathcal{L} = \sum_{j=1}^{n} (\bar{\psi}_j \psi_j)^2$$

$$\mathcal{L} = \sum_{j=1}^{n} (\bar{\psi}_j \gamma_\mu \psi_j \bar{\psi}_j \gamma^\mu \psi_j)$$

Our Work

1. Generalization of these two models to arbitrary nonlinearity

$$\mathcal{L} = (\bar{\psi}\psi)^{\kappa+1}$$

$$\mathcal{L} = (\bar{\psi}\gamma_\mu\psi\bar{\psi}\gamma^\mu\psi)^{(\kappa+1)/2}$$

   Ref. Cooper, Mihaila, Saxena and AK: **PRE 82** (2010) 036604
2. Stability of SS Solitons
   Ref. Shao, Quintero, Mertens, Cooper, Saxena and AK, **PRE 90** (2014) 032915
3. Behaviour of Solitons in the Presence of Forcing Term and Damping
   Ref. Mertens, Quintero, Cooper, Saxena and AK, **PRE 86** (2012) 046602
4. Solitary Waves in PT-symmetric Nonlinear Dirac Equation
   Ref. Cuevas-Maraver, Kevrekidis, Saxena, Cooper, Comech, Bender and AK, arXiv:1508.00852

Nonlinear Dirac Equation:

Plan of the Talk

1. Solitary Wave Solutions for arbitrary κ

(a) SS case (b) VV case

2. Nonrelativistic Limit: Generalized NLSE $+O(1/2m)$

3. Various Stability Criteria (SS)

a. Derrick's Theorem

b. Vakhitov-Kolokokov (VK) Criterion

c. Bogolubsky Criterion: Stability to Changes in frequency at Fixed Charge

4. Numerical Simulation Results

5. PT-Symmetric NLD with SS Interaction

6. Open Questions

The Solitary Wave Solutions

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \mathcal{L}_{\mathcal{I}}$$

SS case:

$$\mathcal{L}_I = \frac{g^2}{\kappa + 1}(\bar{\psi}\psi)^{\kappa+1}$$

VV case:

$$\mathcal{L}_I = \frac{g^2}{\kappa + 1}(\bar{\psi}\gamma_\mu \psi \bar{\psi}\gamma^\mu \psi)^{(\kappa+1)/2}$$

$$D(\psi) = 1/2$$

g dimensionless if $\kappa = 1$

nonrenormalizable if $\kappa > 1$

super renormalizable if $\kappa < 1$

The Solitary Wave Solutions

We treat as effective field theory and ignore the question of renormalizability

We look for stationary solutions of the form

$$\psi(x,t) = e^{-i\omega t}\psi(x)$$

$$\psi(x) = \begin{pmatrix} u \\ v \end{pmatrix}$$

choose $\gamma^0 = \sigma_3\,, \quad \gamma^1 = i\sigma_2$

$$\frac{du}{dx} + (m+\omega)v - g^2(|u|^2 - |v|^2)^\kappa v = 0$$

$$\frac{dv}{dx} + (m-\omega)u - g^2(|u|^2 - |v|^2)^\kappa u = 0$$

The Solitary Wave Solutions

Look for bound states, i.e. $0 < \omega < m$

$\psi(x) - - > 0 \quad \text{as } x \to \pm\infty$

$$\partial_\mu T^{\mu\nu} = 0$$

uniquely leads (in both SS and VV cases)

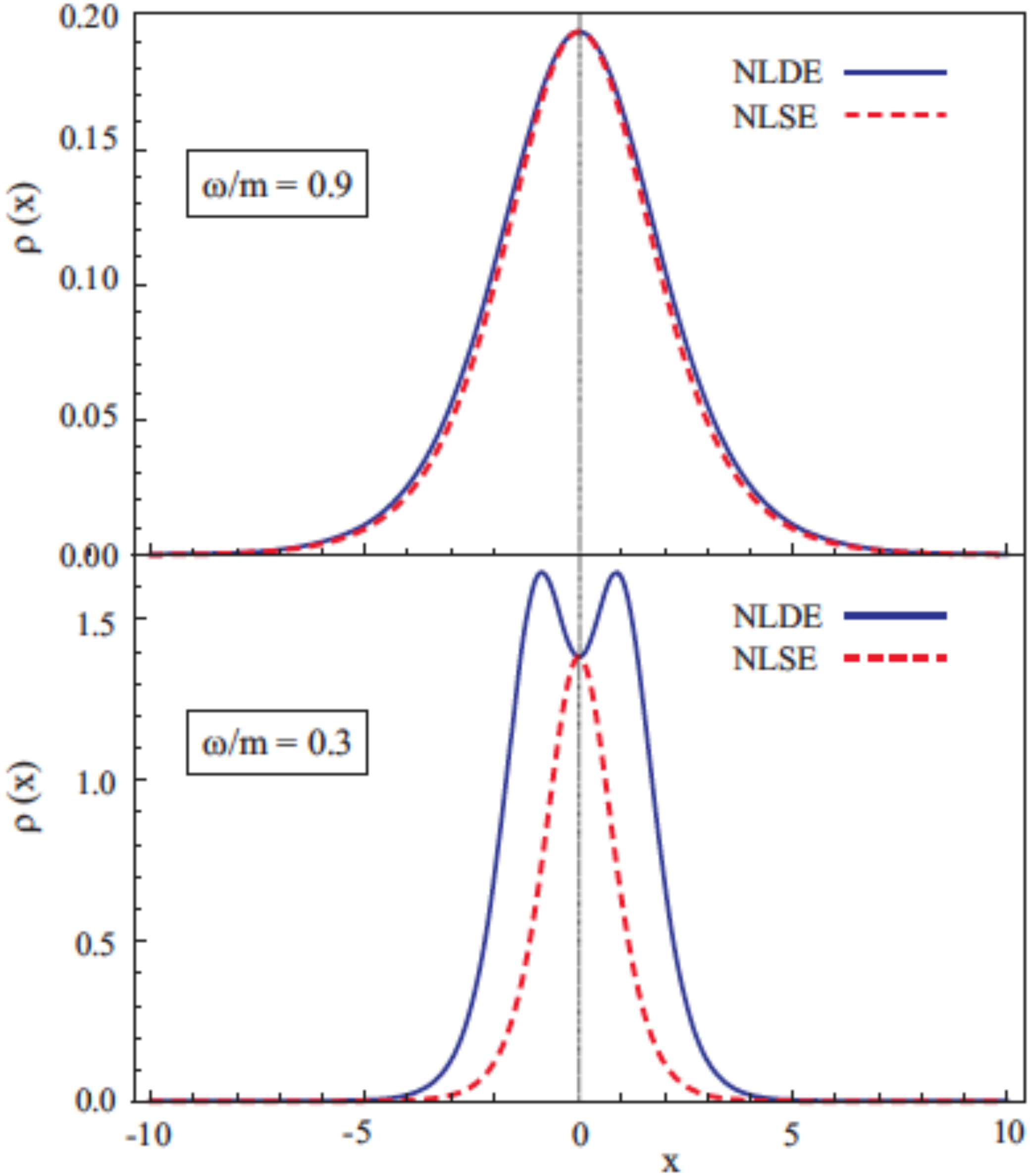
$$\tan\theta(x) = \alpha \tanh(\kappa\beta x)$$

$$u = R\cos\theta\,, \quad v = R\sin\theta$$

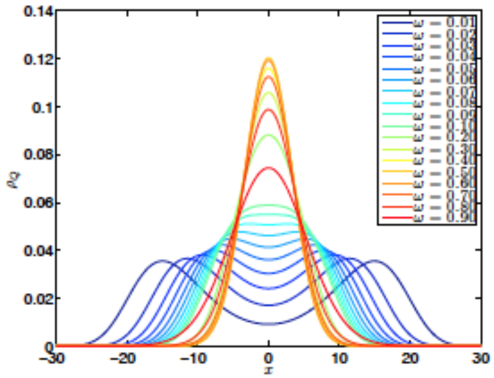
$$\alpha = \sqrt{\frac{m-\omega}{m+\omega}}\,, \quad \beta = \sqrt{m^2-\omega^2}$$

$$R_{SS}^2 = \left[\frac{\omega + m \cosh(2\kappa\beta x)}{m + \omega \cosh(2\kappa\beta x)}\right] \left[\frac{(1+\kappa)\beta^2}{g^2(m + \omega \cosh(2\kappa\beta x)}\right]^{1/\kappa}$$

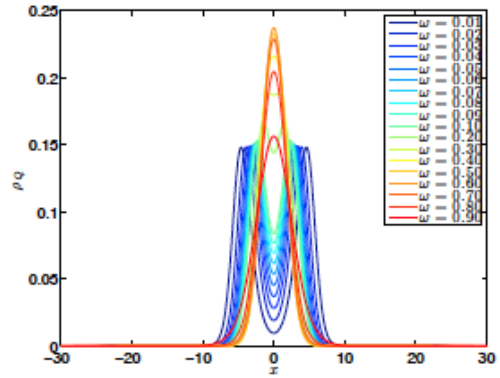
$$R_{VV}^2 = \left[\frac{(1+\kappa)\beta^2}{g^2(\omega + m \cosh(2\kappa\beta x)}\right]^{1/\kappa}$$



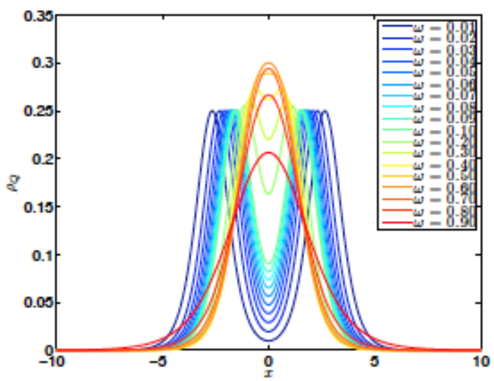
0.20
0.15
0.10
0.05
0.00
ρ (x)
NLDE
NLSE
ω/m = 0.9
1.5
1.0
0.5
0.0
ρ (x)
NLDE
NLSE
ω/m = 0.3
-10
-5
0
5
10
x



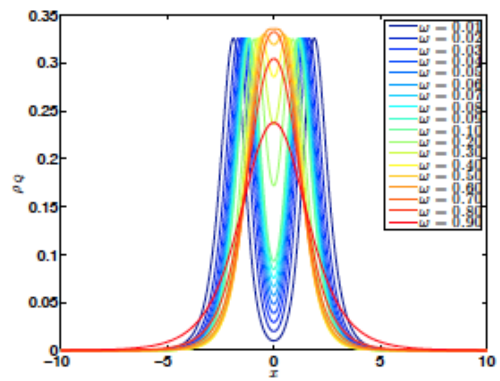
(a)$\kappa = 0.1$



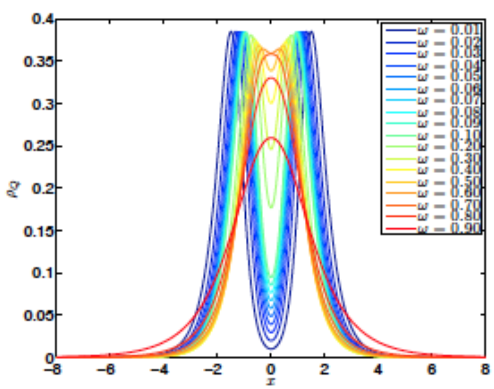
(b)$\kappa = 0.5$



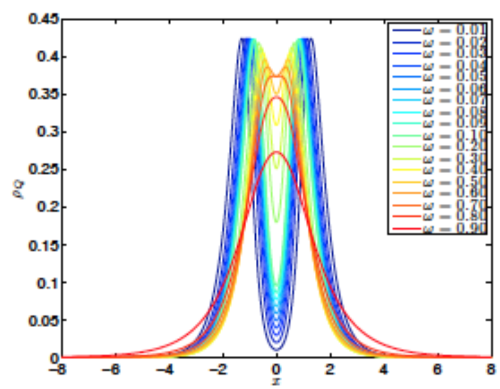
(c)$\kappa = 1.0$



(d)$\kappa = 1.5$



(e)$\kappa = 2.0$



(f)$\kappa = 2.4$

Properties of Solutions

1. In SS case, bound states exist for all values of κ and $g > g_{min}$
2. IN VV case though bound states exist only if $\kappa \leq 2.5$
3. Two conserved quantities Q, H

$$Q = \int_{-\infty}^{+\infty} dx\, \psi^{+}\psi = \int_{-\infty}^{\infty} dx\, R^2$$

$$H = \int_{-\infty}^{\infty} dx\, h(x)$$

$$h = h_1 + h_2 + h_3$$

$$h_1 = -i\bar{\psi}\gamma^1\partial_1\psi$$

$$h_2 = m\bar{\psi}\psi$$

$$h_3 = -L_I$$

For the stationary solutions one can show that $h_3 = \frac{h_1}{\kappa}$

Nonrelativistic Reduction

gNLDE can be written as

$$i\sigma_3\partial_t\psi + i\sigma_2\partial_1\psi - m\psi - V_I\psi = 0$$

$$V_I = -\frac{\partial L_I}{\partial\bar{\psi}} = -g^2(\bar{\psi}\psi)^\kappa$$

Moore's Decoupling Method

$$V_I(\lambda) = (1+\sigma_3)/2V_I + \lambda(1-\sigma_3)/2V_I$$

Perturbation theory in λ

$$gNLDE \to gNLSE \mp O(\frac{m-\omega}{2m})$$

Stability of NLD Solitons

We apply same 3 criteria which work for gNLSE

$$i\psi_t + \psi_{xx} + g^2|\psi|^{2\kappa}\psi = 0$$

Derrick's Theorem

$x \rightarrow \lambda x$

$M = \int_{\infty}^{\infty} |\psi|^2\psi \, dx$ Preserved

$$\psi(x) \rightarrow (\lambda)^{1/2}\psi(\lambda x)$$

gNLSE soliton stable (unstable) if $\kappa < (>)\, 2$

$\kappa = 2$ Marginal case

Blow up if $M > M_c = 2.72$ ...Simulation Result

Our result using exact solution $\mathrm{sech}^{1/\kappa}(x)$: $M_c = 2.7207$

Stability of NLD Solitons

Vakhitov-Kolokolov Criterion

If $\frac{dM}{d\omega} < 0$...Soliton is stable

For gNLSE we get the same result as above

Bogolubsky (PLA 73 (1979) 87) Criterion

If energy increased (decreased) as ω is varied for fixed M

then soliton is stable (unstable)

Again *same* result as above for gNLSE

Thus for gNLSE all three criteria predict

gNLSE soliton stable (unstable) if $\kappa < (>)\, 2$

Stability of NLD Solitons

For NLD, 3 criteria give totally different answers

1. **Derrick Theorem**

   $0 < \kappa < 1$ ($\kappa > 1$) ...soliton stable (unstable) for any ω

2. **Vakhitov-Kolokolov Criterion**

   $0 < \kappa < 2$ Soliton Stable for any ω

   $\kappa > 2$ Soliton unstable except if $\omega < \omega_c$

3. **Bogolubsky Criterion**

   $0 < \kappa < 2$ soliton stable if $\omega > \omega_b = 0.7$

   $\kappa > 2$ soliton is unstable but for a tiny region of ω

Stability of NLD Solitons

Why 3 approaches give different results?

Perhaps because in NLD there is no lower bound on H

Put another way, for NLD, one needs to prove that the stable solutions of Dirac eq. are not merely stationary sols. of variational principle but are actually minimum of H

Results of Comech et al (Ann. Inst. Henri Poincare 31 (2014) 639)

For ω close to m, NLD soliton stable for $0 < \kappa < 2$

Numerical Simulation Results

Using Fourth Order Operator splitting Integration method (Sihong Shao)

$0 < \kappa < 2$ soliton stable if $\omega_c < \omega < m$

$\kappa = 0.1, \quad \omega_c = 0.35$

$\kappa = 0.75, \quad \omega_c = 0.53$

$\kappa = 1.75, \quad \omega_c = 0.89$

For $\kappa > 2$ soliton unstable

PT-symmetric NLD With SS Interaction

PT-symmetry playing Important role not only in QM but also in Optics

Eqs. for PT-invariant NLD with SS interaction are

$$\frac{du}{dx} + (m+\omega)v - g^2(|u|^2 - |v|^2)^\kappa v - i\gamma u = 0$$

$$\frac{dv}{dx} + (m-\omega)u - g^2(|u|^2 - |v|^2)^\kappa u + i\gamma v = 0$$

P: $x \to -x\,, \quad u \to u\,, \quad v \to -v$

T: $t \to -t\,, \quad i \to -i$

PT-symmetric NLD With SS Interaction

Exact Solutions at $m = 0$ and $\kappa = 1$

1. $$u = \frac{1}{2\sqrt{g|\omega|}}[|\omega| + \beta \tanh \beta x - i\gamma]$$
$$v = \frac{1}{2\sqrt{g|\omega|}}[|\omega| - \beta \tanh \beta x - i\gamma]$$

2. $$u = \frac{1}{2\sqrt{g|\omega|}}[\beta - |\omega| \tanh \beta x + i\gamma \tanh \beta x]$$
$$v = \frac{1}{2\sqrt{g|\omega|}}[-\beta - |\omega| \tanh \beta x + i\gamma \tanh \beta x]$$

Not Localized Solitary Wave Solutions

Solutions Break PT-symmetry

Open Problems

1. One criticism of our work is that perhaps the discretization is giving spurious instability for $\kappa < 2\,,\ \omega < \omega_c$ and may be VK criterion is correct, i.e. soliton is stable for $0 < \kappa < 2$ for any ω
2. Stability of VV NLD solitons

Conclusions

- Solitons: balance of dispersion with nonlinearity.
- Breathers: dynamical localization (ILMs).
- Integrability: inverse scattering and conserved quantities.
- NLD systems, solutions and equations.
- Stability of NLD solitons nontrivial.
- External force driving simulaions.