Renormalization Group Theory for Fluid and Plasma Turbulence

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Abstract

For the last several decades, renormalization group (RG) methods have been applied to a wide variety of problems of turbulence in hydrodynamics and plasma physics. A comprehensive review of this work will be presented, covering RG methods in hydrodynamic turbulence and in turbulent systems with coupled fluctuating fields like magnetohydrodynamic (MHD) turbulence. This review will attempt to specifically consider several questions about RG: (1) Does RG provide an improvement over previous analytical theories like the direct interaction approximation, or is RG a useful simplification of those theories? (2) How are nonlocal, or 'sweeping' effects treated in RG formalisms, or are they ignored entirely? (3) Can RG theories treat both local and nonlocal interactions in turbulence?

Key words: Keywords: turbulent flows,, Magnetodydrodynamic (MHD) turbulence, plasma, renormalization group theories, renormalized perturbation theories

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1 Overview

High Reynolds number turbulent flow has found many important applications in science and engineering (Tennekes & Lumley, 1972). The magnetized counterpart is also found in many astrophysical and space plasma applications (Biskamp, 2003; Goldstein *et al.*, 1995; Elmegreen & Scalo, 2004). The challenge in computing such flows can be traced to strong nonlinearities which excite motion on spatial scales that span a vast range of magnitudes (Batchelor, 1953; Monin & Yaglom, 1975; Rose & Sulem, 1978; Frisch, 1995; Pope, 2000; Zhou & Speziale, 1998; Zhou *et al.*, 2004).

This paper will provide a comprehensive, up to date review of renormalization group (RG) methods [1] in fluid and MHD turbulence. These methods raised a number of fundamental problems and controversies, some of which have never perhaps been entirely resolved. One of the underly problems is that we seek to apply RG to systems far from equilibrium.

Although RG remains an active area of research with several papers published every year, many years have elapsed since the Annual Review article by Smith & Woodruff (1998), which remains the only review article on this subject. Moreover, because that article emphasized the Yakhot & Orszag (1986, hereafter YO) ϵ expansion method (ϵ-RG), it did not provide coverage of all available RG methods in hydrodynamic turbulence, and it did not discuss attempts to apply RG to plasma turbulence [2]. In McComb's (1990) book, a summary of previous Forster, Nelson, and Stephen (1976, 1977, hereafter FNS) and De Dominicis & Martin (1979, hereafter DeDM) RG work is immediately followed by an extensive coverage of his own iterative averaging RG (hereafter, i-RG) modeling. It also did not review much work that had already been published when it appeared. In addition, the review paper of McComb (1995) only discussed RG briefly. The book by Adzhemyan et al. (1999) is a translation from Russian and is a resource for its authors' previous publications.

In the case of RG applications to plasma turbulence in particular, the only available review appears in two general review articles (Krommes, 1997, 2002), where RG is only considered very briefly.

The current review will attempt to be more comprehensive, survey more recent

[1] There are many books or review articles on aspects of the RG methods for other scientific applications, among them, see for example, Arnit, 1987; Binney et al., 1993; Fisher, 1998; Hu, 1983; Kadanoff, 1977; Ma & Mazenko, 1975.

[2] It only included a simple scalar model problem (Avellaneda & Majda, 1990, 1992a,b, 1994). Majda & Kramer (1999) provided a detailed review of the work by Majda and co-workers; see also the work by Nayak (1993) for a RG analysis of turbulent transport.

work, and cover the application of RG to a broader range of topics.

This review of RG is organized around three major themes: the relation between RG theories and previous analytical closure theories, the treatment of sweeping and straining effects in RG theories, and the treatment of local and distant interactions in RG theories. To begin, we will describe each of these themes in some more detail.

(1) RG has much in common with renormalized perturbation theories (RPT) like the direct interaction approximation (DIA, Kraichnan, 1959) and its Lagrangian versions (the Lagrangian-history DIA (LHDIA) of Kraichnan (1965a)) and the Lagrangian renormalized approximation (LRA) of Kaneda (1981, 2007; see also Gotoh *et al.*, 1988) [3]. For example, both RG and RPT use the one-loop approximation [4]. Our first major theme is the relation between RG and RPT theories. The broad RG program at first glance suggested that RG offers a significant theoretical improvement over DIA on one hand, and a significant simplification on the other. Indeed, a successful translation of Wilson's idea of expansion about a Gaussian theory (Wilson's ϵ-expansion RG, see Wilson, 1975, 1983; Wilson & Kogut, 1974) to hydrodynamic turbulence would have established RG as a rational approximation, a claim which could never be made for DIA. Moreover, the relative simplicity of all RG schemes permits straightforward analytical calculation of quantities like the Kolmogorov constant, which in full DIA remains a complex numerical undertaking. This analytic simplicity also suggested applications to turbulence modeling, another extension which had apparently eluded analytical theories. Now that the controversy generated by RG has subsided, it is appropriate to revisit these claims and evaluate them impartially. We ask: what if any advantages do RG formulations offer over analytical theories like DIA? Does RG improve, or merely simplify, RPT?

(2) A central discovery of Kraichnan (1959) is that 'sweeping' (random advection of small scales by large scales) and 'straining' (local distortion of small scales by scales of comparable size) (Kolmogorov, 1941; Tennekes & Lumley, 1972) effects both exist in turbulence but each plays a differ-

[3] Kida & Goto (1997) and Goto & Kida (1999, 2002), in their work called the closure Lagrangian DIA (LDIA) and the sparse direct interaction perturbation (SDIP), derived the same set of equations as the LRA for homogeneous isotropic turbulence and passive scalar field. Recently, OGorman & Pullin (2005) reported that the results from these closure agrees well with their direct numerical simulations (DNS) data

[4] Adzhemyan et al., 2002a,b, 2003, 2006 have taken the next step and carried out the two-loop calculations. Note that, however, the values of the Kolmogorov constant evaluated from the two-loop calculations are much higher than those obtained from both experimental measurements and numerical simulations, (≈ 1.6, see for example Monin & Yaglom, 1975, Zhou & Speziale, 1998)

ent role in turbulence dynamics. The distinction between the Ironshikov-Kraichnan (Ironshikov, 1963; Kraichnan, 1965c) spectrum turbulence and the Kolmogorov spectrum for MHD turbulence rests on precisely on this distinction (Zhou *et al.*, 2004). Yet, an RPT model in non-Langragian framework (the local-energy-transfer model, hereafter LET, McComb, 1990) sidesteps sweeping entirely. Furthermore, claims have been made in the RG literature (Yakhot, Orszag & She (YOS), 1987) that sweeping is asymptotically negligible. How then is the distinction between sweeping and straining accounted for by RG theories?

(3) Another important contribution of Kraichnan (1971) which is central to RPT theories is the different roles of local and distant interactions in turbulence, even when energy transfer is formally 'local': although inertial range energy transfer essentially decouples from the production and dissipation mechanisms, interactions between inertial range modes of any scale are possible. It is often claimed that RG theories treat distant interactions as dominant and ignore local interactions. Yet local interactions are clearly not negligible, as they are responsible for the famous 'cusp' in the eddy viscosity(Kraichnan, 1976), among other phenomena. The recursive RG method (Rose, 1977, Zhou et al., 1988, 1989) was developed in an attempt to incorporate both local and nonlocal interactions. The question remains: how do different RG theories account for both local and distant interactions?

We focus our review on how the key physical processes are handled by different RG methodologies. At the same time, since the goal of the renormalization group theory is to reduce the number of degrees of freedom in a turbulent flow, it is entirely natural that the RG procedure might culminate in a physical model that is appropriate for engineering or other scientific applications.

2 Basic description of fluid and MHD turbulence

2.1 Navier-Stokes Equation (NSE)

For high Reynolds number fluid turbulence (Tennekes & Lumley, 1972, Monin & Yaglom, 1975, Lesieur, 1990, Frisch, 1995, Pope, 2000), dimensionless arguments regarding the triadic interaction and the energy transfer process lead to the famous Kolmogorov (1941) $k^{-5/3}$ scaling law [5]. Briefly, a forcing $\mathbf{f}$ is applied to a (dissipative) fluid flow at a certain large scale, L, injecting energy into the system. The fluid motion at scale $r < L$ becomes unstable and loses its energy to neighboring smaller scales without directly dissipating it into heat (i.e., a local inertial energy transfer). This process is assumed to repeat itself until one reaches the so-called Kolmogorov scale L_K, where the energy transfer is directly dissipated into heat by the molecular viscosity. In steady state, the rate of energy input at the large scales and the rate of energy dissipation (denoted $\mathcal{E}$) at the Kolmogorov scale are equal to each other as well as to the energy transfer rate across the spectrum of the intermediate scales [6]. The level of anisotropy and inhomogeneity, present at the large scales, are presumed to diminish at the smaller spatial scales. Local isotropy at the small spatial scales implies one has statistical isotropy and homogeneity on those scales that are much smaller than L .

We consider incompressible turbulence. The fluctuating momentum equation (Navier-Stokes equation, NSE) in usual vector notation is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}. \tag{1}$$

The fluid density ρ and the kinematic viscosity ν, are assumed to be constants for an incompressible fluid [7]. It is interesting to note that the concept of molecular viscosity is itself a type of renormalization: the underlying kinetic collisional degrees of freedom are removed and their effects are represented at the macroscopic level by a transport coefficient (Huang, 1987).

The pressure fluctuation field, p, is obtained by solving the Poisson equation

[5] Here $k = |\mathbf{k}|$ is the wavenumber.
[6] The energy flux, Π.
[7] In this review, we will restrict ourselves to incompressible turbulent flows. There are a few RG application to compressible flows (see for example, Antonov et al., 1999 and Staroselsky et al., 1990). There are some additional efforts on applying the RG procedure to coupled fluctuating fields (see for example, Riahi et al, 1997, 1998a,b)

(hereafter, the summation convention over repeated subscripts is utilized)

$$\nabla^2(\frac{p}{\rho}) = -\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}. \tag{2}$$

2.2 Magnetohydrodynamics (MHD)Equation

A magneto fluid model is a collisional model that can be applied to a plasma or fluid that is a good conductor for electric current in the low frequency limit. Described by magnetohydrodynamics (MHD) (see, for example, Cowling, 1957, Shercliff, 1965, Nicholson, 1983, Biskamp, 2003, Elmegreen & Scalo, 2004, and Zhou *et al.*, 2004), such systems can be found in astrophysical, geophysical, as well as laboratory settings. In particular, MHD turbulence is seen in the Sun, the solar wind, the interstellar medium, galaxy clusters, accretion disks, Jupiter, and molecular clouds.

We will restrict ourselves to locally homogeneous MHD turbulence, which is often invoked in astrophysical and space application so that the problems become tractable. The incompressible MHD model written in terms of the fluid velocity $\mathbf{u}$ and the magnetic field $\mathbf{B}$, includes a momentum equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla\tilde{p} + \frac{1}{4\pi\rho}(\nabla\times\mathbf{B})\times\mathbf{B} + \nu\nabla^2\mathbf{u} \tag{3}$$

and a magnetic induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B}) + \mu\nabla^2\mathbf{B}. \tag{4}$$

The plasma density ρ, the kinematic viscosity ν, and the magnetic diffusivity μ, are assumed to be constants[8]. The velocity and magnetic field are solenoidal, $\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{B} = 0$, and the pressure $\tilde{p}$ is again determined from a Poisson equation resulting from taking the divergence of Eq. (3).

Some immediate quantities of interest are the kinetic and magnetic energies, (which, of course, give the total energy when they are summed) and the cross helicity (see for example, Goldstein *et al.*, 1995; Zhou & Matthaeus, 1990a,b). The dimensionless parameters used in this review are the ratio of twice the cross helicity to the total energy (the so called normalized cross helicity), and the ratio of kinetic to magnetic energy (known as the Alfvén ratio).

[8] see Braginskii, 1965

Hydrodynamic turbulent flows are characterized by high Reynolds numbers [9]. Denoting the root-mean-square (rms) of the fluctuating velocity field by $\tilde{u}$, the macroscopic Reynolds number is defined as

$$R_e = \tilde{u}L/\nu \tag{5}$$

(recall that L is a typical large scale), This non-dimensional parameter is a measure of the relative strength of the non-linear convective term $\mathbf{u} \cdot \nabla \mathbf{u}$ to the dissipative term $\nu \nabla^2 \mathbf{u}$ in Eq. (1).

The energy dissipation rate, $\mathcal{E}$, is another important parameter since it indicates the rate of energy flux from large to small scales. Kolmogorov suggested, and Batchlor (1953) illustrated experimentally, that the energy dissipation rate could be characterized by $\tilde{u}$ and L (for some recent results, see, for example, Sreenivassan, 1984, 1998, Yeung & Zhou, 1997, Antonia & Pearson, 2000, Pearson *et al.*, 2002, Kenada *et al.*, 2003, Burattini *et al.*, 2005). These available data suggest that the non-dimensional parameter

$$\mathcal{D}_{\mathcal{E}} = \frac{\mathcal{E}L}{\tilde{u}^3,} \tag{6}$$

approaches a constant for sufficiently high Reynolds number.

To characterize MHD turbulence, the kinetic and magnetic Reynolds number, R_e and R_m, are useful parameters. Apart from the kinetic Reynolds number, R_e,

$$R_e = \tilde{u}L/\nu \tag{7}$$

which we have already defined (but now $\tilde{u}$ and L must be re-interpreted as the typical velocity field and large length scale of the MHD equations), R_m is the magnetic Reynolds number introduced in a similar manner as

$$R_m = \tilde{u}L/\mu \tag{8}$$

where μ is the magnetic diffusivity. Again, we assume the transport coefficients, ν and μ are constants. The kinetic and magnetic Reynolds numbers

[9] Zhou (2007) and Zhou *et al.* (2003a,b) found that $R_e \geq 1.6 \cdot 10^5$ is need to achieve the so called minimal state, the lowest Reynolds number turbulent state that insures that the energy containing scales are not contaminated by the dissipation scales

indicate the relative importance of the flow's inertial forces to the viscous or magnetic diffusivity effects.

One may start to wonder why the strength of the magnetic field is absent from the definition of the magnetic Reynolds number, R_m. It is usually assumed, but with little broad observational/experimental justification outside of some solar wind observation(see, for example, Goldstein & Roberts, 1999, Goldstein *et al.*, 1995), that the turbulent flow $\tilde{u}$ is on the same order as the typical magnetic field strength $\tilde{B}$. Such an assumption, based on an assumed 'equipartition of energy', will greatly simplify the analysis. The Alfvén ratio (between the kinetic energy E_K and the magnetic energy E_B)

$$r_A = E_K/E_B \tag{9}$$

is order one, and usually $\approx 1/2$ in the inertial range. Similarly, Zhou & Matthaeus (2005) noted that the equality of the kinetic outer scale L_K to the magnetic outer scale L_B is often assumed.

However, for many astrophysical applications, the assumption of energy equipartition is questionable or even disallowed. In Zhou & Mattheaus (2005), the term "non-equipartition" is introduced to refer to cases in which $E_K \neq E_B$ or $L_K \neq L_B$ or both. A dissimilarity in the magnetic and kinetic Reynolds numbers is one indication of possible non-equipartition, or possible departure from symmetry, between the flow and magnetic fields in MHD turbulence. For example in the environments such as the interstellar medium and protogalactic plasma, the magnetic Prandtl number

$$Pr = \frac{\nu}{\mu} \equiv \frac{R_m}{R_e} \tag{10}$$

is very large and can even reach 10^{14} - 10^{22} (Zhou & Mattheaus, 2005). The issues related to non-equipartition in MHD will not be examined here since RG has not been applied to these more complex problems.

3 Straining and Sweeping Motions

The challenge of studying turbulent flow arises due to the very extended range of strongly interacting scales of motion. Indeed, the concept of turbulent flow demands the existence of an inertial range (Kolmogorov, 1941; Batchelor, 1953), which separates the spatial scales between the energy input at large scales and the Kolmogorov scale at very small scales where dissipation takes place.

Suppose energy is injected into the fluid at large scales [10], L, by some external forcing. The resultant fluid motion at these scales becomes unstable leading to large scale vortices or "eddies" (Tennekes & Lumley, 1972; Lumley, 1996). These interact and break up into smaller scale structures. In the straining case, starting from "energy containing eddies" at L, this energy transfer is driven by vortex stretching to smaller and smaller eddies down to the dissipative scale L_k (Batchelor, 1953).

Basically, the Kolmogorov (1941) theory appeals to an effective statistical independence of the one-time probability distribution function of the energy-containing range from that of the inertial-range excitation: at any time instant, the two ranges know about each other only through the rate of energy flux $\mathcal{E}$ (Chen & Kraichnan, 1989).

A turbulent flow with dominant straining motion (i.e., the local distortion of small scales by scales of comparable size, Tennekes & Lumley, 1972; Hamlington, Schumacher, and Dahm, 2008) has its interacting scales local as well as a local energy transfer (Kolmogorov, 1941). This has resulted in the Kolmogorov energy spectrum that has been verified by many experimental measurements (Grant *et al.*, 1962; Chapman, 1979; Saddoughi & Veeravalli, 1994).

A major discovery of Kraichnan (1959) is that 'sweeping' (i.e., the random advection of small scales by large scales, Kraichnan, 1964; Tennekes, 1975) and 'straining' effects both exist in turbulence and play very different roles in the turbulence dynamics. The 'sweeping' effect actually lead to an initial difficulty in Kraichnan's direct interaction approximation (DIA): the failure of this theory to predict the Kolmogorov $k^{-5/3}$ spectrum (Kraichnan, 1959).

Tennekes (1975) pointed out that this implies a statistical form of Taylor's hypothesis (1938) so that inertial-range components of the velocity field suffer advective sweeping by the energy-range excitation, $\tilde{u}$. Consequently, the many-time distribution of the inertial-range excitations must also involve the magnitude of the sweeping as well as rate of energy flux $\mathcal{E}$.

The underlying idea is that the large scale flow sweeps the spatial fluctuations past the observation point faster than local nonlinearities can produce distortions, see Fig. 1 (a) (Zhou *et al.*, 2004). The second-order energy spectrum is independent of the sweeping velocity and maintains a robust Kolmogorov energy spectrum for a wide range of values of the Reynolds number, see Fig. 2

[10] For exmaple, see Eswaran & Pope, 1988, Alvelius, 1999, Jimenez *et al.*, 1993, Chen *et al.*, 1993, Sullivan *et al.*, 1994, Borue & Orszag, 1996, Overholt & Pope, 1998

(Grant *et al.*, 1962; Chapman, 1979, Saddoughi & Veeravalli, 1994). However, as we will discuss in detail in the next subsection, the frequency spectrum and the higher-order kinetic energy spectrum are influenced by the sweeping.

We now make a remark about the relationship between "sweeping" and the well-known Taylor frozen flow hypothesis (Zhou et al., 2004). The Taylor (1938) hypothesis assumes that a large scale flow, with velocity $\mathbf{V}$, simply sweeps the turbulence by the point of observation. This approximation, with a large constant speed V ($\gg$ fluctuation speed $\tilde{u}$), is used in wind tunnel studies and in single spacecraft studies of solar wind turbulence (Jokipii, 1973) to convert time-lagged correlations into spatial correlations[11].

3.2 Fuid turbulence: frequency spectra and time correlations

For fluid turbulence, it is generally accepted that the principal action of the large scales on the small scales is to just convect them, without significant effect on the internal dynamics of these small scales (Tennekes, 1975; Chen & Kraichnan, 1989, 1997).

The form of the Eulerian time correlation tensors for time stationary flows

$$U_{ij}(x,\tau) = < u_i(x,t)u_j(x,t+\tau) > \tag{11}$$

depends on whether it is dominated by large scale sweeping or by local straining. In particular, the sound generated by high Reynolds number isotropic turbulence is very dependent on whether local straining or sweeping are dominant. (Zhou & Rubinstein, 1996).

Consider the isotropic temporal spectrum

$$E(k,\tau) = Q(k,\tau)/(4\pi k^2), \tag{12}$$

where Q_{ij} is the spectrum tensor and $k = |\mathbf{k}|$ is the wavenumber.

Now in order to obtain a frequency (ω) spectrum from the single-point, two-time correlation function, one would naturally appeal to a straightforward dimensional analyis following Kolmogorov

$$E_L(\omega) = \mathcal{E}\omega^{-2} f_L(\omega/\omega_0), \tag{13}$$

[11] Nelkin (1994) has pointed out that this approximation works quite well and the corrections to this approximation are both complicated and not particularly reliable.

where the characteristic frequency $\omega_0 = (\mathcal{E}/\nu)$

Suppose there is a mean flow, V, present. Using the Taylor (1938) frozen turbulence approximation the mean flow will sweep the turbulence by the point of observation without distortion, yielding the frequency spectrum in the form

$$E_E(\omega) = \mathcal{E}^{2/3} V^{2/3} \omega^{-5/3} f_E(\omega/\omega_c) \tag{14}$$

where the cutoff frequency, ω_c, is given by

$$\omega_c = V/L_k \tag{15}$$

with L_k the Kolmogorov dissipation scale.

Tennekes (1975) then applied this to the case when there is no mean flow by assuming a *random* Taylor sweeping hypothesis. In this case, the mean velocity is replaced by the rms fluctuations $\tilde{u}$, and the frequency spectrum becomes

$$E_T(\omega) = \mathcal{E}^{2/3} \tilde{u}^{2/3} \omega^{-5/3} f_T(\omega L_k/\tilde{u}). \tag{16}$$

Now the Kolmogorov theory appeals to an effective statistical independence of the one-time probability distribution of the energy-range from that of the inertial-range excitation: at any instant, the two ranges know about each other only through $\mathcal{E}$. Tennekes (1975) pointed out that since $\omega^{-5/3}$ decay is slower than the ω^{-2} decay, the highest frequencies in the flow are generated by the sweeping of the small scale fluctuations by the large scale turbulent motion past the observation point. The large scale turbulent motion carries most of energy. Consequently, the many-time distribution of the inertial-range excitation should involve the magnitude of the sweep, $\tilde{u}$, as well as $\mathcal{E}$.

3.3 MHD Sweeping

Here we would like to stress a major difference between fluid and MHD turbulence. Unlike fluid turbulence, the nonlocal effect of large scales upon the small scales ('sweeping') is an important issue in MHD turbulence. Beginning with the work of Iroshnikov (1963) and Kraichnan (1965c), it has been argued that such effects play a significant role in MHD turbulence, even in the case of no *DC* magnetic fields. If there is a strong, large-scale magnetic field, the small-scale fluctuations are subject to a sweeping-like effect due to Alfvén wave propagation.

To discuss this point further, it is useful to write the MHD equation in a more symmetric form for $\mathbf{u}$ and $\mathbf{B}$. Introducing the Elsasser fields (Elsasser, 1950,1956),

$$\mathbf{z}^{\pm} = \mathbf{u} \pm \mathbf{B}/\sqrt{4\pi\rho} \tag{17}$$

the MHD equations (Eqs. 3-4) can be rewritten as

$$\frac{\partial \mathbf{z}^{\pm}}{\partial t} \mp \mathbf{V_A} \cdot \nabla \mathbf{z}^{\pm} = -\mathbf{z}^{\mp} \cdot \nabla \mathbf{z}^{\pm} - \frac{1}{\rho} \nabla P^{\pm} + \mu \nabla^2 \mathbf{z}^{\pm} \tag{18}$$

where we have explicitly separated out a term involving the large scale magnetic field (written here in terms of the Alfvén velocity $\mathbf{V_A} = \mathbf{B_0}/\sqrt{4\pi\rho}$. For simplicity, we shall assume here that $\nu = \mu$. The pressures $P^{\pm}$ enforce the constraints $\nabla \cdot \mathbf{z}^{\pm} = 0$ for incompressible MHD.

Kraichnan (1965c) pointed out that the mean magnetic field sweeps the small scale interacting structures and during that sweeping time there is a non-linear transfer of energy between length scales (in the Kraichnan picture: the "wave packets" suffer brief "collisions" during which energy transfer occurs). This is illustrated in Fig 1 b. One can see then that the mean magnetic field acts to inhibit the nonlinear energy cascade (Chen & Kraichnan, 1997).

The existence of sweeping has a profound implication for MHD turbulence. Large scales can have profound effects on the small scales under different equations of motion [12]. The nonzero field is often said to correspond to *wave packets* that propagate along the mean field direction. This description can be misleading because the "packets" may not be spatially localized, and also may not even propagate. Non-propagating fluctuations with wavevectors strictly perpendicular to the mean magnetic field have zero phase speed. In any case, one sees from the MHD equations that both type of fluctuations $\mathbf{z}^{\pm}$ are needed for the nonlinear terms to be non-zero and sustain turbulence. That fact was pointed out by Kraichnan (1965c) and has been discussed in the context of space physics applications (Dobrowolny *et al.* 1980a,b; Zhou et al. 2004).

[12] As one can imagine, the MHD energy spectra may be affected by the sweeping, see subsection 4.5.

4 Inertial Range: Energy and Forcing Spectra

4.1 Velocity correlation and energy spectra

Assuming that the energy transfer and interacting scales are local, Kolmogorov (1941) derived the famous scaling law

$$E(k) = C_K \mathcal{E}^{2/3} k^{-5/3}. \tag{19}$$

Here C_K is the Kolmogorov constant (for a collection of recent results, see Sreenivasan 1995, Yeung & Zhou, 1997; Zhou & Speziale, 1998)[13]. The Kolmogorov energy spectrum has been supported by turbulent measurements over a wide range of Reynolds numbers (again, see Fig. 2).

The energy spectrum is related to the correlation of the velocity field by

$$< u_i(\mathbf{k}, t) u_j(\mathbf{k}', s) >= Q_{ij}(\mathbf{k}, t, s)\delta(\mathbf{k} + \mathbf{k}'). \tag{20}$$

For isotropic stationary turbulence Q_{ij} simplified to

$$Q_{ij}(\mathbf{k}, t, s) = D_{ij}(k) Q(|\mathbf{k}|, t - s). \tag{21}$$

where $D_{ij}(k) = \delta_{ij} - k_i k_j / k^2$ and the equal time correlation. Again, we note that $E(k) = 4\pi k^2 Q(k, t, t)$, (Rose & Sulem, 1978).

According to Kolmogorov (1941), excitation in the energy-containing range does not affect energy transfer within the inertial range. Therefore, the average rate of energy dissipation, $\mathcal{E}$, is identified with the rate of spectral energy transfer and the rate of energy production.

In order to infer the form of the inertial-range spectrum, it is necessary to estimate the magnitude of the transfer function correlations. The transfer function correlations induce turbulent spectral transfer. The time scale for the decay of these correlations, τ_T may depend on any relevant turbulence parameters (Kraichnan, 1965c; Matthaeus & Zhou, 1989; Zhou & Matthaeus, 1990b). Turbulence theories (Batchelor, 1953; Monin & Yaglom, 1975) indicate that the energy flux Π is explicitly proportional to τ_T and depends both on the wavenumber and on the power of the omni-directional energy spectrum. In the inertial range, because energy is conserved by the nonlinear interaction

[13] It is interesting to note that Praskovsky & Oncley (1994) reported that the Kolmogorov constant is found to be weekly dependence on the Reynolds number, using their high Reynolds number atmospheric surface layer data

and a local cascade has been assumed, the energy flux Π becomes independent of the wavenumber k. Since there is no "leak" of energy in the inertial range, the energy flux Π and dissipation rate $\mathcal{E}$ assume the same value. A simple dimensional analysis leads to (Matthaeus & Zhou, 1989; Zhou & Matthaeus, 1990)

$$\mathcal{E} = C_1^2 \tau_T(k) k^4 E^2(k), \tag{22}$$

where C_1 is some constant.

We now show how the well-known Kolmogorov spectrum can be obtained within this framework for homogeneous, isotropic, statistically steady turbulence. In this case, the energy-containing range excitation due to external agents is absent. Therefore, the time scale reponsible for the energy transfer resulting from nonlinear interaction is just the local dynamical time scale (Rose & Sulem, 1978)

$$\tau_{nl}(k) = [k^3 E(k)]^{-1/2}. \tag{23}$$

Here, the wavenumber k is inversely proportional to the length scale in the inertial range and $U(k) = [kE(k)]^{1/2}$ is the characteristic velocity of eddies at wave number k.

Under the Kolmogorov assumption of local energy transfer and local interacting scales, the local dynamical time scale is the only available time scale for homogeneous, isotropic, statistically steady state turbulence. Hence, $\tau_T(k) = \tau_{nl}$. The Kolmogorov $k^{-5/3}$ spectrum is thus reproduced.

4.2 Forcing correlation and white noise

Suppose we wish to consider stationary, isotropic steady-state fluid turbulence - a somewhat artificial but nevertheless interesting problem. In this case we must add a (hypothetical) stirring forces to the Navier-Stokes equations, so that the turbulence level can be sustained against the viscous dissipation. The random force, $f_i(\mathbf{k}, t)$, is usually taken to have a multivariate normal distribution with covariance

$$< f_i(\mathbf{k}, t) f_j(-\mathbf{k}, s) >= D_{ij}(\mathbf{k}) F(k) \delta(t - s), \tag{24}$$

where $F(k)$ has dimensions of $velocity^2/time$, and remains to be specified.[14] Typically, the time correlation is assumed as white noise with zero correlation time.

It has always been the normal practice in turbulence simulations to choose $F(k)$ to be peaked near the origin $k = 0$, so that its arbitrary nature is only of importance at low wavenumbers, and a universal energy spectrum can develop at large wavenumbers (see for example, Eswaran & Pope, 1988; Sullivan *et al.*, 1994).

Yet, the application of RG to stirred fluid motion has produced results which depend strongly on the choice of $F(k)$. Accordingly, questions of how $F(k)$ is chosen and justified are of considerable importance (Forster *et al.*, 1977; De Dominics & Martin, 1979, Fourier & Frisch, 1983). In particular, the question of whether or not $F(k)$ should depend on some characteristic length scale (such as an ultra-violet cutoff Λ) will surface later as an issue.

In RG, the random forcing is often employed rather than the energy spectrum, with typical form

$$F(k) \propto k^{-y}. \tag{25}$$

The exponent y will be a focus of attention.

Stationarity requires that the rate at which the stirring forces do work is the same as the rate of viscous dissipation $\mathcal{E}$, or

$$\int_0^\infty 4\pi k^2 F(k) dk = \mathcal{E}. \tag{26}$$

We shall see that the utilization of this relationship will surface as an issue later.

4.3 Forcing correlation and colored noise

Yuan & Ronis (1992) noted that earlier RG calculations, including those of YO and Ronis (1987), had ignored the actual generation of the turbulence, e.g., at the boundaries of the system, as well as missing the precise nature of the stirring force. In particular, Yuan & Ronis (1992) found that there was

[14] A recent numerical simulation was conducted by Biferale *et. al.*, 2004. Two related previous work should be noted, the low numerical resolution by Sain *al.* (1998) and the case of shell models by Mazzi *et al.*, 2002

no *a priori* theory for the exponents used to characterize the random-force autocorrelation function.

To illustrate the issue of the generation of random force, Yuan & Ronis (1992) consider the Navier-Stokes equation in the form

$$\frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} + \mathbf{v}(\mathbf{x},t)\cdot\nabla\mathbf{v}(\mathbf{x},t) = -\nabla[\hat{p}/\rho] + \nu\nabla^2\mathbf{v}(\mathbf{x},t) + \hat{\mathbf{f}}(x,t), \tag{27}$$

where $\mathbf{v}$ is the kinematic velocity and $\hat{\mathbf{f}}$ is a non-stochastic force that arises from interactions with the boundaries.

Typically, in turbulent fluids, one could introduce an effective force into NSE that acts on inertial range length scales to model the nonlinear (turbulent) transfer from the energy injection scales to the shorter scales. The randomness, in some sense, is expected to counteract the reduction in the degrees of freedom in the problem. Yuan & Ronis (1992) examine this in more detail by introducing a projection operator $\mathcal{P}$ on the velocity field $\mathbf{v}$ (and pressure $\hat{p}$)

$$\mathbf{u}(\mathbf{x},t) \equiv \mathcal{P}\mathbf{v}(\mathbf{x},t), \tag{28}$$

$$p(\mathbf{x},t) \equiv \mathcal{P}\hat{p}(\mathbf{x},t), \tag{29}$$

so that the projected fields $\mathbf{u}$ and p now contain only high-wavenumber information. By applying this projection operator $\mathcal{P}$ to Navier-Stokes equation one finds

$$\frac{\partial \mathbf{u}(\mathbf{x},t)}{\partial t} + \mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t) = -\nabla[p/\rho] + \nu\nabla^2\mathbf{u}(\mathbf{x},t) + \mathbf{f}(x,t) \tag{30}$$

where

$$\mathbf{f}(x,t) \equiv \mathcal{P}\hat{\mathbf{f}}(x,t) - \mathcal{P}[\mathbf{v}(\mathbf{x},t)\cdot\nabla\mathbf{v}(\mathbf{x},t)] + \mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t). \tag{31}$$

They then note that this ‘new’ force, $\mathbf{f}$, contains information about boundary interactions as well as the mode-coupling effects associated with velocity components on the injection scales. Since $\mathcal{P}\hat{\mathbf{f}}(x,t) = 0$ away from boundaries, the random stirring force used in RG studies results from mode coupling between the energy containing and the smaller scales. Since the motion on all scales is expected to be ‘chaotic’, including at the energy containing scale, these authors expected that $\mathbf{f}(x,t)$ will have complicated, chaotic time and space dependences. Therefore, Yuan & Ronis (1992) identified this as the quantity which is actually modeled by a stochastic force in random stirring models of turbulence.

The autocorrelation of the transverse parts of $\mathbf{f}(x, t)$ is assumed to have non-zero correlation time (colored noise). The major problem with such an assumption is that the resulting theory would no longer be Galilean invariant. Yuan & Ronis (1992) argued, however, that there is no *a priori* reason why global Galilean invariance must hold. They reason that the random forces represents the effects of boundaries and these are not included in a Galilean transformation.

Carati (1990b) also looked at the extension of the white noise of the forcing correlation function to a colored noise with a non-zero correlation time. He also argued that the correlation time as well as the correlation length can be important due to the appearance of macroscopic scale structures in turbulence. It is then possible to relate the expansion parameter to the stochastic forcing correlation. Carati (1990b), unfortunately, did not address the issue that the resulting theory would not be invariant under Galilean transformation.

4.4 MHD phenomenologies of Kolmogorov and Iroshnikov-Kraichnan

Montgomery and co-workers (Fyfe *et al.*, 1977) argued that the original Kolmogorov reasoning and its associated $k^{-5/3}$ spectrum are also applicable to MHD. The assumption, as reviewed in Zhou *et al.* (2004), seems to be that the nonlinear distortion of eddies is faster than the decorrelation effects associated with wave propagation. This implies that the nonlinear time scale is dominant over that for random sweeping or propagation. The $k^{-5/3}$ spectrum has received strong support from in situ spacecraft observation of solar wind (Matthaeus & Goldstein, 1982).

The description of incompressible MHD turbulence by Iroshnikov (1963) and Kraichnan (1965c) (hereafter, IK theory) retains the same basic underlying assumptions of Kolmogorov: isotropy and local interactions. The IK theory differs from the Kolmogorov picture in considering the effect of colliding Alfvén wave packets. [15] on the energy cascade for higher wavenumbers.

Indeed, small scale fluctuations are envisioned to behave as Alfvén wave packets traveling along the large-scale magnetic field. The small scale structures suffer brief collisions between two oppositely propogating wave packets which provides a basic mechanism for the energy tranfer.

[15] Goldreich & Sridhar (1995, 1997) and Sridhar & Goldreich (1994) have raised two objections against the IK theory. The first one is the anisotropy induced by large scales. This point, of course, is well known. The second one is that the nonlinear three-wave interactions appeal to by IK are "empty." These objections were addressed by Montgomery & Matthaeus 1995, and Ng & Bhattacharjee, 1996, 1997. For a review, see Zhou *et al.*, 2004.

Kraichnan (1965c) suggested that the Kolmogorov phenomenology should also be modified to include magnetic field effects. Again, the same dimensional analysis (see subsection 4.1) leads to (Matthaeus & Zhou, 1989; Zhou & Matthaeus, 1990b)

$$\mathcal{E} = C_2^2 \tau_T(k) k^4 E^2(k), \tag{32}$$

where C_2 is a constant.

For MHD turbulence, IK theory argued that the triple velocity correlations decay in a time on the order of an Alfvén wave period. Therefore, the IK theory would be setting $\tau_T = \tau_A$, where

$$\tau_A \sim \frac{1}{v_A}, \quad v_A = \tilde{B}/\sqrt{4\pi\rho}, \tag{33}$$

where $\tilde{B}$ is the typical magnetic field strength. As a result, the well known IK $k^{-3/2}$ spectrum is recovered (Matthaeus & Zhou, 1989; Zhou & Matthaeus, 1990).

Two-dimensional simulations (Biskamp & Welter, 1989; Biskamp, 1993) offer support to the IK scaling. High resolution 2D (2048^2) MHD simulations of Galtier *et al.* (1999) indicate the decay is significant slower than for neutral fluids in a way that favours IK over the Kolmogorov picture. Biskamp & Muller (2000) concluded that in 2D, because the swirling motions are weak, as manifested by the steep energy spectrum in 2D fluid turbulence, the Alfvén wave dynamics will also be weak. Hence, sweeping dominates over local straining effect. However, Biskamp & Muller (2000) find the energy spectrum in 3D MHD follows the $k^{-5/3}$ scaling law.

Mininni & Pouquet (2007) presented a numerical analysis of incompressible free-decaying magnetohydrodynamic turbulence run on a grid of 1536^3 points. The Taylor Reynolds number [16], R_λ at the maximum of dissipation is 1100. The initial kinetic and magnetic energies are equal [17], with negligible correlation. The resulting energy spectrum is a combination of two components, each moderately resolved. At small scales, weak turbulence shows a $k_\perp^{-2}$ spectrum, the perpendicular direction being defined relative to the local quasiuniform magnetic field. Isotropy is found at the large scales, with a spectral law compatible with the $k^{-3/2}$ Iroshnikov-Kraichnan theory. The authors found that

[16] The Reynolds number, R_e is approximately 181500, R_e is related to the Taylor Reynolds number by $R_\lambda = (20/3)^{1/2} R_e^{1/2}$ for isotropic flow and $R_\lambda \approx 1.4 R_e^{1/2}$. See Zhou, 2007 for more detail.

[17] This is a good example of typical "equi-partition" approximation, see Zhou and Matthaeus, 2005.

the IK spectrum arises from a weakening of the nonlinear interactions due to the Alfvén waves. Furthermore, the scalings of the structure functions confirm the non-Kolmogorovian nature of the flow in this range.

4.5 MHD Extended phenomenology

Matthaeus & Zhou (1989) and Zhou & Matthaeus(1990) have developed a framework for MHD turbulent flows where both the time scales, τ_{nl} and τ_A co-exist (alternatively, where local and nonlocal interactions are important). The basic idea is that the lifetime of transfer function correlations $\tau_T(k)$ in MHD turbulence might be more accurately estimated by taking into account the possibility that these correlations decay because of the influence of both the external agent (sweeping) as well as the turbulent nonlinear interactions (straining) (Pouquet *et al.* (1976), Matthaeus& Zhou (1989), Zhou & Matthaeus (1990), Zhou *et al.*, 2004)

The simple choice of time scale

$$\tau_T = [\frac{1}{\tau_{nl}(k)} + \frac{1}{\tau_A(k)}]^{-1} \tag{34}$$

satisfies the two limiting cases of zero and strong external agent, since $\tau_A \to \infty$ for zero external agent and $\tau_A \ll \tau_{nl}$ for a strong external agent. A generalized energy spectrum, $E(k) \sim k^{-m}$, can be determined which will have a scaling exponent in the interval $3/2 \leq m \leq 5/3$. The energy spectral law reduces to the well known IK and Kolmogorov laws in the appropriate limits (Matthaeus & Zhou, 1989; Zhou & Matthaeus, 1990b).

Note that the IK theory implicitly assumes the absence of correlation betweeen the velociy and the magnetic field. Grappin *et al.* (1982; 1983), Pouquet *et al.* (1976), and Zhou & Matthaeus (1990b) extended this phenomenological to correlated turbulence.

Pouquet *et al.* (1976) found that, based on EDQNM closure and a strong helical state, the nonlocal interactions are responsible for the system evolving to an equipartition of magnetic and kinetic energy while the local interactions are responsible for the energy transfer.

4.6 MHD forcing spectra

To introduce forcing into the MHD equations is a bit more complicated. In the first place, one must decide on whether to work with the primitive variables

$(\mathbf{u}, \mathbf{B})$, or with the Elsaŝser (Elsaŝser, 1950, 1956) variables $(\mathbf{z}^{\pm})$.

In the pioneering work of Fournier *et al.* (1982), the primitive variables $\mathbf{u}, \mathbf{B} \equiv \mathbf{b}/\sqrt{4\pi\rho}$ are employed. The correlations of the stochastic stirring forces were assumed to increase toward large k. Moreover, they assumed different weights for the magnetic and kinetic nonlinearities, and different coefficients in the correlations of the random forces appearing in the fluid and magnetic evolution equations (i.e., in the spectral k-exponents y_1 and y_2).

On the other hand Camargo & Tasso (1992) treated the full MHD equation using Elsaŝser variables. They inspected the work of Fournier *et al.* (1982), and argued that choosing different forcing weights for the magnetic and kinetic nonlinearities would not be appropriate. They reasoned that in the Elsaŝser representation this would lead to a different rescaling of $\mathbf{z}^+$ and $\mathbf{z}^-$. Also, in Camargo & Tasso (1992), the correlations of the stochastic stirring force were assumed to decrease toward large k and the magnetic and kinetic nonlinearities were weighted in the same way.

Nandy & Bhattacharjee (1998) studied the large-scale long-time properties of turbulent motions in a symmetric miscible binary fluid (a 2-field problem like MHD). The governing equations were written in terms of the velocity and concentration field and driven by random stirring fields. The forces were assumed to have Gaussian statistics, with correlations $k^{-(d-4+y)}$ and $k^{-(d-2+y')}$ (here y and y' are exponent parameters and d is the space dimension under consideration). The stability of a RG fixed point [18] is now determined by both the exponent paramaters y and y'. These authors find that the Kolmogorov spectrum, which occurs for $y = 4$ and $y' = 2$, is stable.

Hnatich *et al.* (2001a) reconsidered an RG treatment of MHD (for dimension $d \geq 2$) using the primitive representation of the fluctuating velocity and magnetic field. The external random forces $\mathbf{f}^u$ and $\mathbf{f}^B$ were assumed to have a zero-mean Gaussian distribution. These authors also assumed that the time correlations of the fields are white noise, while the spatial correlations are controlled by the scalar function, $F(k)$. Hnatich et al. chose uncorrelated kinetic and magnetic driving forces, $< \mathbf{f}^u \mathbf{f}^B >= 0$, because they were interested in arbitrary space dimension $d \geq 2$ and it is not possible to define a nonvanishing correlation function of a vector field with a psuedovector field when d is continuous. In contrast to the claims of Fournier *et al.* and Camargo & Tasso, Hnatich *et al.* (2001a) showed that their choice of uncorrelated kinetic and magnetic driving correlations was not an obstacle for the successful application of the RG.

In a recent work, Jurcisin & Stehlik (2006) considered the d-dimensional de-

[18] The existence of a fixed point is needed for RG procedure to work appropriately, see any standard text book on RG listed in footnote 1

veloped MHD turbulence. The forcing correlations for both the veloicity and magnetic fields consisted of two parameters for the purpose of a double expansion.

The extension from white to colored noise in the forcing correlation functions has not yet been attempted in MHD turbulence.

5 Renormalized perturbation theory (RPT)

To facilitating our comparision between RG and RPT, we now review some of the key physics behind RPT. In closure theories for homogeneous fluid turbulence, one typically works with the Fourier transformed NSE,

$$\Big[\frac{\partial}{\partial t} + \nu_0 k^2\Big] u_i(\mathbf{k},t) = \lambda_0 M_{imn}(k) \int d^3p [u_m(\mathbf{p},t) u_n(\mathbf{k}-\mathbf{p},\mathbf{t}), \tag{35}$$

where $\nu_0 \equiv \nu$.

The incompressibility condition

$$k_n u_n(\mathbf{k},\mathbf{t}) = \mathbf{0} \tag{36}$$

has been employed to eliminate the pressure field on Fourier transforming the Poisson equation

$$\nabla^2 p = -\partial^2 (u_n u_m)/\partial x_n \partial x_m. \tag{37}$$

This results in the quadratic nonlinear coupling coefficient M_{imn}

$$M_{imn}(\mathbf{k}) = k_m D_{in}(\mathbf{k}) + k_n D_{im}(\mathbf{k}) \tag{38}$$

where

$$D_{in}(\mathbf{k}) = \delta_{in} - k_i k_n / k^2. \tag{39}$$

Note the symmetry relation $M_{imn} = M_{inm}$ with $k_i M_{imn}(\mathbf{k}) = 0$.

5.1 Direct interaction approximation (DIA)

Analytically, the direct interaction approximation (DIA) is a coupled system of integro-differential equations for *two* descriptors of homogeneous turbulence:

the *correlation tensor* $\mathsf{U}(\mathbf{k};t,s)$ defined in Eq. (20), and the *response tensor* $\mathsf{G}(\mathbf{k};t,s)$ (Kraichnan, 1959). The correlation tensor is familiar from Taylor's statistical theory of turbulence (Batchelor, 1953), but the response tensor is an entirely new quantity introduced by Kraichnan (1959) to analyze perturbations in a turbulent flow (for a detailed treatment of DIA, the author of this review strongly recommends the book by Leslie (1973) for its readability). We recall the standard kinematics that leads in isotropic turbulence to the description by a scalar correlation $U(k;t,s)$ and response function $G(k;t,s)$ where $k=|\mathbf{k}|$.

Because the response tensor will prove to be particularly significant for the RG theory, we would like to begin by reviewing the reasons that led Kraichnan to introduce it. For our purposes, all that is really relevant is the quadratic nonlinearity in the Navier-Stokes equations, and not the tensorial nature of the equations. Thus, the concept of the response function can be illustrated by the *simple scalar model* equation

$$\dot{u}(t)+\nu k^2 u(t)=Mu(t)u(t) \tag{40}$$

where we have ignored the vector indices and wavevector arguments that would appear in the real problem. In the quasinormal theory, one would simply invert the viscous operator on the left hand side to give

$$u(t)=\int_0^t ds\ \exp[-\nu k^2(t-s)]Mu(s)u(s). \tag{41}$$

We could call $G(t,s)=\exp[-\nu(t-s)]H(t-s)$ where H is the Heaviside function, the *viscous response function*. Of course, Eq. (41) is formally 'exact,' but its use in the perturbative context of analytical theories, like the *quasinormal theory*, causes the modeling of time correlations to depend on the viscosity. This dependence is at variance with the Kolmogorov theory (Kraichnan, 1959) However, this was only one of the problems of the theory. More seriously, it was shown to predict negative values of the energy spectrum (Ogura, 1963, Orszag, 1970), thus violating the principle of realizability (Kraichnan, 1959, Orszag, 1977).

Kraichnan argued that to compute correctly, the effect of perturbations should be analyzed by looking at the effects of adding a forcing term to the Navier-Stokes equations. For the scalar model in Eq. (40), this means introducing a forcing term $f(s)$ and now solving

$$\dot{u}(t)+\nu k^2 u(t)=Mu(t)u(t)+f(s), \tag{42}$$

where $s\leq t$ and the different time argument incorporates any transient effects.

Of course, this problem as it stands is more difficult to solve than the original one! But if we understand that the perturbation is to be 'small,' then it is reasonable to replace f by δf and ask for the small change in u, δu that is produced by δf. This means solving the simpler linear problem in $\delta u(t)$

$$\dot{\delta u}(t) + \nu k^2 \delta u(t) = 2Mu(t)\delta u(t) + \delta f(s). \tag{43}$$

The crucial physical distinction between this equation and the quasinormal approximation based on Eq. (41) is that the frequency scale governing the time evolution of δu is now Mu, which is expected to dominate the viscous frequency scale νk^2 in the regime of strong nonlinearity.

The ratio $\delta u(t)/\delta f(s)$ is an exact, linearized response function. DIA achieves closure at the level of second order statistics by replacing this fluctuating quantity by its average, $G(t,s) = \langle \delta u(t)/\delta f(s) \rangle$, and using this deterministic quantity to compute the response to perturbations. Averaging Eq. (43) leads to the formal equation of motion for G,

$$\dot{G}(t,s) + \nu k^2 G(t,s) = 2M\langle u(t)\delta u(t)/\delta f(s) \rangle \tag{44}$$

This is not a closed equation because of the correlation term on the right hand side. DIA will assume a closure for both this correlation, and for the third order moments in the equation for U.

At this point, one could pose the question: why close at the level of second-order statistics? It is clear that such closure has important limitations. For example, an interesting study by Girimaji (2005) considered the effect of a sudden reversal of sign of the velocity modes in decaying turbulence. This problem, or its obvious generalization in which the phases of all modes are suddenly randomized, cannot be studied with closures that only consider second order statistics. While closure at the level of third-order correlations, along the lines of DIA, is indeed possible, the relevant statistical quantities, in particular the appropriate generalizations of the linear response function, are not at all obvious. Such a theory has been described briefly by Kraichnan (1985). We will return to this point later.

An important qualitative feature of DIA is that it depends on the triad interactions that result from the convolution structure of the quadratic nonlinearity in the Fourier representation; that is, it contains integrals with the structure

$$\int d\mathbf{p} d\mathbf{q} \; \delta(\mathbf{k} - \mathbf{p} - \mathbf{q}) \tag{45}$$

so that the effects of all possible nonlinear interactions are modeled. Kraichnan emphasized the crucial distinction between this type of theory and the earlier heuristic classical closures like Heisenberg's (Heisenberg, 1948, for example, Batchelor, 1953; Monin & Yaglom, 1975), which treated nonlinear interactions as if they took place between pairs of modes rather than triads. This objection to Heisenberg's theory had been raised decades earlier by Batchelor (1953), although the even cruder [19] picture of a *stepwise* cascade remains a mainstay of intuitive discussions of turbulence.

A second important qualitative feature is the nonlocality in time, so that quantities at time t are expressed in terms of their evolution over the entire interval $0 \leq s \leq t$. Kraichnan argued that this dependence is unavoidable in any realistic closure scheme and those closures that are entirely in terms of single-time statistics are necessarily unrealistic. Thus, according to the DIA theory, even if we are interested only in single-time and single-point statistics, appropriate closures are impossible without the mediation of two-time, two-point properties. Thus the physical realism of DIA is bought at the expense of considerable analytic complexity.

5.2 Evaluation of DIA

Perhaps the single most important achievement of DIA, repeatedly emphasized by Kraichnan, is that DIA is the exact statistical description of a nonlinear stochastic problem; it is therefore realizable [20], an obviously important property which the quasinormal theory (Ogura, 1963 and references therein) so conspicuously lacked. Kraichnan gave two model problems for which DIA is exact: a random-coupling model (Kraichnan, 1961) in which realizations of Navier-Stokes turbulence are coupled to each other, and a generalized Langevin model (Kraichnan, 1970, Leith, 1971; Herring & Kraichnan, 1971; Leith & Kraichnan, 1972) which we recall here for later comparison with RG:

$$\dot{u}_i(\mathbf{k},t) + \int_0^t ds\ \eta_{ij}(\mathbf{k};t,s)u_j(\mathbf{k},s) = f_i(\mathbf{k},t) \tag{46}$$

where η is a deterministic damping factor, and f is a random force. To avoid lengthy formulas, we only cite the DIA expression for the damping

$$\begin{aligned}&\eta_{ij}(\mathbf{k};t,s) = M_{imn}(\mathbf{k}) \int d\mathbf{p}d\mathbf{q}\ \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) \times \\ &M_{mrs}(\mathbf{p})P_{ns}(\mathbf{q})P_{rj}(\mathbf{p})U(q;t,s)G(p;t,s),\end{aligned} \tag{47}$$

[19] albeit useful!
[20] Kramer, Majda, and Vanden-Eijnden (2003) claim that the realizability is violated for the case of passive scalar advection with a fluctuating random sweep

where $P_{ij}(\mathbf{k}) \equiv (1/2)D_{ij}(\mathbf{k})$ and note that the statistics of the random force, $f_i(\mathbf{k},t)$, are also determined by the correlation function U and response function G, with

$$
\begin{aligned}
&F(\hat{k}) = \langle f_i(\hat{k}) f_i(\hat{k}')\rangle \\
&= -4M_{imn}(\mathbf{k})M_{irs}(\mathbf{k}) \int d\hat{p}d\hat{q}\; D_{mr}(\mathbf{p})D_{ns}(\mathbf{q})Q(\hat{p})Q(\hat{q})\delta(\mathbf{k}+\mathbf{k}'),
\end{aligned} \tag{48}
$$

where $Q(\hat{k})$ is given by

$$
< u_i(\mathbf{k}, t+\tau)u_j(\mathbf{k}', t) >= Q(k,\tau)D_{ij}(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}'). \tag{49}
$$

We again emphasize that:

(1) the damping is time-history dependent.
(2) the entire model depends on wavevector triad $\mathbf{k}, \mathbf{p}, \mathbf{q}$ satisfying $\mathbf{k} = \mathbf{p} + \mathbf{q}$.

The second requirement is critical since it ensures that the energy balance of the turbulent flow is preserved.

The DIA form of the Langevin equation is indeed 'generalized:' a true Langevin model would have the simpler structure

$$
\dot{u}_i(\mathbf{k},t) + \eta_{ij}(\mathbf{k},t)u_j(\mathbf{k},t) = f_i(\mathbf{k},t) \tag{50}
$$

with Markovian (history-independent) damping and white-noise-in-time random forcing. 'Markovianized' closures like EDQNM (Orszag, 1970; Lesieur, 1990) and TFM (Kraichnan, 1972, Leith & Kraichnan, 1972) are based on simplified models similar to Eq. (50).

Despite incorporating much good physics in the realistic treatment of triad interactions and the role of nonlinear temporal decorrelation, DIA has some important drawbacks. First, DIA does not appear to be a rational approximation; although it is known that closure is possible at the level of n-point moments for any n whatsoever, the DIA with the choice of $n = 2$ appears to give the only realizable theory. A second drawback is the heavy analytical complexity noted earlier.

The most conspicuous problem of DIA is that the predicted spectrum is not the Kolmogorov spectrum, but instead the non-local expression

$$
E(k) = \sqrt{\tilde{u}\mathcal{E}}k^{-3/2} \tag{51}
$$

where $\tilde{u}$ is the rms velocity. As Leslie (1973) remarks, it is this dependence on the large-scale excitation, a departure from Kolmogorov's picture of a local inertial range energy cascade, which is more unsatisfactory even than the small difference in spectral exponent.

5.3 DIA and inertial range

Let the real function $H(\xi)$, defined on the semi-infinite interval $0 \leq \xi < \infty$, have the properties

$$H(0) = 1, \quad H(\xi) < 1 \text{ for } \xi > 0, and \int_0^\infty H(\xi) d\xi \ < \ \infty \tag{52}$$

Then standard properties of a delta function imply

$$\lambda H[\lambda(t-s)] \ \sim \ \delta(t-s) \text{ for } \lambda \to \infty \tag{53}$$

On rewriting Eq. (47) in the time domain, and evaluating the wavevector integrals in the distant interaction approximation in which $k \to 0, \ \ p, q \to \infty$, one obtains

$$\eta(\mathbf{k}, t, s) = \int\limits_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p} d\mathbf{q} \ B(\mathbf{k}, \mathbf{p}, \mathbf{q}) G(p, t, s) Q(q, t, s)$$

$$\sim \int d\mathbf{p} \ \{k_m \frac{\partial B}{\partial q_m}(\mathbf{k}, \mathbf{p}, \mathbf{p}) G(p, t, s) Q(p, t, s)$$

$$-B(\mathbf{k}, \mathbf{p}, \mathbf{p}) G(p, t, s) k_m p_m p^{-1} \frac{dQ}{dp}(p, t, s)\}.$$

$B(\mathbf{k}, \mathbf{p}, \mathbf{q})$ denotes the product of the projection operators in Eq. (47). Assuming time stationary similarity forms

$$G(p, t, s) = G(p^r(t-s))$$
$$Q(p, t, s) = R(p^r(t-s)) Q(p)$$

the properties Eqs. (52) of H may reasonably be postulated of the product GR. Therefore, Eq. (53) implies that in this limit the damping is Markovian

$$\eta(\mathbf{k}, t, s) \ = \ \delta(t-s) \eta(\mathbf{k})$$

and the DIA response equation implies that the Green's function is exponential,

$$G(k, t, s) = \exp\ [(s-t) \eta(\mathbf{k})] \ \text{ for } t \geq s \tag{54}$$

Likewise evaluating the force correlation Eq. (48) in the distant interaction limit implies that the forcing is white noise in time:

$$< f_i(\mathbf{k},t) f_j(\mathbf{k}',s) > = \delta(t-s)\delta(\mathbf{k}+\mathbf{k}')F_{ij}(\mathbf{k}) \tag{55}$$

These simplifications of DIA permit analytical evaluation of the inertial range constants. Although values of these constants could be inferred from numerical solutions of DIA, say for decaying turbulence, it is natural to attempt analytical evaluation as well.

Introducing Eqs. (54) and (55) with the Kolmogorov scaling forms

$$E(k) = C_K \mathcal{E}^{2/3} k^{-5/3} \tag{56}$$

$$\eta(k) = C_D \mathcal{E}^{1/3} k^{2/3}$$

into the DIA response equation integrated over all time separations, the result has the form (Leslie, 1973)

$$\frac{C_D^2}{C_K} \propto \tilde{u} \tag{57}$$

Integrating the single time equation for the correlation function with respect to wavenumber k leads to a second equation, which for Kolmogorov scaling gives (Leslie, 1973)

$$\frac{C_D}{C_K^2} = .1904 \tag{58}$$

Thus, as explained by Leslie (1973), energy transfer in DIA is local, but temporal decorrelation is nonlocal. A simple way to suppress the dependence on the rms velocity $\tilde{u}$ in Eq. (57) and thereby impose the locality of temporal decorrelation, is to restrict the region of integration in the response equation to triads satisfying $p \geq \alpha k$. Values of $C_D(\alpha)$ and $C_K(\alpha)$ have been evaluated by Kabbabe (1970) and tabulated by Leslie (1973); obviously, the intervention of α as a disposable parameter is theoretically not all together satisfactory.

5.4 Sweeping and Lagrangian theories

Kraichnan identified the physical origin of the prediction Eq. (51): at the level of second order statistics, the use of Eulerian correlation induces sensitivity of energy transfer to the rms excitation. This was the problem of 'sweeping' described earlier. Kraichnan concluded that since Lagrangian time correlations are not sensitive to the sweeping effect, a second-order statistical closure must be formulated in terms of Lagrangian variables, not Eulerian variables (Kraichnan, 1964).

The crucial distinction between Eulerian and Lagrangian time correlations in turbulence was rediscovered and popularized by Tennekes (1975). The distinction can be motivated by observing that in the Eulerian frame, temporal decorrelation is due to the Eulerian acceleration, the nonlinear advection term $u_p \partial u_i / \partial x_p$. The u_p term can be sensitive to the rms excitation. In the Lagrangian frame, however, the time decorrelation is determined by the pressure. But since $p = \nabla^{-2} \partial u_i / \partial x_j \partial u_j / \partial x_i$ depends only on velocity *gradients*, it is unaffected by large-scale motions.

The reformulation of DIA in Lagrangian variables led first to the Lagrangian History DIA (LHDIA, Kraichnan, 1965a). This theory restored compatibility with Kolmogorov $k^{-5/3}$ scaling in the inertial range. However, it is an exceedingly complex theory requiring four time arguments instead of two. Moreover, the derivation introduces modifications of the Eulerian formulation that may seem hard to motivate. Further assumptions led to a simplification, the Abridged LHDIA (ALHDIA) which required only two time arguments. But here too, the steps taken may not seem logically compelling. Despite these apparent theoretical drawbacks, computation of steady-state spectra using these theories led to values of the Kolmogorov constant in very good agreement with measurements (Kraichnan, 1965b) – suggesting that these theories are reasonable even if not very attractive analytically.

A major breakthrough in the theory was Kaneda's (1981) derivation of the Lagrangian Renormalized Approximation (LRA), a two time argument theory derived by a straightforward and systematic series reversion procedure. This theory also gave very satisfactory predictions for the Kolmogorov constant. But whereas the Eulerian two-time correlation is trivially a unique statistic, the formulation of two-time argument Lagrangian theories like ALHDIA and LRA, proves to permit several distinct choices of two-time correlations. Unfortunately, this subtle issue cannot even be formulated without considerable explanation; suffice it to say that different versions of the LRA theory can be formulated, differing in the choice of a 'representative' – Kaneda's term for the two-time quantity chosen.

By direct evaluation of alternative closures, Kaneda demonstrated that the Kolmogorov constant is quite robust and not strongly dependent on the choice of the closure scheme; nevertheless, this choice may not seem theoretically necessary. Moreover, the introduction of Lagrangian variables appears to lose the existence of stochastic model representations and the possibility of an *a priori* proof of realizability. Finally, we have the puzzling question of whether the Lagrangian theory is only required because we are specifically seeking closure at the level of second-order statistics.

This not an entirely satisfactory situation and remains the state of the art in turbulence closure theory. Let us summarize the main points:

(1) DIA is realizable but inconsistent with Kolmogorov's theory.
(2) Lagrangian versions of DIA restore consistency with Kolmogorov scaling, but unlike DIA, they are not known to be either exact closures for a statistical model, or to admit Langevin model representations that establish their realizability. Although there is no evidence that Lagrangian DIA in any form is not realizable, neither is there a conclusive proof that it is.
(3) DIA-based theories including the Lagrangian theories, depend on two-time quantities and triad interactions. Both are physical necessities but impose substantial computational burdens. These theories have never been applied outside the simplest problems of decaying and forced steady state turbulence.
(4) DIA-based theories do not seem to be rationally extendable to higher order, more accurate theories. Closure at the level of third order moments, for example, is formulated by Kraichnan (1985). However, even formulation of the corresponding closure equations is very far from straightforward [21]

, and some generalizations for the passive scalar are even demonstrably non-realizable. An interesting alternative formulation of higher order models is the *decimation scheme*) (Kraichnan, 1985)

5.5 Local energy transfer

A direct outcome of Kraichnan's work is that the Eulerian based closure models are fundamentally incapable of avoiding the spurious dynamical effects of sweeping on the energy transfer. As a result, a much more complex Langrangian framework must be employed to recover the Kolmogorov $k^{-5/3}$ spectrum.

This conclusion appeared to be contradicted by McComb & Shanmugasundaram (1984) and McComb (1990). In Fig. 14 of McComb (1995), the one-dimensional spectrum obtained from their Eulerian local-energy transfer (LET) (McComb & Shanmugasundaram, 1984) is compared with the predictions from Lagrangian-history theories (Herring & Kraichnan, 1979), as well as several sets of high Reynolds number experiments (including that of Grant *et al.*,

[21] For the application of this method to the random coupling model of Batchov (1966), see Williams *et al.*, 1987, 1989a,b)

1962). The results of the LET seemed compatible with the Kolmogorov spectrum.

McComb (1995) suggested that this unexpected agreement between his Eulerian theory and the other Lagrangian theories is a result of the cutoffs in the numerical procedures (see also Fig. 15 of McComb (1995)). It is possible that this reduction to a finite number of modes could be imposing the sort of cutoff at large scales that would suppress the sweeping effect, much like the suggestion of Leslie (1973) noted earlier under Eq. (58).

Indeed, in opposition to the view of Kraichnan (1965) that no Eulerian theory can possibly distinguish between convective and inertial transfer effects, McComb & Kiyani (2005) focused on how to suppress the divergence that leads to Eq. (57) in an Eulerian context. To this end, the LET was reformulated using a renormalized response function connecting two-point covariances at different times (Kiyani & McComb, 2004). The resulting relationship was first specialized by chosing the initial conditions in the form of a fluctuation-dissipation ralation. The new derivation of LET was shown to contain a counterterm which removes the singularity of previous propagator equations. Another specialization, assuming exponential two-time dependence, was made to show that the LET closure is well behaved in the limit of infinite Reynolds number.

6 RG Procedure

6.1 A simple cartoon for multiple-scale eliminations

In RG, one partitions the unresolvable subgrid scales $[k_c, k_0]$ into shells, characterized by a scale factor f, $0 < f < 1$ ($f = 1-h$). The portion of the spectral scales that one would like to elmininate is partitioned by the wave number set $k_c \equiv k_N = f^N k_0, k_{N-1} \equiv f^{N-1}k_0, ..., k_1 = fk_0, k_0$. k_0 is typically chosen to be on the order of the Kolmogorov dissipation wave number $(2\pi/L_K)$, while k_c is the wave number which separates the actual resolvable scale $k < k_c$ from the unresolvable scales $k_c < k \leq k_0$.

The RG iterative procedure consists of first eliminating the highest wave number subgrid shell $k_1 < k \leq k_0$ from the Navier-Stokes equation for the remaining "supergrid" wavenumbers $k < k_1$.

$$u_i(\mathbf{k}, t) = \begin{cases} u_i^>(\mathbf{k}, t) & \text{if } k_1 < k < k_0, \\ u_i^<(\mathbf{k}, t) & \text{if } k < k_1, \end{cases} \tag{59}$$

For $k < k_1$, the resolvable scale Navier-Stokes equation is

$$\Big[\frac{\partial}{\partial t} + \nu_0 k^2\Big] u_i^<(\mathbf{k},t) = \lambda_0 M_{imn}(k) \int d^3p [u_m^<(\mathbf{p},t) u_n^<(\mathbf{k}-\mathbf{p},\mathbf{t})$$

$$+ \underbrace{2u_m^>(\mathbf{p},t)u_n^<(\mathbf{k}-\mathbf{p},t)}_{A} + \underbrace{u_m^>(\mathbf{p},t)u_n^>(\mathbf{k}-\mathbf{p},t)}_{B}], \tag{60}$$

The strength of the nonlinear interaction is denoted by λ_0, a formal ordering parameter for perturbation theory, which is eventually set to unity. It is convenient to label the kinematic viscosity $\nu \equiv \nu_0$.

For the subgrid scales, $k_1 < k < k_0$, we have

$$\Big[\frac{\partial}{\partial t} + \nu_0 k^2\Big] u_i^>(\mathbf{k},t) = \lambda_0 M_{imn}(k) \int d^3p [\underbrace{u_m^<(\mathbf{p},t) u_n^<(\mathbf{k}-\mathbf{p},\mathbf{t})}_{I}$$

$$+ \underbrace{2u_m^>(\mathbf{p},t)u_n^<(\mathbf{k}-\mathbf{p},t)}_{II} + \underbrace{u_m^>(\mathbf{p},t)u_n^>(\mathbf{k}-\mathbf{p},t)}_{III}]. \tag{61}$$

The RG multiple-scale elimination schemes can be illustrated by a simple cartoon. The formalism introduces a cutoff wavenumber k_1 which is initially around the Kolmogorov scale. Again dropping the indices and wavevector arguments for the sake of clarity, we rewrite the Navier-Stokes equation (Eqs. (60 and (61))as

$$G^< u^< - M(u^<u^< + u^<u^> + u^>u^< + u^>u^>) = f^< \tag{62}$$

$$G^> u^> - M(u^<u^< + u^<u^> + u^>u^< + u^>u^>) = f^> \tag{63}$$

The resolvable scale propagator is now denoted by $G^< \equiv (\partial/\partial t + \nu k^2)^{-1}$, for k residing in the resolvable scales and the corresponding subgrid scale propagator is given by $G^> \equiv (\partial/\partial t + \nu k^2)^{-1}$, with the wavenumber k in the subgrid shell. ν is the renormalized eddy viscosity (although in the first subgrid shell elimination it is just the molecular viscosity $\nu_0 = \nu$).

Formally, the $u^>$ modes are eliminated from Eq. (62) by solving for $u^>$ using Eq. (63). This is found to result in the replacement of the viscosity in Eq. (62) by an enhanced viscosity. Another effect is the generation of the triple product of the resolvable scale velocity fields, $u^<u^<u^<$.

In the process of removing next subgrid shell, the modification to the Navier-Stokes equation are (a) a renormalized eddy viscosity coefficient, $\nu(k) = \nu^{>>}(k) + \nu^{><}(k)$ from both double and triple velocity products and (b) a new triple nonlinearity in the renormalized Navier-Stokes equation.

One now proceeds iteratively, removing at the i-the step the subgrid shell $k_i < k < k_{i-1}$ and continues with this subgrid shell elimination until one reaches the actual resolvable scales at the N-th step [22].

6.2 Comparison with DIA

It should be noted that the formal RG procedure is essentially identical to the closure procedure of DIA, the so-called 'one-loop' line-renormalization perturbation theory (Orszag, 1977). This procedure is applied incrementally in RG, but 'all at once' in DIA (Kraichnan, 1987a). Symbolically, the steps can be written as

$$M(k)u(p)u(q) \to M(k)G(p)M(p)u(k)u(-q)u(q) = \nu(k)u(k) \tag{64}$$

by substituting $G(p)M(p)u(k)u(-q)$ for $u(p)$. The result is the formal effective viscosity $\nu(k) = \int dpdqM(k)G(p)M(p)u(-q)u(q)$.

Our purpose here is not to justify this formalism, but only to note that it occurs in both theories. In this respect, RG on the one hand contributes nothing new, but on the other hand it does follow a well-established formalism. Nevertheless, it is reasonable to ask whether the *incremental* elimination of modes makes the procedure more rigorous because removing a 'small' range of modes might be thought of as a small perturbation. Although this argument has been denounced repeatedly, it should be noted that a related idea does appear in one of Kraichnan's derivations of DIA, namely, in the argument that any one triad interaction weakly perturbs the totality of triad interactions (Kraichnan, 1961). DIA may be understood as a non-standard perturbation theory in which the basic state is unknown, but is determined self-consistently by requiring the validity of this perturbation hypothesis. Curiously, this argument seems more appropriate if the modes are indeed eliminated all at once rather than incrementally. Nevertheless, it seems reasonable to conclude that the incremental elimination of modes in RG by itself is neither more nor less 'rigorous' than the closure procedure adopted in DIA. [23]

[22] Zero helicity is assumed here. For the effect of the helicity on renormalized eddy viscosity, see Zhou, 1990.

[23] We could perhaps note that if 'rigorous' derivation is understood to require derivation from the Navier-Stokes equations alone, 'rigor' is impossible: one might as well demand the derivation of thermodynamics from Newton's laws alone. That is not to deny the possibility or desirability of more plausible or convincing statistical hypotheses than those of current turbulence theory.

Rose (1977) attempted to offer a rationale for removing the subgrid scale incrementally. While his original discussion was given for a model scalar equation, it will instead be repeated here for the Navier-Stokes equation. Essentially, in order for the RG procedure to work, the effective Reynolds number should be less than one. For a cutoff wavenumber k_c, let V_c^2 denote the energy of the velocity modes in the neighbourhood of this wavenumber. The approximation of RG would be untenable if ν_0 is not small enough, since the effective Reynolds number

$$V_c/k_c\nu_0 \tag{65}$$

would be much greater than one.

Rose (1977) reasoned that if one could solve for the subgrid modes in terms of the resolvable modes in such a way that the eddy viscosity replaces the kinematic viscosity in Eq. (65) then the effective Reynolds number would be significantly lowered. This can be partially realized by regarding the ultimate subgrid model as the product of two subsidiary calculations (as an illustration, $(k_c, (1/2)k_0)$ and $((1/2)k_0, k_0)$).

To remove the first subgrid shell, the k_c in Eq. (65) is replaced by one with an uppper cutoff of $(1/2)k_d$. The molecular viscosity is enhanced by the factor

$$\int_{(1/2)k_d}^{k_d} E(p)dp]^{1/2} \ll V_c,$$

As a result, the effective Reynolds number R_e is reduced.

Next, the second calculation models away the modes between $(1/2)k_d$ and k_0. The effective Reynolds number is again smaller because the molecular viscosity is replaced by the enhanced eddy viscosity.

Rose (1977) pointed out that the above process, whereby the subgrid model is produced in two stages, can be further subdivided by eliminating all of the subgrid modes in several stages. When all the subgrid shells have been removed, the effective viscosity attains its final value, of order V_c/k_c. In this way, the effective Reynolds number for the removal of a particular shell is always of order unity, and the final subgrid model has been constructed through the use of a series of uniformly valid approximation (Rose, 1977). A similar argument, that a 'renormalized Reynolds number' based on an effective viscosity is $O(1)$, is made by YO, although this value is not necessarily numerically small.

7 First Applications of RG to Fluids

This section summarizes several papers that have applied the RG incremental elimination procedure to stirred fluid motion [24]. The correlation function of the forcing offerred these authors the freedom to introduce a parameter [25], ϵ. Now the RG multiple scale elimination procedure describled in the previous subsection will be coupled with the perturbation expansion in this parameter ϵ.

The work of Forster, Nelson, & Stephen (FNS) carried out the RG procedure following Ma & Mazenko (1975) [26]. It should be noted, however, that all three forcing correlation models considered in FNS are not directly relevant to the turbulent flows of high Reynolds numbers. This fact will become clearer when the three forcing models are reviewed in the next subsection. DeDominicis & Martin (DeDm 1979) attempted to alleviate one of the limitations in FNS by connecting the forcing correlation to the Kolmogorov energy spectrum.

Kraichnan (1982) remarked that the RG procedure, as carried out by FNS and DeDM, does not offer anything more than the closure theories such as that of DIA. In response, Fourier & Frisch (1983) (FF henceforth) took one step further by obtaining a direct relationship between the factor in front of the Kolmogorov spectrum and the strength of the forcing correlation function.

7.1 Forster, Nelson, & Stephen

Forster *et al.* (FNS) applied the RG procedure to a stirred fluid. The forcing autocorrelation function was assumed to take the form

$$< f_i(\mathbf{k},\omega) f_j(\mathbf{k}',\omega') >= F(k)\delta(\mathbf{k}+\mathbf{k}')\delta(\omega+\omega')(\delta_{ij} - k_i k_j/k^2), \tag{66}$$

where various choices are made for the function $F(k)$.

Specifically, three models were investigated:

Model A:

[24] Martin *et al.* (1973) and Nelkin (1974) were the first who discussed the feasibility in applying the RG methods to turbulent flows

[25] Depending on the authors, this perturbation parameter may be introduced in a slightly different way

[26] Note that Ma & Mazenko (1975) focused their attention on the dynamics of an isotropic ferromagnet - a system that falls within equilibrium statistical statistical mechanics and the evaluation of partition functions

$$F(k) = \begin{cases} D_0 k^2 & \text{if } |k| < \Lambda, \\ 0 & \text{otherwise,} \end{cases} \tag{67}$$

Model B:

$$F(k) = \begin{cases} D_0 & \text{if } |k| < \Lambda, \\ 0 & \text{otherwise,} \end{cases} \tag{68}$$

and finally,
Model C:

$$F(k) = \begin{cases} D_0 & \text{if } \overline{\Lambda} < |k| < \Lambda, \\ 0. & \text{otherwise,} \end{cases} \tag{69}$$

Here Λ is a cutoff wavenumber and D_0 is a forcing amplitude.

Model A can be considered simply as a Langevin model for a fluid near equilibrium. In this case the fluctuation-dissipation theorem requires the forcing amplitude $D_0 = \nu_0 k_B T/\rho$. It can also be thought of as representing some macroscopic stirring force whose spatial integral vanishes. *Model B* includes a statistically defined force which acts on the fluid even at $k = 0$. While it is perhaps somewhat artificial to imagine exciting a fluid even at $k = 0$, *Model B* does exhibit intriguing behavior below four dimensions [27] . *Model C* is perhaps the most realistic. The fluid is excited in a band in k space below $k = \Lambda$, and one is interested in the resulting correlations near $k = 0$.

FNS showed that the infrared behavior of *Model C* is the same as that of *Model A*, which is a further motivation for considering *Model A* . *Model A* generates the familiar long-time tails in the renormalized viscosity, and produces new singularities at small wavenumber as well.

FNS found that a kind of universality applies. Large classes of models exhibit similar infrared, long-time properties. The most "realistic" models all exhibit a spectral density function $E(k)$ which scales as k^{d-1} for small wavenumbers, where d denotes the dimensionality. This agrees with a result obtained by Saffman (1967) for homogeneous isotropic turbulence. FNS stressed that their considerations refer to the region of effectively small Reynolds' number, and $E(k) \propto k^{d-1}$ is simply a consequence of equipartition and phase-space considerations.

Finally, FNS also remarked that *Model B* leads to rather different results at small k. Here nonlinearities dominate the infrared behavior of $E(k)$ below four dimensions, and lead to logarithmic anomalies for dimension $d = 4$.

[27] Heuristically, it corresponds to a macroscopic "shaking" of the fluid container. FNS credited this interpretation to P.C. Martin.

7.2 DeDominicis & Martin

DeDominicis and Martin (1979, DeDM) considered the singular case of a random stirring force in which equal weight is given to all wave vectors, i.e., a force characterized by a noise correlation essentially proportional to k^{-d}. DeDM showed that this stirring force yielded a Kolmogorov spectrum. This derivation of the Kolmogorov spectrum depended on a special noise force correlation and did not address the central issue of why such a spectrum, or one that does not deviate greatly from it, should be found in experiments on strong turbulence (DeDM). Nevertheless, the model may provide a concrete starting point for quantitatively studying discrepancies from the Kolmogorov predictions, and how universally they apply.

To avoid uninteresting infrared divergences, DeDM take a white noise random force whose only nonvanishing cumulant is

$$< ff > \approx D_0 k^{4-d} (m_0^2 + k^2)^{-y/2} \tag{70}$$

where m_0^{-1} is a stirring length (infrared) cutoff. DeDM paid special attention to the asymptotic domain in which

$$m_0 \ll k \ll \Lambda, \tag{71}$$

and eventually let the untraviolet cutoff Λ tend to infinity. In this limit, for any $y \leq 4$ (and $d > 2$), the energy spectral function of FNS can be generalized to

$$E(k) \approx k^{1-2y/3}. \tag{72}$$

The Kolmogorov behavior for the spectral function is approached as y approaches 4 from below (i.e., $y \to 4_-$), for the region which is ultraviolet with respect to m_0 and infrared with respect to Λ.

7.3 Fourier & Frisch

Fourier and Frisch (hereafter FF) focused their attention on the forcing correlation function, assuming the $F(k)$ in Eq. (24) has the form

$$F(k) = 2Dk^{3-\epsilon} \tag{73}$$

$F(k)$ is the amount of energy injected per wavenumber and D is the forcing amplitude. When ϵ is positive and small, the resulting energy spectrum for the turbulence is

$$E(k) \propto k^{1-2\epsilon/3}, \tag{74}$$

- although this is basically a dimensional scaling result and can also be obtained from DIA. This is exactly the argument offered by Kraichnan (1982), who argued that the RG method (in the work of FNS and DeDM) is in no way superior to standard closure methods.

To counter Kraichnan's point, FF went beyond the previous work by deriving more quantitative information than just determining the scaling of the energy spectrum. As noted by FF, previous RG calculations did not attempt to evaluate the proportionality constant in the power law; indeed, in other physical problems to which RG methods have been applied, these constants were not even universal.

FF wrote down the forcing correlation function in (k, ω), where the energy injection spectrum $F(k)$ has the power law Eq. (73). When $\epsilon > -1$, this corresponds to the generalization of *model B* of FNS. In the borderline case of $\epsilon = -1$, the assumed injection spectrum corresponds to *model A* of FNS. The purpose of FF was to calculate the statistical properties of the solution of NSE at a fixed wavenumber and a fixed viscosity $\nu_0 > 0$ as $\epsilon \to 0$.

FF assumed the equivalence

$$E(k; \nu_0, D, \Lambda_0) = E(k; \nu(\Lambda), D, \Lambda), \tag{75}$$

and required that $\nu(\Lambda)$ and $\nu(\Lambda_0) \equiv \nu_0$ be related by

$$\nu^3(\Lambda) - \nu^3(\Lambda_0) = 3C_3\epsilon^{-1}(\Lambda^{-\epsilon} - \Lambda_0^{-\epsilon}), \tag{76}$$

where $C_3 = 1/10\pi^2$.

In the $\epsilon \to 0$ limit, the renormalized Reynolds number is small and FF calculated $E(k; \nu(\Lambda), D, \Lambda)$ using the linear approximation

$$E(k; \nu(\Lambda), D, \Lambda) \approx \frac{2Dk^{3-\epsilon}}{2\nu(\Lambda)k^2} \tag{77}$$

where $k \leq \Lambda = O(1)$. The final result was

$$E(k; \nu_0, D, \Lambda_0 = \infty) \approx D^{2/3}(3C_3)^{-1/3}\epsilon^{1/3}k^{1-2\epsilon/3} \tag{78}$$

in the limit $\epsilon \to 0$. The amplitude in front of the powerlaw $k^{1-2\epsilon/3}$ was determined explicitly in terms of ϵ and D.

The FF result is universal in the sense that – under the assumptions made – it does not depend on the molecular viscosity or on the small scale forcing.

8 ϵ-RG

8.1 ϵ-RG procedure

Building on the work of FF just described, YO extended both the theory and the applications suggested by these predecessors. In this Section, YO's notation will be used for space-time Fourier transforms

$$\hat{k} = (\mathbf{k}, \Omega) \quad \hat{p} = (\mathbf{p}, \omega) \quad \hat{\mathbf{q}} = (\mathbf{q}, \mathbf{\Omega} - \omega)$$

The triadic condition $\mathbf{k} = \mathbf{p} + \mathbf{q}$ will be assumed to hold throughout our discussion; the notation insures the corresponding frequency matching condition.

The starting point of YO's analysis is the Navier-Stokes equations in d-dimensional space driven by a random force $\mathbf{f}$:

$$(-i\Omega + \nu_0 k^2) u_i(\hat{k}) - M_{imn}(\mathbf{k}) \int d\hat{p}\ d\hat{q}\ u_m(\hat{p}) u_n(\hat{q}) = f_i(\hat{k}) \tag{79}$$

The time Fourier transform indicates that the analysis assumes time stationarity as well as spatial homogeneity. The random force is assumed to be Gaussian with correlation function

$$\langle f_i(\hat{k}) f_j(\hat{k}') \rangle = 2D(2\pi)^{d+1} k^{-y} D_{ij}(\mathbf{k}) \delta(\hat{\mathbf{k}} + \hat{\mathbf{k}}') \tag{80}$$

Note that this force is white noise in time. Its spatial correlation depends on the amplitude D and y is a scaling exponent that is treated as a variable for the purposes of the subsequent ϵ-expansion.

The mode elimination procedure of Section 6 is implemented, ı*without* retaining the triple nonlinearity. Iterative mode elimination then generates a momentum equation of the same form but with an enhanced viscosity which is found to satisfy the recurrence relation

$$\frac{d\nu(k)}{dk} = A \frac{D}{\nu^2 k^{\epsilon - 1}} \tag{81}$$

where A is computed explicitly by the theory and

$$\epsilon = 4 + y - d. \tag{82}$$

The energy spectrum has the form

$$E(k) = C_4 D^{2/3} k^{-5/3+(4-\epsilon)/3}, \tag{83}$$

where C_4 is given explicitly by the theory.

YO's conclusion is that the nonlinear term the Navier-Stokes equation can be replaced, in the limit of infinite Reynolds number, by the combination of random forcing f_i and a scale dependent viscosity $\nu(k)$, so that

$$-i\Omega u_i(\hat{k}) + \nu(k)k^2 u_i(\hat{k}) = f_i(\hat{k}) \tag{84}$$

At this point, the scaling exponent of the forcing y in Eq. (80) must be considered, and this brings us to the heart of the YO theory. Substituting $y = d$ in Eqs. (81) and (83) gives the scaling laws

$$\nu(k) \sim D^{1/3} k^{-4/3} \tag{85}$$
$$E(k) \sim D^{2/3} k^{-5/3} \tag{86}$$

which coincide with Kolmogorov scaling if D can be identified with the energy flux $\mathcal{E}$. Note from the definition Eq. (82) that $y = d$ corresponds to $\epsilon = 4$.

But the crucial observation is that the nonlinear coupling in the suitably rescaled Navier-Stokes equations is proportional to $\sqrt{\epsilon}$ in the limit $k \rightarrow 0$. Turbulence driven by a force for which $\epsilon = 0$ is therefore asymptotically linear (in the largest scales of the motion). An expansion about $\epsilon = 0$ in powers of ϵ appears to be that great *desiridatum* of turbulence theory: a rational expansion in powers of the nonlinear interaction. The introduction of this expansion went well beyond closure theories like DIA. Although the expansion is motivated by analogous ideas in the theory of critical phenomena, where similar RG procedures have achieved remarkable results for several physical systems in equilibrium (see review articles on the applications of RG to Ising model, for example), its application to turbulence (a system far from equilibrium) requires independent justification.

To summarize, $\epsilon = 4$ gives us the Kolmogorov scaling regime of interest, while $\epsilon = 0$ will provide the focal point for perturbation expansions. However, for this case of $\epsilon = 0$

$$\nu(k) \sim k^0 \tag{87}$$
$$E(k) \sim k^1 \tag{88}$$

Eq. (87) restates the observation made earlier: since the effective viscosity $\nu(k)$ that accounts for the effect of nonlinear interactions is *constant* at large scales when $\epsilon = 0$, any nonlinearity must necessarily be very weak. This observation again supports the idea that the ϵ-expansion is an expansion in powers of the strength of the nonlinearity.

We now would like to summarize some of the conclusions resulting from this expansion procedure. They will address the questions raised earlier about DIA-based analytical closures.

8.2 Higher order nonlinearities

Exact mode elimination using Eqs. (62) and (63) will generate an infinite set of nonlinear terms of higher order than appear in the quadratically nonlinear Navier-Stokes equation. However, in the YO theory, the nonlinearity remains quadratic. YO justified this radical simplification on the basis of the ϵ-expansion: it was stated that all the higher order nonlinearities scale as (powers of) $k^{4-\epsilon}$. Hence, if $\epsilon < 4$, these higher order nonlinearities $\to 0$ in the large-scale limit $k \to 0$. Although it is true that when $\epsilon = 4$, these higher order nonlinearities are $O(1)$, YO argued that their effect could be expected to be small as $\epsilon \to 4_-$.

A related statement by YO is that their expansion need not go beyond the quadratic terms in the coupling constant because these terms are of higher order in ϵ. Without presenting the details, we simply point out that this 'one-loop' order of perturbation theory appears in both the YO theory and in DIA. This comment, together with the irrelevance of higher order nonlinearities for $\epsilon < 4$ and $k \to 0$, appears to give some rationality to YO's form of closure: 'higher order' closures of the type sketched by Martin *et al* (1973) only generate small corrections in ϵ. Of course, $\epsilon = 4$ is still a sticking point.

8.3 Triad interactions and time-history dependence

YO's effective viscosity $\nu(k)$, like the DIA factor $\eta(k)$, is an 'eddy damping' factor due to nonlinear interactions. But the first thing to note is that the triad interactions that appear in the definition of η in Eq. (43) do not appear in the YO theory expression Eq. (81), which involves only the wavenumber argument k. Moreover, YO's damping is purely Markovian: in the time stationary context of the YO theory, this is reflected in the absence of any frequency dependence in ν. It should perhaps be noted that such dependence on a single wavenumber is the common property of all RG theories [28].

A vast reduction in analytical complexity is thus achieved. Its origin can again be sought in the ϵ-expansion, because in the limit $\epsilon = 0$, the energy spectrum peaks at small scales. Moreover, as Kraichnan (1987) has observed, this peaking of the energy spectrum at small scales indicates the dominance of asymptotically *distant* triadic interactions.

8.4 Eulerian vs. Lagrangian theory

Finally, YO is strictly an Eulerian theory: no mention of Lagrangian variables is made, and mode elimination is carried out on the complete nonlinear term including the advection term, not merely the pressure term. Yet, the YO theory is consistent with Kolmogorov scaling. YOS (Yakhot *et al.*, 1989) sought to justify this conclusion using the ϵ-expansion: 'sweeping' interactions were claimed to be negligible compared to local straining interactions when $\epsilon = 0$, and the ϵ-expansion was invoked to assume that this same conclusion will continue to hold in the Kolmogorov limit $\epsilon = 4$ (see Sec. 10 for a detailed study on this claim).

8.5 Evaluation of inertial range constants

An important application of the ϵ-expansion was in the evaluation of 'inertial range constants:

$$E(k) = C_K \varepsilon^{2/3} k^{-5/3}, \tag{89}$$
$$\eta(k) = C_D \varepsilon^{1/3} k^{2/3}. \tag{90}$$

Here, C_K and C_D are the constants that enter the Kolmogorov scaling. C_K in Eq. (89) defines the famous 'Kolmogorov constant'. Its derivation is rightly regarded as an important test of a turbulence theory: while dimensional analysis alone can give the $k^{-5/3}$ scaling law, no similar elementary consideration will recover the value of C_K. Eq. (90) defines a second constant, which unfortunately has no standard name or notation and a dearth of measured values. It is related to the *two-time* properties of homogeneous isotropic turbulence. These constants had been computed earlier by Kraichnan (1964) and Kaneda

[28] Note that the eddy viscosity here is computed after the distant interaction limit. The spectral eddy viscosity of the recursive RG in Sec. 12, without the distant interaction limit, results from triadic interactions

(1981) by numerical integration of Lagrangian closure theories. We also noted Kabbabe (1970) and Leslie's (1973) computation using DIA with an *ad hoc* cutoff at large scales. On the other hand, the calculation in YO is particularly simple.

9 Perspectives on ϵ-RG

This section will review some of the criticism that have been raised against the YO theory.

9.1 Higher order nonlinearities

YO's claim that the higher order nonlinearities are negligible was contradicted by Eyink (1994), who determined that, on the contrary, higher nonlinearities always scale as k^0 rather than $k^{4-\epsilon}$. In this case, these higher nonlinearities cannot be said to be small in any perturbative sense: in the statistical mechanical language of this debate, the higher order nonlinearities are always 'marginal,' never 'irrelevant'. The crucial claim is that this conclusion holds, even at $\epsilon = 0$. Readers can evaluate the arguments for themselves by reference to the original articles. But if higher order nonlinearities are indeed marginal even when $\epsilon = 0$, we return to the situation of DIA-based closures: higher order nonlinearities are simply ignored (or retained) depending on how the theory is constructed; no general principle governs their inclusion or exclusion. The neglect of these nonlinearities in the YO theory simply reflect an assumption which cannot be further justified.

9.2 Triad interactions and time history dependence

The role of triad interactions in the YO theory is rather subtle. On the one hand, the eddy viscosity is constructed without explicit consideration of triadic interactions; this type of approximation was shown by Kraichnan (1987a) to occur when replacing general nonlinear interactions by *distant* interactions such that $k \ll p \approx q$. On the other hand, the energy balance in YO was taken from RPT (Dannevik *et al.*, 1987) in which triad interactions appear explicitly [29].

[29] Specifically, Dannevik *et al.* linked YO ϵ-RG with the eddy-damped-quasi-normal Markovian (EDQNM) approximation

Indeed, the energy flux balance cannot be obtained correctly without the consideration of triadic interactions: this point will be discussed in Section 9.4. The prominence given by the YO theory to distant interactions suggests analogies to the classical Heisenberg model (Batchelor, 1953) and various other proposals (Canuto *et al.*, 1987; Canuto & Dubovikov, 1996; Rubinstein & Clark, 2004). Kraichnan (1987a) had already shown that such approximations result from replacing general nonlinear interactions by *distant* interactions such that $k \ll p \approx q$.

Yet both analytical closure (Kraichnan, 1976) and DNS data (Zhou, 1993a,b, Gotoh & Watanabe, 2005) suggest that asymptotically distant interactions are not in fact dominant, but instead it is the slightly elongated wavevector triads with the maximum to the minimum side ≈ 2 which are responsible for most of the energy transfer. This led Kraichnan (1987b) to ask whether the Kolmogorov constant, apparently computed by YO using a theory based on distant interactions, might be insensitive to local interactions.

Thus, one observes that even if the nearly local interactions dominate energy transfer this does not necessarily rule out a reasonably accurate description of the effective energetics of turbulence using a simplified one variable model [30]. The models of Canuto & Dubovikov (1996) and Rubinstein & Clark (2004) both attempt a compromise by adding a 'backscatter' term to represent local interactions. The recursive RG model of Zhou et al. (1988, 1989) and Zhou & Vahala (1993a,b) is a different attempt to represent the effect of local interactions by introducing a higher-order nonlinearity. This fundamentally different formulation of the renormalization group will be discussed in more detail elsewhere in this review.

9.3 The 'correspondence principle'

Whereas it is generally agreed that Eq. (79) provides a plausible model of isotropic turbulence provided the random force f is concentrated at large scales and therefore provides an energy source, the introduction of a force acting on all inertial range scales appears to lack fundamental justification. YO's model Eqs. (79)–(80) can be compared to the DIA Langevin model (Kraichnan, 1976), written here in a time stationary form,

$$-i\Omega u(\hat{k}) + \eta(\hat{k}) u_i(\hat{k}) = f_i(\hat{k}) \tag{91}$$

[30] see Kraichnan and Spiegel, 1962

where the damping function and force correlation are expressed in terms of the DIA response and correlation functions G and Q by

$$\eta(\hat{k}) = 2iM_{rmn}(\mathbf{k}) \int d\hat{p}\ d\hat{q}\ \delta(\hat{k} - \hat{p} - \hat{q}) D_{mr}(\mathbf{p}) D_{ns}(\mathbf{q}) G(\hat{p}) Q(\hat{q}) \tag{92}$$

and

$$\begin{aligned} F(\hat{k}) = \langle f_i(\hat{k}) f_i(\hat{k}') \rangle = -4M_{imn}(\mathbf{k}) M_{irs}(\mathbf{k}) \\ \int d\hat{p}\ d\hat{q}\ \delta(\hat{k} + \hat{k}') D_{mr}(\mathbf{p}) D_{ns}(\mathbf{q}) Q(\hat{p}) Q(\hat{q}) \delta(\mathbf{k} + \mathbf{k}') \end{aligned} \tag{93}$$

Eq. (91) is a generic model in statistical mechanics which replaces the effects of an infinity of nonlinear interactions on any one mode by a random force acting against a generalized damping. The "fixed point" RG model Eqs. (80) and (84) formally resembles the DIA Langevin equation model. But the damping $\nu(k)k^2$ in the RG model Eq. (80) is Markovian, so that

$$\eta(\hat{k}) = \eta(k) \tag{94}$$

only and the forcing in Eq. (84) is white noise in time, so that

$$F(\hat{k}) = F(k) \tag{95}$$

only. Neither of these conditions holds for the DIA Langevin model.

To investigate the connection between these models, we follow Kraichnan to write

$$Q(\hat{p}) = Q(p) R(\hat{p}), \tag{96}$$

where R is the time correlation function. We now perform the frequency integration in Eq. (93) and evaluate the result in the long time limit in which $\Omega = 0$. This limit corresponds to observing the system over times long compared to any characteristic correlation time of the true DIA random force. The result is

$$F(\hat{k}) = -4M_{imn}(\mathbf{k}) M_{jrs}(\mathbf{k}) \int d\hat{p}\ d\hat{q}\ D_{mr}(\mathbf{p}) D_{ns}(\mathbf{q}) Q(p) Q(q) \Theta(k, p, q) \tag{97}$$

where

$$\Theta(k, p, q) = \int_{-\infty}^{\infty} d\omega R(\hat{p}) R(\hat{q}) \mid_{\Omega=0} \tag{98}$$

In this limit, the random force is white in time. Further, in Kolmogorov scaling,

$$Q(\lambda p) = \lambda^{-11/3} Q(p) \tag{99}$$

$$\Theta(\lambda k, \lambda p, \lambda q) = \lambda^{-2/3} \Theta(k, p, q) \tag{100}$$

consequently, the scaling dimension of the random force is found to be -3:

$$F(\lambda k) = \lambda^{-3} F(k) \tag{101}$$

Formally, in the long time limit, the random force in the DIA Langevin model has the same space-time correlation as the force postulated at the outset by YO.

It should be noted that the power counting which leads to Eq. (101) is purely formal, since the actual force correlation integral in Eq. (93), like the integral of Eq. (92), diverges when evaluated by itself for an infinite Kolmogorov inertial range. In the energy flux balance, these divergences cancel, as required by the locality of energy transfer in the theory.

That the -3 force is natural in the context of any steady state far from equilibrium with a constant flux of some invisicid invariant is also suggested by the derivation (Rubinstein, 1994a) of Bolgiano scaling inertial range for buoyant turbulence by applying the YO formalism with forcing of the temperature equation only. From this point of view, the introduction by Lam (1992) of a distinguished infrared scale in the RG force is not necessary: locality means, as in the computation of DeDM, that when this scale becomes infinite, the results of the theory remain finite. This is simply the analytic statement of Kolmogorov's idea of locality of the inertial range.

9.4 Energy balance via the forcing correlation

As we have mentioned before, Fourier & Frisch (FF) were the first to determine the coefficient in the energy spectrum. Specifically, recall that FF found that the forcing strength D appears with the 2/3 exponent ($D^{2/3}$) in the energy spectrum. While this energy spectrum has a $-5/3$ scaling ($k^{-5/3}$) in spectral space, technically it is not yet a Kolmogorov spectrum because the coefficient is not propotional to the dissipation rate $\mathcal{E}$ with the 2/3 exponent ($\mathcal{E}^{2/3}$). This missing link between FF and the Kolmogorov spectrum was provided in YO's evaluation of the Kolmogorov constant, through a relation between the stirring force amplitude D and the dissipation rate. This was also done in YO, where the DIA energy balance was invoked. Later, Dannevik, Yakhot

& Orszag (1987) re-established this result by showing that EDQNM could be deduced as a correction to the lowest order description given by Eq. (79). The required proportionality is

$$D = 15.633\mathcal{E} \tag{102}$$

McComb (1990) objected that the direct integration of the correspondence equation model does not lead to this result because integration of the k^{-3} force leads to a logarithmic integral. According to McComb (1990), this leads to an unrealistically small interval of validity of the theory. From this viewpoint, the necessity to correct Eq. (79) in order to obtain the correct energy flux balance perhaps requires an explanation.

In Dannevik, Yakhot & Orszag (1987), the relationship between the forcing strength and the dissipation rate is obtained by constructing an energy transfer function. However, the starting point for building this energy transfer equation is from the ϵ RG generated Navier-Stokes equation. Recall that the eddy viscosity was obtained by both ϵ expansion and by neglecting local interactions. To get the second order energy equation, another expansion of the velocity field was made in term of order parameters $(O(\epsilon^0, \epsilon^{1/2}, ...)$. The final energy transfer equation is the same as that of the EDQNM model. The issue of why the energy balance is not obtained in the YO theory directly from the correspondence principle will be discussed further later.

9.5 The ϵ-expansion and the distant interaction approximation

The ϵ-expansion has been reconsidered in a number of references including Ronis (1987), Teodorovich (1987, 1993, 1994), Lam (1992), and Wang & Wu (1993). In YO's original presentation, the ϵ-expansion is an expansion about a logarithmically divergent theory. An interesting alternative was suggested by Carati (1990a), who suggested expanding about a theory with vanishing energy transfer (Fournier & Frisch, 1978). Here, this expansion will be considered, following Woodruff (1992, 1994, 1995) and Rubinstein (1994b), as an approximation in DIA.

To complete the transition from the DIA Langevin model to the YO theory, further approximations are required. They are

(1) evaluate the DIA integrals in the distant interaction limit in which $k/p, k/q \to 0$
(2) Markovianize the damping

(3) introduce an infrared cutoff so that the integrals in Eqs. (92),(93) are restricted to $p \geq k$ and $q \geq k$ only
(4) evaluate the amplitudes using the ϵ-expansion

It has been emphasized by Woodruff that these approximations are closely related. First, as noted by Kraichnan (1987), the ϵ-expansion is an expansion about a theory in which distant interactions are dominant; accepting this point provisionally, we outline how the distant interaction limit brings about the Markovianization of the damping and forcing.

The ϵ-expansion can be considered as a method of infrared regularization by analytic continuation. Namely, we now replace Eq. (56) by the general form

$$E(k) = C_K D^{2/3} k^{1-2\epsilon/3} \tag{103}$$

The scale independence of the integrated response equation demands

$$\eta(k) = C_D D^{1/3} k^{2-\epsilon/3} \tag{104}$$

The units of D, consistent with Eq. (80), make these equations dimensionally correct. Substituting these scalings in the integrated response equation gives the ϵ-dependent form of Eq. (57),

$$\frac{C_D^2}{C_K} = \int d\hat{p}\ d\hat{q}\ 2iM_{rmn}(\mathbf{k})D_{mr}(\mathbf{p})D_{ns}(\mathbf{q})\frac{p^{-1-2\epsilon/3}}{(p^{2-\epsilon/3}+q^{2-\epsilon/3})k^{2-\epsilon/3}} \tag{105}$$

The integral in Eq. (105) is ultraviolet divergent when $\epsilon < 0$ and is logarithmic when $\epsilon = 0$. Woodruff observes that it is reasonable to evaluate Eq. (105) for $\epsilon > 0$ by asymptotic expansion about $\epsilon = 0$. This expansion greatly simplifies the integration since the ultraviolet divergence for $\epsilon < 0$ implies that the integral is dominated by distant interactions, namely by wavevector triangles such that $p, q \to \infty$. In this limit, a simple analytical evaluation of the integrals is possible. The calculation gives

$$\frac{C_D^2}{C_K} = \frac{1}{\epsilon}A(\epsilon) = \frac{A_{-1}}{\epsilon} + A_0 + A_1\epsilon + \cdots \tag{106}$$

where

$$A_{-1} = \frac{3}{5}$$

The constant A_{-1} is distinguished since it is the only one in the series Eq. (106) which has been evaluated exactly in two senses. First, increasing the number of "loops," that is, considering terms in the perturbative solution of the Navier-Stokes equations with a larger number of force correlations, will correct A_n only for $n \geq 0$. It can also be shown (Rubinstein, 1994b) that even at the one loop level, correcting the distant interaction approximation by power series expansions in k/p also only corrects A_n for $n \geq 0$. Accordingly, it is reasonable to evaluate Eq. (106) by taking the leading term only. Setting $\epsilon = 4$,

$$\frac{C_D^2}{C_K} = \frac{3}{20}$$

which is equivalent to YO's calculation.

It is sometimes claimed that YO evaluate the inertial range constants by setting $\epsilon = 4$ and $\epsilon = 0$ at different places in the same equation. However, it must be emphasized that *in the calculation given here, ϵ is never set to any value but 4*. The analytical procedure which leads to Eq. (106) is entirely routine: it is the evaluation of the leading term in an asymptotic expansion, not a novel procedure unique to YO.

The ϵ-expansion can be considered to be a regularization necessary to evaluate the right side of Eq. (57), which diverges when $p \to 0$. Triads with $p \sim 0, q \sim k$ correspond to sweeping of modes with wavevector $\mid \mathbf{k} \mid = \mathbf{k}$ by modes of much larger scale. The dynamic significance of this divergence has been elucidated by Kraichnan (1982). This divergence is removed in YO, and indeed in several renormalization group approaches by focusing exclusively on interactions for which $p, q \geq k$.

However, as Woodruff notes, the integral becomes infrared divergent when $\epsilon = 3$, and so the analytic continuation from $\epsilon = 0$ to $\epsilon = 4$ in the YO theory becomes quite problematic: this observation can also be attributed to DeDM. Thus, although it is satisfying to be able to compute the inertial range constants, and even to obtain satisfactory values by a straightforward computation, the fact remains that the analytic continuation which underlies the calculation lacks serious justification. Moreover, Woodruff also suggests that one might attempt an ϵ expansion about this infrared divergence. Not unexpectedly, the results are quantitatively unsatisfactory, but this possibility suggests that the expansion about $\epsilon = 0$ may not be the only possibility. We can conclude that the idea of finding a good point about which to expand is potentially fruitful, it may be that the choice in YO is by no means optimal.

Another objection to this procedure can be raised in connection with Eq. (58): the constant has been obtained by exact evaluation of the triangle integrals making neither the ϵ-expansion nor the distant interaction approximation. Again one should also look back at the observations of McComb (1990) noted earlier (see subsection 9.4). Now the integral can be shown to be ultraviolet divergent for $\epsilon < 4$ and logarithmic exactly when $\epsilon = 4$. Thus, there is no possibility of an ϵ expansion for this integral. There is no alternative but to evaluate it exactly.

Another expansion method was proposed by Carati (1990b), where the parameter ϵ is no longer treated as a small parameter but is fixed at its "final" value $\epsilon = 4$. Indeed, the frequency correlation provides a required free parameter through the powerlaw exponent $\epsilon'/3$ where $-1 < \epsilon' < 1$. The original YO's ϵ-expansion is now replaced by arguing that the frequency integral is nearly divergent because of the assumed value of ϵ'. Thus the ϵ-expansion has now been replaced by an ϵ'-expansion where $\epsilon' = 1 - \delta$. This δ-expansion was interpreted as a scheme in which a large amount of energy is injected into the system (Carati, 1990b).

Although Carati viewed this colored forcing correlation as a good way to sidestep the problems with the ϵ-expansion, he did not consider that the introduction of this colored noise would violate the Galilean invariance. This important issue was pointed out by Yuan and Ronis (1992) in their analysis (see subsection 4.3).

9.6 Kraichnan's Interpretation of YO using the Distant Interaction Algorithm (DSTA)

Kraichnan (1987a,b) found that the principal physical content of YO can be more directly expressed by an approximation, the Distant Interaction Algorithm (DSTA), that involves neither the RG procedure of the elimination of successive infinitesimal shells in wave vector space nor the ϵ expansion.

From the closure theories, Kraichnan noted that the eddy viscosity at wavenumber k is influenced by all triad interactions involving a mode of wavenumber $q \geq p > k$, and has the asymptotic form

$$\nu(k|p,t) = \frac{1}{15}\int_p^\infty dq\theta(q)(5E(q,t) + q\frac{\partial E(q,t)}{\partial q}), \tag{107}$$

where the characteristic time for build up of triple correlations is

$$\theta(q,t) = [2q^2\nu(q) + 2q^2\nu]^{-1}. \tag{108}$$

The next assumption of Kraichnan is that the total viscosity $\nu(k|p,t)$ satisfies

$$\nu(k,t) = \nu(k|\xi k,t). \tag{109}$$

Here $\xi \geq 1$ is a cutoff ratio and $\nu(k|p,t)$ is defined as the contribution to $\nu(k,t)$ arising from all triad interactions $(\mathbf{k}, \mathbf{q}, \mathbf{q}')$ such that either q, q', or both are $> p$. Thus, above equation implies that the dynamical damping arises solely from interactions of k with sufficiently higher wavenumbers.

The additional assumption is that

$$\nu(k|p,t) = \nu(0|p,t), (p \geq \xi k). \tag{110}$$

This equation states that the total dynamical viscosity experienced at k from interactions with modes of wavenumber $> \xi k$, with $\xi \geq 1$, may be approximated by the asymptotic dynamical viscosity exerted at very low wavenumbers by the same modes. Using this model, one obtains a ξ-dependent Kolmogorov constant $C_K(\xi)$ for a steady state inertial range spectrum,

$$E(k) = C(\xi)\mathcal{E}^{2/3}k^{-5/3}. \tag{111}$$

With this form of the energy spectrum, beta-dependent eddy viscosity scaling can deduced from that obtained from the closure theories

$$\nu(k) \equiv \nu(0|k,t) = A(\xi)\mathcal{E}^{1/3}k^{-4/3}. \tag{112}$$

The coefficient of the eddy viscosity is given

$$A(\xi) = [7C(\xi)/60]^{1/2}\xi^{2/3}. \tag{113}$$

Kraichnan (1987) inspected the logical distinction between two procedures, RG and Distant Interaction Algorithm (DSTA): First of all, he inspected the multiple scale elimination. The evaluation of eddy viscosity by perturbative elimination of successive small spherical shells of high-wavenumber modes can be carried out by specifying an actual energy spectrum (Rose, 1977, Zhou *et al.*, 1988, 1989) rather than introducing a forcing spectrum. Next, Kraichnan discussed the method of "ϵ-expansion". In this procedure, the properties of a $E(k) \sim k^{1-2\epsilon/3}$ spectrum range are examined through an expansion, in powers of ϵ, about the properties of a spectrum $E(k) \sim k$.

Kraichnan's analysis is based on an estimation of the qualitative nature of eddy damping in an energy spectrum of the form

$$E(k) \propto k^s \qquad (k_c < k < k_d). \ \ with \ \ s = 1 - \epsilon \qquad (0 < \epsilon < 4), \tag{114}$$

The corresponding eddy damping is given by RG as

$$\nu(k|p) \propto (p/k)^{-\epsilon/3} \tag{115}$$

For the Kolmogorov inertial range spectrum, $\epsilon = 4$, the eddy damping is local for any positive ϵ since $\nu(k|p) \to 0$ as $p \to \infty$.

In YO, the explicit calculation of the eddy damping is first made in the near neighborhood of this reference spectrum. Now distant interactions are actually weakly dominant for $E(k) \sim k$. At $s = 1$ or $\epsilon = 0$, Eq. (115) is replaced by

$$\nu(k|p) \propto \ln(k_c/p). \tag{116}$$

The results are then mapped to the Kolmogorov spectrum by extrapolating to the limit $\epsilon \to 4$, although such a numerical value of 4 is not small and the validity of the extrapolation is by no means clear.

10 Eulerian vs. Lagrangian theory in ϵ-RG: Sweeping vs Straining

As noted earlier, a central discovery of Kraichnan was that both 'sweeping' (random advection of small scales by large scales) and 'straining' (local distortion of small scales by scales of comparable size) exist in turbulence and have different roles in turbulence dynamics. We saw that analytically, sweeping interactions occur when either 'leg' $\mathbf{p}$ or $\mathbf{q}$ of the triad $(\mathbf{k}, \mathbf{p}, \mathbf{q})$, with $\mathbf{k} = \mathbf{p} + \mathbf{q}$, approaches zero. In this arrangement, it is possible for the rms excitation at large scales to become dynamically significant. We recall that the dependence of the Eulerian two-time correlations on sweeping leads to the DIA prediction of a $k^{-3/2}$ energy spectrum (see Sections 3); the dynamic processes leading to the Ironshikov-Kraichnan spectrum turbulence and the Kolmogorov spectrum for MHD turbulence rests on the analogous possibility of dependence on the rms magnetic field.

The issue of sweeping was also studied by YOS (Yakhot, Orszag, and She (1989)) in the context of the scaling of the kinetic energy fluctuations and Kraichnan's principle (Kraichnan, 1964) of 'random Galilean invariance.' They made the observation that 'sweeping' interactions are negligible when $\epsilon = 0$, because the corresponding energy spectrum $E(q) \sim q$ vanishes at $q = 0$.

Within the ϵ-RG framework, YOS concluded that the kinetic energy fluctuations scales as $-7/3$, after extrapolating $\epsilon = 0$ to arbitrary ϵ. It should be noted that this $-7/3$ scaling was found previously when the Kolmogorov dimensional analyis is applied to higher order powers of velocity fluctions. In fact, Dutton & Deaven (1972) obtained the spectra of all even powers of the velocity scale according to their dimensional analysis based on an extended Kolmogorov scaling.

Yet, both Chen & Kraichnan (1989) and Nelkin & Tabor (1990) have argued that the YOS attempt to extrapolate to arbitrary ϵ proves to lead to an incorrect scaling of kinetic energy fluctuations. In particular, Nelkin & Tabor (1990) carefully inspected the spectra of kinetic energy fluctuations and found them scale as $k^{-5/3}$. These authors pointed out that the controversy on whether the kinetic energy fluctuations should scale as $-5/3$ or $-7/3$ can be conclusively resolved. Van Atta and Wyngaard (1975) documented strong evidence in support of the $-5/3$ scaling from their geophysical turbulence experimental data [31] . Zhou *et al.*, 1993 also offered clear evidence based on high Reynolds number laboratory experimental data obtained from both the return channal and mixing layer (Praskovsky et al., 1993)

A useful comparison can be made with the dimensionally equivalent pressure fluctuation spectrum (for a classical analysis, see Batchelor (1951). for a analysis without the joint Gaussian assumption, see Hill and Wilczak (1995)). Analytically, it is found that substitution of a Kolmogorov spectrum extending over all scales into the expression for the pressure spectrum yields a finite result, for which dimensional analysis correctly predicts the scaling exponent, $-7/3$ (for DNS result, see Gotoh and Fukayama, 2001). However, the same calculation for the kinetic energy fluctuation spectrum yields a divergent result. However if one replaces the infinite Kolmogorov spectrum by a Kolmogorov spectrum cut off at some scale k_0, the pressure spectrum gives a result proportional to $k_0^{-2/3}k^{-5/3}$. This dependence on k_0 reflects the dependence on the rms velocity, which also diverges as $k_0^{-2/3}$ for a Kolmogorov energy spectrum. Intuitively, the difference comes about because the pressure spectrum depends on velocity gradients, which are insensitive to sweeping by large scales. This is the same mechanism responsible for the successful prediction of the Kolmogorov spectrum by Lagrangian versions of DIA.

Chen & Kraichnan (1989) argued that in the YO and YOS theory, the mode elimination preferentially focuses on triads $\mathbf{k}, \mathbf{p}, \mathbf{q}$ for which $p \gg k$ and $q \gg k$. The sweeping interactions with $q \approx 0$ do not appear in the mode elimination procedure. In this sense, Chen & Kraichnan remarked that the sweeping is 'excluded at the outset' in ϵ-RG analysis.

It is perhaps more precise to say that in YO's formalism, local interactions are represented by the random force (Smith and Woodruff, 1990). But here too, the random force is *defined* to be local, that is, independent of the rms excitation. If local interactions are modeled this way, then it is again correct to say that sweeping is excluded from the outset.

These arguments make the actual suppression of sweeping when $\epsilon = 0$ dynamically irrelevant. This issue can be inspected in the context of Woodruff's

[31] see also, Van Atta and Chen (1970), Van Atta and Park (1972) and Van Atta (1996)

(1994) application of the ϵ-expansion to DIA. Woodruff found that sweeping is suppressed when $\epsilon < 3$, but becomes dominant when $\epsilon > 3$. While one can argue that this is a property of DIA, not necessarily of turbulence itself, it nevertheless suggests that extrapolation from $\epsilon = 0$ is by no means straightforward and not necessarily valid. This observation has also been made by Sukoriansky et al. (2003).

11 Iterative conditional averaging RG approach

11.1 i-RG procedure

We now summarize a method of eliminating turbulent modes which is based on the use of a conditional average to distinguish between amplitude and phase correlation effects (i-RG hereafter).[32] It has its roots in the method of iterative averaging of McComb (1982) and McComb & Shanmugasundaram, (1993, 1994)[33], which was developed over a number of years as a possible method of applying the renormalization group approach to real fluid turbulence (McComb 1982, 1986, 1990). However, an essential feature of the more recent work in this area is the formal treatment of the conditional average and the development of methods of approximating its relationship to the usual ensemble average (McComb & Watt 1990, 1992; McComb et al., 1992, 1993).

The basic idea is that the turbulent velocity field in wavenumber space may be decomposed into two distinct fields. One is a purely *chaotic field*; while the other is a *correction field*, and carries all the phase information. Application of this decomposition to a thin shell of wavenumbers in the dissipation range allows the elimination of modes in that shell; with the usual mode-coupling problems being circumvented by the use of a conditional average. The (conditional) mean effect of the eliminated modes appears as an increment to the viscosity. An iteration (with appropriate rescaling) to successively lower shells, reaches a fixed point, corresponding to a renormalized turbulent viscosity.

While several interpretations have been presented, in the final analysis, the net effect of the iteractive averaging method is to neglect both the term (A) in Eq. (60) and the term (I) in Eq. (61). We now illustrate the i-RG procedure in some detail.

[32] The iterative condtional averaging method, also called a Two-Fields method, is discussed in McComb, 1990, 1995. see also, Zhou, McComb, and Vahala, 1997
[33] Zhou & Hossain (1990) offered a perspective on the iterative RG

The resolvable scale Navier-Stokes equation ($k < k_1$) is

$$\Big[\frac{\partial}{\partial t} + \nu_0 k^2\Big] u_i^<(\mathbf{k},t) = \lambda_0 M_{imn}(k) \int d^3p [u_m^<(\mathbf{p},t) u_n^<(\mathbf{k}-\mathbf{p},\mathbf{t})$$

$$+ \underbrace{u_m^>(\mathbf{p},t) u_n^>(\mathbf{k}-\mathbf{p},t)}_{B}], \qquad (117)$$

Again, the strength of the nonlinear interaction is denoted by λ_0, a formal ordering parameter for perturbation theory, which is eventually set to unity.

The subgrid scales Navier=Stokes equation ($k_1 < k < k_0$) is

$$\Big[\frac{\partial}{\partial t} + \nu_0 k^2\Big] u_i^>(\mathbf{k},t) = \lambda_0 M_{imn}(k) \int d^3p [\underbrace{2u_m^>(\mathbf{p},t) u_n^<(\mathbf{k}-\mathbf{p},t)}_{II}]. \qquad (118)$$

We now provide some detail on the i-RG procedure for removing the first subgrid shell $k_1 < k \leq k_0$. The subgrid equation is substituted into the resolvable Navier-Stokes equation and then an average is performed over the subgrid scales, keeping terms only to $O(\lambda_0^2)$.

On substituting term (II) of Eq. (118), we obtain a term which on subgrid scale averaging yields a renormalized eddy viscosity, $\delta\nu_0^{>>}(k)$ which consists only contributions from subgrid-subgrid interactions.

After removing the first subgrid shell, the renormalized Navier-Stokes equation (replacing $u^<$ by u) reads

$$\Big[\frac{\partial}{\partial t} + \nu_1(k) k^2\Big] u_i(\mathbf{k},t) = \lambda_0 M_{imn}(k) \int d^3p [u_m(\mathbf{p},t) u_n(\mathbf{k}-\mathbf{p},\mathbf{t}), \qquad (119)$$

where $k < k_1$. The renormalized eddy viscosity is given by

$$\nu_1(k) = \nu_0 + \delta\nu_0^{>>}(k). \qquad (120)$$

As a result, i-RG procedure does not introduce new triple velocity product in the resulting renormalized Navier-Stokes equation. This is especially significant when the remaining subgrid shells are removed, since the enhanced eddy viscosity will only be contributed from the usual Navier-Stokes quadratic interactions.

The renormalized eddy viscosity, $\nu_*^{>>}(k)$ is the fixed point of the recursion relation

$$\nu_{n+1}(k) = \nu_n(k) + \delta\nu_n^{>>}(k). \qquad (121)$$

As shown in Fig. 4, the i-RG spectral eddy viscosity does not show a cusp-like behaviour as $k \to k_c$. This is in disagreement with that from closure theory (Kraichnan, 1976; Leslie & Quarini, 1979; Chollet & Lesieur, 1981)

The renormalized Navier-Stokes equation is given by

$$\Big[\frac{\partial}{\partial t} + \nu_*^{>>}(k)k^2\Big]u_i(\mathbf{k},t) = \lambda_0 M_{imn}(k)\int d^3p[u_m(\mathbf{p},t)u_n(\mathbf{k}-\mathbf{p},\mathbf{t}), \quad (122)$$

where $k < k_c$.

This is a good time to summarize the key findings of i-RG:

(1) Unlike YO, the renormalized eddy viscosity obtained from i-RG has $k \subset [0, k_c]$

(2) At $k \to 0$, the renormalized eddy viscosity approached a constant and this asymptotic result should be compared with that of YO.

(2) The renormalized eddy viscosity, $\nu_*(k)$, decreases as $k \to k_c$. Here, $\nu_*(k)$ represents only these contributions from subgrid-subgrid interactions. The resolvable-subgrid scales interactions are neglected in i-RG.

(4) The renormalized Navier-Stokes equation has the same structure with only quadratic nonlinear interactions, but with (i) $k \subset [0, k_c]$ instead of $k \subset [0, k_0)$ and (ii) $\nu_*(k)$ instead of ν_0.

11.2 Some work using the iterative averaging RG procedure

Nagano & Itazu (1997b) have applied the iterative averaging RG method to derive an eddy viscosity. Since the subgrid-resolvable scale interactions are not considered in this method, it not surprising the resulting eddy visosity is essentially that obtained from the Boussinesq approximation. Nagano & Itazu found that the proportionality constant takes a suitable value when the Kolmogorov constant is chosen near its accepted value.

Lin et al. (2001) applied iterative averaging RG to turbulent thermal transport and Chang & Lin (2002) implemented the same scheme to MHD. Cao & Chow (2004) also followed the procedure of McComb et al. and carried out their calculations for eddy viscosity and thermal eddy diffusivity [34].

[34] For reasons not properly understood by this author, Cao & Chow (2004) were able to obtain an eddy viscosity with a strong cusp when $k \to k_c$.

While their RG procedure is essentially that of McComb's i-RG, Chang *et al.* (2003) did make two changes from McComb's i-RG. First, the assumed energy spectrum is given in the form of Pao (1965) and Leslie & Quarini (1979) spectrum instead of the Kolmogorov spectrum. Second, the recursive relation for the eddy viscosity is computed[35] by an ordinary differential equation (o.d.e.) which is a function of resolvable scale wavenumber dependence k.

12 Recursive RG

Recursive renormalization group (r-RG) procedures were introduced because of problems encountered in the development of ϵ-RG. In particular, one of the problem faced by YO was a small ϵ expansion followed by an extrapolation from $\epsilon \ll 1$ to $\epsilon \to 4$, yet passing through a divergence at $\epsilon = 3$. The $\epsilon \to 4$ limit is crucial in order to recover the Kolmogorov energy spectrum. Also, in ϵ-RG one must invoke the distant interaction limit $k \to 0$. This makes it nearly impossible to make any comparison between the transport coefficients of ϵ-RG and the the wave-number dependent transport coefficients of closure theories (Kraichnan, 1976; Leslie & Quarini, 1979; Chollet & Lesieur, 1981)

12.1 r-RG procedure

For $k < k_1$, the resolvable scale Navier-Stokes equation is

$$\Big[\frac{\partial}{\partial t} + \nu_0 k^2\Big] u_i^<(\mathbf{k},t) = \lambda_0 M_{imn}(k) \int d^3p [u_m^<(\mathbf{p},t) u_n^<(\mathbf{k}-\mathbf{p},\mathbf{t})$$

$$+ \underbrace{2u_m^>(\mathbf{p},t)u_n^<(\mathbf{k}-\mathbf{p},t)}_{A} + \underbrace{u_m^>(\mathbf{p},t)u_n^>(\mathbf{k}-\mathbf{p},t)}_{B}], \tag{123}$$

The strength of the nonlinear interaction is denoted by λ_0, a formal ordering parameter for perturbation theory, which is eventually set to unity. It is convenient to label the kinematic viscosity $\nu \equiv \nu_0$.

For the subgrid scales, $k_1 < k < k_0$, we have

$$\Big[\frac{\partial}{\partial t} + \nu_0 k^2\Big] u_i^>(\mathbf{k},t) = \lambda_0 M_{imn}(k) \int d^3p [\underbrace{u_m^<(\mathbf{p},t) u_n^<(\mathbf{k}-\mathbf{p},\mathbf{t})}_{I}$$

[35] Note that the structure of the o.d.e. is similar to that of Rose (1977) and Zhou & Vahala (1993a).

$$+\underbrace{2u^>_m(\mathbf{p},t)u^<_n(\mathbf{k}-\mathbf{p},t)}_{II}+\underbrace{u^>_m(\mathbf{p},t)u^>_n(\mathbf{k}-\mathbf{p},t)}_{III}]. \tag{124}$$

We now provide some detail on the RG procedure for removing the first subgrid shell $k_1 < k \leq k_0$. The subgrid equation is substituted into the resolvable Navier-Stokes equation and then an average is performed over the subgrid scales, keeping terms only to $O(\lambda_0^2)$.

Now we consider term by term the effect of this substitution

(a) Term (A) Eq. (60):

The effect of term (I) is to introduce a new triple nonlinearity.

In his application of RG procedure to a passive scalar equation, Rose (1977) discussed the role of the triple nonlinear terms in physical space. He pointed out that it represents the possibility of an exchange of scalar eddies between the resolvable and subgrid scales. This effect is an inherent property of measurements made on the passive scalar system with instruments which have a spatial resolution limited to an eddy size greater than $1/k_c$.

The effect of term (II) in Eq. (60) is zero after performing the subgrid scale averaging [under subgrid scale averaging $< u^> >= 0$].

Term (III) in Eq. (60) is also zero on averaging over the homogeneous subgrid scales since $u^>$ and $u^<$ are connected by the same vertex. This can be readily seen algebraically: for $\mathbf{p}$ in the subgrid shell, term (III) becomes, on subgrid averaging

$$\int d\mathbf{p}d\mathbf{p}' < u^>(\mathbf{p}-\mathbf{p}')u^>(\mathbf{p}')u^<(\mathbf{k}-\mathbf{p}) > \; = \int d\mathbf{p}d\mathbf{p}'Q(\mathbf{p}-\mathbf{p}')\delta(\mathbf{p})u^<(\mathbf{k}-\mathbf{p}) = 0 \tag{125}$$

since $\mathbf{p}$ is in the subgrid $k_0 < p < k_1$, and so cannot satisfy $|\mathbf{p}| = 0$.

(b) Term (B) Eq. (60):

Working only to $O(\lambda_0)$, the substitution of term (I) in Eq. (61) into term (B) of Eq. (60) yields a term of the form $< u^<u^<u^> >$. Under subgrid scale averaging this term vanishes, since $< u^> >= 0$

On substituting term (II) of Eq. (61), we obtain a term which on subgrid scale averaging yields a renormalized eddy viscosity.

Again, as is invariably done in RG theories, one neglects the effect of substituting term (III) in Eq. (61) by arguing that it is a higher order effect. Neglecting this term is, of course, a closure approximation. +

12.2 Renormalized momentum equation

After the first subgrid shell is removed, the renormalized Navier-Stokes equation reads

$$[\partial/\partial t + \nu_1(k)k^2]u_i(\mathbf{k},t) = M_{imn}(\mathbf{k}) \int d^3p u_m(\mathbf{p},t)u_n(\mathbf{k}-\mathbf{p},t)$$
$$+M_{imn}(\mathbf{k}) \int d^3pd^3p' M_{mm'n'}(\mathbf{p})\Gamma_1 u_{m'}(\mathbf{p}',t)u_{n'}(\mathbf{p}-\mathbf{p}',t)u_n(\mathbf{k}-\mathbf{p},t), \quad (126)$$

where $\nu_1(k)$ is the enhanced eddy viscosity (see next subsection) and Γ_1 as a wavenumber dependent factor.

After the second subgrid shell is removed, the renormalized Navier-Stokes equation reads

$$[\partial/\partial t + \nu_2(k)k^2]u_i(\mathbf{k},t) = M_{imn}(\mathbf{k}) \int d^3p u_m(\mathbf{p},t)u_n(\mathbf{k}-\mathbf{p},t)$$
$$+M_{imn}(\mathbf{k})\Sigma_1^2 \int d^3pd^3p' M_{mm'n'}(\mathbf{p})\Gamma_1 u_{m'}(\mathbf{p}',t)u_{n'}(\mathbf{p}-\mathbf{p}',t)u_n(\mathbf{k}-\mathbf{p},t), \quad (127)$$

where $\nu_1(k)$ is the enhanced eddy viscosity (see next subsection) and Γ_1 as a wavenumber dependent factor.

We obtain the final renormalized Navier-Stokes equation

$$[\partial/\partial t + \nu_*(k)k^2]u_i(\mathbf{k},t) = M_{imn}(\mathbf{k}) \int d^3p u_m(\mathbf{p},t)u_n(\mathbf{k}-\mathbf{p},t)$$
$$+M_{imn}(\mathbf{k})\Sigma_1^N \int d^3pd^3p' M_{mm'n'}(\mathbf{p})\Gamma u_{m'}(\mathbf{p}',t)u_{n'}(\mathbf{p}-\mathbf{p}',t)u_n(\mathbf{k}-\mathbf{p},t) \quad (128)$$

by removing all the subgrid shells iteratively. For simplicity, we denoted Γ as a wavenumber dependent factor [36] .

We now turn our attention to the question of the Galilean invariance of the renormalized Navier-Stokes equations of r-RG. The importance of Galilean

[36] Interesting readers are referred to Zhou *et al.* (1988, 1989) and Zhou & Vahala (1993a,b) for detail

invariance in turbulence modelling has been emphasized by Speziale (1985). To be consistent with the basic physics, it is required that the description of the turbulence be the same in all inertial frames of reference. The appearance of the triple nonlinear term, which is a function of the resolvable scales velocity fields, makes the property of the Galilean invariance of r-RG not transparent. However, Zhou & Vahala (1993b) have demonstrated that the renormalized Navier -Stokes equation of r-RG is Galilean invariant.

12.3 Difference equation for the renormalized eddy viscosity

After the removal of the first subgrid shell, the spectral eddy viscosity becomes

$$\nu_1(k) = \nu_0(k) + \delta\nu_0^{>>}(k), \tag{129}$$

where $\nu_0^{>>}(k)$ is the contribution from the quadratic term of Navier-Stokes equation. Note that the triple velocity product term in the renormalized Navier-Stokes equation was just generated in this step.

After the removal of the second subgrid shell, the spectral eddy viscosity in the renormalized momentum equation is

$$\nu_2(k) = \nu_1(k) + \delta\nu_1^{>>}(k) + \delta\nu_1^{><}(k), \tag{130}$$

here the $\delta\nu_1^{>>}(k)$ term is due to the quadratic nonlinearity and the $\delta\nu_1^{><}(k)$ is due to the triple nonlinearity.

After the removal of the $(n+1)^{th}$ subgrid shell, the spectral eddy viscosity in the renormalized momentum equation is determined by the recursion relation

$$\nu_{n+1}(k) = \nu_n(k) + \delta\nu_n^{>>}(k) + \Sigma\delta\nu_n^{><}(k). \tag{131}$$

This recursion relation has contributions from both the quadratic and the triple nonlinear terms in the renormalized Navier-Stokes equation. This difference equation, after rescaling, has been solved by Zhou et al. (1988, 1989) and fixed points were readily determined for finite $f \leq 0.7$, where f measures the width of the range of modes being eliminated (sec. 6).

As shown in Fig. 4 (Zhou *et al.*, 1988), the r-RG spectral eddy viscosity exhibits a mild cusp as $k \to k_c$, in qualitative agreement with that from closure theory (Kraichnan, 1976; Leslie & Quarini, 1979; Chollet & Lesieur, 1981). The test-field model result of Kraichnan (1976), exhibiting the cusp behavior near the cutoff wavenumber, k_c, is reproduced in the unmarked curve. The curve marked by triangle is the r-RG model without triple nonlinearity. McComb's i-RG result(which is claimed to be able to somewhat avoid the triple nonlinearity) is shown by the dash-dot curve.

12.4 Total r-RG spectral eddy viscosity based on the energy transport

At a glance, the strength of the eddy viscosity cusp as $k \to k_c$ is a major discrepancy between r-RG and the closure theories. This discrepancy is resolved by examining the r-RG energy transfer equation (Zhou and Vahala, 1993a).

In fact, there is no reason the eddy viscosity obtained from r-RG and closure theories should agree at the first place. Indeed, Leslie & Quarini (1979) elegantly illustrated how the spectral eddy viscosity can be evaluated from the energy transfer equation. Since the renormalized Navier-Stokes equation now contains the triple velocity products, one should expect that these cubic nonlinearities will contribute to the energy transfer process and enhanced eddy viscosity.

The time evolution of the energy spectrum, $E(\mathbf{k}, t)$, was constructed from the renormalized Navier-Stokes equation and reads

$$\frac{\partial E(\mathbf{k}, t)}{\partial t} = -2\nu_*(k)k^2E(k)(\mathbf{k}, t) + T^D(\mathbf{k}, t) + T^T(\mathbf{k}, t). \tag{132}$$

In this equation, $T^D(\mathbf{k}, t)$ is the standard energy transfer from the quadratic nonlinearity. In contrast, $T^T_{ii}(\mathbf{k}, t) = -2\nu_T(k)k^2E(k)$ is the energy transfer arising from the RG induced triple nonlinearity.

In Fig. 5, we comparie the net r-RG eddy viscosity ($\nu_{net}(k) = \nu^*(k) + \nu_T(k)$ arising in the r-RG energy transport equation (Zhou and Vahala, 1993a) with that arising from closure theories (Kraichnan, 1976; Leslie & Quarini, 1979; Chollet & Lesieur, 1981). Here $\nu^*(k)$ is the r-RG eddy viscosity and $\nu_T(k)$ is the drain eddy viscosity in the energy transport equation arising from the triple nonlinearities of the renormalized r-RG Navier-Stockes equation. The net r-RG eddy viscosity $n_{net}(k)$ is plotted from various vaules for $r = k_c/K_p$, where K_p is a constant which is directly correlated to the location of the maximum in a production-type energy spectrum (Leslie & Quarini, 1979, for a plot of such production energy spectrum, see Fig. 1 of Zhou and Vahala, 1993a). It is clear that $\nu_T(k)$ is the major contributor to the cusp-like behavior of the spectral eddy viscosity as $k \to k_c$.

The closure theories (Kraichnan, 1976; Leslie & Quarini, 1979; Chollet & Lesieur, 1981) have shown that the major contribution to the strong cusp behavior in the spectral eddy viscosity was the local interactions (namely, the resolvable-subgrid scales interactions. Now returning to the r-RG framework, these triple velocity products in the renormalized Navier-Stokes equation were also resulted from the resolvable-subgrid scale interactions. When the energy transfer equation is formed, these cubic nonlinearities should and have shown to exhibit the cusp-like behaviour as $k \to k_c$.

In summry, in recursive RG, no attempt is made to introduce a special form of overlapping as in conditional averaging, and one proceeds directly with standard averaging and handles the triple nonlinearity directly. The basic differences between the r-RG and $\epsilon-$-RG as well as i-RG procedures are that in r-RG:

(1) In contrast with $\epsilon-$-RG but in agreement with i-RG, ϵ-expansion is not performed.

(2) The turbulent transport coefficients are determined for the whole resolvable wavenumber scales (both local and nonlocal interactions are taken into account), but

(i) In $\epsilon-$-RG, only $k \to 0$ limit is considered

(ii) In i-RG, only subgrid-subgrid interactions are kept when $k \to k_c$

(3) Triple nonlinearities are generated in the renormalized momentum equation and play a critical role in determining the transport coefficients. This is a major departure point when compared with both $\epsilon-$-RG and i-RG.

13 Measurements of the locality of interactions and subgrid/resolvable interactions using numerical simulation databases

13.1 Disparity parameter

The essential aspects of these two fundamental assumptions of Kolmogorov (1941) – the local energy transfer and local interactions – have received support from studies that use both direct numerical simulations and Eddy-damped Quasi-Normal (EDQNM) closures. The local energy transfer was confirmed by Domaradzki & Rogallo (1990), Yeung & Brasseur (1991), Ohkitani & Kida (1992) and Zhou (1993a,b). While there were some suggestions that the Kolmogorov's local intereaction assumption may not correct (Domaradzki & Rogallo, 1990; Yeung & Brasseur 1991), it has been shown that the local interaction are in fact dominant by the introduction of the *scale disparity* parameter

(Zhou, 1993a,b; Zhou et al., 1996)

$$s = \frac{max(k,p,q)}{min(k,p,q)} \tag{133}$$

where $\mathbf{k} = \mathbf{p} + \mathbf{q}$ forms the standard triads. The introduction of this disparity parameter permits a separation of the local from the nonlocal interactions. Noting that the flux rate of energy, $\Pi(k)$, across a spectral scale k is the most basic measure of the energy transfer process and within the framework of Kolmogorov universal equilibrum range, it is the only link between the enegetic and dissipative scales of motion. It was found that the dependence upon the scale disparity parameter is the same for all inertial ranges scales. The measured fraction energy flux $\Pi(k,s)/\Pi(k)$ is essentially independent of k (Fig. 4) as would be expected in a scale-similar inertial range (Zhou, 1993a,b).

13.2 Numerically evaluated subgrid-resolvable and subgrid-subgrid eddy viscosities

We consider directly the contributions of subgrid-resolvable and subgrid-subgrid terms in Eq. (60) to the eddy viscosity.

The time evolution of $E^{<<}(\mathbf{k},t)$ for $k < k_c$ is

$$\frac{\partial E^{<<}(\mathbf{k},t)}{\partial t} = -2\nu k^2 E^{<<}(\mathbf{k},t) + T^{<<}(\mathbf{k},t) + T^{><}(\mathbf{k},t) + T^{>>}(\mathbf{k},t). \tag{134}$$

In this equation, $T_{ij}^{<<}(\mathbf{k},t)$ is the standard resolvable scale energy transfer from the quadratic nonlinearity. In contrast, $T^{><}$ and $T^{>>}(\mathbf{k},t)$ are the energy transfer arising from the subgrid-resolvable scales (the RG induced triple nonlinearity) and the subgrid-subgrid scale (the RG quadratic nonlinearity) interactions.

Energy transfer (between resolvable and subgrid scales) and spectral eddy viscosity can be analyzed using results from direct numerical simulations by introducing an artificial cut at a wavenumber k_c that is smaller than the maximum resolved wavenumber k_m of the simulation. With this fictitious separation between the subgrid and resolvable scales, it is possible to evaluate the effect of the subgrid $k_c < k < k_m$ on the resolved scales $k < k_c$ (Domaradzki et al., 1987, Lesieur &ogallo, 1989, Zhou & Vahala, 1993a). We form an energy equation from the momentum equation and introduce the following notation: $T^{><}(k)$ and $T^{>>}(k)$ represent the spectrum of energy transfer to mode $\mathbf{k}$ resulting from interactions with *one* and *both* modes above the cutoff k_c respectively.

Measurements of numerical simulation databases indicate the following (Zhou & Vahala, 1993a) (see Fig. 5):

• $T^{>>}(k)$ removes energy throughout the resolvable scales in a manner consistent with the notion of eddy viscosity.

• $T^{><}(k)$ removes energy from the last resolved octave that was transferred there by the resolved scale transfer; that is, it allows the local flow of energy through k_c. It is the most important subgrid effect near k_c and accounts for most of the energy flow from the resolved scales.

The subgrid spectral eddy viscosity $\nu^{>>}(k)$ and $\nu^{><}(k)$ can be determined from $T^{>>}(k)$ and $T^{><}(k)$ for a given energy spectrum, $E(k)$. Specifically, $\nu^{>>}(k) = -T^{>>}(k)/2k^2E(k)$ and $\nu^{><}(k) = -T^{><}(k)/2k^2E(k)(see Fig.6)$.

Two important features of the quadratic contribution $\nu^{>>}(k)$ should be stressed. First, its positive constant asymptote at small k indicates that the concept of modeling this contribution as an eddy viscosity in analogy to the molecular viscosity is plausible, and second, its value decreases monotonically as k increases toward k_c. This indicates that if we include only the contribution of quadratic velocity products, there is no eddy viscosity cusp at the cutoff k_c. The most important feature of $\nu^{><}(k)$ is the sharp increase at $k \to k_c$.

13.3 Dynamical measurements of the locality of interactions and subgrid/resolvable interactions

13.3.1 Description of the simulation models

To investigate the influence of the nonlinear interaction terms (A) and (B) in Eq. (60) on the time evolution of the resolvable scales $u_\alpha^<(\mathbf{k})$, Dubois, Jauberteau, and Zhou (1997, hereafter DJZ) have implemented two models.

The first model, model^A, computes $u_i^{<,A}(\mathbf{k},t)$ ignoring term (B) in Eq. (60):

$$\Big[\frac{\partial}{\partial t} + \nu_0 k^2\Big] u_i^{<,A}(\mathbf{k},t) = M_{imn}(k)\int d^3p[u_m^{<,A}(\mathbf{p},t)u_n^{<,A}(\mathbf{k}-\mathbf{p},t)$$

$$+2u_m^{>,DNS}(\mathbf{p},t)u_n^{<,A}(\mathbf{k}-\mathbf{p},t)]. \tag{135}$$

Here, $\mathbf{u}^{>,\mathbf{DNS}}$ corresponds to the small scales of the DNS velocity field $\mathbf{u}^{\mathbf{DNS}}$ that is obtained by solving the full system, Eqs. (60) and (61), without making any approximation regarding the nonlinear terms.

The second method, model^B, evaluates $u_\alpha^{<,B}(\mathbf{k},t)$ with the equation (1) in which the term (A) is neglected in Eq. (60):

$$\Big[\frac{\partial}{\partial t} + \nu_0 k^2\Big] u_i^{<,B}(\mathbf{k}, t) = M_{imn}(k) \int d^3p [u_m^{<,B}(\mathbf{p}, t) u_n^{<,B}(\mathbf{k}-\mathbf{p}, \mathbf{t})$$

$$+ u_m^{>,DNS}(\mathbf{p}, t) u_n^{>,DNS}(\mathbf{k}-\mathbf{p}, \mathbf{t})]. \tag{136}$$

These two models can be summarized as following. Model^A keeps only the cross interactions between the resolvable and subgrid scales while neglecting the influence of the interactions among small scales on the evolution of the large ones. Model^B, on the other hand, keeps the subgrid interaction term, while neglecting the interactions between the subgrid-resolvable scales.

As mentioned before, the "modeled LES" fields should be statistically the same as the large scales of the DNS field. Hence, in order to check the validity of Model^A and Model^B, the solutions $u_\alpha^{<,A}$ and $u_\alpha^{<,B}$ are compared with $u_\alpha^{<,DNS}$, the filtered DNS solution (fDNS).

13.3.2 Comparison of velocity spectra

We first consider the results of the so-called simulation S_1. Figure 7 shows the energy and enstrophy spectra corresponding to DNS, Model^A and Model^B at the intermediate time $t = 13.5\tau$. It is clear that Model^A provides a better resolution of the resolvable scale energy spectrum. We note that an energy pile-up appears near the cut-off wavenumber k_1 on the Model^B spectrum. This energy pile-up is not dissipated but tends to accumulate and to modify the slope of the spectrum for $k < k_1$, even at short times. This energy pile-up is not seen in the Model^A spectrum. Thus, by taking into account the cross interaction term (A) in Eq. (60), one achieves a better description of the energy transfers.

13.3.3 Examination of a correlation coefficient

Figure 8 shows the time evolution of the correlation coefficient, defined in (12), for the simulation S_1. As time increases, the correlation coefficient between the fDNS and Model^B decreases more rapidly than that between the fDNS and Model^A. After 10 eddy turnover times, the Model^A correlation coefficient remains on the order of 1, while the Model^B coefficient is close to 0.75. After this period (i.e. for $t > 10\tau$), both coefficients then have a similar behavior and decay as $t/3$, before reaching a short plateau. This result shows the importance of the cross-interaction term (A) in subgrid modeling: the cross interaction term provides not only the correct magnitude for the energy, but also the needed phase information from the subgrid scale.

In both Model^A and Model^B, a part of the interaction terms is kept and computed directly with the small scales $\mathbf{u}^{>,\mathbf{DNS}}$ of the full DNS. As a result, the velocity fields $\mathbf{u}^{<,\mathbf{A}}$ and $\mathbf{u}^{<,\mathbf{B}}$ remain correlated with $\mathbf{u}^{<,\mathbf{DNS}}$ over a longer period of time.

13.4 Additional Evidence for the Importance of Subgrid-Resolvable Scale Interaction: The work by Laval et al.

In an interesting paper, Laval, Dubrulle & McWilliam (2003) studied several approximations to the Navier-Stokes equations, in the spirit of rapid distortion theory. The empirical eddy viscosities obtained from the Langevin model are in good agreement with that obtained from the DIA of Kraichnan. Furthermore, the so-called Langevin Rapid Distortion Theory (RDR) model, which keep the interactions between local and nonlocal scales, is able to reproduce the correct spectrum shape, intermittecy statistics, and coherent flow structures for both the resolved and the largest sub-grid scales. These results are consistent with those found in Dubois, Jauberteau & Zhou (1997). Finally, Laval et al. plotted the correlation coefficient between their model and DNS. Again, as in Dubois et. al., it seems (see their Fig. 22) that the model incorporates the effects of the interaction between local and nonlocal scales with filtered DNS for much longer times than that arising from simulations with only a spectral eddy viscosity in the form of Chollet & Lesieur (1981).

13.5 Additional Evidence for the Importance of Subgrid-Resolvable Scale Intgeraction: The work by McComb et al.

We recall that in all of the iterative RG procedure by McComb and co-workers, the single focus was to eliminate the interaction between the subgrid and resolvable scales (as well as the triple nonlinear terms that are resulted form such interactions). However, McComb, Hunter & Johnson (2001) now state that the large-eddy simulations based on iterative averaging RG (now refered by them as a *two-field theory*), and which now only contains the effects of subgrid-subgrid stress, perform reasonably well when compared to other approaches, presumably those of Smagorinsky type. This fact was previously demonstrated in Dubois et. al.. Based on their numerical experiments, again similar to those of Dubois et. al., and Laval et al., the spectral correlations of subgrid and resolved scales dominate the momentum transfers in the equation of motion. Numerical simulations of McComb et al. (2001) also show that the subgrid-subgrid interaction only plays a significant role for very large scale separation. This result was reported previously in Zhou & Vahala (1993a).

So what one can say about the iterative RG (or "two-field theory")? McComb et al., themselves offered a candid assesement: i-RG can provide a partial model to represent the dissipation rate and that a promising way forward would be to adopt a hybrid approach. This is clearly true, since it is the subgrid-subgrid interactions that are responsible for the dissipative effects (see comments on this issue in Zhou & Vahala, 1993a, Zhou, 1991).

14 Selected work on plasma turbulence

14.1 Hasegawa-Mima Equation

Large-scale flows, such as zonal flows, play an important role in steady state turbulence. On many occasions, such flows are believed to develop out of the turbulence through the nonlinear interactions between the fluctuations themselves (Kim, 2004). The Hasegawa-Mima (1978) equation is a widely used model for electrostatic fluctuations, and has a close resemblance to the two-dimensional Navier-Stokes equation for an incompressible fluid.

The Hasegawa-Mima equation has only one nonlinear term which originates from the nonlinear polarization drift (Hasegawa & Mima, 1978), but has demonstrated over the years its capacity of yielding useful information with reasonable cost. The distribution of the energy anisotropically in two directions has been revealed in computer simulatioins (Horton, 1999). Using this equation, Hasegawa & Mima obtained the frequency integrated k spectral density as well as the width of the ω spectrum, assuming the coexisting of a large amplitude long wavelength potential fluctuation. Since the result does not depend on any particular mode of the system, the spectral density obtained is considered to be universal for a magnetized nonuniform collisionless plasma (Hasegawa & Mima, 1978). Krommes & Kim (2000) also calculated the growth rate of the large-scale fluctuation due to the interactions between the short-scale fluctuations.

Kim (2004) applied the field-theoretical RG method to the Hasegawa-Mima equation. Since small-scale drift-wave turbulence may drive the large scale fluctuations anisotropically, it is modeled as a random anisotropic external forcing peaked at high k.

Two dimensional turbulence, as will be shown in the next subsection, introduced higher level of complexity than the three-dimensional problem. Following Honkonen & Nalimov (1996), a new small parameter was introduced by Kim (2004) to avoid a divergent diagram by noise renormalization.

As usual, the RG procedure attempts to find a stable fixed point of the RG flows in the limit of long-time and large-scale limit. Of course, one assumes that the scaling behavior is well established up to this limit (Kim, 2004). Local noise is added so that the divergence can be regulated. The drift-wave frequency and the gyroradius were found to be irrelevant parameters and the corrections to their values were of higher order. In the limit of $k \to 0$ and $\omega \to 0$, a stable fixed point was found as a function of forcing. The renormalized coupling constants for the nonlocal and local noise as well as the renormalized anisotropy as a function of the external forcing. Up to one loop order, large-scale fluctuations were found to grow under the anisotropic forcing drive, in agreement with the results from numerical simulations.

14.2 Alfvén turbulence

The theory of compressible magnetohydrodynamic MHD, e.g., Alfvénic! turbulence has been a topic of interest. As summarized by Medvedev and Diamond (1997), Alfvén wave turbulence presents several novel challenges, due to the fact that the $k\omega$ selection rules preclude three Alfvén-wave resonance. Thus, in incompressible MHD, two Alfven waves can interact only with the vortex (i.e., eddy) mode. Compressibility relaxes this constraint by allowing interaction with accoustic and ionballistic modes (i.e., Landau damping), along with waveform steepening.

Chen & Mahajan (1985) have considered a model Alfvén wave turbulence problem and under certain assumptions showed that the Alfvén wave spectrum developed a power law inertial range. The model shear Alfvén wave equation was derived from ideal magnetohydronamics for equilibria with no fluid flow, uniform density and constant toroidal magnetic field.

Now in working with such a nonlinear equation, one may ask: in shear Alfven wave turbulence, how important can the small scale structures be if an inverse cascade exists as it does in two-dimensional Navier-Stokes turbulence (Kraichnan & Montgomery, 1980)? Note that in the presence of an external magnetic field the usual arguments used to suggest the possibility of inverse cascades no longer apply and there is no longer any a priori reason to expect an inverse cascade or any enhanced transfer to the longest wavelengths in the spectrum of any particular quantity. The numerical results (Hossain et al., 1985) for forced two-dimensional magnetohydrodynamic turbulence in a uniform external magnetic field showed that the magnetic spectrum back-transfer ceases well before the longest wavelength modes contain the same high fraction of the total mean square vector potential as in the case for no external magnetic field. Moreover, these dominant long wavelength modes do not constitute a resonantly coupled triad, but are coupled by the smaller-scale turbulence.

In considering the effect of the truncated subgrid scales on the tractable supergrid scales one must assume some properties of the subgrid scales [37] . Here, backed by the simulation results of Chen and Mahajan the subgrid Lorenzian wavenumber and frequency spectra are assumed. (There is some similarity here to the Kolmogorov inertial range energy spectrum of fluid turbulence.)

Applying the RG procedure, Zhou & Vahala (1987, 1988) evaluated the effect of small, unresovable subgrid scales on the large scales in Alfvén wave turbulence. The removal of the subgrid scales leads to a renormalized response function, which can be calculated analytically (Again, there is a similarity to the renormalized eddy viscosity in the RG application to fluid turbulence). To find an explicit solution for the response function, a Lorenz frequency spectrum is assumed while the subgrid wavenumber spectrum was found by Chen and Mahajan.

With these spectra for the subgrid scales, the response function has been computed as a function of the normalized frequency ω for various values of the Lorentz spectrum parameter. For broad Lorenz frequency spectra, the removed subgrid scales have only a very small effect on the response function. On the other hand, for peaked Lorenzians, there is a considerable frequency variation in both real and imaginary part of the response function. Strong absorption can occur around the Alfvén frequency for sharply peaked subgrid frequency spectra.

Medvedev and Diamond (1997) presented an analytical study of the noisy derivative nonlinear Schrodinger (noisy KNLS) equation (Kennel *et al*, 1990; Malkov *et al.*, 1991) as a generic model of collisionless, largeamplitude Alfvénic shocklet turbulence. Stationarity is maintained via the balance of noise and dissipative nonlinearity. Dissipation here results from ion Landau damping, which balances the parallel ponderomotive force produced by modulations of the compressible Alfvén wave train. A one-loop RG! calculation calculation was carried out. In contrast to more familiar paradigms of turbulence, dissipation arises from Landau damping, enters via nonlinearity, and is distributed over all scales. The theory predicts that two different regimes or phases of turbulence are possible, depending on the ratio of steepening to damping coefficient

[37] An alternative, of course, is to assume a forcing and define its correlation function. The application of RG to the forcing model Afvén turbulence have also been carried out, with and without the ϵ-expansion method, see Zhou & Vahala, 1988, 1989

In the RG work of Longcope & Sudan (1991), the equation of reduced MHD (Strauss, 1977, Montgomery, 1982, Zank & Matthaeus, 1992) are introduced and recast using the Elsasser (1956) variables. Ignoring curvature, gravity, and density stratification the coronal loop can be modeled as a uniform magnetic field. If one is interested in typical perpendicular scale structures, the equations of resistive MHD can be reduced to a simpler set of equation involving the streamfunction and the z component of the vector potential.

To elucidate the behavior of these equations the linear propagator is first identified and characterized (Longcope & Sudan, 1991). The methods of Forster *et al.* (1977) and YO are used to determine the effect of the removal of an infinitesimal shell on the dissipation coefficients. Following rescaling, a set of ordinary differential equations (ODEs) are derived.

These ODEs describe the change of model parameters as successive shells of modes are removed. Longcope & Sudan (1991) have examined the fixed point of their RG procedures and determined effective diffusion coefficients representing that range of small scales.

As expected, the resulting resistive and viscous dissipation is represented by differential operators, whose coefficients depend upon the amplitudes of the large-scale quantities being determined. The diffusion coefficient is also cast in terms of large-scale quantities. Liang & Diamond (1993) obtained the subgrid model in a closed form.

15 Application of RG to 2D MHD turbulence

The application of RG to 2D MHD turned out to be quite complicated and challenging. Fournier *at al.* (1982) included 2D MHD turbulence as part of their application of RG to d-dimensional MHD turbulence. Over the intervening years, conflicting results have been obtained with the final chapter yet to be written.

Liang & Diamond (1993) considered the feasibility of applying RG method to 2D MHD. They noted, as was reviewed in detail earlier by Kraichnan & Montgomery (1980), that in comparison with 3D fluids, 2D systems have more conserved quantities (in the ideal limit). These "rugged invariants" complicate the ensuing dynamics of the real dissipative systems. In 2-D MHD turbulence, two conserved quantities, namely the total energy and total mean square magnetic flux, flow in different direction in k space. Just as 2D fluid turbulence, the enstrophy flows to small spatial scales while the energy flows to large spatial scales.

Another important phenomena in 2D MHD turbulence, which has no counterpart in 2D fluid turbulence, is the Alfven effect. The Alfven effect accounts for the strong influence of large-scale root-mean-square (rms) magnetic fields on smaller eddies. This intrinsically nonlocal effect significantly modified the spectra of both kinetic and magnetic energies.

Liang & Diamond (1993) have commented on the 2D MHD work of Fourier *et al.* (1982). They argued that in that work, neither the dual-direction transfer – which implies that the renormalized viscosity should have different signs – nor the Alfven effect – which accounts for the effect of large-scale magnetic fields on smaller scales – were addressed. Also, the exponents in the forcing functions were left as unknown variables throughout the calculation. Thus, in Fourier *et al.* (1982), the stability of fixed points in 2D MHD could not be determined since this depended on the actual numerical values of the forcing exponents.

Liang & Diamond claimed that there does not exist an RG fixed point for 2D MHD turbulence. They attributed this to the coexistence of the dual cascade transfers. The absence of a fixed point, of course, would render the RG method incapable of describing 2D MHD. Liang & Diamond reached a similar conclusion for the application of RG to 2D hydrodynamic turbulence because of the inverse energy cascade.

Kim & Yang (1999) offered a critique of Liang & Diamond and disagreed with their basic findings on the non-existance of a fixed point. Through the identification of the primitively divergent vertex functions, Kim & Yang (1999) were able to shown systematically that there exist fixed point solutions of the RG equations. Kim & Yang stated that this conclusion is reached by identifying one important term that they claimed was missing from Liang & Diamond.

Kim and Yang (1999) also noted several other points:

(i) The dominant role of the power exponent of the external driving noise correlation in determining the correlation functions as well as the energy spectrum, that is so apparent in hydrodynamic turbulence, persists when applying RG to MHD.

(2) In the application of the epsilon expansion in critical (near equilibrium) phenomena there are no problems in extrapolating the expansions to $\epsilon \approx 1$. However, in 2D MHD turbulence, one requires $\epsilon = 15/4$. This can cast some doubt on the validity of RG results for MHD.

Hnatich et al (2001a) criticized previous 2D MHD RG work. They remarked that it was only conjectured that in 2D the magnetic scaling regime does not exist due to the instability of the magnetic fixed point. Moreover, Hnatich et al (2001a) claimed flaws in the renormalization applied to 2D fluid turbulence, and even more serious shortcomings in their investigation of 2D MHD turbulence. In their analysis they utilized a stream function and magnetic potential.

Hnatich et al (2001a) argued that the structure of a renormalization should always be analyzed separately. In particular, for the solution of the stochastic MHD equations, it was not at all obvious that the quadratic nonlinear terms should be excluded from a renormalization. These authors showed that the Lorentz-force term in the momentum equation is renormalized. Hence, Hnatich *et al.* claim that Liang & Dimond (1993) incorrectly neglected the renormalization of the quadratically nonlinear terms by simply assuming they were higher-order effect. Also, Hnatich *et al.* argued that Kim & Yang (1999), in their field-theoretic treatment of the same problem, ignored the renormalization of the Lorentz force without any justification and also neglected the renormalization of the forcing correlations by effectively considering renormalization of the model at $d > 2$. Clearly, Kin & Yang's renormalized treatment at $d > 2$ does not seem to be appropriate in a framework based the stream function and magnetic potential – quantities that are strictly valid for 2D.

Hnatich et al (2001a) performed an RG analysis of the large-scale asymptotic behavior of the solution of stochastically forced magnetohydrodyanmic equations for all space dimensions $d \geq 2$. In particular, for the first time, they took proper account of the additional divergences appearing in 2D. In a two-expansion scheme, these authors found three infrared-stable fixed points in the physically relevant region of the parameter space spanned by the forcing parameters and the inverse magnetic Prandtl number.

16 Application of RG to 3d MHD

16.1 Camargo & Tasso and Hnatich et al.

The application of the RG technique to MHD has brought several new features that were absent in the case of Navier-Stokes equations. The first issue in applying the RG procedure is to make a determination on how the two forcing functions should be introduced. In their calculation, Fournier *et al.* weighted the inertial nonlinearity and Lorentz force differently. They also considered two different coefficients for the correlations of the forces (y_1 and y_2).

In their study of 3d MHD turbulence, Camargo and Tasso (1992) found that the selection of two different coefficients is not possible in their case, because the scaling of (z^+ and z^-) would have to be different from each other, and by virture of their definitions, this does not make sense. Indeed, as it is noted already in Fournier *et al.*, that the magnetic field is not a passive scalar. Camargo and Tasso (1992) believed that this obligates them to renormalize simultaneously both the resistivity and viscosity.

Camargo and Tasso (1992) treated the full MHD equations in the manner of YO, using Elsasser (1950, 1956) variables, and in contrast to Fournier *et al.* (1982), they weight all nonlinearities in the same way. They argued that since the MHD equations contain resistivity and viscosity, both must be simultaneously renormalized. The renormalized Prandtl number also deserves special attention and its range of values can be determined by the RG technique.

Camargo and Tasso were able to determine the asymptotic behavior and determine the effective resistivity and viscosity. In particular, they determine the values of the turbulent Prandtl number as the function of a paramater which characterizes the relative correlation (but not on the absolute values) strength of the kinetic and magnetic stirring forces. Negative effective viscosity is not possible in their result; instead, the tendency is to have zero effective viscosity. In certain cases, with an extended interpretation of the calculations, negative effective resistivity and in others zero effective restivity are obtained.

In the papers by Hnatich et al (2001a), the existence of two different anomalous scaling regimes (kinetic and magnetic) in three dimensions was established corresponding to two nontrivial infrared stable fixed point of renormalization group.

The authors emphasized, as we mentioned in a previous section, that the structure of renormalization should always be analyzed separately and it is not at all obvious that the nonlinear terms are not renormalized in the solution of the stochastic MHD equations. In fact, direct calculation shows that the Lorentz-force term is renormalized. There seems to be a certain amount of confusion about this point in the recent literature. Hnatich et al (2001a) stated that Camargo and Tasso erroneously neglect renormalization of nonlinear terms as high-order effect. We will not provide further detail but simply refer the reader to their original paper for the 3d MHD RG work by Hnatich et al (2001a).

16.2 Verma's justification of the Kolmogorov spectrum for MHD

Verma's work (1999) requires some attention. Here, the RG averaging has been performed for small wave numbers, in contrast to other RG approaches in which higher wavenumbers were averaged out. The effective mean magnetic field at large wave numbers is obtained.

We are uneasy about the procedure of averaging over the small wavenumber (large-scales) for fluid or MHD turbulence. Here, it should be stressed that the small scale of MHD is dominated by the large-scale fields (for example, Shebalin *et al.*, 1983, Oughton *et al.*, 1994, Kinney and McWilliams, 1998; Zhou *et al.*, 2004). The dominance of either sweeping or straining time scales is directly linked to the large-scales. Therefore, the RG procedure for small wavenumber may not be appropriate for MHD turbulence.

Verma (1999) also stated that a simple power counting suggests the Kolmogorov spectrum, and the $k^{-3/2}$ prediction of the IK phenomenology does not satisfy the RG equations (Verma, 2001, a,b). From these arguments Verma claimed that Kraichnans power law is ruled out for strong MHD turbulence, but may still be considered in the framework of weak turbulence theory.

Verma (2001a) assumed that the mean magnetic field is zero, and so are magnetic and kinetic helicities (nonhelical plasma). The calculation of the renormalized parameters is quite complex for arbitrary cross helicity, σ_C and Alfén ratio, r_A. For simplicity, two limiting cases were considerred. Furthermore, the effects of triple nonlinearity is not also included. In order to compute the renormalized eddy viscosity and magnetic diffusivity, the Kolmogorovs power law is taken for the energy spectrum (2001a,b) in the recursive RG procedure.

The final renormalized coeficients are constant for small wavenumber but shifts toward zero for larger k. As noted by Verma (200a), similar behavior has been seen by McComb and co-workers (McComb, 1990) for fluid turbulence; this behavior is attributed to the neglect of triple nonlinearity, and the corrective procedure has been prescribed by Zhou *et al.* (1988,1989).

Restricted to nonhelical turbulence, Verma (2000b) obtained various fluxes and Kolmogorovs constant. The impact of the approximation of dropping the triple nonlinearity is not clear and has not been assesed. Using the flux equations, and assume the Kolmogorov spectrumd, he found that the Kolmogorov constant does not vary significantly with the variation of r_A , and it is close to its values for fluid turbulence (Zhou and Speziale, 1998).

17 RG based turbulence modeling

17.1 The nonlinear eddy viscosity model

The two-equation turbulence model was developed to overcome a major shortcoming of mixing-length models, namely that in practical turbulent flows in realistic geometries, the choice of a mixing length is often not obvious, and attempts to construct general length scales from the spatial derivatives of the mean velocity field have not proven successful. The two-equation model also has its shortcoming in that it depends on empirically specified model constants, and requires various *ad hoc* modifications for low-Reynolds number flows (Speziale, 1991).

The two-equation model expresses the eddy viscosity ν_t as a ratio of two important turbulence parameters K and $\mathcal{E}$:

$$\nu_t = C_\nu \frac{K^2}{\mathcal{E}} \tag{137}$$

K is the turbulent kinetic energy and $\mathcal{E}$ the turbulent transfer rate. This approach eliminates the need for an empirical specification of a mixing length, because the model also provides transport equations for K and $\mathcal{E}$. However, as Eq. (137) shows, although exogenous empirical content is significantly reduced, it is not eliminated entirely, because the model has no way to determine the constant C_ν. Typically it is determined by requiring the model to reproduce the results for some relatively simple standard flow.

One important claim of YO is that the YO theory can bring much needed rationality to two-equation modeling by providing theoretically derived values for the model constants, and deductive forms for low Reynolds number turbulence.

The calculation of viscosity has effectively been already given by Eq. (81), which exhibits both features just noted: the calculation is entirely self-contained and introduces no empirical constants. By letting the wavenumber k be arbitrarily close to the Kolmogorov scale, the low Reynolds number modifications of the viscosity, including reduction to the molecular viscosity when k actually equals the Kolmogorov scale, are naturally included in the model. It should perhaps be noted that even this theory does not entirely escape empirical input, because it proves to require the constant of proportionality relating the scale at which molecular viscosity is actually dominant to the Kolmogorov scale $(\nu^3/\mathcal{E})^{1/4}$.

In turbulence modeling, the eddy viscosity is used to express the unknown

Reynolds stress through the simple proportionality

$$\tau_{ij} = \nu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \tag{138}$$

where U_i is the mean velocity. However, for problems like the turbulent flow in a square duct, this simple relation is inadequate. A generalization, the *algebraic Reynolds stress model*, expresses the stress by a more complex function of mean velocity gradients. The generalization of the calculation leading to Eq. (137) to a model in which the stresses are quadratic functions of the mean gradients was given by Speziale, (1987, 1991), Horiuti, 1990, Rubinstein & Barton, 1990. Here too, the low Reynolds number effects, in this case, the vanishing of the quadratic contribution as k approaches the Kolmogorov scale, is naturally included in the model.

De Langhe, Merci, and Dick (2005) recently developed a Hybrid RANS/LES model with an approximate RG method.

17.2 The cubic eddy viscosity model

In Zhou *et al.* (1994), a formal expression for the Reynolds stress is given and it includes both the local and nonlocal interactions was obtained based on r-RG (section 12). It is of some interest to note that integrity basis representations are commonly employed to represent the anisotropic part of the Reynolds stress tensor for three dimensional turbulent flows based on a systematic derivation from a hierarchy of second-order closure models (Gatski & Speziale, 1993). It can readily be shown that the tensors that constitute the integrity basis are recovered for most part when the proposed r-RG model is recast appropriately (Zhou *et al.*, 1994).

The RG theory is utilized to develop Reynolds stress closure models for the prediction of turbulent separated flows. The combined model includes both the traditional and nonlinear eddy viscosity models. The ability of the proposed model to accurately predict separated flows is analyzed from a combined theoretical and computational standpoint by considering turbulent flow past a backward facing step as a test case (Zhou *et al.*, 1994). The final model for the Reynolds stress, including both the nolocal and local interactions, is obtained for the mean velocity field $\overline{u}$,

$$\begin{gathered}\tau_{ij} = \nu_T[\partial\overline{u}_i/\partial x_j + \partial\overline{u}_j/\partial x_i] \\ -K^3/\mathcal{E}^2[C_{\tau1}[\partial\overline{u}_i/\partial x_m \partial\overline{u}_j/\partial x_m]^* + C_{\tau2}[\partial\overline{u}_i/\partial x_m \partial\overline{u}_m/\partial x_j + \partial\overline{u}_j/\partial x_m \partial\overline{u}_m/\partial x_i]^* \\ + C_{\tau3}[\partial\overline{u}_m/\partial x_i \partial\overline{u}_m/\partial x_j]^*]\end{gathered}$$

$$+C_{R1}K^4/\mathcal{E}^3[\partial\overline{u}_i/\partial x_m \partial\overline{u}_j/\partial x_n \partial\overline{u}_n/\partial x_m + i \longleftrightarrow j] + C_{R2}K^7/\mathcal{E}^5[\partial\overline{u}_i/\partial x_m$$

$$+\frac{\partial}{\partial x_j}[\partial\overline{u}_n/\partial x_h \frac{\partial^2\overline{u}_h}{\partial x_m \partial x_n}] + i \longleftrightarrow j]. \tag{139}$$

The model coefficients for the quadratic nonlinear terms agree with those of Rubinstein & Barton (1990) while the model coefficients for the cubic nonlinear terms are determined in Zhou *et al* (1994). It should be noted that the nonlinear Reynolds stress can be recast (Zhou *et al.*, 2004) into a form-invariant integrity basis representation (Spencer, 1971, Pope, 1975, Gatski & Speziale, 1993).

The results (Zhou *et. al.*, 1994), based on detailed computations, demonstrate that the RG model can yield very good predictions for the turbulent flow of an incompressible viscous fluid over a backward-facing step. Thus, in spite of its well known deficiencies, provided the anisotropy of the turbulent stresses are properly accounted for, the two-equation turbulence models can be quite effective for the prediction of turbulent separated flows.

Craft, Launder, and Suga (1996) also proposes a cubic relation between the strain and vorticity tensor and the stress tensor, which does much better than a conventional eddy-viscosity scheme in capturing effects of streamline curvature over a range of flows. They considered the flows range from simple shear at high strain rates and pipe flow, to flows involving strong streamline curvature and stagnation.

17.3 The $\mathcal{E}$ equation

YO derived the $\mathcal{E}$ equation model using ϵ-RG method. However, Speziale (1990) found that the original YO model performed quite poorly in homogeneous shear flow. The value $C_{\epsilon 1}$ derived in YO yields excessively large growth rate for the turbulent kinetic energy in homogeneous shear flow in comparision to both physical and numerical experiments (Speziale, 1991).

Briefly,[38] closure is achieved in the exact equation for $\mathcal{E}$ by using iterated mode band elimination and then applying the ϵ-expansion to evaluate the final amplitudes. Smith & Reynolds (1992) found some algebraic error in the original derivation of YO. and noted that the coefficient of the dissipative term in $\mathcal{E}$ equation was not in good agreement with thegenerally accepted value. Furthermore, YO's derivation did not yield a term responsible for production

[38] This section largely repeats the contents of a NASA contract report (Zhou et al. 1997)

in the $\mathcal{E}$ equation The original derivation of YO was revised by Yakhot & Smith (YS) (1992) by the following features:

(1) An 'infrared cutoff' of the random force was introduced so that $< f^> f^> >= 0$ for $0 < k < \Lambda_L$: i.e.,

$$< f_i(k,t) f_j(k',t') >= D_0 k^{-y} D_{ij}(k) \delta(k+k') \delta(t-t'),$$

$$\Lambda_L < k < \infty = 0, \quad 0 < k < \Lambda_L = 2\frac{\pi}{L} \tag{140}$$

This property is needed in the derivation of the equation for the mean rate of energy dissipation $\mathcal{E}$ (YS).

(2) The input of energy spectrum for the interval $0 < k < \Lambda_L$

$$E_(k) \sim k^M \tag{141}$$

is required to evaluate the integrals (with choice of exponent $M = 2$).

(3) Performing a Reynolds decomposition of $T_1 = -2\nu_0(\nabla_j u_i)(\nabla_j u_l)(\nabla_l u_i)$ into mean $\mathbf{U}$ and fluctuating $\mathbf{u}$ velocities.

The derivation of YO and YS starts from dynamical equations for the homogeneous part of the instantaneous rate of energy dissipation per unit mass $\mathcal{E} \equiv \nu_0(\nabla_j u_i)^2$

$$\frac{\partial \mathcal{E}}{\partial t} + u_i \nabla_i \mathcal{E} = \overbrace{2\nu_0(\nabla_j u_i)(\nabla_j f_i)}^{\mathcal{P}_\mathcal{E}} - \overbrace{2\nu_0(\nabla_j u_i)(\nabla_j u_l)(\nabla_l u_i)}^{T_1}$$

$$- \overbrace{2\nu_0^2(\nabla_j \nabla_l u_i)^2}^{T_2} - 2\nu_0(\nabla_j u_i)(\nabla_i \nabla_j p) + \nu_0 \nabla_i \nabla_i \mathcal{E} \tag{142}$$

After some work, for stirred fluids in the long-time and large-distance limit, the ϵ-RG dissipation equation (YS, 1992) is found to be

$$D_t \mathcal{E} = C_{\epsilon 1}(\mathcal{E}/K)\tau_{ij}\partial_j u_i - C_{\epsilon 2}\mathcal{E}^2/K + \partial_i(\alpha \nu_T \partial_i \mathcal{E}) - \mathcal{R} \tag{143}$$

where $C_{\epsilon 1} = 1.42$, $C_{\epsilon 2} = 1.68$ and

$$\mathcal{R} = 2\nu_0 S_i j \overline{\frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_j}} \tag{144}$$

The above ϵ-RG dissipation equation is not closed because of the $\mathcal{R}$-term. Now the neglect of $\mathcal{R}$ is only formally justified at high Reynolds number if local isotropy is assumed. But Durbin & Speziale (1991) have questioned the validity of local isotropy in strongly strained turbulence flows. Yakhot et al. (1992) have proposed a model where $\mathcal{R} = \mathcal{R}(\zeta)$, where the standard form of the model is recovered for $\mathcal{R} \to 0$ in the limit of weak straining. Note that Durbin (1990) has already developed a model for the production of dissipation along these lines that was quadratic in the ratio of production to the dissipation and, hence, quartic in ζ. Lam (1994) has published a critique of the YS derivation [39] .

Iterating the expression for $\mathcal{R}$ using the Navier-Stokes equation will generate a power series

$$\mathcal{R} = \nu_T S^3 \sum_{n=0}^{\infty} r_n (\frac{SK}{\mathcal{E}})^n \tag{145}$$

where $S = (2S_{ij}S_{ij})^{1/2}$. It is not possible to evalute the summation since the values of coefficients, r_n, are unknown.

The $\mathcal{R}$ is modeled via three steps:

1. The summation is performed assuming a *geometric series* for *every third term*, thereby reducing the number of unknown coefficients to *one*, $\tilde{\beta}$.

$$\mathcal{R}^0 = \nu_T S^3 \sum_{n=0}^{\infty} (-\tilde{\beta})^n (\frac{SK}{\mathcal{E}})^{3n} = \frac{\nu_T S^3}{1 + \tilde{\beta}\zeta^3}. \tag{146}$$

2. For homogeneous shear flows, there is a fixed point at $\zeta_0 = 4.38$. Yakhot et. al. (1992) then assume that this fixed point is invariant to the neglecting of all terms but those retained in Eq. (146) above and generalize Eq. (146) to

$$\mathcal{R} = \frac{\nu_T S^3}{1 + \tilde{\beta}\zeta^3}(1 - \zeta/\zeta_0). \tag{147}$$

3. One now further assumes with the isotropic Reynolds stress $\tau_{ij} = -2C_\nu K \zeta_{ij}$ ($\zeta_{ij} = S_{ij}K/\mathcal{E}$)

$$\mathcal{R} = \frac{C_\nu \zeta^3 (1 - \zeta/\zeta_0)}{1 + \tilde{\beta}\zeta^3} \frac{\mathcal{E}^3}{K} = \frac{\zeta(1 - \zeta/\zeta_0)}{1 + \tilde{\beta}\zeta^3} \frac{\mathcal{E}}{K} \tau_{ij} S_{ij}. \tag{148}$$

The undertermined constant $\tilde{\beta} = 0.012$ for the von Kármán constant 0.4. The final ϵ-RG dissipation rate transport equation is then given by

$$D_t \mathcal{E} = C^*_{\epsilon 1} (\mathcal{E}/K) \tau_{ij} \partial_j u_i - C_{\epsilon 2} \mathcal{E}^2/K + \partial_i (\nu_T \partial_i \mathcal{E}) \tag{149}$$

[39] For a perspective of Lam's work, see Zhou, 1995

where the coefficient $C^*_{\epsilon 1}$ is given by

$$C^*_{\epsilon 1} = C_{\epsilon 1} - \frac{\zeta(1 - \zeta/\zeta_0)}{1 + \tilde{\beta}\zeta^3} \tag{150}$$

This model of Yakhot et al. (1992) has been tested for homogeneous shear flows and for flow over a backward facing step. Excellent results are obtained in both cases. Recently, Yakhot & Orszag have extended the model further and applied it to complex flows using the FLUENT code.

17.4 Nagano & Itazu's comment on YO modeling

A strong negative assesement was offerred by Nagano & Itazu (1997a) who wade through the details of the YO and Yakhot & Smith (1992). These authors stated that "it becomes evident" that the Yakhot et al. $K - \mathcal{E}$ model is not directly obtained from the renormalization group theory. This conclusion is consistent with the previous subsection. Nagano & Itazu noted that from their analysis the numerical constants claimed by Yakhot et. al. are invalid, the modeling of $\mathcal{E}$ is estimated in the low wavenumber range, and the turbulent energy dissipation rate itself is also underestimated.

Nagano & Itazu stressed that the direct numerical simulation by Kim et al.(1987) has reveal that there is a term that dominates the production of $\mathcal{E}$ - and that this term was eliminated by Yahkot et al. Not unexpectedly, this problematic term is nothing but a triple velocity product term which arises in the construction of the transport equation for $\mathcal{E}$.

18 Summary and conclusions

In comparing the application of RG methods to hydrodynamics and to MHD, it seems clear that the first application is much more developed, much like the general theory. In MHD, basic questions about the proper generalization of the energy spectrum – whether it should be the Iroshnikov-Kraichnan spectrum or a direct generalization of the Kolmogorov spectrum itself – remain unanswered. Recently, Zhou *et al.* (2004) advanced a perspective that the central issue is the time scale for decay of the transfer correlation functions. In MHD turbulence there is a smooth variation between such spectral limits. Observation of distinct spectral indices in various cases is indicative of the enhanced effect of sweeping effects versus straining effects, or of local effects versus nonlolcal effects, in accordance with how the prevailing conditions impact the relevant time scales (Zhou *et al.*, 2004). We cannot say that RG has had much success in resolving this issue theoretically. At this point, we must await further clarification from experiments and numerical simulations. It is fair to say, however, that RG provides an interesting approach to inspecting both theoretical and modeling aspects of MHD and plasma turbulence.

In hydrodynamics, the defining characteristic of RG, at least in the perhaps overly restricted sense of this review of some kind of iterated mode elimination, is the replacement of the integrations in DIA-based analytical closures over a two-dimensional wavenumber slot by much simpler one-dimensional integrations over a single wavenumber argument. One way or another, this replacement arises from simplification of the triad interaction. It is likely that this simplification will remain attractive for applications regardless of its status from the viewpoint of fundamental theory [40].

As a fundamental theory, RG has led, in the ϵ-RG (Yakhot & Orszag, 1986, Yakhot, Orszag, and She, 1989, Dannevik, Yakhot, and Orszag, 1987) formulation, to the idea of an expansion about some particularly simple theory. Even if the correct expansion point remains uncertain, the idea itself continues to inspire new research [41].

There is no question that there are some advantages to working around the limit state at which $\epsilon \to 0$. This is also the state where many of the first applications of RG to turbulence by Foster et al. (1977) and Fourier & Frisch (1983) were made. These advantages, if they could actually be realized in high Reynolds number flows, could be indeed viewed as offering a significant advance over renormalized perturbation theories (RPT) because this state corresponds to a flow with weak nonlinear interactions that permits particularly simple analysis.

However, the key question is whether these desirable features could be maintained in the extrapolation from $\epsilon \to 0$ to $\epsilon = 4$ in order to reproduce the Kolmogorov energy spectrum (Kraichnan, 1987a,b), especially in view of the apparent change to dominant sweeping when $\epsilon \to 3$. Moreover, the controversy over the role of higher order nonlinearities (Eyink, 1994) suggests that $\epsilon \to 0$ is not quite so simple a turbulent state as it might at first seem. If so, the advantages of perturbing around that state may not be so great.

To further compare and contrast RPT and ϵ-RG, it is interesting to note that the RPT include both local and nonlocal interactions. This allows the flexibility of a RPT model to either consider the complete range of interactions or take the special limit of large distant where only nonlocal interactions are kept. This capability is appealing since the DNS has shown that the local interactions are dominant (Zhou, 1993a,b; Zhou and Vahala, 1993a; Dubois,

[40] Recent examples include, for example, (Sukoriansky *et al.*, 2005)
[41] see, for example, Yakhot, 2001; L'vov & Procaccia, 2000

Jauberteau, and Zhou, 1997). ϵ-RG, on the other hand, is constructed under the limit of distant interaction. Certainly, this fact does not rule out an adequate account of overall turbulence energetics, suitable say for modeling, on the basis of this simplification of the dynamics. Indeed, Kraichnan made a profound study of the question of what might be unchanged under the suppression of local interactions. Clearly, more studies are needed to better quantify the accuracy of modeling the effects of mode coupling with a forcing function and an eddy viscosity at very large distant limit.

The r-RG (Zhou *et al.*, 1988, 1989; Zhou & Vahala, 1993a,b) is an effort to incorporate both local and nonlocal interactions. No ϵ-expansion is envisaged or introduced and the RG procedure is carried out directly with the assumption of an inertial range energy spectrum or its correponding force correlations. Without the benefit of distant interaction limit, the resulting renormalized Navier-Stokes equation now involves both the quadratic and triple nonlinearities (Rose, 1977). In the framework of the energy transfer equations (Leslie and Quarini, 1979), the total spectral eddy viscosity obtained from r-RG and closure theories are in very good agreement. The eddy viscosity also depends on all resolved spectral space (and reproduces the cusp-like behavior of DNS and RPT), with contributions from both quadratic and triple nonlinearities. The price one must pay for capturing these correct physics is the significant additional complexity.

The i-RG of McComb (McComb, 1982, 1990; McComb and Shanmugasundaram, 1983) attempts to stake out a middle ground: no ϵ parameter or expansion is used and the triple velocity products are dropped. By avoiding the controversial aspects of the ϵ expansion and maintaining the quadratic nonlinearity of the Navier-Stokes, the i-RG has attracted some followers. However, without the local interactions associated with the triple nonlinearities, the i-RG eddy viscosity, like that of YO, does not have the cusp-like behavior identified by Kraichnan (1976) and Chollet & Lesieur (1981). Our numerical studies have conclusively identified the crucial role of the local interactions that is associated with the triple nonlinearities. McComb and co-workers (McComb *et al.*, 2001) have recently reached similar conclusions on the basis of their own numerical studies.

It therefore appears that RG theories, like all effectively computable turbulence theories, are compromises – compromises in the interest of analytical tractability. Continued interest in this kind of compromise is certainly justified provided that the simplifications of the dynamics that are introduced are well understood.

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List of Figures and Figure captions

Fig. 1 Comparison of hydrodynamics and magnetohydrodynamics: (a) In fluid turbulence, a mean or large-scale flow sweeps the small-scale eddies without affecting the energy transfer between length scales; (b) in magnetohydrodynamics a mean or large-scale magneti field sweep oppositely propagating fluctuations $\mathbf{z}^+$ and $\mathbf{z}^-$, which affects the energy transfer (illustrated as distortions after the two types of fluctuation have passed through each other) (Zhou et al., 2004)

Fig. 2 Kolmogorov's universal scaling for one-dimensional longitudinal power spectra. This compilation is from Chapman (1979) with late additions. (Reprinted from Saddoughi and Veeravalli, J. Fluid Mech., 1994 with permission from Cambridge Univ. Press)

Fig. 3 Ratios of higher-order to modal energy spectra in the return channel (a) and in the mixing layer (b). Square, n=2, diamond, n=3. Solid symbols and open symbols represent longitudinal and transversal components. Vertical arrow correpond to the inertial range bounds (from Zhou et al., 1993)

Fig. 4. Scaled renormalization eddy viscosity as a function of the scaled wavenumber for a relatively fine sugrdi partition (f=0.7). The unmarked curve, exhibiting the cusp behavior near the cutoff wavenumber, k_c, is the test-field model result of Kraichnan while the curve marked by squre is r-RG result. The curve marked by triangle is the r-RG model without triple nonlinearity. McComb's i-RG result(which is claimed to be able to somewhat avoid the triple nonlinearity) is shown by the dash-dot curve (Zhou *et al.*, 1988).

Fig. 5. A comparison of the net r-RG eddy viscosity ($\nu_{net}(k) = \nu^*(k) + \nu_T(k)$ arising in the r-RG energy transport equation with that arising from closure theories for free-decaying turbulence. Here $\nu^*(k)$ is the renormalized momentum eddy viscosity and $\nu_T(k)$ is the drain eddy viscosity in the energy transport equation arising from the triple nonlinearities of the renormalized r-RG Navier-Stockes equation. The net r-RG eddy viscosity $n_{net}(k)$ is plotted fro various vaules for $r = k_c/K_p$, where K_p is a constant which is directly correlated to the location of the maximum in the energy spectrum (Zhou and Vahala, 1993a).

Fig. 6 Scale Disparity parameters for Interacting scales, $s = max(k, p, q)/min(k, p, q)$. Fractional contribution $\Pi(k, s)/\Pi(k)$ to the energy flux. The straight lines indicates $s^{-2/3}$ and $s^{-4/3}$ behaviors (Zhou, 1993b)

Fig. 7. The energy transfer functions, $T^{<<}(k), T^{>>}(k), T^{><}(k)$, resulted from the resolvable-resolvable, subgrid-subgrid, and resolvable-subgrid interactions (Zhou and Vahala, 1993a)

Fig. 8 Spectral Eddy viscosity for the subgrid-subgrid and resolvable-subgrid interactions (Zhou and Vahala, 1993a)

Fig. 9 Energy spectra from filtered DND vs the LES models A and model B (Dubois, Jauberteau and Zhou, 1997)

Fig. 10 Time evolution of the correlation coefficient (LES Models A and B with filtered DNS) (Dubois, Jauberteau and Zhou, 1997)