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YOUNGS-TYPE MATERIAL STRENGTH MODEL IN THE BESNARD-HARLOW-RAUENZAHN TURBULENCE EQUATIONS

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Abstract. Youngs [AWE Report Number 96/96, 1992] has augmented a two-phase turbulence model to account for material strength. Here we adapt the model of Youngs to the turbulence model for the mixture developed by Besnard, Harlow, and Rauenzahn [LANL Report LA-10911, 1987].

1. Introduction

For applications to mixing driven by acceleration instabilities, Youngs [6] has devised a model of material strength within the context of a two-fluid turbulence model. In the present work, we adapt the model of Youngs to the turbulence model of Besnard, Harlow, and Rauenzahn [1] (BHR), which solves for the Favre-mean conservation equations and assumes species-mass flux via turbulent diffusion only.

After introducing the model of Youngs, the approach we take to the derivation of the governing equations is to generalize the BHR model successively for (1) flow with fluctuating source terms, (2) flow of the individual phases in a multiphase fluid, and (3) flow of a multiphase fluid mixture. Step (1) is carried out in Secs. 2 and 3, whereas Steps (2) and (3) are carried out in Sec. 4. Using these results, we specialize the BHR equations for a mixture to account for material strength in analogy to Youngs model in Sec. 5.

Youngs begins with the two-fluid turbulence model in Ref. [5] and adds two additional terms to account for material strength. The first is an additional dissipation term in the turbulent kinetic energy equation (using BHR nomenclature).

$$\partial_t(\bar{\rho}K) = -c_4' \frac{K^{1/2}Y}{S} \tag{1.1}$$

where K is the turbulent kinetic energy, Y the yield stress and S the turbulent length scale. The second addition is an inter-species drag term which is added to the species momentum equations:

$$D_{12}' = -c_1' \frac{\alpha_1 \alpha_2}{S} Y \frac{U}{|U|} \tag{1.2}$$

where α_i are the volume fractions and U is the difference is species velocity. Thus, the Youngs model says the effect of material strength is two-fold. It increases the dissipation rate, and reduces fluid interpenetration. Our goal is to include both of these terms in the BHR framework. To preview the work ahead, the end result is two sets of terms in the BHR equations:

$$\partial_t(\overline{\rho}\, K) + \ldots = -c_1' \, \frac{\overline{\rho}}{|\overline{\rho}_1 - \overline{\rho}_2|} \, \frac{Y}{S} \, |\boldsymbol{a}| - c_4' \frac{K^{1/2}Y}{S} \tag{1.3}$$

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2. Reynolds-Averaged Conservation Laws

Besnard, Harlow, and Rauenzahn [1], and these authors together with Zemach [2], have developed a system of evolution equations governing the flow of a turbulent two-phase mixture. This system comprises: (a) the Reynolds-averaged conservation laws for the mass of each of the two fluid constituents, for the mixture momentum, and for the mixture energy; and (b) evolution equations for certain flow quantities that arise in the conservation laws as a result of averaging. In this section, we derive the averaged conservation laws; *cf.* Refs. [2, 4]. Our attention is focused on two issues: the constitutive assumption for the stress tensor that accounts for material strength, and the changes to the conservation laws that result from fluctuating source terms in the momentum and energy equations.

We employ the following notation, close to that of Ref. [2], for the flow quantities: ρ, $\boldsymbol{u}$, I, $\boldsymbol{\sigma}$, $\boldsymbol{q}$, $\boldsymbol{f}$, and r denote the mass density, particle velocity vector, specific internal energy, Cauchy stress tensor, energy (*i.e.*, heat) flux vector, specific body force vector, and specific energy source, respectively. We allow $\boldsymbol{f}$ and r to exhibit turbulent fluctuations. Also, the two constituents are labeled by $k = 1,\, 2$, and ρ_k, c_k, α_k, and $\boldsymbol{j}_k$ denote the intrinsic mass density, mass fraction, volume fraction, and drift (*i.e.*, diffusive) flux vector. (In particular, $\rho\, c_k = \alpha_k\, \rho_k$.) Other notation is defined as it is introduced.

2.1. Conservation laws.

The conservation laws for mass, momentum, energy, and species (for $k = 1,\, 2$) are

$$\partial_t \rho + \nabla \cdot (\rho\, \boldsymbol{u}) = 0, \tag{2.1}$$

$$\partial_t(\rho\, \boldsymbol{u}) + \nabla \cdot (\rho\, \boldsymbol{u}\, \boldsymbol{u} - \boldsymbol{\sigma}) = \rho\, \boldsymbol{f}, \tag{2.2}$$

$$\partial_t(\rho\, E) + \nabla \cdot (\rho\, E\, \boldsymbol{u} - \boldsymbol{u} \cdot \boldsymbol{\sigma} + \boldsymbol{q}) = \rho\, \boldsymbol{u} \cdot \boldsymbol{f} + \rho\, r, \tag{2.3}$$

$$\partial_t(\rho\, c_k) + \nabla \cdot (\rho\, c_k\, \boldsymbol{u} + \boldsymbol{j}_k) = 0 \tag{2.4}$$

with $E := \frac{1}{2}|\boldsymbol{u}|^2 + I$ being the specific total energy. By definition, $\sum_k c_k = 1$ and $\sum_k \boldsymbol{j}_k = 0$; hence summing Eq. (2.4) produces Eq. (2.1).

In regions where the flow is smooth (*i.e.*, away from shock waves), the mass and momentum equations imply the kinetic energy balance law

$$\partial_t(\rho\, \tfrac{1}{2}|\boldsymbol{u}|^2) + \nabla \cdot \big(\rho\, \tfrac{1}{2}|\boldsymbol{u}|^2\, \boldsymbol{u}\big) = \boldsymbol{u} \cdot (\nabla \cdot \boldsymbol{\sigma}) + \rho\, \boldsymbol{u} \cdot \boldsymbol{f}. \tag{2.5}$$

Subtracting Eq. (2.5) from Eq. (2.3) shows that the specific internal energy satisfies

$$\partial_t(\rho\, I) + \nabla \cdot (\rho\, I\, \boldsymbol{u} + \boldsymbol{q}) = \boldsymbol{\sigma} : \nabla \boldsymbol{u} + \rho\, r. \tag{2.6}$$

2.2. Constitutive assumptions.

The stress tensor decomposes as

$$\boldsymbol{\sigma} = -P\, \boldsymbol{I} + \boldsymbol{\tau}, \tag{2.7}$$

where $P := -\frac{1}{3}\operatorname{tr} \boldsymbol{\sigma}$ is the mean pressure and $\boldsymbol{\tau} := \operatorname{dev} \boldsymbol{\sigma}$ is the deviatoric stress tensor. The thermal and caloric equations of state specify, respectively, the pressure P and the specific internal energy I as functions of the mass density ρ, temperature T, and fluid composition, determined by c_1. For example, for an ideal gas, $P \overset{\mathrm{m}}{=} \rho\, R\, T$ and $I \overset{\mathrm{m}}{=} c_v\, T$, where $R = \sum_k c_k\, R_k$ and $c_v = \sum_k c_k\, c_{v,k}$; here $R_k = N_A k_B / M_k$ and $c_{v,k}$ are constants for $k = 1,\, 2$.

For a Newtonian viscous fluid, as treated in Refs. [2, 4], the deviatoric stress $\boldsymbol{\tau}$ is modeled by

$$\boldsymbol{\tau}_{\text{visc}} := 2\, \mu\, \operatorname{dev} \boldsymbol{d} \qquad \text{(Newtonian viscous fluid)}, \tag{2.8}$$

with $\mu \geq 0$ being the shear viscosity. Here $\boldsymbol{d} := \operatorname{sym}(\nabla \boldsymbol{u})$ is the rate of deformation tensor. In contrast, for an isotropic rigid-plastic or rigid-viscoplastic solid undergoing plastic yielding, as considered by Youngs [6], $\boldsymbol{\tau}$ is modeled by

$$\boldsymbol{\tau}_{\text{plast}} := \sqrt{\tfrac{2}{3}}\, Y \, \frac{\operatorname{dev} \boldsymbol{d}}{\|\operatorname{dev} \boldsymbol{d}\|} \qquad \text{(rigid-plastic solid)}, \tag{2.9}$$

where Y is the dynamic yield strength of the solid. In the present work, we model $\boldsymbol{\tau}$ by

$$\boldsymbol{\tau} \overset{\text{m}}{=} \boldsymbol{\tau}_{\text{visc}} + \boldsymbol{\tau}_{\text{plast}} \tag{2.10}$$

with $\mu = \sum_i \alpha_i \, \mu_i$ and $Y = \sum_i \alpha_i \, Y_i$ being the volume-averaged shear viscosity and dynamic yield strength.

Fourier's law provides the model $\boldsymbol{q} \overset{\text{m}}{=} -\kappa \, \nabla T$ for the energy flux vector, with κ being the thermal conductivity, assumed constant. Similarly, Fick's law gives the model $\boldsymbol{j}_k \overset{\text{m}}{=} -\rho \, D \, \nabla c_k$ for the drift flux vector, with D being the mass diffusivity, also assumed to be constant.

2.3. **Reynolds-averaged balance laws.** We adopt a Reynolds averaging operator $\langle \cdot \rangle$, such as ensemble averaging. The associated notation employed for averages and fluctuations of flow quantities f and g is $\overline{f} := \langle f \rangle$, $f' := f - \overline{f}$, $\overline{\rho}\, \tilde{g} := \langle \rho \, g \rangle$, and $g'' := g - \tilde{g}$. Application of $\langle \cdot \rangle$ to Eqs. (2.1)–(2.4) yields

$$\partial_t \overline{\rho} + \nabla \cdot (\overline{\rho}\, \tilde{\boldsymbol{u}}) = 0 \tag{2.11}$$

$$\partial_t (\overline{\rho}\, \tilde{\boldsymbol{u}}) + \nabla \cdot (\overline{\rho}\, \tilde{\boldsymbol{u}}\, \tilde{\boldsymbol{u}} - \underline{\boldsymbol{\sigma}}) = \overline{\rho}\, \tilde{\boldsymbol{f}}, \tag{2.12}$$

$$\partial_t (\overline{\rho}\, \tilde{E}) + \nabla \cdot \Big(\overline{\rho}\, \tilde{E}\, \tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}} \cdot \underline{\boldsymbol{\sigma}} + \underline{\boldsymbol{q}}\Big) = \overline{\rho}\, \tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{f}} + \overline{\rho}\, \underline{r}, \tag{2.13}$$

$$\partial_t (\overline{\rho}\, \tilde{c}_k) + \nabla \cdot \big(\overline{\rho}\, \tilde{c}_k\, \tilde{\boldsymbol{u}} + \underline{\boldsymbol{j}_k}\big) = 0, \tag{2.14}$$

with $\tilde{E} := \frac{1}{2}|\tilde{\boldsymbol{u}}|^2 + \underline{I}$. Appearing in these Reynolds-averaged equations are the turbulence-enhanced stress tensor, specific internal energy, energy flux vector, specific energy source, and drift flux vector

$$\underline{\boldsymbol{\sigma}} := \overline{\boldsymbol{\sigma}} - \boldsymbol{R}, \tag{2.15}$$

$$\underline{I} := \tilde{I} + K, \tag{2.16}$$

$$\underline{\boldsymbol{q}} := \overline{\boldsymbol{q}} + \boldsymbol{q}^I + \boldsymbol{q}^{\boldsymbol{\sigma}} + \boldsymbol{q}^K, \tag{2.17}$$

$$\underline{r} := \tilde{r} + r^{\boldsymbol{f}}, \tag{2.18}$$

$$\underline{\boldsymbol{j}_k} := \overline{\boldsymbol{j}_k} + \boldsymbol{j}_k^c. \tag{2.19}$$

Here

$$\boldsymbol{R} := \langle \rho\, \boldsymbol{u}''\, \boldsymbol{u}'' \rangle, \tag{2.20}$$

$$\overline{\rho}\, K := \langle \rho\, \tfrac{1}{2} |\boldsymbol{u}''|^2 \rangle, \tag{2.21}$$

$$\boldsymbol{q}^I := \langle \rho\, I''\, \boldsymbol{u}'' \rangle, \tag{2.22}$$

$$\boldsymbol{q}^{\boldsymbol{\sigma}} := -\langle \boldsymbol{u}'' \cdot \boldsymbol{\sigma} \rangle, \tag{2.23}$$

$$\boldsymbol{q}^K := \langle \rho\, \tfrac{1}{2} |\boldsymbol{u}''|^2\, \boldsymbol{u}'' \rangle, \tag{2.24}$$

$$\overline{\rho}\, r^{\boldsymbol{f}} := \langle \rho\, \boldsymbol{u}'' \cdot \boldsymbol{f}'' \rangle, \tag{2.25}$$

$$\boldsymbol{j}_k^c := \langle \rho\, c_k''\, \boldsymbol{u}'' \rangle \tag{2.26}$$

define the Reynolds stress tensor $-\boldsymbol{R}$, the turbulence kinetic energy (TKE) K, the turbulence energy flux vectors associated with the internal energy ($\boldsymbol{q}^I$), stress ($\boldsymbol{q}^{\boldsymbol{\sigma}}$), and TKE ($\boldsymbol{q}^K$), respectively, the turbulence specific energy source $r^{\boldsymbol{f}}$, and the turbulence species flux $\boldsymbol{j}_k^c$.

In regions where the flow is smooth, the Reynolds-averaged mass and momentum equations imply that the mean-flow kinetic energy satisfies

$$\partial_t(\rho\,\tfrac{1}{2}|\tilde{\boldsymbol{u}}|^2) + \nabla\cdot\big(\rho\,\tfrac{1}{2}|\tilde{\boldsymbol{u}}|^2\,\tilde{\boldsymbol{u}}\big) = \tilde{\boldsymbol{u}}\cdot(\nabla\cdot\underline{\boldsymbol{\sigma}}) + \overline{\rho}\,\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{f}}. \tag{2.27}$$

Therefore the turbulence-enhanced specific internal energy satisfies

$$\partial_t(\overline{\rho}\,\underline{I}) + \nabla\cdot\big(\overline{\rho}\,\underline{I}\,\tilde{\boldsymbol{u}} + \underline{\boldsymbol{q}}\big) = \underline{\boldsymbol{\sigma}} : \nabla\tilde{\boldsymbol{u}} + \overline{\rho}\,\underline{r}. \tag{2.28}$$

On the other hand, applying the Reynolds averaging operator to Eq. (2.6) reveals that

$$\partial_t(\overline{\rho}\,\tilde{I}) + \nabla\cdot\Big(\overline{\rho}\,\tilde{I}\,\tilde{\boldsymbol{u}} + \overline{\boldsymbol{q}} + \boldsymbol{q}^I\Big) = \langle\boldsymbol{\sigma} : \nabla\boldsymbol{u}\rangle + \overline{\rho}\,\tilde{r}. \tag{2.29}$$

Following Ref. [2], define the vector $\boldsymbol{a} := -\langle\boldsymbol{u}''\rangle$, which appears in the identities

$$\bar{\boldsymbol{u}} = \tilde{\boldsymbol{u}} - \boldsymbol{a}, \tag{2.30}$$

$$\boldsymbol{u}' = \boldsymbol{u}'' + \boldsymbol{a}. \tag{2.31}$$

Then

$$\begin{aligned}\langle\boldsymbol{\sigma} : \nabla\boldsymbol{u}\rangle &= \overline{\boldsymbol{\sigma}} : \nabla\tilde{\boldsymbol{u}} - \overline{\boldsymbol{\sigma}} : \nabla\boldsymbol{a} + \langle\boldsymbol{\sigma}' : \nabla\boldsymbol{u}'\rangle \\ &= \overline{\boldsymbol{\sigma}} : \nabla\tilde{\boldsymbol{u}} - \nabla\cdot(\boldsymbol{a}\cdot\overline{\boldsymbol{\sigma}}\,) + \boldsymbol{a}\cdot(\nabla\cdot\overline{\boldsymbol{\sigma}}\,) + \langle\boldsymbol{\sigma}' : \nabla\boldsymbol{u}'\rangle.\end{aligned} \tag{2.32}$$

As a result, the averaged specific internal energy is governed by

$$\partial_t(\overline{\rho}\,\tilde{I}) + \nabla\cdot\Big(\overline{\rho}\,\tilde{I}\,\tilde{\boldsymbol{u}} + \overline{\boldsymbol{q}} + \boldsymbol{q}^I + \boldsymbol{a}\cdot\overline{\boldsymbol{\sigma}}\Big) = \overline{\boldsymbol{\sigma}} : \nabla\tilde{\boldsymbol{u}} + \mathcal{D}^K + \overline{\rho}\,\tilde{r}, \tag{2.33}$$

where

$$\mathcal{D}^K := \boldsymbol{a}\cdot(\nabla\cdot\overline{\boldsymbol{\sigma}}\,) - \overline{\rho}\,\pi + \overline{\rho}\,\epsilon, \tag{2.34}$$

$$\overline{\rho}\,\pi := \langle P'\,\nabla\cdot\boldsymbol{u}'\rangle, \tag{2.35}$$

$$\overline{\rho}\,\epsilon := \langle\boldsymbol{\tau}' : \nabla\boldsymbol{u}'\rangle. \tag{2.36}$$

Subtracting Eq. (2.29) from Eq. (2.28), we find that K satisfies the equation

$$\partial_t(\overline{\rho}\,K) + \nabla\cdot\big(\overline{\rho}\,K\,\tilde{\boldsymbol{u}} + \boldsymbol{q}^K - \langle\boldsymbol{u}'\cdot\boldsymbol{\sigma}'\rangle\big) = -\boldsymbol{R} : \nabla\tilde{\boldsymbol{u}} - \mathcal{D}^K + \overline{\rho}\,r^{\boldsymbol{f}}. \tag{2.37}$$

Hence $\mathcal{D}^K$ is the rate of transfer of energy from K to $\tilde{I}$.

2.4. **Closure.** The system (2.11), (2.12), (2.14), and (2.33) involves several quantities that require modeling. We refer to Ref. [2, Sec. 3.8] for a discussion of the averaged thermal and caloric equations of state that determine $\overline{P}$ and $\tilde{I}$ from $\overline{\rho}$, $\tilde{T}$, and $\tilde{c}_1$. A derivation of, and closure model for, an evolution equation governing $\boldsymbol{R}$ (and hence K) is presented in Sec. 3. Following Ref. [2], we adopt the models

$$\overline{\boldsymbol{q}} \overset{\mathrm{m}}{=} -\kappa\,\nabla\tilde{T}, \tag{2.38}$$

$$\boldsymbol{q}^I \overset{\mathrm{m}}{=} -C_{DI}\,\tau_{\mathrm{t}}\,(\nabla\tilde{I})\cdot\boldsymbol{R}, \tag{2.39}$$

$$\overline{\boldsymbol{j}_k} \overset{\mathrm{m}}{=} -\overline{\rho}\,D\,\nabla\tilde{c}_k, \tag{2.40}$$

$$\boldsymbol{j}_k^c \overset{\mathrm{m}}{=} -C_{Dc}\,\tau_{\mathrm{t}}\,(\nabla\tilde{c}_k)\cdot\boldsymbol{R}, \tag{2.41}$$

$$\pi \overset{\mathrm{m}}{=} 0. \tag{2.42}$$

Here $\tau_{\mathrm{t}} := K/\epsilon$ is the turbulence time scale and C_{DI} and C_{Dc} are nondimensional constants.

In the case of a fluid, ϵ can be modeled by

$$\epsilon_{\mathrm{visc}} := K^{3/2}/L, \tag{2.43}$$

where L is the turbulence length scale, which is governed by a balance law analogous to Eq. (2.37) for K. The correct non-dimensional grouping for the inclusion of Yield strength is:

$$\epsilon := \frac{K^{3/2}}{L} + c' \frac{K^{1/2} Y}{\overline{\rho} L} \tag{2.44}$$

3. BHR Equations

In Refs. [1, 2], evolution equations are developed (in the case without fluctuating source terms) for $\boldsymbol{R}$ and for auxiliary flow quantities such as $\boldsymbol{a} := -\langle \boldsymbol{u}'' \rangle$ and $b := \overline{\rho}\, \langle \upsilon'' \rangle$ (with $\upsilon := 1/\rho$ being the specific volume). These equations are equivalent (modulo some minor changes explained below) to the following ones:

$$\begin{aligned} &\partial_t \boldsymbol{R} + \nabla \cdot \left(\boldsymbol{R}\, \tilde{\boldsymbol{u}} - 3\, C_{DR}\, \tau_{\mathrm{t}}\, \mathrm{sym}_3 \left\{ \left[\nabla \left(\tilde{\upsilon}\, \boldsymbol{R}\right)\right] \cdot \boldsymbol{R} \right\}\right) \\ &\quad = -2\, \mathrm{sym} \left[(\nabla \tilde{\boldsymbol{u}}) \cdot \boldsymbol{R} \right] + 2\, C_{2R}\, \mathrm{dev}\, \mathrm{sym} \left[(\nabla \tilde{\boldsymbol{u}}) \cdot \boldsymbol{R} \right] \\ &\qquad - 2\, \mathrm{sym} \left(\boldsymbol{a}\, \nabla \cdot \overline{\boldsymbol{\sigma}} \right) + 2\, C_{3R}\, \mathrm{dev}\, \mathrm{sym} \left(\boldsymbol{a}\, \nabla \cdot \overline{\boldsymbol{\sigma}} \right) \\ &\qquad - C_{1R}\, \mathrm{dev}\, \boldsymbol{R}/\tau_{\mathrm{t}} - \tfrac{2}{3}\, \overline{\rho}\, \epsilon\, \boldsymbol{I}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\partial_t \boldsymbol{a} + \nabla \cdot \left(\boldsymbol{a}\, \tilde{\boldsymbol{u}} - 2\, C_{Da}\, \tau_{\mathrm{t}}\, \mathrm{sym} \left\{ \left[\nabla \left(\tilde{\upsilon}\, \boldsymbol{a} \right) \right] \cdot \boldsymbol{R} \right\} \right) \\ &\quad = (\nabla \tilde{\upsilon}) \cdot \boldsymbol{R} + (\nabla \cdot \tilde{\boldsymbol{u}})\, \boldsymbol{a} + (C_{2a} - 1)\, (\nabla \tilde{\boldsymbol{u}}) \cdot \boldsymbol{a} \\ &\qquad + (C_{3a} - 1)\, \tilde{\upsilon}\, b\, \nabla \cdot \overline{\boldsymbol{\sigma}} - C_{1a}\, \boldsymbol{a}/\tau_{\mathrm{t}}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\partial_t (\tilde{\upsilon}\, b) + \nabla \cdot \left(\tilde{\upsilon}\, b\, \tilde{\boldsymbol{u}} - C_{Db}\, \tau_{\mathrm{t}} \left[\nabla \left(\tilde{\upsilon}_2\, b \right) \right] \cdot \boldsymbol{R} \right) \\ &\quad = 2\, (\nabla \tilde{\upsilon}) \cdot \boldsymbol{a} + 2\, \tilde{\upsilon}\, b\, \nabla \cdot \tilde{\boldsymbol{u}} - C_{1b}\, \tilde{\upsilon}\, b/\tau_{\mathrm{t}}. \end{aligned} \tag{3.3}$$

Here the C coefficients are nondimensional constants. In particular, one-half of the trace of Eq. (3.1) is the equation for K:

$$\partial_t(\overline{\rho}\, K) + \nabla \cdot \left(\overline{\rho}\, K\, \tilde{\boldsymbol{u}} - C_{DK}\, \overline{\rho}\, K\, \tau_{\mathrm{t}}\, \nabla K \right) = -\boldsymbol{R} : \nabla \tilde{\boldsymbol{u}} - \boldsymbol{a} \cdot (\nabla \cdot \overline{\boldsymbol{\sigma}}) - \overline{\rho}\, \epsilon. \tag{3.4}$$

(In this model for the diffusive flux of K, we have neglected $\mathrm{dev}\, \boldsymbol{R}$ relative to $\overline{\rho}\, K$ and set $C_{DK} := (10/9)\, C_{DR}$. As in Ref. [2], C_{DK} is identified with C_μ.)

The derivation of these equations is now recounted and generalized to a minor extent. Our purpose is to determine the changes in the BHR equations that result from the fluctuating source terms $\rho\, \boldsymbol{f}$ and $\rho\, r$.

3.1. Evolution of a covariance.

Occurring frequently in Eqs. (2.20)–(2.26) is the Favre covariance

$$\widetilde{\mathrm{cov}}(g,\, \boldsymbol{u}) := \langle \rho\, g''\, \boldsymbol{u}'' \rangle / \overline{\rho} \tag{3.5}$$

between a tensorial flow quantity g and the velocity $\boldsymbol{u}$. For example, the covariance corresponding to $g = \boldsymbol{u}$ is given by

$$\overline{\rho}\, \widetilde{\mathrm{cov}}(\boldsymbol{u}, \boldsymbol{u}) = \boldsymbol{R}; \tag{3.6}$$

and the covariance corresponding to $g = \upsilon$ is given by

$$\overline{\rho}\, \widetilde{\mathrm{cov}}(\upsilon, \boldsymbol{u}) = \langle \rho\, \upsilon''\, \boldsymbol{u}'' \rangle = \langle \rho\, \upsilon\, \boldsymbol{u}'' \rangle = \langle \boldsymbol{u}'' \rangle = -\boldsymbol{a}. \tag{3.7}$$

Suppose that g satisfies a balance law of the form

$$\partial_t(\rho\, g) + \nabla\cdot(\rho\, g\, \boldsymbol{u}) = \sigma_g, \tag{3.8}$$

with σ_g denoting the source of g. For example, if $g = \boldsymbol{u}$, then by Eq. (2.2), $\sigma_g = \nabla\cdot\boldsymbol{\sigma} + \rho\,\boldsymbol{f}$; and if $g = \upsilon$, then $\sigma_g = \nabla\cdot\boldsymbol{u}$. As the Reynolds average of Eq. (3.8) is

$$\partial_t(\overline{\rho}\,\tilde{g}) + \nabla\cdot[\overline{\rho}\,\tilde{g}\,\tilde{\boldsymbol{u}} + \overline{\rho}\,\widetilde{\mathrm{cov}}(g,\, \boldsymbol{u})] = \overline{\sigma_g}, \tag{3.9}$$

the covariance $\overline{\rho}\,\widetilde{\mathrm{cov}}(g,\, \boldsymbol{u})$ represents the extra flux of $\tilde{g}$ associated with the turbulent fluctuations, *i.e.*, the turbulence flux of g.

3.2. **Evolution of a covariance with the velocity.** We seek the evolution equation governing $\widetilde{\mathrm{cov}}(g,\, \boldsymbol{u})$. To this end, start with the equations for g and $\tilde{g}$, revised using Eqs. (2.1) and (2.11) to take the forms

$$\rho\,\partial_t g + \rho\,(\nabla g)\cdot\boldsymbol{u} = \sigma_g, \tag{3.10}$$

$$\overline{\rho}\,\partial_t\tilde{g} + \overline{\rho}\,(\nabla\tilde{g})\cdot\tilde{\boldsymbol{u}} + \nabla\cdot[\overline{\rho}\,\widetilde{\mathrm{cov}}(g,\, \boldsymbol{u})] = \overline{\sigma_g}, \tag{3.11}$$

respectively. Noting that $\tilde{\upsilon} = 1/\,\overline{\rho}$, subtract $\tilde{\upsilon}\,\rho$ times the second equation above from the first:

$$\rho\,\partial_t g'' + \rho\,(\nabla g'')\cdot\boldsymbol{u} = -\rho\,(\nabla\tilde{g})\cdot\boldsymbol{u}'' + \tilde{\upsilon}\,\rho\,\nabla\cdot[\overline{\rho}\,\widetilde{\mathrm{cov}}(g,\, \boldsymbol{u})] + \sigma_g - \tilde{\upsilon}\,\rho\,\overline{\sigma_g}. \tag{3.12}$$

Example 3.1. If g is taken to be $\boldsymbol{u}$, then $\overline{\rho}\,\widetilde{\mathrm{cov}}(g,\, \boldsymbol{u})$ is $\boldsymbol{R}$ and, according to Eq. (2.2), σ_g is $\nabla\cdot\boldsymbol{\sigma} + \rho\,\boldsymbol{f}$. In this case Eq. (3.12) reads

$$\rho\,\partial_t\boldsymbol{u}'' + \rho\,(\nabla\boldsymbol{u}'')\cdot\boldsymbol{u} = -\rho\,(\nabla\tilde{\boldsymbol{u}})\cdot\boldsymbol{u}'' + \nabla\cdot\boldsymbol{\sigma} + \tilde{\upsilon}\,\rho\,\nabla\cdot(-\overline{\boldsymbol{\sigma}} + \boldsymbol{R}) + \rho\,\boldsymbol{f}''. \tag{3.13}$$

Next add Eq. (3.12), multiplied on the right by $\boldsymbol{u}''$, to Eq. (3.13), multiplied on the left by g''. Also add Eq. (2.1) multiplied by $g''\,\boldsymbol{u}''$. Combine the differentiated terms on the left-hand side to find that

$$\begin{aligned}\partial_t(\rho\, g''\,\boldsymbol{u}'') + \nabla\cdot(\rho\, g''\,\boldsymbol{u}''\,\boldsymbol{u}) &= -(\nabla\tilde{g})\cdot\rho\,\boldsymbol{u}''\,\boldsymbol{u}'' + \tilde{\upsilon}\,\nabla\cdot[\overline{\rho}\,\widetilde{\mathrm{cov}}(g,\, \boldsymbol{u})]\,\rho\,\boldsymbol{u}'' \\ &\quad + \sigma_g\,\boldsymbol{u}'' - \tilde{\upsilon}\,\overline{\sigma_g}\,\rho\,\boldsymbol{u}'' - \rho\, g''\,(\nabla\tilde{\boldsymbol{u}})\cdot\boldsymbol{u}'' + g''\,\nabla\cdot\boldsymbol{\sigma} \\ &\quad + \tilde{\upsilon}\,\rho\, g''\,\nabla\cdot(-\overline{\boldsymbol{\sigma}} + \boldsymbol{R}) + \rho\, g''\,\boldsymbol{f}''.\end{aligned} \tag{3.14}$$

Finally, substitute $\boldsymbol{u} = \tilde{\boldsymbol{u}} + \boldsymbol{u}''$ in the second term on the left-hand side and apply the Reynolds averaging operator:

$$\begin{aligned}&\partial_t\,[\overline{\rho}\,\widetilde{\mathrm{cov}}(g,\, \boldsymbol{u})] + \nabla\cdot[\overline{\rho}\,\widetilde{\mathrm{cov}}(g,\, \boldsymbol{u})\,\tilde{\boldsymbol{u}} + \langle\rho\, g''\,\boldsymbol{u}''\,\boldsymbol{u}''\rangle] \\ &\quad = -(\nabla\tilde{g})\cdot\boldsymbol{R} + \langle\sigma_g\,\boldsymbol{u}''\rangle - \overline{\rho}\,\widetilde{\mathrm{cov}}(g,\, (\nabla\tilde{\boldsymbol{u}})\cdot\boldsymbol{u}) + \langle g''\,\nabla\cdot\boldsymbol{\sigma}\rangle + \overline{\rho}\,\widetilde{\mathrm{cov}}(g,\, \boldsymbol{f}).\end{aligned} \tag{3.15}$$

Example 3.2. If g is taken to be $\boldsymbol{u}$, as in the preceding example, then Eq. (3.15) becomes

$$\begin{aligned}&\partial_t\boldsymbol{R} + \nabla\cdot[\boldsymbol{R}\,\tilde{\boldsymbol{u}} + \langle\rho\,\boldsymbol{u}''\,\boldsymbol{u}''\,\boldsymbol{u}''\rangle] \\ &\qquad = -2\ \mathrm{sym}\,[(\nabla\tilde{\boldsymbol{u}})\cdot\boldsymbol{R}] + 2\ \mathrm{sym}\langle\boldsymbol{u}''\,\nabla\cdot\boldsymbol{\sigma}\rangle + 2\,\overline{\rho}\ \mathrm{sym}\,\widetilde{\mathrm{cov}}(\boldsymbol{u},\, \boldsymbol{f}).\end{aligned} \tag{3.16}$$

As a consequence of the identity

$$\langle\boldsymbol{u}''\,\nabla\cdot\boldsymbol{\sigma}\rangle = -\boldsymbol{a}\,\nabla\cdot\overline{\boldsymbol{\sigma}} + \langle\boldsymbol{u}''\,\nabla\cdot\boldsymbol{\sigma}'\rangle = -\boldsymbol{a}\,\nabla\cdot\overline{\boldsymbol{\sigma}} + \nabla\cdot\langle\boldsymbol{u}'\,\boldsymbol{\sigma}'\rangle - \langle(\nabla\boldsymbol{u}')\cdot\boldsymbol{\sigma}'\rangle, \tag{3.17}$$

this equation becomes

$$\begin{aligned}&\partial_t\boldsymbol{R} + \nabla\cdot[\boldsymbol{R}\,\tilde{\boldsymbol{u}} + \langle\rho\,\boldsymbol{u}''\,\boldsymbol{u}''\,\boldsymbol{u}''\rangle] - 2\ \mathrm{sym}\,[\nabla\cdot\langle\boldsymbol{u}'\,\boldsymbol{\sigma}'\rangle] \\ &\qquad = -2\ \mathrm{sym}\,[(\nabla\tilde{\boldsymbol{u}})\cdot\boldsymbol{R}] - \boldsymbol{\mathcal{D}}^{\boldsymbol{R}} + 2\,\overline{\rho}\ \mathrm{sym}\,\widetilde{\mathrm{cov}}(\boldsymbol{u},\, \boldsymbol{f}),\end{aligned} \tag{3.18}$$

where the energy transfer rate tensor is

$$\boldsymbol{\mathcal{D}}^{\boldsymbol{R}} := 2\,\mathrm{sym}\,[\boldsymbol{a}\,(\nabla\cdot\overline{\boldsymbol{\sigma}}\,)] + 2\,\mathrm{sym}\,\langle(\nabla\boldsymbol{u}')\cdot\boldsymbol{\sigma}'\rangle\,. \tag{3.19}$$

This equation governing the Reynolds tensor $\boldsymbol{R}$ coincides with the equation at the top of page 14 in Ref. [2]. Also, half of the trace of this equation for $\boldsymbol{R}$ is Eq. (2.37) for K.

Example 3.3. If g is taken to be υ, then $\overline{\rho}\,\widetilde{\mathrm{cov}}(g,\,\boldsymbol{u}) = -\boldsymbol{a}$ and σ_g is $\nabla\cdot\boldsymbol{u}$. Hence Eq. (3.15) is the governing equation for $\boldsymbol{a}$:

$$\begin{aligned}\partial_t\boldsymbol{a} + \nabla\cdot[\boldsymbol{a}\,\tilde{\boldsymbol{u}} - \langle\rho\,\upsilon''\,\boldsymbol{u}''\,\boldsymbol{u}''\rangle] = (\nabla\tilde{\upsilon})\cdot\boldsymbol{R} + (\nabla\cdot\tilde{\boldsymbol{u}})\,\boldsymbol{a} - \langle(\nabla\cdot\boldsymbol{u}'')\,\boldsymbol{u}''\rangle \\ - (\nabla\tilde{\boldsymbol{u}})\cdot\boldsymbol{a} - \tilde{\upsilon}\,b\,\nabla\cdot\overline{\boldsymbol{\sigma}} - \langle\upsilon'\,\nabla\cdot\boldsymbol{\sigma}'\rangle - \overline{\rho}\,\widetilde{\mathrm{cov}}(\upsilon,\,\boldsymbol{f}).\end{aligned} \tag{3.20}$$

(We have decomposed $\boldsymbol{u}$ as $\tilde{\boldsymbol{u}} + \boldsymbol{u}''$ and $\boldsymbol{\sigma}$ as $\overline{\boldsymbol{\sigma}} + \boldsymbol{\sigma}'$.)

Remark. Multiplying Eq. (3.20) by $\overline{\rho}$, adding $\boldsymbol{a}$ times Eq. (2.11), and invoking the identities

$$\langle(\nabla\cdot\boldsymbol{u}'')\,\boldsymbol{u}''\rangle = \langle(\nabla\cdot\boldsymbol{u}')\,\boldsymbol{u}'\rangle + (\nabla\cdot\boldsymbol{a})\,\boldsymbol{a}, \tag{3.21}$$

$$\langle\rho\,\upsilon''\,\boldsymbol{u}''\,\boldsymbol{u}''\rangle = -\tilde{\upsilon}\,\langle\rho'\,\boldsymbol{u}'\,\boldsymbol{u}'\rangle + 2\,\boldsymbol{a}\,\boldsymbol{a} \tag{3.22}$$

yields an alternative form of the governing equation for $\boldsymbol{a}$:

$$\begin{aligned}\partial_t(\overline{\rho}\,\boldsymbol{a}) + \nabla\cdot(\overline{\rho}\,\boldsymbol{a}\,\tilde{\boldsymbol{u}}) + \overline{\rho}\,(\nabla\bar{\boldsymbol{u}})\cdot\boldsymbol{a} = -\overline{\rho}\,\nabla\cdot[\tilde{\upsilon}\,\langle\rho'\,\boldsymbol{u}'\,\boldsymbol{u}'\rangle] - \tilde{\upsilon}\,(\nabla\overline{\rho})\cdot\boldsymbol{R} + \overline{\rho}\,\nabla\cdot(\boldsymbol{a}\,\boldsymbol{a}) \\ - b\,\nabla\cdot\overline{\boldsymbol{\sigma}} - \overline{\rho}\,\langle\upsilon'\,\nabla\cdot\boldsymbol{\sigma}'\rangle - \overline{\rho}\,\langle(\nabla\cdot\boldsymbol{u}')\,\boldsymbol{u}'\rangle - \overline{\rho}\,\langle\boldsymbol{f}''\rangle.\end{aligned} \tag{3.23}$$

If $\boldsymbol{f} = 0$, this equation is the combination of equations (28) and (29) in Ref. [2] under the assumption, made therein, that $\nabla\cdot\boldsymbol{u}' = 0$.

3.3. **Evolution of a general covariance.** More generally, the Favre covariance between the flow quantities g and h is

$$\widetilde{\mathrm{cov}}(g,\,h) := \langle\rho\,g''\,h''\rangle/\,\overline{\rho}. \tag{3.24}$$

Manipulations similar to those in Sec. 3.2 show that

$$\begin{aligned}\partial_t(\rho\,g''\,h'') + \nabla\cdot(\rho\,g''\,h''\,\boldsymbol{u}) = -(\nabla\tilde{g})\cdot\rho\,\boldsymbol{u}''\,h'' - \rho\,g''\,(\nabla\tilde{h})\cdot\boldsymbol{u}'' \\ + \tilde{\upsilon}\,\nabla\cdot[\overline{\rho}\,\widetilde{\mathrm{cov}}(g,\,\boldsymbol{u})]\,\rho\,h'' + \tilde{\upsilon}\,\rho\,g''\,\nabla\cdot[\overline{\rho}\,\widetilde{\mathrm{cov}}(h,\,\boldsymbol{u})] \\ + \sigma_g\,h'' - \tilde{\upsilon}\,\overline{\sigma_g}\,\rho\,h'' + g''\,\sigma_h - \tilde{\upsilon}\,\rho\,g''\,\overline{\sigma_h}.\end{aligned} \tag{3.25}$$

Averaging then yields the equation governing $\widetilde{\mathrm{cov}}(g,\,h)$:

$$\begin{aligned}\partial_t\,[\overline{\rho}\,\widetilde{\mathrm{cov}}(g,\,h)] + \nabla\cdot[\overline{\rho}\,\widetilde{\mathrm{cov}}(g,\,h)\,\tilde{\boldsymbol{u}} + \langle\rho\,g''\,h''\,\boldsymbol{u}''\rangle] \\ = -(\nabla\tilde{g})\cdot\overline{\rho}\,\widetilde{\mathrm{cov}}(\boldsymbol{u},\,h) - \overline{\rho}\,\widetilde{\mathrm{cov}}(g,\,(\nabla\tilde{h})\cdot\boldsymbol{u}) + \langle\sigma_g\,h''\rangle + \langle g''\,\sigma_h\rangle.\end{aligned} \tag{3.26}$$

Example 3.4. If g and h are both taken to be υ, then Eq. (3.26) is the equation governing the Favre variance $\mathrm{var}(\upsilon) := \widetilde{\mathrm{cov}}(\upsilon,\,\upsilon) = \tilde{\upsilon}^2\,b$:

$$\partial_t(\tilde{\upsilon}\,b) + \nabla\cdot[\tilde{\upsilon}\,b\,\tilde{\boldsymbol{u}} + \langle\rho\,\upsilon''\,\upsilon''\,\boldsymbol{u}''\rangle] = 2\,(\nabla\tilde{\upsilon})\cdot\boldsymbol{a} + 2\,\tilde{\upsilon}\,b\,\nabla\cdot\tilde{\boldsymbol{u}} + 2\,\langle\upsilon''\,\nabla\cdot\boldsymbol{u}''\rangle. \tag{3.27}$$

(Again we have decomposed $\boldsymbol{u}$ as $\tilde{\boldsymbol{u}} + \boldsymbol{u}''$.)

Remark. Multiplying Eq. (3.27) by $\overline{\rho}$, adding $\tilde{\upsilon}\,b$ times Eq. (2.11), and invoking the identities

$$\langle\upsilon''\,\nabla\cdot\boldsymbol{u}''\rangle = \langle\upsilon'\,\nabla\cdot\boldsymbol{u}'\rangle - \tilde{\upsilon}\,b\,\nabla\cdot\boldsymbol{a}, \tag{3.28}$$

$$\langle\rho\,\upsilon''\,\upsilon''\,\boldsymbol{u}''\rangle = \langle\upsilon'\,\boldsymbol{u}'\rangle + (-b+1)\,\tilde{\upsilon}\,\boldsymbol{a} \tag{3.29}$$

along with $\tilde{\boldsymbol{u}} = \bar{\boldsymbol{u}} + \boldsymbol{a}$ yields an alternative form of the governing equation for b:

$$\partial_t b + \bar{\boldsymbol{u}} \cdot \nabla b + (b+1)\, \tilde{\upsilon}\, \nabla \cdot (\overline{\rho}\, \boldsymbol{a}) + \overline{\rho}\, \nabla \cdot \langle \upsilon'\, \boldsymbol{u}' \rangle - 2\, \overline{\rho}\, \langle \upsilon'\, \nabla \cdot \boldsymbol{u}' \rangle = 0, \tag{3.30}$$

which is Eq. (34) in Ref. [2].

3.4. **Closure.** The equations governing $\boldsymbol{R}$, $\boldsymbol{a}$, and b, *viz.*, Eqs. (3.18), (3.20), and (3.27), are not closed because of the appearance of several unknown averages. Closures for these quantities were proposed in Refs. [1, 2].

3.4.1. *Closure for the equation governing* $\boldsymbol{R}$. As in Refs. [1, 2] we close Eq. (3.18) for $\boldsymbol{R}$ by modeling the second term in the energy transfer rate tensor (3.19) by

$$\begin{aligned} 2\,\mathrm{sym}\,\langle (\nabla \boldsymbol{u}') \cdot \boldsymbol{\sigma}' \rangle \overset{\mathrm{m}}{=} \mathrm{dev}\,\{ & C_{1R}\, \boldsymbol{R}/\tau_{\mathrm{t}} - 2\, C_{2R}\, \mathrm{sym}\, [(\nabla \tilde{\boldsymbol{u}}) \cdot \boldsymbol{R}] - 2\, C_{3R}\, \mathrm{sym}\, (\boldsymbol{a}\, \nabla \cdot \overline{\boldsymbol{\sigma}}) \} \\ & + \tfrac{2}{3}\, (\epsilon - \pi)\, \boldsymbol{I} \end{aligned} \tag{3.31}$$

and adopting the models (2.42) and (2.44) along with

$$\langle \rho\, \boldsymbol{u}''\, \boldsymbol{u}''\, \boldsymbol{u}'' \rangle \overset{\mathrm{m}}{=} -3\, C_{DR}\, \tau_{\mathrm{t}}\, \mathrm{sym}_3\, \{ [\nabla\, (\tilde{\upsilon}\, \boldsymbol{R})] \cdot \boldsymbol{R} \}, \tag{3.32}$$

$$\mathrm{sym}\, [\nabla \cdot \langle \boldsymbol{u}'\, \boldsymbol{\sigma}' \rangle] \overset{\mathrm{m}}{=} 0. \tag{3.33}$$

Here C_{1R}, C_{2R}, C_{3R}, C_{DR}, and C_μ are nondimensional constants. Thus we obtain Eq. (3.1) augmented by the term $2\, \overline{\rho}\, \mathrm{sym}\, \widetilde{\mathrm{cov}}(\boldsymbol{u}, \boldsymbol{f})$. In particular, the term $\langle \rho\, \boldsymbol{u}'' \cdot \boldsymbol{f}'' \rangle$ is added to the right-hand side of Eq. (3.4).

3.4.2. *Closure for the equation governing* $\boldsymbol{a}$. We close Eq. (3.20) for $\boldsymbol{a}$ in a manner similar to that of Refs. [1, 2]:

$$-\langle \upsilon'\, \nabla \cdot \boldsymbol{\sigma}' \rangle \overset{\mathrm{m}}{=} -C_{1a}\, \boldsymbol{a}/\tau_{\mathrm{t}} + C_{2a}\, (\nabla \tilde{\boldsymbol{u}}) \cdot \boldsymbol{a} + C_{3a}\, \tilde{\upsilon}\, b\, \nabla \cdot \overline{\boldsymbol{\sigma}}, \tag{3.34}$$

$$\langle \rho\, \upsilon''\, \boldsymbol{u}''\, \boldsymbol{u}'' \rangle \overset{\mathrm{m}}{=} 2\, C_{Da}\, \tau_{\mathrm{t}}\, \mathrm{sym}\, \{ [\nabla\, (\tilde{\upsilon}\, \boldsymbol{a})] \cdot \boldsymbol{R} \}, \tag{3.35}$$

$$\langle (\nabla \cdot \boldsymbol{u}'')\, \boldsymbol{u}'' \rangle \overset{\mathrm{m}}{=} 0. \tag{3.36}$$

Here C_{1a}, C_{2a}, C_{3a}, and C_{Da} are nondimensional constants. Thus we obtain Eq. (3.2) augmented by the term $-\overline{\rho}\, \widetilde{\mathrm{cov}}(\upsilon, \boldsymbol{f})$.

Remark. In effect, the closure model of Ref. [2], which assumes that $\nabla \cdot \boldsymbol{u}' = 0$, takes

$$\langle \rho\, \upsilon''\, \boldsymbol{u}''\, \boldsymbol{u}'' \rangle \overset{\mathrm{m}}{=} 2\, C_{Da}\, \tau_{\mathrm{t}}\, \tilde{\upsilon}\, \mathrm{sym}\, [(\nabla \boldsymbol{a}) \cdot \boldsymbol{R}] + 2\, \boldsymbol{a}\, \boldsymbol{a}, \tag{3.37}$$

$$\langle (\nabla \cdot \boldsymbol{u}'')\, \boldsymbol{u}'' \rangle \overset{\mathrm{m}}{=} (\nabla \cdot \boldsymbol{a})\, \boldsymbol{a} \tag{3.38}$$

instead of Eqs. (3.35) and (3.36). Notice that in this formulation the diffusing flow quantity is $\boldsymbol{a}$ instead of $\tilde{\upsilon}\, \boldsymbol{a}$.

3.4.3. *Closure for the equation governing* b. We also close Eq. (3.27) for b in a manner similar to that of Refs. [1, 2]:

$$2\, \langle \upsilon''\, \nabla \cdot \boldsymbol{u}'' \rangle \overset{\mathrm{m}}{=} -C_{1b}\, \tilde{\upsilon}\, b/\tau_{\mathrm{t}}, \tag{3.39}$$

$$\langle \rho\, \upsilon''\, \upsilon''\, \boldsymbol{u}'' \rangle \overset{\mathrm{m}}{=} -C_{Db}\, \tau_{\mathrm{t}}\, \left[\nabla \left(\tilde{\upsilon}^2\, b \right) \right] \cdot \boldsymbol{R}, \tag{3.40}$$

Here C_{1b} and C_{Db} are nondimensional constants. Thus we obtain Eq. (3.3).

Remark. In effect, the closure model of Ref. [2] takes

$$2\,\langle \upsilon''\,\nabla\cdot\boldsymbol{u}''\rangle \overset{\mathrm{m}}{=} -C_{1b}\,\tilde{\upsilon}\,b/\tau_{\mathrm{t}} - 2\,\tilde{\upsilon}\,b\,\nabla\cdot\boldsymbol{a}, \tag{3.41}$$

$$\langle \rho\,\upsilon''\,\upsilon''\,\boldsymbol{u}''\rangle \overset{\mathrm{m}}{=} -C_{Db}\,\tau_{\mathrm{t}}\,\tilde{\upsilon}\,\left[\nabla\left(\frac{1+b}{\overline{\rho}}\right)\right]\cdot\boldsymbol{R} + (-b+1)\,\tilde{\upsilon}\,\boldsymbol{a} \tag{3.42}$$

instead of Eqs. (3.39) and (3.40). In this formulation the diffusing flow quantity is $\tilde{\upsilon}\,(1+b)$ instead of $\tilde{\upsilon}^2\,b$.

3.5. **Summary.** Thus the effects of the fluctuating source terms $\rho\,\boldsymbol{f}$ and $\rho\,r$ on the BHR equations are that (a) Eq. (3.1) governing $\boldsymbol{R}$ is augmented by the term $2\,\overline{\rho}\,\operatorname{sym}\widetilde{\operatorname{cov}}(\boldsymbol{u},\,\boldsymbol{f})$ and (b) Eq. (3.2) governing $\boldsymbol{a}$ is augmented by the term $-\overline{\rho}\,\widetilde{\operatorname{cov}}(\upsilon,\,\boldsymbol{f})$. In particular, Eq. (3.4) governing $\overline{\rho}\,K$ is augmented by the term $\langle \rho\,\boldsymbol{u}''\cdot\boldsymbol{f}''\rangle$.

4. Multiphase Fluid Flow

This section concerns a flowing continuum that is a mixture of multiple fluid phases. Our goal is to relate flow quantities associated with each phase to a corresponding quantity for the mixture.

4.1. **Phase-specific quantities.** The phases are labeled by the subscript k. Flow quantities associated with the phases are constructed with the aid of the phase indicator function β_k: for each realization of the flow, $\beta_k(\boldsymbol{x},t)$ equals 1 if $\boldsymbol{x}$ belongs to the spatial region occupied by phase k at time t, and it equals 0 otherwise. For a flow quantity f, the Reynolds average $\langle \beta_k\,f\rangle$ weighted by β_k is specific to phase k.

The notation we employ for averaging is $\alpha_k\,\overline{f}_k := \langle \beta_k\,f\rangle$ and $\alpha_k\,\overline{\rho}_k\,\tilde{g}_k := \langle \beta_k\,\rho\,g\rangle$, where $\alpha_k := \langle \beta_k\rangle$ is the volume fraction of phase k and $\overline{\rho}_k := \langle \beta_k\,\rho\rangle/\langle \beta_k\rangle$ is the intrinsic mass density of phase k. We recognize the Favre average $\langle \rho\,\beta_k\rangle/\overline{\rho}$ of β_k as the Favre averaged mass fraction $\tilde{c}_k$ of phase k. Therefore

$$\overline{\rho}\,\tilde{c}_k = \langle \beta_k\,\rho\rangle = \alpha_k\,\overline{\rho}_k. \tag{4.1}$$

is the mass density of phase k within the mixture. Notice that, because $\sum_k \beta_k = 1$,

$$\sum_k \alpha_k = 1 \qquad \text{and} \qquad \sum_k \tilde{c}_k = 1. \tag{4.2}$$

In addition, $f'_k := f - \overline{f}_k$ and $g''_k := g - \tilde{g}_k$ denote the fluctuations within phase k, and $\alpha_k\,\overline{\rho}_k\,\widetilde{\operatorname{cov}}_k(g,h) := \langle \beta_k\,\rho\,g''_k\,h''_k\rangle$ defines the Favre covariance $\widetilde{\operatorname{cov}}_k(g,h)$ within phase k of flow quantities g and h.

4.2. **Sum rules.** A flow quantity associated with the mixture, regarded as a single continuum, can be related to the corresponding phase-specific quantities. If f and g are flow quantities, then

$$\sum_k \alpha_k\,\overline{f}_k = \sum_k \langle \beta_k\,f\rangle = \overline{f}, \tag{4.3}$$

$$\sum_k c_k\,\tilde{g}_k = \sum_k \langle \beta_k\,\rho\,g\rangle/\overline{\rho} = \tilde{g}. \tag{4.4}$$

The sum rules for covariances are somewhat more complicated. Let us adopt the notation

$$g_k^{\mathrm{D}} := \tilde{g}_k - \tilde{g} \tag{4.5}$$

for the variation in g associated with drift. Then if g and h are flow quantities,

$$\sum_k \tilde{c}_k \, \widetilde{\mathrm{cov}}_k(g,h) = \sum_k \langle \beta_k \, \rho \, (g - \tilde{g}_k) \, (h - \tilde{h}_k) \rangle / \overline{\rho} = \langle \rho \, g \, h \rangle / \overline{\rho} - \sum_k \tilde{c}_k \, \tilde{g}_k \, \tilde{h}_k \tag{4.6}$$

and

$$\sum_k \tilde{c}_k \, g_k^{\mathrm{D}} \, h_k^{\mathrm{D}} = \sum_k \tilde{c}_k \, \tilde{g}_k \, \tilde{h}_k - \tilde{g} \, \tilde{h}. \tag{4.7}$$

Adding these two equations reveals that

$$\sum_k \underbrace{\tilde{c}_k \, \widetilde{\mathrm{cov}}_k(g,\, h)}_{\text{intra-phase}} + \underbrace{\sum_k \tilde{c}_k \, g_k^{\mathrm{D}} \, h_k^{\mathrm{D}}}_{\text{inter-phase}} = \widetilde{\mathrm{cov}}(g, h). \tag{4.8}$$

Thus the covariance in the mixture is the sum over intra-phase covariances plus an inter-phase covariance associated with drift.

Remark. If there are only two phases, then because $\tilde{g} = \tilde{c}_1 \, \tilde{g}_1 + \tilde{c}_2 \, \tilde{g}_2$,

$$g_1^{\mathrm{D}} = \quad \tilde{c}_2 \, (\tilde{g}_1 - \tilde{g}_2), \tag{4.9}$$

$$g_2^{\mathrm{D}} = -\tilde{c}_1 \, (\tilde{g}_1 - \tilde{g}_2). \tag{4.10}$$

Therefore

$$\sum_k \tilde{c}_k \, g_k^{\mathrm{D}} \, h_k^{\mathrm{D}} = \tilde{c}_1 \, \tilde{c}_2 \, (\tilde{g}_1 - \tilde{g}_2) \, (\tilde{h}_1 - \tilde{h}_2). \tag{4.11}$$

4.3. **Source terms arising from $\boldsymbol{f}$.** According to Eq. (3.18), the source term for $\boldsymbol{R}$ arising from $\boldsymbol{f}$ is $2 \, \overline{\rho} \; \mathrm{sym} \, \widetilde{\mathrm{cov}}(\boldsymbol{u}, \, \boldsymbol{f})$, which can be related to intra- and inter-phase quantities through the rule (4.8):

$$2 \, \overline{\rho} \; \mathrm{sym} \, \widetilde{\mathrm{cov}}(\boldsymbol{u}, \, \boldsymbol{f}) = 2 \, \overline{\rho} \; \mathrm{sym} \sum_k \tilde{c}_k \, \widetilde{\mathrm{cov}}_k(\boldsymbol{u}, \, \boldsymbol{f}) + 2 \, \overline{\rho} \; \mathrm{sym} \sum_k \tilde{c}_k \, \boldsymbol{u}_k^{\mathrm{D}} \, \boldsymbol{f}_k^{\mathrm{D}}. \tag{4.12}$$

Similarly, Eq. (3.20) shows that the source term for $\boldsymbol{a}$ arising from $\boldsymbol{f}$ is

$$-\, \overline{\rho} \, \widetilde{\mathrm{cov}}(\upsilon, \, \boldsymbol{f}) = -\overline{\rho} \sum_k \tilde{c}_k \, \widetilde{\mathrm{cov}}_k(\upsilon, \, \boldsymbol{f}) - \overline{\rho} \sum_k \tilde{c}_k \, \upsilon_k^{\mathrm{D}} \, \boldsymbol{f}_k^{\mathrm{D}}. \tag{4.13}$$

On the other hand, Eq. (2.12) implies that the source term for $\overline{\rho} \, \tilde{\boldsymbol{u}}$ arising from $\boldsymbol{f}$ is

$$\overline{\rho} \, \tilde{\boldsymbol{f}} = \overline{\rho} \sum_k \tilde{c}_k \, \tilde{\boldsymbol{f}}_k \tag{4.14}$$

by virtue of rule (4.4). Finally, by Eq. (2.13), the source term for $\overline{\rho} \, \tilde{E}$ arising from $\boldsymbol{f}$ is

$$\overline{\rho} \, \tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{f}} + \overline{\rho} \, r^{\boldsymbol{f}} = \overline{\rho} \, \tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{f}} + \sum_k \langle \beta_k \, \rho \, \boldsymbol{u}_k'' \cdot \boldsymbol{f}_k'' \rangle + \overline{\rho} \sum_k \tilde{c}_k \, \boldsymbol{u}_k^{\mathrm{D}} \cdot \boldsymbol{f}_k^{\mathrm{D}}. \tag{4.15}$$

The source term for $\overline{\rho} \, K$ arising from $\boldsymbol{f}$, obtained by forming one-half of the trace of the source term for $\boldsymbol{R}$, is

$$\langle \rho \, \boldsymbol{u}'' \cdot \boldsymbol{f}'' \rangle = \sum_k \langle \beta_k \, \rho \, \boldsymbol{u}_k'' \cdot \boldsymbol{f}_k'' \rangle + \overline{\rho} \sum_k \tilde{c}_k \, \boldsymbol{u}_k^{\mathrm{D}} \cdot \boldsymbol{f}_k^{\mathrm{D}}. \tag{4.16}$$

No other balance laws involve $\boldsymbol{f}$.

5. Material Strength Model

In this section, we apply the foregoing results to model the effect of material strength on turbulent flow. The development starts with the material strength model of Youngs [6] for two-phase flow and obtains the corresponding turbulent mixture equations.

5.1. **Material strength model for two-phase flow.** Youngs [6] has developed, within the context of a two-phase turbulence model, a model for the effect of material strength. In essence, this model involves a viscoplasticity-related drag term added to the turbulent flow equations. In the notation of Sec. 4, it takes

$$\alpha_1\,\overline{\rho}_1\,\tilde{\boldsymbol{f}}_1 \overset{\mathrm{m}}{=} -C_1'\,\alpha_1\,\alpha_2\,\frac{Y}{L}\,\frac{\boldsymbol{U}}{|\boldsymbol{U}|}, \tag{5.1}$$

$$\alpha_2\,\overline{\rho}_2\,\tilde{\boldsymbol{f}}_2 \overset{\mathrm{m}}{=} \quad C_1'\,\alpha_1\,\alpha_2\,\frac{Y}{L}\,\frac{\boldsymbol{U}}{|\boldsymbol{U}|} \tag{5.2}$$

along with $\tilde{r} \overset{\mathrm{m}}{=} 0$. Here $\boldsymbol{U} := \tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2$ is the two-phase flow velocity difference,

$$Y = \alpha_1\,Y_1 + \alpha_2\,Y_2 \tag{5.3}$$

is the volume-weighted average of the yield strengths Y_k of the two phases, and C_1' is a nondimensional constant. In particular,

$$\tilde{\boldsymbol{f}}_1 - \tilde{\boldsymbol{f}}_2 = -C_1'\left(\frac{\alpha_2}{\overline{\rho}_1} + \frac{\alpha_1}{\overline{\rho}_2}\right)\frac{Y}{L}\,\frac{\boldsymbol{U}}{|\boldsymbol{U}|} = -C_1'\,\frac{\overline{\rho}}{\overline{\rho}_1\,\overline{\rho}_2}\,\frac{Y}{L}\,\frac{\boldsymbol{U}}{|\boldsymbol{U}|}. \tag{5.4}$$

5.2. **Material strength model for the mixture.** The material strength model that we propose for a two-phase mixture derives from Youngs' model under the assumption that each intra-phase covariance is negligible compared to the inter-phase covariance.

For Youngs' model, Eq. (4.14) shows that

$$\tilde{\boldsymbol{f}} = 0, \tag{5.5}$$

and because there are only two phases,

$$\sum_k \tilde{c}_k\,\boldsymbol{u}_k^{\mathrm{D}}\,\boldsymbol{f}_k^{\mathrm{D}} = \tilde{c}_1\,\tilde{c}_2\,(\tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2)\,(\tilde{\boldsymbol{f}}_1 - \tilde{\boldsymbol{f}}_2) = -C_1'\,\tilde{c}_1\,\tilde{c}_2\,\frac{\overline{\rho}}{\overline{\rho}_1\,\overline{\rho}_2}\,\frac{Y}{L}\,\frac{\boldsymbol{U}\,\boldsymbol{U}}{|\boldsymbol{U}|}, \tag{5.6}$$

$$\sum_k \tilde{c}_k\,\upsilon_k^{\mathrm{D}}\,\boldsymbol{f}_k^{\mathrm{D}} = \tilde{c}_1\,\tilde{c}_2\,(\tilde{\upsilon}_1 - \tilde{\upsilon}_2)\,(\tilde{\boldsymbol{f}}_1 - \tilde{\boldsymbol{f}}_2) = -C_1'\,\tilde{c}_1\,\tilde{c}_2\,(\tilde{\upsilon}_1 - \tilde{\upsilon}_2)\,\frac{\overline{\rho}}{\overline{\rho}_1\,\overline{\rho}_2}\,\frac{Y}{L}\,\frac{\boldsymbol{U}}{|\boldsymbol{U}|}. \tag{5.7}$$

We neglect each intra-phase covariance in Eqs. (4.12), (4.13), (4.15), and (4.16); therefore we preliminarily take the source terms for $\boldsymbol{R}$, $\boldsymbol{a}$, $\overline{\rho}\,\tilde{E}$, and $\overline{\rho}\,K$ to be

$$2\,\overline{\rho}\;\mathrm{sym}\,\widetilde{\mathrm{cov}}(\boldsymbol{u},\,\boldsymbol{f}) \overset{\mathrm{m}}{=} -2\,C_1'\,\tilde{c}_1\,\tilde{c}_2\,\frac{(\overline{\rho})^2}{\overline{\rho}_1\,\overline{\rho}_2}\,\frac{Y}{L}\,\frac{\boldsymbol{U}\,\boldsymbol{U}}{|\boldsymbol{U}|}, \tag{5.8}$$

$$-\overline{\rho}\,\widetilde{\mathrm{cov}}(\upsilon,\,\boldsymbol{f}) \overset{\mathrm{m}}{=} +C_1'\,\tilde{c}_1\,\tilde{c}_2\,(\tilde{\upsilon}_1 - \tilde{\upsilon}_2)\,\frac{(\overline{\rho})^2}{\overline{\rho}_1\,\overline{\rho}_2}\,\frac{Y}{L}\,\frac{\boldsymbol{U}}{|\boldsymbol{U}|}, \tag{5.9}$$

$$\overline{\rho}\,\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{f}} + \overline{\rho}\,r^{\boldsymbol{f}} \overset{\mathrm{m}}{=} -C_1'\,\tilde{c}_1\,\tilde{c}_2\,\frac{(\overline{\rho})^2}{\overline{\rho}_1\,\overline{\rho}_2}\,\frac{Y}{L}\,\left|\boldsymbol{U}\right|, \tag{5.10}$$

$$\langle\rho\,\boldsymbol{u}''\cdot\boldsymbol{f}''\rangle \overset{\mathrm{m}}{=} -C_1'\,\tilde{c}_1\,\tilde{c}_2\,\frac{(\overline{\rho})^2}{\overline{\rho}_1\,\overline{\rho}_2}\,\frac{Y}{L}\,\left|\boldsymbol{U}\right|, \tag{5.11}$$

respectively. Note that $\tilde{\upsilon}_k = 1/\overline{\rho}_k$.

5.3. **Correspondence with Two-Phase Flow.** The preliminary model of the previous section involves the quantity $\boldsymbol{U} := \tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2$, which is not calculated by the governing equations for the mixture. To address this issue, we develop a correspondence between these quantities and ones that are, in fact, calculated in the BHR model.

As $\boldsymbol{a} = -\overline{\rho}\,\widetilde{\mathrm{cov}}(\upsilon, \boldsymbol{u})$, neglect of each intra-phase covariance in Eq. (4.8) results in the approximation

$$\boldsymbol{a} \approx -\overline{\rho}\,\tilde{c}_1\,\tilde{c}_2\,(\tilde{\upsilon}_1 - \tilde{\upsilon}_2)\,(\tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2). \tag{5.12}$$

Similarly, as $b = (\overline{\rho})^2\,\widetilde{\mathrm{cov}}(\upsilon, \upsilon)$, we make the approximation that

$$b \approx (\overline{\rho})^2\,\tilde{c}_1\,\tilde{c}_2\,(\tilde{\upsilon}_1 - \tilde{\upsilon}_2)^2. \tag{5.13}$$

Hence $\boldsymbol{U} := \tilde{\boldsymbol{u}}_1 - \tilde{\boldsymbol{u}}_2$ and $\tilde{\upsilon}_1 - \tilde{\upsilon}_2$ can be calculated, modulo an undetermined sign, from $\boldsymbol{a}$ and b:

$$\boldsymbol{U} \overset{\mathrm{m}}{=} -\frac{1}{\overline{\rho}\,\tilde{c}_1\,\tilde{c}_2\,(\tilde{\upsilon}_1 - \tilde{\upsilon}_2)}\,\boldsymbol{a}, \tag{5.14}$$

$$|\tilde{\upsilon}_1 - \tilde{\upsilon}_2| \overset{\mathrm{m}}{=} \frac{1}{\overline{\rho}}\left(\frac{b}{\tilde{c}_1\,\tilde{c}_2}\right)^{1/2}. \tag{5.15}$$

With the aid of this correspondence, we arrive at our material strength model: the source terms for $\boldsymbol{R}$, $\boldsymbol{a}$, $\overline{\rho}\,\tilde{E}$, and $\overline{\rho}\,K$ are

$$2\,\overline{\rho}\;\mathrm{sym}\,\widetilde{\mathrm{cov}}(\boldsymbol{u},\,\boldsymbol{f}) \overset{\mathrm{m}}{=} -2\,C_1'\,\frac{\overline{\rho}}{|\overline{\rho}_1 - \overline{\rho}_2|}\,\frac{Y}{L}\,\frac{\boldsymbol{a}\,\boldsymbol{a}}{|\boldsymbol{a}|}, \tag{5.16}$$

$$-\overline{\rho}\,\widetilde{\mathrm{cov}}(\upsilon,\,\boldsymbol{f}) \overset{\mathrm{m}}{=} -C_1'\,\frac{b}{|\overline{\rho}_1 - \overline{\rho}_2|}\,\frac{Y}{L}\,\frac{\boldsymbol{a}}{|\boldsymbol{a}|}, \tag{5.17}$$

$$\overline{\rho}\,\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{f}} + \overline{\rho}\,r^{\boldsymbol{f}} \overset{\mathrm{m}}{=} -C_1'\,\frac{\overline{\rho}}{|\overline{\rho}_1 - \overline{\rho}_2|}\,\frac{Y}{L}\,|\boldsymbol{a}|, \tag{5.18}$$

$$\langle \rho\,\boldsymbol{u}''\cdot\boldsymbol{f}''\rangle \overset{\mathrm{m}}{=} -C_1'\,\frac{\overline{\rho}}{|\overline{\rho}_1 - \overline{\rho}_2|}\,\frac{Y}{L}\,|\boldsymbol{a}|, \tag{5.19}$$

respectively.

The codes are able to compute these terms from the BHR and hydro equations. The species densities are computed from the equations-of-state via the pressure/temperature equilibration assumption.

6. FLAG Implementation

The model is implemented in the FLAG hydrocode in the same framework described in Ref. [3]. The equations are, in Lagrangian form:

$$
\begin{aligned}
\bar{\rho}\frac{DK}{Dt} &= a_j\frac{\partial\bar{p}}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\frac{\mu_T}{\sigma_k}\frac{\partial K}{\partial x_j}\right) - R_{ij}\frac{\partial\tilde{u}_i}{\partial x_j} - \frac{K}{S}\left(\bar{\rho}\sqrt{K} + \frac{c_{s4}\bar{Y}}{\sqrt{K}}\right) - c_{s1}\frac{\bar{\rho}}{|\rho_1-\rho_2|}\frac{\bar{Y}}{S}|a_i| \\
\bar{\rho}\frac{DS}{Dt} &= \frac{S}{K}\left[\left(\frac{3}{2}-C_4\right)a_j\frac{\partial\bar{p}}{\partial x_j} - \left(\frac{3}{2}-C_1\right)R_{ij}\frac{\partial\tilde{u}_i}{\partial x_j}\right] + \frac{\partial}{\partial x_j}\left(\frac{\mu_T}{\sigma_S}\frac{\partial S}{\partial x_j}\right) \\
&\quad - \left(\frac{3}{2}-C_2\right)\left(\bar{\rho}\sqrt{K} + \frac{c_{s4S}\bar{Y}}{\sqrt{K}}\right) - c_{s1}\frac{\bar{\rho}}{|\rho_1-\rho_2|}\frac{\bar{Y}}{K}|a_i| \\
\bar{\rho}\frac{Da_i}{Dt} &= b\frac{\partial\bar{p}}{\partial x_i} + \bar{\rho}\frac{\partial a_i a_j}{\partial x_j} - \bar{\rho}a_j\frac{\partial\tilde{u}_i - a_i}{\partial x_j} - \frac{R_{ij}}{\bar{\rho}}\frac{\partial\bar{\rho}}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\frac{\mu_T}{\sigma_a}\frac{\partial a_i}{\partial x_j}\right) \\
&\quad - \frac{C_a a_i}{S}\left(\bar{\rho}\sqrt{K} + \frac{c_{s4a}\bar{Y}}{\sqrt{K}}\right) - c_{s1}\frac{b}{|\rho_1-\rho_2|}\frac{\bar{Y}a_i}{S|a_i|} \\
\bar{\rho}\frac{Db}{Dt} &= 2\bar{\rho}a_j\frac{\partial b}{\partial x_j} - 2(b+1)a_j\frac{\partial\bar{\rho}}{\partial x_j} + \bar{\rho}^2\frac{\partial}{\partial x_j}\left(\frac{\mu_T}{\bar{\rho}^2\sigma_b}\frac{\partial b}{\partial x_j}\right) - C_b\frac{b}{S}\left(\bar{\rho}\sqrt{K} + \frac{c_{s4b}\bar{Y}}{\sqrt{K}}\right)
\end{aligned}
$$

with the new terms highlighted in red. $\bar{Y}$ is computed by the volume averaging the material contributions. For completeness, the dissipation terms were added to all equations. The relevant parameter values are taken from Youngs where possible. Two equations do not have analogous terms in the Youngs model. For the b equation the strength-based dissipation is set to zero. This is the assumption that dissipation based on yield does not contribute to molecular mixing. In the S equations, various assumptions are possible. The c_{s4S} parameter can be set to zero, which mean the yield strength does not affect the turbulent length scale. Alternatively it can be set such that the strength-based dissipation does not change the classical viscous dissipation rate ϵ. This can be derived from the definition $S = K^{3/2}/\epsilon$ and for standard BHR parameters gives $c_{s4S} = -2.381$. This choice is found to be the most stable.

c_{s1}	c_{s4}	c_{s4S}	c_{s4a}	c_{s4b}
2.0	1.0	-2.381	1.0	0.0

Table 6.1. Parameter Values

The pressure/temperature equilibrium assumption sets volume fractions of the species, which allow us to compute species-specific densities. The mean yield stress is computed using these volume fractions.

$$\bar{Y} = \alpha_1 Y_1 + \alpha_2 Y_2$$

This creates a distribution of Yield strength across the mixing layer.

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