An Emergent Conductivity Relationship for Water Flow Based on Minimized Energy Dissipation: From Landscapes to Unsaturated Soils

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Abstract

Optimality principles have been widely used in many areas. Based on an optimality principle that a flow field will tend toward a minimum in the energy dissipation rate, this work shows that there exists a unified form of conductivity relationship for two different flow systems: landscapes and unsaturated soils. The conductivity, the ratio of water flux to water head (energy) gradient, is a power function of water flux where the exponent value is system dependent. This relationship indicates that to minimize energy dissipation rate for a whole system, water flow has a small resistance (or a large conductivity) at a location of large water flux. Empirical evidence supports validity of the relationship for landscape and unsaturated soils (under gravity dominated conditions). Especially, it is of interest that according to this relationship, hydraulic conductivity for gravity-dominated unsaturated flow, unlike that defined in the classic theories, depends on not only capillary pressure (or saturation), but also the water flux. Use of the optimality principle allows for determining useful results that may be applicable to a broad range of areas involving highly non-linear processes and may not be possible to obtain from classic theories describing water flow processes.

Key words. Optimality, unsaturated flow, surface hydrology

1. Introduction

Optimality principles refer to the state of a physical process that is controlled by an optimal condition subject to physical and/or resource constraints. These principles have been used in many different areas, including evolution of vegetation coverage under water-limited conditions (Eagleson, 2002; Liu 2011a), tree-like paths for liquid flow and heat transfer (Bejan, 2000), and application of the maximum entropy production (MEP) principle, in a heuristic sense, to the prediction of steady states of a wide range of systems (e.g., Tondeur and Kvaalen, 1987; Bejan and Tondear, 1998; Nieven, 2010; Kleidon, 2009). However, the theoretical connections between these optimality principles and the currently existing fundamental laws are not fully established. Bejan (2000) argued that these principles are actually self-standing and do not follow from other known laws (Bejan, 2000). It is our belief that these principles are probably results of characters of chaotic (non-linear dynamical) systems. For a chaotic system, details of system behavior on a small scale are not predictable. However, emergent patterns, as a result of self-organization, often occur on macroscopic scales. More importantly, these patterns are self-organized in such a way that they are efficiently adapted to conditions of the relevant environment (Heylighen, 2008). This adaption feature may correspond to the optimality principles. Our argument is supported by an observation that the involved processes are generally highly non-linear when these principles are found useful.

The role of optimality principles in forming complex natural patterns has been recognized for many years in the surface hydrology community (Leopold and Langbein, 1962; Howard, 1990; Rodriguez-Iturbe et al., 1992; Rinaldo et al., 1992; Liu, 2011b). For example, Leopold and Langbein (1962) proposed a maximum entropy principle for studying the formation of landscapes. Rodriguez-Iturbe et al (1992) postulated principles of optimality in energy expenditure at both local and global scales for channel networks. However, application of these principles in the subsurface hydrology has been very limited, probably because flow patterns in the subsurface are difficult to observe and characterize to motivate research activities based on the related emergent patterns.

Most recently Liu (2011b,c), based on the optimality principle that energy dissipation rate (or flow resistance) is minimized for the entire flow system, demonstrated that conductivity for water flow is a power function of water flux for both landscapes and unsaturated soils (under gravity-dominated conditions). That development is supported by experimental observations and empirical relations. The conductivity herein is defined as water flux divided by the head (energy) gradient along the flux direction. Water head refers to energy per unit weight of water. This study extends the work of Liu (2011 b,c) in two aspects. First, the development in Liu (2011b) relied on Manning's equation for describing water flow over landscapes. We will show that Manning's equation is not needed for obtaining a conductivity relationship for overland flow. Second, we will extend the results of Liu (2011c) for unsaturated flow in homogeneous soils to heterogeneous cases. The ultimate objective of this contribution is to show that power-function relationships between conductivity and water flux, resulting from the optimality principle, seems to be common for different natural flow systems, although the power value is system dependent.

2. Steady-state optimal landscape

Liu (2011b) showed that using an optimality principle leads to a conclusion that conductivity for an optimal landscape is a power function of water flux. However, his work is based on a notion that overland water flow can be described by the well-known Manning's equation. In this section, we demonstrate that the notion is not needed to derive the conductivity relationship. Also note that mathematical derivation procedure of Liu (2011b) is closely followed here with a key difference that a more general expression for water flux is used in this work.

Because this paper describes results for two different systems (landscapes and unsaturated soils), the definitions and physical meanings of a same symbol (denoting a variable or function) may be different in different sections herein unless the same physical meanings are explicitly indicated.

Following Liu (2011b), we consider a landscape involving steady-state water flow and surface evolution processes. This steady-state assumption has been implicitly employed in previous studies on topological structures of channel networks (Leopold and Maddock, 1953; Howard, 1990; Rodriguez-Iturbe et al., 1992; Rinaldo et al., 2006). A more detailed justification of this treatment can be found, for example, in Liu (2011b).

Based on the above simplifications, coupled water-flow (over a land surface) and surface-elevation equations can be derived from the principle that global energy expenditure rate is at the minimum. From the water mass (volume) conservation, the steady-state water flow equation is given by

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = Q \tag{1}$$

where x and y are two horizontal coordinate axes, q_x and q_y are water fluxes (water velocity multiplied by water depth) along x and y directions, respectively, and Q is the rainfall rate.

Accordingly, the energy expenditure rate for a unit land-surface area, Δ_E, can be expressed as (based on energy conservation)

$$\Delta_E = \frac{\partial(q_x E)}{\partial x} + \frac{\partial(q_y E)}{\partial y} - QE \tag{2}$$

The above equation simply states that for a given unit area, the energy expenditure rate at that location is equal to the energy carried by water flowing into the area minus the energy carried by water flowing out of the area. The rainfall is assumed to have the same energy as water at the location where the rain falls. The head, E (a function of x and y), refers to water energy per unit weight, including both potential (corresponding to elevation z) and kinetic energy, and given by:

$$E = z + \frac{v^2}{2g} \tag{3}$$

where g is gravitational acceleration and v is water velocity. Note that the second term is generally small and has been ignored in some previous studies (e.g., Howard, 1990; Rinaldo et al., 2006). For completeness, this term is included here.

A combination of Equations (1) and (2) yields

$$\Delta_E = q_x \frac{\partial E}{\partial x} + q_y \frac{\partial E}{\partial y} \tag{4}$$

The water flux is considered to be given by

$$q_x = -K(h, S_*) \frac{\partial E}{\partial x} \tag{5a}$$

$$q_y = -K(h, S_*)\frac{\partial E}{\partial y} \quad (5b)$$

where

$$S_* = S^2 = \left(\frac{\partial E}{\partial x}\right)^2 + \left(\frac{\partial E}{\partial y}\right)^2 \quad (5c)$$

In Equations (5a) and (5b), K(h, S_*) is conductivity and *h* (m) is water depth. Note that in Liu (2011b), the specific formulation of conductivity is given based on the Manning's equation. In this study, we assume *h* to be a function of local slope *S* only (Gupta and Waymire, 1989). Many studies indicate that on average a number of hydraulic parameters can be considered as functions of local slope (Leopold and Maddock, 1953). In this case, K is a function of S_* only.

When we combine Equations (4) and (5), the global energy expenditure rate throughout water-flow domain Ω is given by

$$\iint_\Omega \Delta_E dxdy = \iint_\Omega (-KS_*)dxdy \quad (6)$$

The optimality principle in our problem is to minimize the absolute value of the above integral. To do so, we employ the calculus of variations that seeks optimal (stationary) solutions to a functional (a function of functions) by identifying unknown functions (Weinstock, 1974). For example, the former corresponds to the integral defined in Equation (6) and the latter to land-surface elevation distribution *z*(*x*,*y*).

Furthermore, we employ the following constraint for the optimization problem (Liu, 2011b):

$$\iint_\Omega E dxdy = C \quad (7)$$

where *C* is a constant. Since *E* is mainly composed of potential energy *z*, the above equation essentially states that the average elevation throughout the model domain (or total volume of the landscape under consideration) remains unchanged, which is consistent with the steady-state assumption made in this study. It should be emphasized that the optimality principle corresponds to minimization of the global energy expenditure rate, not the total energy within the model domain. Under steady state conditions, the global energy expenditure is equal to the difference between the latter and energy carried by water flowing out of the system.

Based on Equations (5), (6) and (7), the Lagrangian for the given problem is given by

$$L = -KS_* + \lambda_1^*[S_* - \left(\frac{\partial E}{\partial x}\right)^2 - \left(\frac{\partial E}{\partial y}\right)^2] + \lambda_2^*[E - C] \quad (8)$$

Note that the first term is from Equation (6) and other terms are constraints from Equations (5) and (7). The Lagrange multipliers λ_1^* and λ_2^* are a function of location and a constant, respectively. The last term on the right hand side of Equation (8) corresponds to the constraint defined in Equation (7). Note that the constraint related to water flow, Equation (1), is not included in Equation (8), but will be handled later for mathematical convenience.

The following Euler-Lagrange equation is used to determine an unknown function w associated with *L* to minimize the integral defined in Equation (8) (Weinstock, 1974; Pike, 2001):

$$\frac{\partial L}{\partial w} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial w_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial w_y}\right) = 0 \quad (9)$$

where w_x and w_y are partial derivatives with respect to *x* and *y*, respectively. In this study, *w* corresponds to S_* and *E*, respectively.

Applying the Euler-Lagrange Equation (9) to S_* in (8) gives

$$\lambda_1^* = \frac{d(KS_*)}{dS_*} \quad (10)$$

Applying the Euler-Lagrange Equation (9) to *E* yields

$$\lambda_2^* + \frac{\partial}{\partial x}(2\lambda_1^* \frac{\partial E}{\partial x}) + \frac{\partial}{\partial y}(2\lambda_1^* \frac{\partial E}{\partial y}) = 0 \quad (11)$$

For the optimization results to be physically valid, they must satisfy the water flow equation (1). A direct comparison between Equations (1) and (11) and consideration of (5) reveal that Equations (1) and (11) are identical under the following conditions

$$\lambda_1^* = \left(\frac{\lambda_2^*}{2Q}\right)K \quad (12)$$

Combining Equations (10) and (12), we can obtain

$$K \propto S_*^{-1+\frac{\lambda_2^*}{2Q}} = S^{-2+\frac{\lambda_2^*}{Q}} \quad (13)$$

From Equations (5) and (13), water flux q and local slope S has the following relationship:

$$q \propto S^{-1+\frac{\lambda_2 *}{Q}} \tag{14}$$

where

$$q = \sqrt{q_x^2 + q_y^2} \tag{15}$$

The power-function relationship between water flux (or discharge) and local slope has been intensively investigated and validated in the literature (Rodriguez-Iturbe et al., 1992; Rinaldo et al., 2006; Banavar et al., 2001). Also note that the power value in the power-function relationship varies with different site conditions. However, previous studies (Leopold and Maddock, 1953; Rodriguez-Iturbe et al., 1992; Rinaldo et al., 2006; Banavar et al., 2001) indicate that the averaged exponent value is about -2 in (14), suggesting that $\frac{\lambda_2 *}{Q}$ is close to -1 in an average sense.

Equations (13) and (14) lead to a relationship between the flow conductivity K (flux divided by head gradient) and flux:

$$K \propto q^{\frac{2-\frac{\lambda_2 *}{Q}}{1-\frac{\lambda_2 *}{Q}}} \tag{16}$$

When $\frac{\lambda_2 *}{Q}$ is close to -1, the exponent value in the above equation is about 1.5. The equation indicates that under optimal conditions, locations where a relatively large water flux occurs correspond to a relatively small resistance (or large conductance). As demonstrated in the sections to follow, that flow conductivity is a power function of water flux seems to be a common rule under optimal flow conditions.

3. Unsaturated flow in porous media

For a homogeneious soil, Liu (2011c) reported that unsaturated conductivity, similar to Equation (16), is a power function of water flux under gravity-dominant and optimal flow conditions. This section will show that the similar power-function relationship can be applied to heterogeneous soils as well. We also like to emphasize that our mathematical development closely follows Liu (2011c) except that conductivity here is spatially variable.

We consider a steady state unsaturated flow system associated with a heterogeneous and isotropic porous medium. From the water mass (volume) conservation, the steady-state water flow equation is given by

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} = 0 \qquad (17)$$

where x and y are two horizontal coordinate axes, z is the vertical axis, and q_x, q_y and q_z are volumetric fluxes of water along x , y and z directions, respectively.

We use E (a function of x, y and z) to represent the hydraulic head for unsaturated flow:

$$E = z + \frac{P}{\rho g} = z + h \qquad (18)$$

Where z is elevation, g is gravitational acceleration, P is capillary pressure, ρ is water density, and h is capillary pressure head. Accordingly, the energy expenditure rate for a unit control volume, Δ_E, can be expressed as

$$\Delta_E = \frac{\partial(q_x E)}{\partial x} + \frac{\partial(q_y E)}{\partial y} + \frac{\partial(q_z E)}{\partial z} \qquad (19)$$

The above equation simply states that for a given unit volume, the energy expenditure rate at that location is equal to the energy carried by water flowing into the volume minus the energy carried by water flowing out of the volume.

A combination of Equations (17) and (19) yields

$$\Delta E = q_x \frac{\partial E}{\partial x} + q_y \frac{\partial E}{\partial y} + q_z \frac{\partial E}{\partial z} \qquad (20)$$

Throughout this development, Darcy's law is assumed to applied to unsaturated flow:

$$q_x = -K \frac{\partial E}{\partial x} \qquad (21a)$$

$$q_y = -K \frac{\partial E}{\partial y} \qquad (21b)$$

$$q_z = -K \frac{\partial E}{\partial z} \qquad (21c)$$

where K is hydraulic conductivity and given by

$$K = K_s(x,y,z)k_r(h^*,S_*) \qquad (21d)$$

$$S_* = \left(\frac{\partial E}{\partial x}\right)^2 + \left(\frac{\partial E}{\partial y}\right)^2 + \left(\frac{\partial E}{\partial z}\right)^2 \qquad (21e)$$

$$h^* = h\sqrt{\frac{K_{s,ref}}{K_s}} \qquad (21f)$$

In Equations (21d) and (21f), K_s is the saturated hydraulic conductivity that is a function of location and $K_{s,ref}$ refers to the K_s value at a reference location. The relative permeability k_r is assumed to be a function of both normalized capillary pressure head (h*) and the square of the energy gradient (S_*). To be able to get closed-form results, we further assume that the function form of relative permeability is independent of location. This treatment is based on a widely used miller-miller similarity that pore-space geometry is similar at different locations of heterogeneous porous media (Miller and Miller, 1956). When the similarity is satisfied, the functional form of k_r, in terms of normalized variables, remains the same at different locations.

Assuming k_r to be a function of water flux is equivalent to assuming it to be a function of hydraulic head gradient, because water flux, hydraulic head gradient and K are related through Darcy's law. Our theory here is developed for a macroscopic scale that may include a number of fingering or preferential flow paths. Local scale refers to the continuum scale within each finger. Unsaturated flow process on a local scale is mainly controlled by pore-scale physics.

It should be noted that Darcy's law was initially developed for water flow in saturated porous media. Buckingham (1907) extended Darcy's law to unsaturated conditions, although it is an issue of debate regarding whether he was aware of Darcy's law when developing his relationship. An excellent historic review of Edgar Buckingham and his scientific contributions to unsaturated flow in soils was recently published by Nimmo and Landa (2005). It may be more appropriate to call Darcy law for unsaturated flow Darcy-Buckingham law. In his extension, Buckingham (1907) used an unsaturated hydraulic conductivity, a function of water saturation or capillary pressure, to replace hydraulic conductivity in Darcy's law. Although not explicitly stated in Buckingham (1907), this development is based on the local equilibrium assumption whose limitations will be discussed later.

When we combine Equations (20) and (21), the global energy expenditure rate is the same as Equation (6). The corresponding Lagrangian for the given problem is given by

$$L = -KS_* + \lambda_1 [S_* - \left(\frac{\partial E}{\partial x}\right)^2 - \left(\frac{\partial E}{\partial y}\right)^2 - \left(\frac{\partial E}{\partial z}\right)^2] \tag{22}$$

Replacing *w* with S_* in the Euler-Lagrangian Equation (8) yields

$$\lambda_1 = \frac{\partial (KS_*)}{\partial S_*} \tag{23}$$

Replacing *w* with *h* (or E) in Equation (22) and using (23) and the continuity equation, we have

$$\frac{\partial\left(\frac{\partial K}{\partial(\log S_*)}\frac{\partial E}{\partial x}\right)}{\partial x} + \frac{\partial\left(\frac{\partial K}{\partial(\log S_*)}\frac{\partial E}{\partial y}\right)}{\partial y} + \frac{\partial\left(\frac{\partial K}{\partial(\log S_*)}\frac{\partial E}{\partial z}\right)}{\partial z} = \frac{S_*}{2}\frac{\partial K}{\partial h} \tag{24}$$

In general, it is difficult to obtain an analytical solution to the above equation. However, for some special case in which the term on the right hand side is small compared to other terms, a closed-form solution can be obtained. This is true for a gravity-dominant flow (Liu 2011c). A comparison between Equation (24) (without the term on the right hand side) with the continuity equation (Equations (17) and (21)) yields

$$\frac{\partial K}{\partial(\log S_*)} = AK \tag{25}$$

where *A* is a constant.

To get practically useful results, we consider K to be further expressed by

$$K = K_s(x,y,z)k_r(h^*, S_*) = K_s(x,y,z)f(h^*)g(S_*) \tag{26}$$

Substituting (26) into (25) results in (for a given location)

$$g(S_*) \propto S_*^{\ A} \tag{27}$$

Based on Darcy's law, (27) can be rewritten as

$$g(S_*) \propto \left(\frac{|q|}{K}\right)^{A/2} \tag{28}$$

where $|q|$ is the magnitude of water flux given by

$$|q| = \left[q_x{}^2 + q_y{}^2 + q_z{}^2 \right]^{1/2} \tag{29}$$

Combining (28) and (26) gives our final conductivity relationship as follows

$$K = K_s(x, y, z) F(h^*) \left(\frac{|q|}{K_s} \right)^a \tag{30-1}$$

where *a* is a constant exponent for the conductivity-flux power law. Equation (30-1) may also be rewritten as

$$k_r(h, S_*) = F(h^*) \left(\frac{|q|}{K_s} \right)^a \tag{30-2}$$

There may be different interpretations of Equation (30). One interpretation is that F(h) is the local-scale relative permeability within the fingering-flow zone and that the power function of flux in the equation represents the fraction of fingering flow zone in an area normal to water flux direction. The validity of (30) for homogeneous soils was demonstrated in Liu (2011c) with both laboratory-experimental observations of vertical fingering flow (Wang et al. 1998) and field observations (Sheng et al. 2009). Based on these observations, Liu (2011c) also speculated that parameter *a* may have a universal value of 0.5, which, however, needs further verification.

Equation (30) clearly shows that for a gravity-dominated unsaturated flow, relative permeability is not only a function of water potential (or saturation), but also a power function of water flux. Treating relative permeability as a function of water potential (or saturation) only has been widely used in the literature of vadose zone hydrology. This needs to be revisited, because the treatment is based on a local-equilibrium assumption that capillary pressure is uniform within a representative elementary volume (REV). This is obviously violated at a large scale by the existence of fingering flow. The power-function term in (30) largely reflects the self-organization of flow patterns driven by the minimization of total flow resistance.

An analogue of the corresponding relation between our relative-permeability expression and the classic one is that between some parameters for turbulent and laminar water flow processes in a pipe. When water flow in a pipe is laminar, the product of friction factor and average water velocity, corresponding to K in this work, is proportional to water viscosity and independent of the velocity (Moody, 1944). However, when water flow becomes turbulent, the product is related to both viscosity and Reynolds number that is a function of water velocity. Note that the turbulent-flow case corresponds to our new expression for the relative permeability. This analogue highlights the needs to develop different theories for different flow regimes. While the classic unsaturated-flow theory was developed based on the local equilibrium assumption, our theory intends to deal with water flow when the assumption is not valid anymore.

Application of (30) to model unsaturated flow can follow the procedure used in the so-called active region model (ARM) (Liu et al., 1998, 2003, 2005; Sheng et al., 2009). ARM divides the flow domain into two parts: active and inactive. Water flow occurs within the active region only, and inactive region is simply bypassed. The formulation to calculate the volumetric fraction of the active region can be related to (30), as shown in Liu (2011b). Specifically, F(h*) and power-function term in (30) can be considered as the relative permeability within the active region and the volumetric fraction of that region, respectively. Also note that the derivation of (30) does not consider the physics of water flow at pore scale. To get physically valid results, the relative permeability calculated from (30) needs to be limited by a upper limit of S_e and a lower limit of $k_r^*(S_e)$, where S_e is the effective saturation defined as water volume (excluding residual water) divided by porosity (excluding pore space occupied by residual water) and $k_r^*(S_e)$ is the local relative permeability at saturation S_e. The rational for these limits and detailed numerical procedure of ARM can be found in Liu et al. (2003) and Sheng et al. (2009).

We also need to emphasize that the theoretical development in this section is based on an assumption that saturated hydraulic conductivity (K_s) is spatially variable, but the relative permeability relation (in terms of normalized variables) is not and can be described by Miller-Miller similarity (Miller and Miller, 1956). While it is an important step forward compared with the work of Liu (2011c), future studies are needed to more rigorously explore the impact of heterogeneity within the optimality framework.

4. Discussions

Optimality principles have been widely used in different areas. This study is based on a particular optimality principle that energy dissipation rate (or flow resistance) is minimized for the entire flow system. This allows us to unify the conductivity relationships for two seemingly different systems: landscape and unsaturated soil. (This may partially explain why flow patterns in these systems are similar.) There is a considerable amount of experimental evidence to support the validly of our results (Liu 2011 b, c). This in turn demonstrates the validity of the optimality principle that we have employed. It is of interest to note that the systems under consideration have the similar feature that water flow processes and formation of flow patterns are strongly coupled in highly non-linear ways. For example, landscapes are formed by water-flow-induced erosion and fingering pattern formation in unsaturated soils are also closely related to non-linear water flow processes. In the other words, the existence of strong feed backs between flow pattern formation and the flow process itself may likely be the underlying reason that the optimality principle is valid for these different circumstances. As indicated in the introduction section, it seems to us that these optimality principles may result from behavior of chaotic systems.

While there are a number of studies on applications of the optimality in the literature, our study is unique in revealing that conductivity is a power function of water flux in two water-flow systems. This interesting finding has several important implications. First, it makes sense within the context of resource (or conductance) allocation. The power-function relationship with a positive power value always gives a small flow resistance at a location with a large flux, such that flow in the whole system is the most efficient. This allocation strategy is consistent with our daily-life experience. For example, in a highway system, locations with high traffic flux are generally wider or have larger conductivities (Liu 2011c). Second, while complex partial differential equations are involved in our derivation procedure, the form of our results (power function) is very simple. This form likely has something to do with a fractal pattern that has been observed and studied intensively in the literature (e.g., Feder 1988). In general, a fractal has many features that can be characterized by power functions and is related to chaotic systems. A detailed exploration of linkage between our finding and fractals is beyond the scope of this paper and left to future investigations. Third, one grand challenge facing us in the area of hydrogeology is the need to develop physical laws for large-scale multiphase-flow problems. At a local-scale, fluid distribution is mainly controlled by capillarity and not sensitive to flow conditions. That is why relative permeability at local scale can be successfully described as a function of saturation (or capillary pressure) only. At a large scale, this is not the case anymore, although local-scale relationship has been widely used at large scale because alternatives are unavailable. It is fair to say that as a result of the high non-linearity involved, how to model large-scale multiphase flow is an issue that has not been resolved at a fundamental level. Our results (Eq. (30)) suggests that functional forms of large-scale relationships to describe multiphase flow are very likely different from their counterparts at local scales, which cannot be resolved from upscaling based on the same functional forms as those at local scales. It is our hope that the optimality approach may provide an important way to obtain such large-scale relationships.

5. Concluding remarks

Optimality principles have been widely used in many areas. Based on an optimality principle that the water-flow energy dissipation rate is minimal, this work shows that there exists a unified form of conductivity relationship for two different flow systems: landscapes and unsaturated soils. The conductivity is a power function of water flux although the power value is system dependent. This relationship indicates that to minimize energy dissipation for a whole system, water flow has a small resistance (or a large conductivity) at a location of large water flux. Empirical evidence supports the relationship for landscape and unsaturated soils (under gravity dominated conditions). Especially, according to this relationship, hydraulic conductivity for gravity-dominated unsaturated flow, unlike that defined in the classic theories, depends on not only capillary pressure (or saturation), but also the water flux. Use of the optimality principle allows for determining useful results that are applicable to a broad range of areas involving highly non-linear processes and that may not be possible to obtain from classic theories describing water flow processes.

Finally, we need to emphasize that this study represents the first step to develop a unified theoretical framework to describe flow processes in different systems. Since the optimality principle may hold the key to dealing with complex processes related to water flow, more work is highly desirable along the line discussed here, including the further confirmation of results from this study, exploration of the usefulness of our results in other flow systems, and application of the results to modeling flow processes in the relevant systems.

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