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Title: Orthogonal Transformations for the Ensemble Kalman Filter

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Orthogonal Transformations for the Ensemble Kalman Filter

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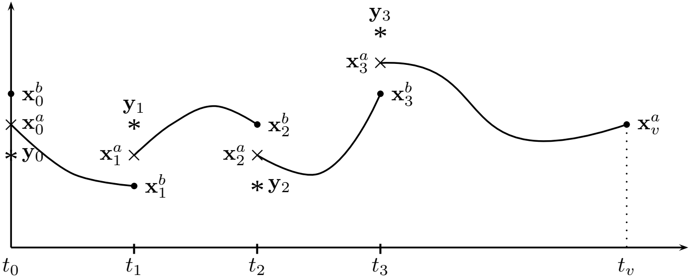
Ensemble Kalman Filter (EnKF)



The Ensemble Kalman Filter (EnKF) was first introduced by Evensen (1994) as a Monte Carlo approximation to Kalman filtering and has gained wide acceptance in data assimilation applications

- EnKF is a sequential data assimilation method that uses an ensemble of model forecast to approximate the model mean and covariance matrix
- The ensemble is updated with every analysis to reflect information provided by the observations, and is evolved using the forecast model between analysis.

EnKF formulation

Let $\mathcal{M}_{t_k \to t_{k+1}}$ be the forecast model,

$$\mathbf{x}(t_{k+1}) = \mathcal{M}_{t_k \to t_{k+1}}(\mathbf{x}(t_k)) \tag{1}$$

For an vector of observations $\mathbf{y}^o \in \mathbb{R}^m$ and an ensemble of N forecast $\mathbf{x}_i^f \in \mathbb{R}^n, i = 1, \ldots, N$ the EnKF analysis equation are given by:

$$\mathbf{x}_i^a = \mathbf{x}_i^f + \mathbf{K}\left(\mathbf{y}_i^o - \mathbf{H}\mathbf{x}_i^f\right), \quad i = 1, \ldots, N \tag{2}$$

$$\mathbf{K} = \mathbf{P}^f\mathbf{H}^T\left(\mathbf{H}\mathbf{P}^f\mathbf{H}^T + \mathbf{R}\right)^{-1}. \tag{3}$$

In the EnKF the forecast error covariance matrix is obtained through the ensemble of model forecast, using the relation

$$\mathbf{P}^f = \frac{1}{N-1}\sum_{i=1}^{N}\left(\mathbf{x}_i^f - \overline{\mathbf{x}}^f\right)\left(\mathbf{x}_i^f - \overline{\mathbf{x}}^f\right)^T, \tag{4}$$

where $\overline{\mathbf{x}}^f$ is the forecast ensemble average

EnKF with Orthogonal Transformation

Let $\mathbf{z}_i = \mathbf{x}_i^f - \bar{\mathbf{x}}^f$ be the centered state vector, and

$$\mathbf{Z} = [\mathbf{z}_1, \cdots, \mathbf{z}_N]$$

be the matrix of centered state vectors. Consider an SVD of this matrix, $\mathbf{Z} = \mathbf{U}\Sigma\mathbf{V}^T$, and define

$$\mathbf{B} = \frac{1}{\sqrt{N-1}}\mathbf{U}_k\Sigma_k,$$

where $\mathbf{U}_k = [\mathbf{u}_1 \cdots \mathbf{u}_k]$, $\Sigma_k = \text{diag}\,(\sigma_1 \cdots \sigma_k)$, $k \ll n$. $\mathbf{B}$ is a set of orthogonal basis vectors, project onto this space and do assimilate.
We can project the deviations to weights for a linear combination in the orthogonal space

$$\mathbf{w}_i = \left(\mathbf{B}^T\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{z}_i, i = 1\ldots, N$$

Notice that $\mathbf{z}_i = \mathbf{B}\mathbf{w}_i$. Also, since $\mathbf{z} \sim N(\mathbf{0}, \mathbf{P})$, then by construction $\mathbf{w} \sim N(\mathbf{0}, \mathbf{I})$

Let $\hat{\mathbf{H}} = \mathbf{HB}$, and let $\mathbf{z}^o = \mathbf{y}^o - \mathbf{H}\overline{\mathbf{x}}^f$, then the projection of the observations is

$$\mathbf{w}^o = \left(\hat{\mathbf{H}}^T \hat{\mathbf{H}}\right)^{-1} \hat{\mathbf{H}}^T \mathbf{z}^o$$

Use Kalman filter equation to update weights for the ensemble

$$\mathbf{w}_i^a = \mathbf{w}_i + \mathbf{K}\left(\mathbf{w}^o - \mathbf{w}_i\right)$$

where the Kalman gain matrix is given by

$$\mathbf{K} = \left(\mathbf{I} + \hat{\mathbf{R}}\right)^{-1}$$

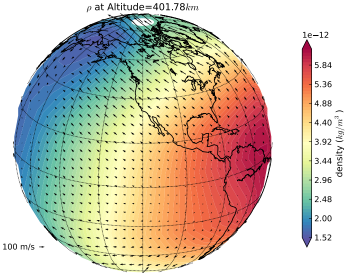
where

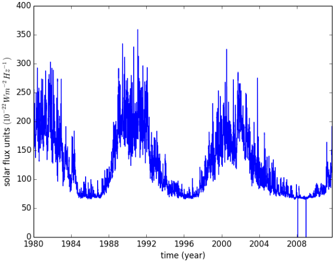
$$\hat{\mathbf{R}} = \left(\hat{\mathbf{H}}^T \hat{\mathbf{H}}\right)^{-1} \hat{\mathbf{H}}^T \mathbf{R} \hat{\mathbf{H}} \left(\hat{\mathbf{H}}^T \hat{\mathbf{H}}\right)^{-1}$$

The final analysis update is given by

$$\mathbf{x}_i^a = \overline{\mathbf{x}} + \mathbf{B}\mathbf{w}_i^a$$

Assimilation for Inosphere-Thermosphere Model





- GITM: physics based model that solves the full Navier-Stokes equations for density, velocity, and temperature for a number of neutral and charged components
- F10.7 measures solar flux at wavelength of 10.7 cm
- This parameter is used as proxy for the Sun activity over the Earth
- Used in GITM to account for solar heating

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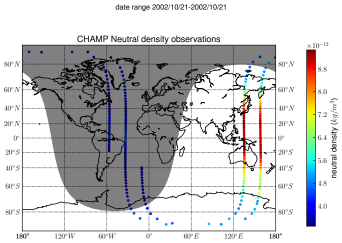
Assimilation for $F_{10.7}$ "Calibration" in GITM

The calibration of $F_{10.7}$ is to estimate the appropriate coupling between the observed $F_{10.7}$ and the model.

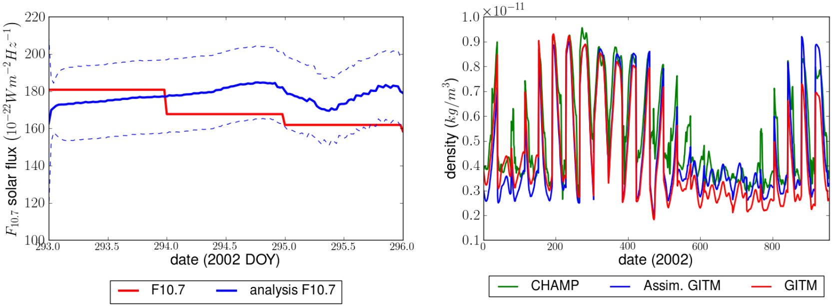
$$p^m_{F10.7} = p^o_{F10.7} + \delta p_{F10.7}$$

This calibration is performed with data assimilation of density observations. In particular, a ensemble-based assimilation method is utilized.
Localized Transform Kalman Filter (LETKF)



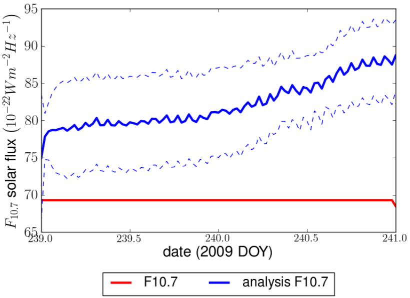
- CHAMP and GRACE provide neutral density derived observations
- assimilation done with 20 ensemble member
- observations assimilated every 30 minutes
- "calibrated" $F_{10.7}$ parameter
- ensemble generated with mean $F_{10.7} = 165.89$ and standard deviation of 10.0

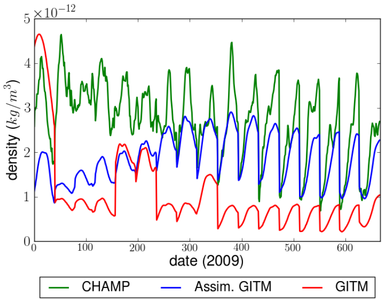
Assimilation during Solar Maximum



- Assimilation performed for October 21–24, 2002, during solar max
- GITM provides an accurate estimate for the ionosphere-thermosphere
- assimilated $F_{10.7}$ oscillating closely to measured $F_{10.7}$, not much correction is needed for GITM match observed density from CHAMP

Assimilation during Solar Minimum





- Assimilation performed for August 28–31, 2009, during solar minimum.
- During solar minimum, more complex internal processes dominate ionosphere-thermosphere, GITM unable to provide accurate representation
- Assimilation provides significant changes to $F_{10.7}$, GITM seems to get closer to observed density from CHAMP

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Conclusions/Questions

- An ensemble Kalman filter in orthogonal space was presented
- method allows simplification of analysis, and reduction of cross-correlation noise
- implemented to a global ionosphere-thermosphere model to estimate total neutral density
- results are promising, additional work needed to fully validate

Questions/Comments?