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Optimality conditions for the numerical solution of optimization problems with PDE constraints

Theoretical framework and applications to parameter identification and optimal control problems

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Abstract

A theoretical framework for the numerical solution of partial differential equation (PDE) constrained optimization problems is presented in this report. This theoretical framework embodies the fundamental infrastructure required to efficiently implement and solve this class of problems. Detail derivations of the optimality conditions required to accurately solve several parameter identification and optimal control problems are also provided in this report. This will allow the reader to further understand how the theoretical abstraction presented in this report translates to the application.

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Contents

Chapter 1

Theoretical Framework

Let $\mathcal{Z}$,$\mathcal{U}$, $\mathcal{Y}$ be Hilbert spaces and $\mathcal{Z}$, $\mathcal{U}$ are reflexive, i.e. $z \sim z \quad \forall \quad z \in \mathcal{Z}$ and $u \sim u \quad \forall \quad u \in \mathcal{U}$. Furthermore, let $\mathcal{J} \colon \mathcal{U} \times \mathcal{Z} \rightarrow \mathbb{R}$ and $g \colon \mathcal{U} \times \mathcal{Z} \rightarrow \mathcal{Y}$. Lets consider an optimization problem of the form

$$\begin{aligned} &\underset{(u,z) \; \in \; \mathcal{U}\times\mathcal{Z}}{\text{minimize}} \qquad \mathcal{J}(u, z) \\ \text{subject to} \\ &g(u, z) = 0, \end{aligned} \tag{1.1}$$

where $u \in \mathcal{U}_{ad} \subset \mathcal{U}$ and $z \in \mathcal{Z}_{ad} \subset \mathcal{Z}$. $\mathcal{U}_{ad}$ and $\mathcal{Z}_{ad}$ denote admissible subsets of the state and control spaces $\mathcal{U}$ and $\mathcal{Z}$, respectively. If the following conditions are met:

1. $\mathcal{Z}_{ad} \subset \mathcal{Z}$ is convex, bounded and closed;
2. $\mathcal{U}_{ad} \subset \mathcal{U}$ is convex, closed, and contains a feasiblepoint; i.e. $g(u, z) = 0$ has a bounded solution operator, $u \colon \mathcal{Z} \rightarrow \mathcal{U}$;
3. the mapping $(u, z) \mapsto g(u, z)$ is continuous under weak convergence; and
4. $\mathcal{J}$ is sequentially lower semicontinuous;

there exists a solution to the optimization problem defined in Eq. 1.1 [2, 1]. The above result ensures the existence of an optimal solution to the optimization problem defined in Eq. 1.1. However, the uniqueness of the solution is problem dependent.

To efficiently solve the optimization problem in Eq. 1.1, first order necessary optimality conditions and second order sufficient conditions are required to find the optimal solution. These conditions involve the gradient of the objective function being zero at the optimal solution and the Hessian operator being positive semidefinite at the optimal solution. These conditions can be derived from Lagrangian multiplier theory [4].

Full-Space Formulation

Consider the optimization problem defined in Eq. 1.1 and assume that the objective function $\mathcal{J}(u, z)$ and constraint $g(u, z)$ are twice continuously differentiable with Lipschitz continuous second derivatives. We define a Lagrangian functional $\mathcal{L}\colon \mathcal{U} \times \mathcal{Z} \to \mathbb{R}$, given by

$$\mathcal{L}(u, z, \lambda) = \mathcal{J}(u, z) + \langle \lambda, g(u, z)\rangle_{\mathcal{U}^*,\mathcal{U}},$$

If $(\hat{u}, \hat{z}) \in \mathcal{U} \times \mathcal{Z}$ is a local solution of Eq. 1.1, then there exists a Lagrange multiplier $\hat{\lambda} \in \mathcal{U}$ such that the first-order necessary optimality conditions

$$\begin{aligned}
\langle \mathcal{L}_u(u, z, \lambda), \delta u\rangle &= \langle \mathcal{J}_u(u, z) + g_u(u, z)^*\lambda, \delta u\rangle = 0\\
\langle \mathcal{L}_z(u, z, \lambda), \delta z\rangle &= \langle \mathcal{J}_z(u, z) + g_z(u, z)^*\lambda, \delta z\rangle = 0\\
\langle \mathcal{L}_\lambda(u, z, \lambda), \delta \lambda\rangle &= \langle g\left(u, z\right), \delta \lambda\rangle = 0
\end{aligned} \tag{1.2}$$

are satisfy for any $(\delta u, \delta z, \delta\lambda) \in \mathcal{U} \times \mathcal{Z} \times \mathcal{U}$. Here, $*$ denotes the adjoint of an operator.

If $(\tilde{u}, \tilde{z}, \tilde{\lambda})$ satisfy Eq. 1.2, and

$$\nabla^2\mathcal{L}(\tilde{u}, \tilde{z}, \tilde{\lambda})\,[s, s] \geq \delta\,\|s^2\| \quad \forall\, s \in \ \mathrm{Null}\left(\nabla g(u, z)\,[\delta u, \delta z]\right),$$

for a given $\delta > 0$, then $(\tilde{u}, \tilde{z})$ is a strict local minimum. This statement denotes the second-order sufficient conditions.

Applying Newton's method to the first-order necessary optimality conditions in Eq. 1.2 results in the following optimality system

$$\begin{pmatrix} \mathcal{L}_{uu}(u, z, \lambda) & \mathcal{L}_{uz}(u, z, \lambda) & g_u(u, z)^* \\ \mathcal{L}_{zu}(u, z, \lambda) & \mathcal{L}_{zz}(u, z, \lambda) & g_z(u, z)^* \\ g_u(u, z) & g_z(u, z) & 0 \end{pmatrix} \begin{pmatrix} \delta u \\ \delta z \\ \delta\lambda \end{pmatrix} = - \begin{pmatrix} \mathcal{J}_u(u, z) + g_u\left(u, z\right)^*\lambda \\ \mathcal{J}_z(u, z) + g_z\left(u, z\right)^*\lambda \\ g\left(u, z\right) \end{pmatrix}, \tag{1.3}$$

where, $\mathcal{L}_{uu}(u, z, \lambda)$, $\mathcal{L}_{uz}(u, z, \lambda)$, $\mathcal{L}_{zz}(u, z, \lambda)$, and $\mathcal{L}_{zu}(u, z, \lambda)$ are respectively defined as

$$\mathcal{L}_{uu}(u, z, \lambda) = J_{uu}(u, z) + (g_{uu}(u, z)\cdot)^*\lambda, \tag{1.4}$$

$$\mathcal{L}_{uz}(u, z, \lambda) = J_{uz}(u, z) + (g_{uz}(u, z)\cdot)^*\lambda, \tag{1.5}$$

$$\mathcal{L}_{zz}(u, z, \lambda) = J_{zz}(u, z) + (g_{zz}(u, z)\cdot)^*\lambda, \tag{1.6}$$

$$\mathcal{L}_{zu}(u, z, \lambda) = J_{zu}(u, z) + (g_{zu}(u, z)\cdot)^*\lambda. \tag{1.7}$$

In practice, this problem is also known as an equality constrained optimization problem.

Reduced-Space Formulation

We define a Lagrangian functional

$$\mathcal{L}(u(z), z, \lambda) = \mathcal{J}(u(z), z) + \langle \lambda, g(u(z), z)\rangle_{\mathcal{U}^*,\mathcal{U}},$$

where $\langle\cdot,\cdot\rangle\colon \mathcal{U}^* \times \mathcal{U} \to \mathbb{R}$ and $u = u(z)$ denotes the solution to the constraint equation $g(u, z) = 0$ by the implicit function theorem. Then, the objective functional becomes

$$\widehat{\mathcal{J}}(z) = \mathcal{J}(u(z), z)$$

Computing the Fréchet derivative of $\widehat{\mathcal{J}}(z)$ and equating to zero, one has that for any $\delta z \in \mathcal{Z}$,

$$\langle \nabla \widehat{\mathcal{J}}(z), \delta z\rangle_{\mathcal{Z}^*,\mathcal{Z}} = \langle \mathcal{L}_u(u(z), z, \lambda), u_z(z)\delta z\rangle + \langle \mathcal{L}_z(u(z), z, \lambda), \delta z\rangle.$$

Then, for any $\delta z \in \mathcal{Z}$ and $\delta u \in \mathcal{U}$ we have

$$\begin{aligned} \langle \delta z, \mathcal{L}_z(u(z), z, \lambda)\rangle &= \langle \mathcal{J}_z(u(z), z), \delta z\rangle_{\mathcal{Z}^*,\mathcal{Z}} + \langle \lambda, g_z(u(z), z)\delta z\rangle_{\mathcal{Y}^*,\mathcal{Y}} \\ &= \langle \delta z, \mathcal{J}_z(u(z), z)^* + g_z(u(z), z)^*\lambda\rangle_{\mathcal{Z}^*,\mathcal{Z}} \end{aligned} \tag{1.8}$$

$$\begin{aligned} \langle \delta u, \mathcal{L}_u(u(z), z, \lambda)\rangle &= \langle \mathcal{J}_u(u(z), z), \delta u\rangle_{\mathcal{U}^*,\mathcal{U}} + \langle \lambda, g_u(u(z), z)\delta u\rangle_{\mathcal{Y}^*,\mathcal{Y}} \\ &= \langle \delta u, \mathcal{J}_u(u(z), z)^* + g_u(u(z), z)^*\lambda\rangle_{\mathcal{U}^*,\mathcal{U}}, \end{aligned} \tag{1.9}$$

where $\delta u = u_z(z)\delta z$.

Let λ: $\mathcal{Z} \to \mathcal{U}$. Then, by the implicit function theorem, a $\lambda \equiv \lambda(z)$ is attained such that $\mathcal{L}_u(u(z), z, p) = 0$. Notice that λ can be obtained by solving Eq. 1.9 as follows

$$\lambda = -g_u(u(z), z)^{-*}\mathcal{J}(u(z), z). \tag{1.10}$$

Then, the reduced gradient is calculated by substituting λ, which is obtained by solving Eq. 1.10, into Eq. 1.8 as follows

$$\begin{aligned} \nabla \widehat{\mathcal{J}}(z) &= \mathcal{J}_z(u(z), z) + g_z(u(z), z)^*(-g_u(u(z), z)^{-*}\mathcal{J}_u(u(z), z)) \\ &= \mathcal{J}_z(u(z), z) + g_z(u(z), z)^*\lambda(z) \end{aligned} \tag{1.11}$$

The application of the Hessian operator to direction δz is computed by differentiating $\mathcal{L}_z(u(z), z, \lambda(z))$ with respect to $u(z)$, z, and $\lambda(z)$ as follows:

$$\nabla^2\mathcal{J}(z)\delta z = \mathcal{L}_{zu}(u(z), z, \lambda(z))\delta u + \mathcal{L}_{zz}(u(z), z, \lambda(z))\delta z + \mathcal{L}_{z\lambda}(u(z), z, \lambda(z))\delta\lambda, \tag{1.12}$$

where $\delta\lambda \equiv \lambda_z(z)\delta z$. Thus, in order to find $\nabla^2\mathcal{J}(z)\delta z$ we need to find $\delta u \in \mathcal{U}$ and $\delta\lambda \in \mathcal{U}$.

Notice that $g(u(z), z) = 0$ for all $z \in \mathcal{Z}$. This means that $g_z(u(z), z)\delta z = 0$ for all $\delta z \in \mathcal{Z}$. Exapanding this equality yields

$$g_z(u(z), z)\delta z = g_u(u(z), z)\delta u + g_z(u(z), z)\delta z = 0. \tag{1.13}$$

Solving Eq. 1.13 for δu, gives

$$\delta u = -g_u(u(z), z)^{-1}g_z(u(z), z)\delta z. \tag{1.14}$$

Then, $\delta\lambda$ is found by differentiating $\mathcal{L}_u(u(z), z, \lambda(z))$ with respect to $u(z)$, z, and $\lambda(z)$ as follows

$$\mathcal{L}_{uu}(u(z), z, \lambda(z))\delta u + \mathcal{L}_{uz}(u(z), z, \lambda(z))\delta z + \mathcal{L}_{u\lambda}(u(z), z, \lambda(z))\delta\lambda = 0. \tag{1.15}$$

Differentiating Eq. 1.9 with respect to $\lambda(z)$ yields

$$\mathcal{L}_{u\lambda}(u(z), z, \lambda(z)) = g_u(u(z), z)^*. \tag{1.16}$$

Substituting Eq. 1.16 into Eq. 1.15 yields

$$\mathcal{L}_{uu}(u(z), z, \lambda(z))\delta u + \mathcal{L}_{uz}(u(z), z, \lambda(z))\delta z + g_u(u(z), z)^* \delta\lambda = 0. \tag{1.17}$$

Solving Eq. 1.17 for $\delta\lambda$ gives

$$\delta\lambda = -g_u(u(z), z)^{-*}\big[\mathcal{L}_{uu}(u(z), z, \lambda(z))\delta u + \mathcal{L}_{uz}(u(z), z, \lambda(z))\delta z\big], \tag{1.18}$$

where δu is given by Eq. 1.14. Lastly, differentiating Eq. 1.8 with respect to $\lambda(z)$ yields

$$\mathcal{L}_{z\lambda}(u(z), z, \lambda(z)) = g_z(u(z), z)^* \tag{1.19}$$

After substituting Eq. 1.19 into Eq. 1.12, the application of the Hessian to direction $\delta z \in \mathcal{Z}$ is defined as

$$\nabla^2 \widehat{\mathcal{J}}(z)\delta z = \mathcal{L}_{zu}(u(z), z, \lambda(z))\delta u + \mathcal{L}_{zz}(u(z), z, \lambda(z))\delta z + g_z(u(z), z)^* \delta\lambda, \tag{1.20}$$

where δu and $\delta\lambda$ are respectively defined in Eqs. 1.14 and 1.18.

To summarize, the application of the Hessian operator to direction $\delta z \in \mathcal{Z}$ is calculated as follows:

1. Solve $g(u(z), z) = 0 \quad \text{for} \quad u(z)$
2. Solve $g_u(u(z), z)^* \, \lambda = -\mathcal{J}_u(u(z), z) \quad \text{for} \quad \lambda$
3. Solve $g_u(u(z), z)\, \delta u = g_z(u(z), z)\, \delta z \quad \text{for} \quad \delta u$
4. Solve $g_u(u(z), z)^* \, \delta\lambda = -\big[\mathcal{L}_{uu}(u(z), z, \lambda(z))\, \delta u + \mathcal{L}_{uz}(u(z), z, \lambda(z))\, \delta z\big] \quad \text{for} \quad \delta\lambda$
5. Compute

$$\nabla^2 \widehat{\mathcal{J}}(z)\, \delta z = \mathcal{L}_{zu}(u(z), z, \lambda(z))\, \delta u + \mathcal{L}_{zz}(u(z), z, \lambda(z))\, \delta z + g_z(u(z), z)^* \, \delta\lambda$$

The reduced-space formulation allows us to reformulate the optimization problem defined in Eq. 1.1 as an unconstrained optimization problem of the form

$$\underset{z \in \mathcal{Z}}{\text{minimize}} \qquad \widehat{\mathcal{J}}(z),$$

where the reduced gradient is given by Eq. 1.11 and the application of the Hessian operator to direction δz is given by Eq. 1.20.

Main Operators

Althought in practice, the algorithms utilized to solve full-space and reduced-space problems can differ, there are common features that are required to effectively solve these problems. For instance, optimality conditions are derived and implemented to find the optimal solution to the optimization problem defined in Eq. 1.1. From the previous theoretical discussion, the reader can notice that these optimality conditions rely on the same operators to calculate both the gradient and Hessian. These common operators are

$$\begin{gathered}
J(u,z),\ J_u(u,z),\ J_z(u,z),\ J_{uu}(u,z),\ J_{uz}(u,z),\ J_{zz}(u,z),\ J_{zu}(u,z) \\
g(u,z),\ g_u(u,z),\ g_z(u,z),\ (g_u(u,z))^*,\ (g_z(u,z))^*, \\
(g_{uu}(u,z)\cdot)^*,\ (g_{uz}(u,z)\cdot)^*,\ (g_{zz}(u,z)\cdot)^*,\ (g_{zu}(u,z)\cdot)^*.
\end{gathered} \tag{1.21}$$

By defining these sixteen operators, the algorithms used to solve full-space and reduced-space problems can share the same interface. Allowing practitioners to easily apply full-space and reduced-space formulations to solve PDE-constrained optimization problems.

Several PDE constrained optimization problems are presented in Chapter 2. First order optimality conditions and second order sufficient conditions are derived and presented for each problem. These examples will allow readers to understand the theoretical framework presented herein as well as enable readers to bridge the gap between theory and application.

Chapter 2

Optimality Conditions

Parameter Identification in the Poisson Equation

A prototypical parameter identification (inverse) problem is to estimate the coefficient z from measurements $\widehat{u}$ related to the solution u of the elliptic boundary value problem:

$$\begin{aligned} -(z\, u_i)_{,i} &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{2.1}$$

Here, $\Omega \subseteq \mathbb{R}^d$, $d \in \{1, 2, 3\}$, is the computational domain with boundary $\partial\Omega$. For a steady-state heat equation, z is the coefficient of thermal diffusion, u is the temperature distribution, $\widehat{u}$ are the temperature measurements, and f is a given heat source. In the following we use standard tensor notation with Einstein summation. The indices i and j take on the values $1, \ldots, d$. Partial differentiation is denoted by a comma.

We consider the nonlinear programming problem (NLP)

$$\begin{aligned} &\underset{(u,z)\, \in\, \mathcal{U}\times\mathcal{Z}}{\text{minimize}} \qquad \frac{\beta}{2}\langle u - \widehat{u}, u - \widehat{u}\rangle + R(z) \\ &\text{subject to} \\ &\quad -(z\, u_i)_{,i} = f \quad \text{in } \Omega, \end{aligned} \tag{2.2}$$

where $\mathcal{U} = \{u\colon u \in H^1(\Omega),\ u = 0 \text{ on } \partial\Omega\}$, $\mathcal{Z} = \{z\colon z \in L^2(\Omega),\ z > 0\}$, and $R(\cdot)$ denotes a regularization functional. $H^m(\Omega)$ are Sobolev spaces of square integrable functionals whose m-th derivatives are also square integrable. Whenever $m = 0$ we shall keep the notation with $L^2(\Omega)$. The L^2 inner products for the space $\mathcal{U}$ is defined as

$$\langle u, v\rangle := \int_\Omega u(x)\, v(x) dx,$$

First-Order Necessary Optimality Conditions. The Lagrangian associated with the NLP defined in Eq. 2.2 is given by

$$\mathcal{L}(u, z, \lambda) = \frac{\beta}{2}\langle u - \widehat{u}, u - \widehat{u}\rangle + R(z) + \langle \lambda, -(z\, u_i)_{,i} - f\rangle$$

If $\{\tilde{u}, \tilde{z}\} \in \mathcal{U} \times \mathcal{Z}$ is a local solution to the NLP in Eq. 2.2, there exists Lagrange multipliers $\lambda \in \mathcal{U}$, such that the first-order necessary optimality conditions hold at $\{\tilde{u}, \tilde{z}\}$, i.e.

$$\nabla \mathcal{L}(u, z, \lambda)(\delta u, \delta z, \delta \lambda) = \mathcal{L}_u(u, z, \lambda)\delta u + \mathcal{L}_z(u, z, \lambda)\delta z + \mathcal{L}_\lambda(u, z, \lambda)\delta \lambda = 0 \tag{2.3}$$

Second-Order Sufficient Conditions. If $(\tilde{u}, \tilde{z}, \lambda) \in \mathcal{U} \times \mathcal{Z} \times \mathcal{U}$ satisfy the first-order necessary optimality conditions and the Hessian operator exist and is positive semidefinite, then $(\tilde{u}, \tilde{z})$ is a strict local minimum.

Operators. The required operators for the paramater identification problem are defined as

$$J(u, z) = \frac{\beta}{2}\langle u - \widehat{u}, u - \widehat{u}\rangle + R(z) \tag{2.4}$$

$$\begin{aligned} \langle J_u(u, z), \delta u\rangle &= \frac{\partial}{\partial u}\left(\frac{\beta}{2}\langle u - \widehat{u}, u - \widehat{u}\rangle + R(z)\right) \\ &= \beta\langle u - \widehat{u}, \delta u\rangle \end{aligned} \tag{2.5}$$

$$\begin{aligned} \langle J_z(u, z), \delta z\rangle &= \frac{\partial}{\partial z}\left(\frac{\beta}{2}\langle u - \widehat{u}, u - \widehat{u}\rangle + R(z)\right) \\ &= \langle R_z(z), \delta z\rangle \end{aligned} \tag{2.6}$$

$$\begin{aligned} \langle J_{uu}(u, z), \delta u\rangle &= \frac{\partial}{\partial u}\Big(\beta\langle \mathbf{1}, u - \widehat{u}\rangle\Big) \\ &= \beta\langle \mathbf{1}, \delta u\rangle \end{aligned} \tag{2.7}$$

$$\langle J_{uz}(u, z), \delta z\rangle = \frac{\partial}{\partial z}\Big(\beta\langle \mathbf{1}, u - \widehat{u}\rangle\Big) = \mathbf{0} \tag{2.8}$$

$$\begin{aligned} \langle J_{zz}(u, z), \delta z\rangle &= \frac{\partial}{\partial z}\Big(\langle \mathbf{1}, R_z(z)\rangle\Big) \\ &= \langle R_{zz}(z), \delta z\rangle \end{aligned} \tag{2.9}$$

$$\langle J_{zu}(u, z), \delta u\rangle = \frac{\partial}{\partial u}\Big(\langle \mathbf{1}, R_z(z)\rangle\Big) = \mathbf{0} \tag{2.10}$$

$$g(u, z) = -(z\, u_i)_{,i} - f = 0 \quad \text{in} \quad \Omega \tag{2.11}$$

$$\begin{aligned} \langle g_u(u, z), \delta u\rangle &= \langle \lambda, -(z\, u_i)_{,i} - f, \rangle = 0 \quad \forall \quad \lambda \in \mathcal{U} \\ &= \langle \lambda_i, z\, u_i\rangle - \langle \lambda, u_i n_i\rangle_{\partial\Omega} - \langle f, \lambda\rangle \\ &= \langle \lambda_i, z\, u_i\rangle - \langle \lambda, f\rangle \\ &= \frac{\partial}{\partial u}\Big(\langle \lambda_i, z\, u_i\rangle - \langle \lambda, f\rangle\Big) \\ &= \langle \lambda_i, z\, \delta u_i\rangle \\ &= \langle -(z\, \lambda_i)_{,i}, \delta u\rangle + \langle \lambda_i n_i, \delta u\rangle_{\partial\Omega} \\ &= \langle -(z\, \lambda_i)_{,i}, \delta u\rangle \end{aligned} \tag{2.12}$$

The application of the divergence theorem results in a set of homogeneous Dirichlet boundary condition in Eq. 2.12. since $u \in \mathcal{U}$ and $\lambda \in \mathcal{U}$. Therefore, $u = \lambda = 0$ on $\partial\Omega$ by definition of the space $\mathcal{U}$. This definition will be used in the subsequent derivations.

$$\begin{aligned}
\langle g_z(u,z), \delta z\rangle &= \langle \lambda, -(z\, u_i)_{,i} - f\rangle = 0 \quad \forall \quad \lambda \in \mathcal{U} \\
&= \langle \lambda_i, z\, u_i\rangle - \langle \lambda, u_i n_i\rangle_{\partial\Omega} - \langle \lambda, f\rangle \\
&= \langle \lambda_i, z\, u_i\rangle - \langle \lambda, f\rangle \\
&= \frac{\partial}{\partial z}\Big(\langle \lambda_i, z\, u_i\rangle - \langle \lambda, f\rangle\Big) \\
&= \langle \lambda_i u_i, \delta z\rangle
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
\langle \delta u, (g_u(u,z))^*\rangle &= \langle \lambda, -(z\, u_i)_{,i} - f\rangle = 0 \quad \forall \quad \lambda \in \mathcal{U} \\
&= \langle \lambda_i, z\, u_i\rangle - \langle \lambda, u_i n_i\rangle_{\partial\Omega} - \langle \lambda, f\rangle \\
&= \langle \lambda_i . z\, u_i\rangle - \langle \lambda, f\rangle \\
&= \langle -(z\,\lambda_i)_{,i}, u\rangle + \langle \lambda_i n_i, u\rangle_{\partial\Omega} - \langle \lambda, f\rangle \\
&= \langle -(z\,\lambda_i)_{,i}, u\rangle - \langle \lambda, f\rangle \\
&= \frac{\partial}{\partial u}\Big(\langle u, -(z\,\lambda_i)_{,i}\rangle^* - \langle f, \lambda\rangle^*\Big) \\
&= \langle \delta u, -(z\,\lambda_i)_{,i}\rangle^*
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
\langle \delta z, (g_z(u,z))^*\rangle &= \langle \lambda, -(z\, u_i)_{,i} - f\rangle = 0 \quad \forall \quad \lambda \in \mathcal{U} \\
&= \langle \lambda_i, z\, u_i\rangle - \langle \lambda, u_i n_i\rangle_{\partial\Omega} - \langle \lambda, f\rangle \\
&= \frac{\partial}{\partial z}\Big(\langle \lambda_i, z\, u_i\rangle - \langle \lambda, f\rangle\Big) \\
&= \langle \lambda_i\, u_i, \delta z\rangle \\
&= \langle \delta z, u_i\, \lambda_i\rangle^*
\end{aligned} \tag{2.15}$$

$$\langle (g_{uu}(u,z))^*, \delta u\rangle = \frac{\partial}{\partial u}\Big(\langle \mathbf{1}, -(z\,\lambda_i)_{,i}\rangle\Big) = \mathbf{0} \tag{2.16}$$

$$\begin{aligned}
\langle (g_{uz}(u,z))^*, \delta z\rangle &= \langle w, -(z\,\lambda_i)_{,i}\rangle = 0 \quad \forall \quad w \in \mathcal{U} \\
&= \frac{\partial}{\partial z}\Big(\langle w_i, z\,\lambda_i\rangle - \langle w, \lambda\rangle\Big) \\
&= \langle \lambda_i\, w_i, \delta z\rangle
\end{aligned} \tag{2.17}$$

$$\langle (g_{zz}(u,z))^*, \delta z\rangle = \frac{\partial}{\partial z}\Big(\langle \mathbf{1}, u_i\,\lambda_i\rangle\Big) = \mathbf{0} \tag{2.18}$$

$$\begin{aligned}
\langle (g_{zu}(u,z))^*, \delta u\rangle &= \langle v, u_i\,\lambda_i\rangle \quad \forall \quad v \in \mathcal{Z} \\
&= \langle -(v\lambda_i)_{,i}, u\rangle + \langle \lambda_i n_i, u\rangle_{\partial\Omega} \\
&= \frac{\partial}{\partial u}\Big(\langle -(v\lambda_i)_{,i}, u\rangle\Big) \\
&= \langle -(v\lambda_i)_{,i}, \delta u\rangle
\end{aligned} \tag{2.19}$$

Discretization. We define finite-dimensional subspaces $\mathcal{U}^h \subset \mathcal{U}$ with basis $\mathcal{U}^h = \text{span}\{\phi^m\}_{m=1}^M$ and $\mathcal{Z}^h \subset \mathcal{Z}$ with basis $\mathcal{Z}^h = \text{span}\{\psi^m\}_{m=1}^M$. This leads to $u^h = \sum_{m=1}^M u_i^m \phi_i^m$, $z^h = \sum_{n=1}^N z^n \psi^n$, and $\lambda^h = \sum_{m=1}^M \lambda^m \phi^m$. Furthermore, the quantities δu^h and δz^h can be respectively approximated as follows: $\delta u^h = \sum_{m=1}^M \delta u^m \phi^m$ and $\delta z^h = \sum_{n=1}^N \delta z^n \psi^n$. Subsequently, the Galerkin approximation of the required operators to solve the parameter identification problem are defined as

$$J(u^h, z^h) = \frac{\beta}{2}\langle u^h - \widehat{u}, u^h - \widehat{u}\rangle + R(z^h) \tag{2.20}$$

$$J_u(u^h, z^h) = \sum_{e=1}^{N_{elem}} \left(\sum_{a=1}^{M} \beta\langle \phi^a, u^h - \widehat{u}\rangle \right) \tag{2.21}$$

$$J_z(u^h, z^h) = \sum_{e=1}^{N_{elem}} \left(\sum_{a=1}^{N} \langle \psi^a, R_z(z^h)\rangle \right) \tag{2.22}$$

$$J_{uu}(u^h, z^h) = \sum_{e=1}^{N_{elem}} \left(\sum_{a=1}^{M}\sum_{b=1}^{M} \beta\langle \phi^a, \phi^b\rangle \right) \tag{2.23}$$

$$J_{uz}(u^h, z^h) = \mathbf{0} \tag{2.24}$$

$$J_{zz}(u^h, z^h) = \sum_{e=1}^{N_{elem}} \left(\sum_{a=1}^{N}\sum_{b=1}^{N} \langle \psi^a, R_{zz}(z^h)\psi^b\rangle \right) \tag{2.25}$$

$$J_{zu}(u^h, z^h) = \mathbf{0} \tag{2.26}$$

$$g(u^h, z^h) = \sum_{e=1}^{N_{elem}} \left(\sum_{a=1}^{M} \langle \phi_i^a, z^h u_i^h\rangle - \langle \phi^a, f\rangle \right) \tag{2.27}$$

$$g_u(u^h, z^h) = \sum_{e=1}^{N_{elem}} \left(\sum_{a=1}^{M} \langle \phi_i^a, z^h \lambda_i^h\rangle \right) \tag{2.28}$$

$$g_z(u^h, z^h) = \sum_{e=1}^{N_{elem}} \left(\sum_{a=1}^{N} \langle \psi^a, u_i^h\, \lambda_i^h\rangle \right) \tag{2.29}$$

$$(g_u(u^h, z^h))^* = \sum_{e=1}^{N_{elem}} \left(\sum_{a=1}^{M} \langle z^h \lambda_i^h, \phi_i^a\rangle \right) \tag{2.30}$$

$$(g_z(u^h, z^h))^* = \sum_{e=1}^{N_{elem}} \left(\sum_{a=1}^{M} \langle \lambda_i^h\, u_i^h, \psi^a\rangle \right) \tag{2.31}$$

$$(g_{uu}(u^h, z^h))^* = \mathbf{0} \tag{2.32}$$

$$(g_{uz}(u^h, z^h))^* = \sum_{e=1}^{N_{elem}} \left(\sum_{a=1}^{M}\sum_{b=1}^{N} \langle \psi^b \lambda_i^h, \phi_i^a\rangle \right) \tag{2.33}$$

$$(g_{zz}(u^h, z^h))^* = \mathbf{0} \tag{2.34}$$

$$(g_{zu}(u^h, z^h))^* = \sum_{e=1}^{N_{elem}} \left(\sum_{a=1}^{M} \sum_{b=1}^{N} \langle \phi_i^a \lambda_i^h, \psi^b \rangle \right) \tag{2.35}$$

Parameter Identification in Linear Elastostatics

A parameter identification problem in linear elastostatics is to estimate the shear modulus μ and bulk modulus κ from displacement measurements $\{\widehat{u}_i\}_{i=1,\ldots,d}$. Let $\Omega \subseteq \mathbb{R}^d$, $d \in \{1, 2, 3\}$, be the computational domain with boundary $\partial\Omega = \partial\Omega_u \cup \partial\Omega_\tau$. The regions $\partial\Omega_u \subset \partial\Omega$ and $\partial\Omega_\tau \subset \partial\Omega$ are the boundaries where Dirichlet and Neumann conditions are respectively applied. We use standard tensor notation with Einstein summation. The indices i, j, k, and l take on the values $1, \ldots, d$. Partial differentiation is denoted by a comma.

The partial differential equations (PDEs) of linear elastostatics are given by

$$\begin{aligned} -(\mathbf{C}_{ijkl}\, \epsilon_{kl})_{,j} &= 0 \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega_u \\ (\mathbf{C}_{ijkl}\, \epsilon_{kl})\, n_j &= \tau_i \quad \text{on} \quad \partial\Omega_\tau, \end{aligned} \tag{2.36}$$

where $\{u_i\}_{i=1,\ldots,d}$ denotes the displacement in the i-th direction and τ_i denotes the surface traction in the i-th direction. The unit outward normal component in the i-th direction on $\partial\Omega_\tau$ with respect to the region Ω is denoted by n_j.

The fourth-order tensor of elastic moduli is given by

$$\mathbf{C}_{ijkl} = \kappa\, \delta_{ij}\, \delta_{kl} + \mu \left(\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk} - \frac{2}{3}\, \delta_{ij}\, \delta_{kl} \right) \tag{2.37}$$

and the strain tensor is defined as the symmetric part of the displacement gradient, as follows

$$\epsilon_{kl} = \frac{1}{2} \left(u_{k,l} + u_{l,k} \right). \tag{2.38}$$

Lets consider the following NLP

$$\begin{aligned} &\underset{(u,\mu,\kappa)\, \in\, \mathcal{U}_i \times \mathcal{G} \times \mathcal{B}}{\text{minimize}} \qquad \frac{\beta}{2} \langle u_i - \widehat{u}_i, u_i - \widehat{u}_i \rangle + R(\mu) + R(\kappa) \\ &\text{subject to} \\ &\qquad -(\mathbf{C}_{ijkl}\, \frac{1}{2}(u_{k,l} + u_{l,k}))_{,j} = 0 \quad \text{in} \quad \Omega, \\ &\qquad (\mathbf{C}_{ijkl}\, \frac{1}{2}(u_{k,l} + u_{l,k}))\, n_j = \tau_i \quad \text{in} \quad \partial\Omega_\tau \end{aligned} \tag{2.39}$$

where $\mathcal{U}_i = \{u_i \colon u_i \in H^1(\Omega),\ u_i = 0 \text{ on } \partial\Omega\}$, $\mathcal{G} = \{\mu \colon \mu \in L^2(\Omega),\ \mu > 0\}$, and $\mathcal{B} = \{\kappa \colon \kappa \in L^2(\Omega),\ \kappa > 0\}$. $H^m(\Omega)$ are Sobolev spaces of square integrable functionals whose m-th derivatives are also square integrable. Whenever $m = 0$ we shall keep the notation with $L^2(\Omega)$. The L^2 inner products for the space $\mathcal{U}_i$

$$\langle u_i, v_i \rangle := \int_\Omega u_i(x)\, v_i(x) dx. \tag{2.40}$$

Similarly, the L^2 inner products for the spaces $\mathcal{G}$ and $\mathcal{B}$ is defined as

$$\langle u, v \rangle := \int_\Omega u(x)\, v(x) dx, \tag{2.41}$$

First-order neccesary optimality conditions. The Lagrangian associated with the NLP defined in Eq. 2.39 is given by

$$\begin{aligned}\mathcal{L}(u_i, \mu, \kappa, \lambda_i) = &\frac{\beta}{2}\langle u_i - \widehat{u}_i, u_i - \widehat{u}_i\rangle + R(\mu) + R(\kappa)\\ &+ \langle \lambda_i, -\left(\mathbf{C}_{ijkl}\,\epsilon_{kl}\right)_{,j}\rangle - \langle \lambda_i, \left(\left(\mathbf{C}_{ijkl}\,\epsilon_{kl}\right) n_j - \tau_i\right)\rangle_{\partial\Omega_\tau}\end{aligned} \tag{2.42}$$

If $(\tilde{u}_i, \tilde{\mu}, \tilde{\kappa}) \in \mathcal{U}_i \times \mathcal{G} \times \mathcal{B}$ is a local solution to the NLP in Eq. 2.39, there exists Lagrange multipliers $\lambda_i \in \mathcal{U}_i$, such that the first-order necessary optimality conditions hold at $(\tilde{u}_i, \tilde{\mu}, \tilde{\kappa})$, i.e.

$$\begin{aligned}\nabla\mathcal{L}(u_i, \mu, \kappa, \lambda_i)(\delta u_i, \delta\mu, \delta\kappa, \delta\lambda_i) = &\mathcal{L}_u(u_i, \mu, \kappa, \lambda_i)\delta u_i + \mathcal{L}_\kappa(u_i, \mu, \kappa, \lambda_i)\delta\mu\\ &+ \mathcal{L}_\kappa(u_i, \kappa, \lambda_i)\delta\kappa + \mathcal{L}_\lambda(u_i, \mu, \kappa, \lambda_i)\delta\lambda_i = 0\end{aligned}$$

Second-Order Sufficient Conditions. If $(\tilde{u}_i, \tilde{\mu}, \tilde{\kappa}, \lambda_i) \in \mathcal{U}_i \times \mathcal{G} \times \mathcal{B} \times \mathcal{U}_i$ satisfy the first-order necessary optimality conditions and the Hessian operator exist and is positive semidefinite, then $(\tilde{u}_i, \tilde{\mu}, \tilde{\kappa})$ is a strict local minimum.

Operators. Let $z \equiv \{\mu, \kappa\} \in \mathcal{G} \times \mathcal{B}$, then the operators required to solve the paramater identification problem are defined as

$$J(u_i, z) = \frac{\beta}{2}\langle u_i - \widehat{u}_i, u_i - \widehat{u}_i\rangle + R(\mu) + R(\kappa) \tag{2.43}$$

$$\begin{aligned}\langle J_u(u_i, z), \delta u_i\rangle &= \frac{\partial}{\partial u_i}\Big(\beta\langle u_i - \widehat{u}_i, u_i - \widehat{u}_i\rangle + R(\mu) + R(\kappa)\Big)\\ &= \beta\langle u_i - \widehat{u}_i, \delta u_i\rangle\end{aligned} \tag{2.44}$$

$$\begin{aligned}\langle J_z(u_i, z), \delta z\rangle &= \frac{\partial}{\partial z}\Big(\frac{\beta}{2}\langle u_i - \widehat{u}_i, u_i - \widehat{u}_i\rangle + R(\mu) + R(\kappa)\Big)\\ &= \begin{bmatrix} \langle R_\mu(\mu), \delta\mu\rangle \\ \langle R_\kappa(\kappa), \delta\kappa\rangle \end{bmatrix}\end{aligned} \tag{2.45}$$

$$\begin{aligned}\langle J_{uu}(u_i, z), \delta u_i\rangle &= \frac{\partial}{\partial u_i}\Big(\beta\langle \mathbf{1}, u_i - \widehat{u}_i\rangle\Big)\\ &= \beta\langle \mathbf{1}, \delta u_i\rangle\end{aligned} \tag{2.46}$$

$$\langle J_{uz}(u_i, z), \delta z\rangle = \frac{\partial}{\partial z}\Big(\beta\langle \mathbf{1}, u_i - \widehat{u}_i\rangle\Big) = \mathbf{0} \tag{2.47}$$

$$
\begin{aligned}
\langle J_{zz}(u_i,z),\delta z\rangle &= \frac{\partial}{\partial z}\left(\begin{bmatrix} \langle \mathbf{1}, R_\mu(\mu)\rangle \\ \langle \mathbf{1}, R_\kappa(\kappa)\rangle \end{bmatrix}\right) \\
&= \begin{bmatrix} \langle R_{\mu\mu}(\mu),\delta\mu\rangle \\ \langle R_{\kappa\kappa}(\kappa),\delta\kappa\rangle \end{bmatrix}
\end{aligned}
\tag{2.48}
$$

$$
\langle J_{zu}(u_i,z),\delta u_i\rangle = \frac{\partial}{\partial u_i}\left(\begin{bmatrix} \langle \mathbf{1}, R_\mu(\mu)\rangle \\ \langle \mathbf{1}, R_\kappa(\kappa)\rangle \end{bmatrix}\right) = \mathbf{0}
\tag{2.49}
$$

$$
g(u_i,z) = \begin{bmatrix} -(\mathbf{C}_{ijkl}\,\tfrac{1}{2}(u_{k,l}+u_{l,k}))_{,j} = 0 \quad \text{in} \quad \Omega \\ (\mathbf{C}_{ijkl}\,\tfrac{1}{2}(u_{k,l}+u_{l,k}))\,n_j - \tau_i = 0 \quad \text{on} \quad \partial\Omega_\tau \end{bmatrix}
\tag{2.50}
$$

$$
\begin{aligned}
\langle g_u(u_i,z),\delta u_i\rangle &= \langle \lambda_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))_{,j}\rangle \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\;\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j - \tau_i\rangle_{\partial\Omega_\tau} = 0 \quad \forall \quad \lambda_i \in \mathcal{U}_i \\
&= \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k})\rangle \\
&- \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j\rangle_{\partial\Omega_\tau} \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j - \tau_i\rangle_{\partial\Omega_\tau} \\
&= \frac{\partial}{\partial u}\left(\langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k})\rangle - \langle \lambda_i, \tau_i\rangle_{\partial\Omega_\tau}\right) \\
&= \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})\,\mathbf{C}_{ijkl}, \frac{1}{2}(\delta u_{k,l}+\delta u_{l,k})\rangle \\
&= \langle -(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}, \delta u_i\rangle + \langle (\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j, \delta u_i\rangle_{\partial\Omega_\tau} \\
&= \langle -(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}, \delta u_i\rangle
\end{aligned}
\tag{2.51}
$$

The boundary in Eq. 2.51 is defined as $\partial\Omega = \partial\Omega_\tau \cup \partial\Omega_u$. The application of the divergence theorem yields a set of homogeneous Dirichlet boundary condition on $\partial\Omega_u$ since $\lambda_i \in \mathcal{U}_i$ and $u_i \in \mathcal{U}_i$. Therefore, $u_i = \lambda_i = 0$ on $\partial\Omega_u$ by definition of the space $\mathcal{U}_i$. Finally, in order to ensure the validity of the set of equations defined in Eq. 2.51, proper Neumann boundary condition need to be defined. Therefore, a set of free Neumann boundary conditions is defined on $\partial\Omega_\tau$, i.e. $(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j = 0$ on $\partial\Omega_\tau$. These definitions will be used in the subsequent derivations.

$$
\begin{aligned}
\langle g_z(u_i,z),\delta z\rangle &= \langle \lambda_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))_{,j}\rangle \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j - \tau_i\rangle_{\partial\Omega_\tau} = 0 \quad \forall \quad \lambda_i \in \mathcal{U}_i \\
&= \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k})\rangle \\
&- \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j\rangle_{\partial\Omega_\tau} \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j - \tau_i\rangle_{\partial\Omega_\tau} \\
&= \frac{\partial}{\partial z}\Big(\langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k})\rangle - \langle \lambda_i, \tau_i\rangle_{\partial\Omega_\tau}\Big) \\
&= \begin{bmatrix} \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}), \delta\mu\rangle \\ \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}), \delta\kappa\rangle \end{bmatrix}
\end{aligned}
\tag{2.52}
$$

$$
\begin{aligned}
\langle \delta u_i, (g_u(u_i,z))^*\rangle &= \langle \lambda_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))_{,j}\rangle \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j - \tau_i\rangle_{\partial\Omega_\tau} = 0 \quad \forall \quad \lambda_i \in \mathcal{U}_i \\
&= \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k})\rangle \\
&- \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j\rangle_{\partial\Omega_\tau} \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j - \tau_i\rangle_{\partial\Omega_\tau} \\
&= \frac{\partial}{\partial u}\Big(\langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k})\rangle - \langle \lambda_i, \tau_i\rangle_{\partial\Omega_\tau}\Big) \\
&= \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})\,\mathbf{C}_{ijkl}, \frac{1}{2}(\delta u_{k,l}+\delta u_{l,k})\rangle \\
&= \langle -(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}, \delta u_i\rangle \\
&+ \langle (\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j, \delta u_i\rangle_{\partial\Omega_\tau} \\
&= \langle -(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}, \delta u_i\rangle \\
&= \langle \delta u_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}\rangle^*
\end{aligned}
\tag{2.53}
$$

The boundary in Eq. 2.53 is defined as $\partial\Omega = \partial\Omega_\tau \cup \partial\Omega_u$. The application of the divergence theorem yields a set of homogeneous Dirichlet boundary condition on $\partial\Omega_u$ since $\lambda_i \in \mathcal{U}_i$ and $u_i \in \mathcal{U}_i$. Therefore, $u_i = \lambda_i = 0$ on $\partial\Omega_u$ by definition of the space $\mathcal{U}_i$. Finally, in order to ensure the validity of the set of adjoint equations defined in Eq. 2.53, proper Neumann boundary condition are required. Therefore, a set of free Neumann boundary conditions is defined as $(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j = 0$ on $\partial\Omega_\tau$. These definitions ensure a self-adjoint operator in Eq. 2.53 [3].

$$\begin{aligned}\langle \delta z, (g_z(u_i,z))^* \rangle &= \langle \lambda_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))_{,j}\rangle \\ &+ \langle \lambda_i, (\mathbf{C}_{ijkl}\;\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j - \tau_i\rangle_{\partial\Omega_\tau} = 0 \quad \forall \quad \lambda_i \in \mathcal{U}_i \\ &= \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k})\rangle \\ &- \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j\rangle_{\partial\Omega_\tau} \\ &+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j - \tau_i\rangle_{\partial\Omega_\tau} \\ &= \frac{\partial}{\partial z}\left(\langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k})\rangle - \langle \lambda_i, \tau_i\rangle_{\partial\Omega_\tau}\right) \\ &= \begin{bmatrix} \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})\,\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}), \delta\mu\rangle \\ \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})\,\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}), \delta\kappa\rangle \end{bmatrix} \\ &= \begin{bmatrix} \langle \delta\mu, \frac{1}{2}(u_{i,j}+u_{j,i})\,\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k})\rangle^* \\ \langle \delta\kappa, \frac{1}{2}(u_{i,j}+u_{j,i})\,\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k})\rangle^* \end{bmatrix}\end{aligned} \tag{2.54}$$

$$\langle (g_{uu}(u_i,z))^*, \delta u_i\rangle = \frac{\partial}{\partial u_i}\left(\langle \mathbf{1}, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}\rangle\right) = \mathbf{0} \tag{2.55}$$

$$\begin{aligned}\langle (g_{uz}(u_i,z))^*, \delta z\rangle &= \langle w_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}\rangle = 0 \quad \forall \quad w \in \mathcal{U}_i \\ &= \langle \frac{1}{2}(w_{i,j}+w_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k})\rangle \\ &- \langle w_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j)\rangle_{\partial\Omega_\tau} \\ &= \frac{\partial}{\partial z}\left(\langle \frac{1}{2}(w_{i,j}+w_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k})\rangle\right) \\ &= \begin{bmatrix} \langle \frac{1}{2}(w_{i,j}+w_{j,i})\mathbf{C}^{\mu}_{ijkl}\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}), \delta\mu\rangle \\ \langle \frac{1}{2}(w_{i,j}+w_{j,i})\mathbf{C}^{\kappa}_{ijkl}\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}), \delta\kappa\rangle \end{bmatrix}\end{aligned} \tag{2.56}$$

$$\langle (g_{zz}(u_i,z))^*, \delta z\rangle = \frac{\partial}{\partial z}\left(\begin{bmatrix} \langle \mathbf{1}, \frac{1}{2}(u_{i,j}+u_{j,i})\,\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k})\rangle \\ \langle \mathbf{1}, \frac{1}{2}(u_{i,j}+u_{j,i})\,\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k})\rangle \end{bmatrix}\right) = \mathbf{0} \tag{2.57}$$

$$\begin{aligned}\langle (g_{zu}(u_i,z))^*, \delta u_i\rangle &= \begin{bmatrix} \langle \mathbf{1}, \frac{1}{2}(u_{i,j}+u_{j,i})\,\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k})\rangle \\ \langle \mathbf{1}, \frac{1}{2}(u_{i,j}+u_{j,i})\,\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k})\rangle \end{bmatrix} \\ &= \begin{bmatrix} \langle \mathbf{1}, -u_i(\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}\rangle \\ +\langle \mathbf{1}, u_i(\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j\rangle_{\partial\Omega_\tau} \\ \langle \mathbf{1}, -u_i(\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}\rangle \\ +\langle \mathbf{1}, u_i(\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j\rangle_{\partial\Omega_\tau} \end{bmatrix} \\ &= \frac{\partial}{\partial u_i}\left(\begin{bmatrix} \langle -(\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}, u_i\rangle \\ \langle -(\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}, u_i\rangle \end{bmatrix}\right) \\ &= \begin{bmatrix} \langle -(\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}, \delta u_i\rangle \\ \langle -(\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}, \delta u_i\rangle \end{bmatrix}\end{aligned} \tag{2.58}$$

In the preceding equations, $\mathbf{C}^{\mu}_{ijkl}$ and $\mathbf{C}^{\kappa}_{ijkl}$ are defined as

$$\mathbf{C}^{\mu}_{ijkl} = \delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk} - \frac{2}{3}\,\delta_{ij}\,\delta_{kl} \tag{2.59}$$

$$\mathbf{C}^{\kappa}_{ijkl} = \delta_{ij}\,\delta_{kl} \tag{2.60}$$

Discretization. We define finite-dimensional subspaces $\mathcal{U}^h_i \subset \mathcal{U}_i$ with basis $\mathcal{U}^h_i = \text{span}\{\phi^m_i\}^M_{m=1}$, $\mathcal{G}^h \subset \mathcal{G}$ with basis $\mathcal{G}^h = \text{span}\{\psi^n\}^N_{n=1}$, and $\mathcal{B}^h \subset \mathcal{B}$ with basis $\mathcal{B}^h = \text{span}\{\chi^o\}^O_{o=1}$. This leads to $u^h_i = \sum^M_{m=1} u^m_i \phi^m_i$, $\mu^h = \sum^N_{n=1} \mu^n \psi^n$, $\kappa^h = \sum^O_{o=1} \kappa^o \chi^o$,and $\lambda^h_i = \sum^M_{m=1} \lambda^m_i \phi^m_i$. Furthermore, the quantities δu^h_i, $\delta\mu^h$, $\delta\kappa^h$, and $\delta\lambda^h_i$ can be respectively approximated as follows: $\delta u^h_i = \sum^M_{m=1} \delta u^m_i \phi^m_i$, $\delta\mu^h = \sum^N_{n=1} \delta\mu^n \psi^n$, and $\delta\kappa^h = \sum^O_{o=1} \delta\kappa^o \chi^o$. Subsequently, the Galerkin approximation of the required operators to solve the parameter identification problem are defined as

$$J(u^h_i, z^h) = \frac{\beta}{2}\langle u^h_i - \widehat{u}_i, u^h_i - \widehat{u}_i\rangle + R(\mu^h) + R(\kappa^h) \tag{2.61}$$

$$J_u(u^h_i, z^h) = \sum^{N_{elem}}_{e=1}\left(\sum^{M}_{a=1} \beta\langle \phi^a_i, u^h_i - \widehat{u}_i\rangle\right) \tag{2.62}$$

$$J_z(u^h_i, z^h) = \sum^{N_{elem}}_{e=1}\left(\begin{bmatrix} \sum^N_{a=1} \beta\langle\psi^a, R_\mu(\mu^h)\rangle \\ \sum^O_{a=1} \beta\langle\chi^a, R_\kappa(\kappa^h)\rangle \end{bmatrix}\right) \tag{2.63}$$

$$J_{uu}(u^h_i, z^h) = \sum^{N_{elem}}_{e=1}\left(\sum^{M}_{a=1}\sum^{M}_{b=1} \beta\langle\phi^a, \phi^b\rangle\right) \tag{2.64}$$

$$J_{uz}(u^h_i, z^h) = \mathbf{0} \tag{2.65}$$

$$J_{zz}(u^h_i, z^h) = \sum^{N_{elem}}_{e=1}\left(\begin{bmatrix} \sum^N_{a=1}\sum^N_{b=1} \beta\langle\psi^a, R_{\mu\mu}(\mu^h)\psi^b\rangle \\ \sum^O_{a=1}\sum^O_{b=1} \beta\langle\chi^a, R_{\kappa\kappa}(\kappa^h)\chi^b\rangle \end{bmatrix}\right) \tag{2.66}$$

$$J_{zu}(u^h_i, z^h) = \mathbf{0} \tag{2.67}$$

$$g(u^h_i, z^h) = \sum^{N_{elem}}_{e=1}\left(\sum^{M}_{a=1}\langle\frac{1}{2}(\phi^a_{i,j} + \phi^a_{j,i}), \mathbf{C}_{ijkl}\frac{1}{2}(u^h_{k,l} + u^h_{l,k})\rangle - \sum^{M}_{a=1}\langle\phi^a_i, \tau_i\rangle_{\partial\Omega_\tau}\right) \tag{2.68}$$

$$g_u(u^h_i, z^h) = \sum^{N_{elem}}_{e=1}\left(\sum^{M}_{a=1}\langle\frac{1}{2}(\lambda^h_{i,j} + \lambda^h_{j,i})\mathbf{C}_{ijkl}, \frac{1}{2}(\phi^a_{k,l} + \phi^a_{l,k})\rangle\right) \tag{2.69}$$

$$g_z(u^h_i, z^h) = \sum^{N_{elem}}_{e=1}\left(\begin{bmatrix} \sum^N_{a=1}\langle\psi^a, \frac{1}{2}(\lambda^h_{i,j} + \lambda^h_{j,i})\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(u^h_{k,l} + u^h_{l,k})\rangle \\ \sum^O_{a=1}\langle\chi^a, \frac{1}{2}(\lambda^h_{i,j} + \lambda^h_{j,i})\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(u^h_{k,l} + u^h_{l,k})\rangle \end{bmatrix}\right) \tag{2.70}$$

$$(g_u(u^h_i, z^h))^* = \sum^{N_{elem}}_{e=1}\left(\sum^{M}_{a=1}\langle\frac{1}{2}(\phi^a_{i,j} + \phi^a_{j,i}), \mathbf{C}_{ijkl}\frac{1}{2}(\lambda^h_{k,l} + \lambda^h_{l,k})\rangle\right) \tag{2.71}$$

$$(g_z(u_i^h, z^h))^* = \sum_{e=1}^{N_{elem}} \left(\left[\begin{array}{c} \sum_{a=1}^{N} \langle \psi^a, \frac{1}{2}(u_{i,j}^h + u_{j,i}^h) \mathbf{C}_{ijkl}^{\mu} \, \frac{1}{2}(\lambda_{k,l}^h + \lambda_{l,k}^h) \rangle \\ \sum_{a=1}^{O} \langle \chi^a, \frac{1}{2}(u_{i,j}^h + u_{j,i}^h) \mathbf{C}_{ijkl}^{\kappa} \, \frac{1}{2}(\lambda_{k,l}^h + \lambda_{l,k}^h) \rangle \end{array} \right] \right) \tag{2.72}$$

$$(g_{uu}(u_i^h, z^h))^* = \mathbf{0} \tag{2.73}$$

$$(g_{uz}(u_i^h, z^h))^* = \sum_{e=1}^{N_{elem}} \left(\left[\begin{array}{c} \sum_{a=1}^{M} \sum_{b=1}^{N} \langle \frac{1}{2}(\phi_{i,j}^a + \phi_{j,i}^a) \mathbf{C}_{ijkl}^{\mu} \, \frac{1}{2}(\lambda_{k,l}^h + \lambda_{l,k}^h), \psi^b \rangle \\ \sum_{a=1}^{M} \sum_{b=1}^{O} \langle \frac{1}{2}(\phi_{i,j}^a + \phi_{j,i}^a) \mathbf{C}_{ijkl}^{\kappa} \, \frac{1}{2}(\lambda_{k,l}^h + \lambda_{l,k}^h), \chi^b \rangle \end{array} \right] \right) \tag{2.74}$$

$$(g_{zz}(u_i^h, z^h))^* = \mathbf{0} \tag{2.75}$$

$$(g_{zu}(u_i^h, z^h))^* = \sum_{e=1}^{N_{elem}} \left(\left[\begin{array}{c} \sum_{a=1}^{M} \sum_{b=1}^{N} \langle \frac{1}{2}(\lambda_{i,j}^h + \lambda_{j,i}^h) \mathbf{C}_{ijkl}^{\mu} \psi^b, \frac{1}{2}(\phi_{k,l}^a + \phi_{l,k}^a) \rangle \\ \sum_{a=1}^{M} \sum_{b=1}^{O} \langle \frac{1}{2}(\lambda_{i,j}^h + \lambda_{j,i}^h) \mathbf{C}_{ijkl}^{\kappa} \chi^b, \frac{1}{2}(\phi_{k,l}^a + \phi_{l,k}^a) \rangle \end{array} \right] \right) \tag{2.76}$$

Parameter Identification in Linear Elastodynamics

An example of a parameter estimation problem in linear elastodynamics is to identify the shear modulus μ and the bulk modulus κ from complex-valued displacement measurements $\widehat{u}_{i=1,\ldots,d}$. We define $\Omega \subseteq \mathbb{R}^d$, $d \in \{1, 2, 3\}$, to be the computational domain with boundary $\partial\Omega = \partial\Omega_u \cup \partial\Omega_\tau$, where $\partial\Omega_u$ and $\partial\Omega_\tau$ are the regions where Dirichlet and Neumann conditions are respectively applied. In the subsequent derivations we use standard tensor notation with Einstein summation for linear elastodynamics equations given in frequency domain. The indices i, j and k take on the values $1, \ldots, d$. Partial differentiation is denoted by a comma.

The PDEs of linear elastodynamics in frequency domain are given by

$$\begin{aligned} -(\mathbf{C}_{ijkl}\, \epsilon_{kl})_{,j} &= \omega^2 \rho\, u_i && \text{in} \quad \Omega, \\ u_i &= 0 && \text{on} \quad \partial\Omega_u, \\ (\mathbf{C}_{ijkl}\, \epsilon_{kl})\, n_j &= \tau_i && \text{on} \quad \partial\Omega_\tau, \end{aligned} \tag{2.77}$$

where $\{u_i\}_{i=1,\ldots,d}$ denotes the complex-valued displacement in the i-th direction and τ_i denotes the surface traction in the i-th direction. The unit outward normal component in the i-th direction on $\partial\Omega_\tau$ with respect to the region Ω is denoted by n_j. Mass density is denoted by ρ, while ω stands for the angular frequency. The fourth-order tensor of elastic moduli, $\mathbf{C}_{ijkl}$, is defined in Eq. 2.37 and the symmetric part of the displacement gradient, ϵ_{kl}, is defined in Eq. 2.38.

Lets consider the following NLP

$$\begin{aligned} &\underset{(u_i,\mu,\kappa)\, \in\, \mathcal{U}_i \times \mathcal{G} \times \mathcal{B}}{\text{minimize}} \qquad \frac{\beta}{2} \langle u_i - \widehat{u}_i, u_i - \widehat{u}_i \rangle + R(\mu) + R(\kappa) \\ &\text{subject to} \\ &\qquad -(\mathbf{C}_{ijkl}\, \frac{1}{2}(u_{k,l} + u_{l,k}))_{,j} = \omega^2 \rho\, u_i \quad \text{in} \quad \Omega, \\ &\qquad (\mathbf{C}_{ijkl}\, \frac{1}{2}(u_{k,l} + u_{l,k}))\, n_j = \tau_i \quad \text{on} \quad \partial\Omega_\tau \end{aligned} \tag{2.78}$$

where $\mathcal{U}_i = \{u_i \colon u_i \in H^1(\Omega),\ u_i = 0 \text{ on } \partial\Omega_u\}$, $\mathcal{G} = \{\mu \colon \mu \in L^2(\Omega),\ \mu > 0\}$, and $\mathcal{B} = \{\kappa \colon \kappa \in L^2(\Omega),\ \kappa > 0\}$. $H^m(\Omega)$ are Sobolev spaces of square integrable functionals whose m-th derivatives are also square integrable. Whenever $m = 0$ we shall keep the notation with $L^2(\Omega)$. The L^2 inner products for the space $\mathcal{U}_i$ is defined as

$$\langle u_i, v_i \rangle := \int_\Omega u_i(x)\overline{v}_i(x)dx, \tag{2.79}$$

where the overline denotes complex conjugation. The L^2 inner products for the spaces $\mathcal{G}$ and $\mathcal{B}$ is given by Eq. 2.41.

First-order neccesary optimality conditions. The Lagrangian associated with the NLP defined in Eq. 2.39 is given by

$$\begin{aligned} \mathcal{L}(u_i, \mu, \kappa, \lambda_i) = {} & \frac{\beta}{2}\langle u_i - \widehat{u}_i, u_i - \widehat{u}_i \rangle + R(\mu) + R(\kappa) \\ & + \Re\Big[\langle \lambda_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l} + u_{l,k}))_{,j} - \omega^2 \rho\, u_i \rangle \\ & - \langle \lambda_i, ((\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l} + u_{l,k})\, n_j - \tau_i)\rangle_{\partial\Omega_\tau}\Big] \end{aligned} \tag{2.80}$$

If $(\tilde{u}_i, \tilde{\mu}, \tilde{\kappa}) \in \mathcal{U}_i \times \mathcal{G} \times \mathcal{B}$ is a local solution to the NLP in Eq. 2.39, there exists Lagrange multipliers $\lambda_i \in \mathcal{U}_i$, such that the first-order necessary optimality conditions hold at $(\tilde{u}_i, \tilde{\mu}, \tilde{\kappa})$, i.e.

$$\begin{aligned} \nabla\mathcal{L}(u_i, \mu, \kappa, \lambda_i)(\delta u_i, \delta\mu, \delta\kappa, \delta\lambda_i) = {} & \mathcal{L}_u(u_i, \mu, \kappa, \lambda_i)\delta u_i + \mathcal{L}_\kappa(u_i, \mu, \kappa, \lambda_i)\delta\mu \\ & + \mathcal{L}_\kappa(u_i, \kappa, \lambda_i)\delta\kappa + \mathcal{L}_\lambda(u_i, \mu, \kappa, \lambda_i)\delta\lambda_i = 0 \end{aligned}$$

Second-Order Sufficient Conditions. If $(\tilde{u}_i, \tilde{\mu}, \tilde{\kappa}, \lambda_i) \in \mathcal{U}_i \times \mathcal{G} \times \mathcal{B} \times \mathcal{U}_i$ satisfy the first-order necessary optimality conditions and the Hessian operator exist and is positive semidefinite, then $(\tilde{u}_i, \tilde{\mu}, \tilde{\kappa})$ is a strict local minimum.

Operators. Let $z \equiv \{\mu, \kappa\} \in \mathcal{G} \times \mathcal{B}$, then the operators required to solve the paramater identification problem are defined as

$$J(u_i, z) = \frac{\beta}{2}\langle u_i - \widehat{u}_i, u_i - \widehat{u}_i \rangle + R(\mu) + R(\kappa) \tag{2.81}$$

$$\begin{aligned} \langle J_u(u_i, z), \delta u_i \rangle &= \frac{\partial}{\partial u_i}\Big(\beta\langle u_i - \widehat{u}_i, u_i - \widehat{u}_i \rangle + R(\mu) + R(\kappa)\Big) \\ &= \beta\langle u_i - \widehat{u}_i, \delta u_i \rangle \end{aligned} \tag{2.82}$$

$$\begin{aligned} \langle J_z(u_i, z), \delta z \rangle &= \frac{\partial}{\partial z}\Big(\frac{\beta}{2}\langle u_i - \widehat{u}_i, u_i - \widehat{u}_i \rangle + R(\mu) + R(\kappa)\Big) \\ &= \begin{bmatrix} \langle R_\mu(\mu), \delta\mu \rangle \\ \langle R_\kappa(\kappa), \delta\kappa \rangle \end{bmatrix} \end{aligned} \tag{2.83}$$

$$\begin{aligned} \langle J_{uu}(u_i, z), \delta u_i \rangle &= \frac{\partial}{\partial u_i}\Big(\beta\langle \mathbf{1}, u_i - \widehat{u}_i \rangle\Big) \\ &= \beta\langle \mathbf{1}, \delta u_i \rangle \end{aligned} \tag{2.84}$$

$$\langle J_{uz}(u_i,z),\delta z\rangle = \frac{\partial}{\partial z}\bigg(\beta\langle \mathbf{1}, u_i - \widehat{u}_i\rangle\bigg) = \mathbf{0} \tag{2.85}$$

$$\begin{aligned}\langle J_{zz}(u_i,z),\delta z\rangle &= \frac{\partial}{\partial z}\left(\begin{bmatrix}\langle \mathbf{1}, R_\mu(\mu)\rangle\\ \langle \mathbf{1}, R_\kappa(\kappa)\rangle\end{bmatrix}\right)\\ &= \begin{bmatrix}\langle R_{\mu\mu}(\mu),\delta\mu\rangle\\ \langle R_{\kappa\kappa}(\kappa),\delta\kappa\rangle\end{bmatrix}\end{aligned} \tag{2.86}$$

$$\langle J_{zu}(u_i,z),\delta u_i\rangle = \frac{\partial}{\partial u_i}\left(\begin{bmatrix}\langle \mathbf{1}, R_\mu(\mu)\rangle\\ \langle \mathbf{1}, R_\kappa(\kappa)\rangle\end{bmatrix}\right) = \mathbf{0} \tag{2.87}$$

$$g(u_i,z) = \begin{bmatrix} -(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))_{,j} = \omega^2\rho\,u_i \quad \text{in} \quad \Omega\\ (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j - \tau_i = 0 \quad \text{on} \quad \partial\Omega_\tau\end{bmatrix} \tag{2.88}$$

$$\begin{aligned}\langle g_u(u_i,z),\delta u_i\rangle &= \langle\lambda_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))_{,j} - \omega^2\rho\,u_i\rangle\\ &+ \langle\lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j - \tau_i\rangle_{\partial\Omega_\tau} = 0 \quad \forall \quad \lambda_i\in\mathcal{U}_i\\ &= \langle\frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})\,\mathbf{C}_{ijkl}, \frac{1}{2}(u_{k,l}+u_{l,k})\rangle\\ &- \langle\lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j\rangle_{\partial\Omega_\tau}\\ &- \langle\lambda_i,\omega^2\rho\,u_i\rangle + \langle\lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j - \tau_i\rangle_{\partial\Omega_\tau}\\ &= \langle\frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})\,\mathbf{C}_{ijkl}, \frac{1}{2}(u_{k,l}+u_{l,k})\rangle - \langle\lambda_i,\omega^2\rho\,u_i\rangle - \langle\lambda_i,\tau_i\rangle_{\partial\Omega_\tau}\\ &= \langle -(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}, u_i\rangle - \langle\lambda_i,\omega^2\rho\,u_i\rangle - \langle\lambda_i,\tau_i\rangle_{\partial\Omega_\tau}\\ &+ \langle(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j, u_i\rangle_{\partial\Omega_\tau}\\ &= \frac{\partial}{\partial u_i}\bigg(\langle -(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}, u_i\rangle - \langle\lambda_i,\omega^2\rho\,u_i\rangle - \langle\lambda_i,\tau_i\rangle_{\partial\Omega_\tau}\bigg)\\ &= \langle -(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j} - \omega^2\rho\,\lambda_i, \delta u_i\rangle\end{aligned} \tag{2.89}$$

The boundary in Eq. 2.89 is defined as $\partial\Omega = \partial\Omega_\tau \cup \partial\Omega_u$. The application of the divergence theorem yields a set of homogeneous Dirichlet boundary condition on $\partial\Omega_u$ since $\lambda_i \in \mathcal{U}_i$ and $u_i \in \mathcal{U}_i$. Therefore, $u_i = \lambda_i = 0$ on $\partial\Omega_u$ by definition of the space $\mathcal{U}_i$. Finally, in order to ensure the validity of the set of equations defined in Eq. 2.89, proper Neumann boundary condition need to be defined. Therefore, a set of free Neumann boundary conditions is defined on $\partial\Omega_\tau$, i.e. $(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j = 0$ on $\partial\Omega_\tau$. These definitions will be used in the subsequent derivations.

$$
\begin{aligned}
\langle g_z(u_i,z),\delta z\rangle &= \langle \lambda_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k})),_j - \omega^2\rho\, u_i\rangle \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j - \tau_i\rangle_{\partial\Omega_\tau} = 0 \quad \forall \quad \lambda_i \in \mathcal{U}_i \\
&= \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})\,\mathbf{C}_{ijkl}, \frac{1}{2}(u_{k,l}+u_{l,k})\rangle \\
&- \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j\rangle_{\partial\Omega_\tau} \\
&- \langle \lambda_i, \omega^2\rho\, u_i\rangle + \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j - \tau_i\rangle_{\partial\Omega_\tau} \\
&= \frac{\partial}{\partial z}\bigg(\langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})\,\mathbf{C}_{ijkl}, \frac{1}{2}(u_{k,l}+u_{l,k})\rangle - \langle \lambda_i, \omega^2\rho\, u_i\rangle - \langle \lambda_i, \tau_i\rangle_{\partial\Omega_\tau}\bigg) \\
&= \begin{bmatrix} \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})\,\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}), \delta\mu\rangle \\ \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})\,\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}), \delta\kappa\rangle \end{bmatrix}
\end{aligned}
\tag{2.90}
$$

$$
\begin{aligned}
\langle \delta z, (g_z(u_i,z))^*\rangle &= \langle \lambda_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k})),_j - \omega^2\rho\, u_i\rangle \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j - \tau_i\rangle_{\partial\Omega_\tau} = 0 \quad \forall \quad \lambda_i \in \mathcal{U}_i \\
&= \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})\,\mathbf{C}_{ijkl}, \frac{1}{2}(u_{k,l}+u_{l,k})\rangle \\
&- \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j\rangle_{\partial\Omega_\tau} \\
&- \langle \lambda_i, \omega^2\rho\, u_i\rangle + \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j - \tau_i\rangle_{\partial\Omega_\tau} \\
&= \frac{\partial}{\partial z}\bigg(\langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})\,\mathbf{C}_{ijkl}, \frac{1}{2}(u_{k,l}+u_{l,k})\rangle \\
&- \langle \lambda_i, \omega^2\rho\, u_i\rangle - \langle \lambda_i, \tau_i\rangle_{\partial\Omega_\tau}\bigg) \\
&= \begin{bmatrix} \langle \delta\mu, \frac{1}{2}(u_{i,j}+u_{j,i})\,\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k})\rangle^* \\ \langle \delta\kappa, \frac{1}{2}(u_{i,j}+u_{j,i})\,\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k})\rangle^* \end{bmatrix}
\end{aligned}
\tag{2.91}
$$

$$
\begin{aligned}
\langle \delta u_i, (g_u(u_i,z))^* \rangle &= \langle \lambda_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))_{,\,j} - \omega^2 \rho\, u_i \rangle \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j - \tau_i \rangle_{\partial\Omega_\tau} = 0 \quad \forall \quad \lambda_i \in \mathcal{U}_i \\
&= \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})\,\mathbf{C}_{ijkl}, \frac{1}{2}(u_{k,l}+u_{l,k}) \rangle \\
&- \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j \rangle_{\partial\Omega_\tau} \\
&- \langle \lambda_i, \omega^2 \rho\, u_i \rangle + \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j - \tau_i \rangle_{\partial\Omega_\tau} \\
&= \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})\,\mathbf{C}_{ijkl}, \frac{1}{2}(u_{k,l}+u_{l,k}) \rangle - \langle \lambda_i, \omega^2 \rho\, u_i \rangle - \langle \lambda_i, \tau_i \rangle_{\partial\Omega_\tau} \\
&= \langle -(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,\,j}, u_i \rangle - \langle \lambda_i, \omega^2 \rho\, u_i \rangle - \langle \lambda_i, \tau_i \rangle_{\partial\Omega_\tau} \\
&+ \langle (\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\, n_j, u_i \rangle_{\partial\Omega_\tau} \\
&= \frac{\partial}{\partial u_i}\bigg(\langle -(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,\,j}, u_i \rangle - \langle \lambda_i, \omega^2 \rho\, u_i \rangle - \langle \lambda_i, \tau_i \rangle_{\partial\Omega_\tau} \bigg) \\
&= \langle \delta u_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j} - \omega^2 \rho\, \lambda_i \rangle^*
\end{aligned}
\tag{2.92}
$$

The boundary in Eq. 2.92 is defined as $\partial\Omega = \partial\Omega_\tau \cup \partial\Omega_u$. The application of the divergence theorem yields a set of homogeneous Dirichlet boundary condition on $\partial\Omega_u$ since $\lambda_i \in \mathcal{U}_i$ and $u_i \in \mathcal{U}_i$. Therefore, $u_i = \lambda_i = 0$ on $\partial\Omega_u$ by definition of the space $\mathcal{U}_i$. Finally, in order to ensure the validity of the set of adjoint equations defined in Eq. 2.92, proper Neumann boundary condition are required. Therefore, a set of free Neumann boundary conditions is defined as $(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\, n_j = 0$ on $\partial\Omega_\tau$. These definitions ensure a self-adjoint operator in Eq. 2.92.

$$
\langle (g_{uu}(u_i,z))^*, \delta u_i \rangle = \frac{\partial}{\partial u}\bigg(\langle \mathbf{1}, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j} - \omega^2 \rho\, \lambda_i \rangle \bigg) = \mathbf{0} \tag{2.93}
$$

$$
\begin{aligned}
\langle (g_{uz}(u_i,z))^*, \delta z \rangle &= \bigg(\langle w_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j} - \omega^2 \rho\, \lambda_i \rangle \bigg) \quad \forall \quad w_i \in \mathcal{U}_i \\
&= \langle \frac{1}{2}(w_{i,j}+w_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}) \rangle - \langle w_i, \omega^2 \rho\, \lambda_i \rangle \\
&- \langle w_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\, n_j \rangle_{\partial\Omega_\tau} \\
&= \frac{\partial}{\partial z}\bigg(\langle \frac{1}{2}(w_{i,j}+w_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}) \rangle - \langle w_i, \omega^2 \rho\, \lambda_i \rangle \bigg) \\
&= \begin{bmatrix} \langle \frac{1}{2}(w_{i,j}+w_{j,i})\,\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}), \delta\mu \rangle \\ \langle \frac{1}{2}(w_{i,j}+w_{j,i})\,\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}), \delta\kappa \rangle \end{bmatrix}
\end{aligned}
\tag{2.94}
$$

$$
\langle (g_{zz}(u_i,z))^*, \delta z \rangle = \frac{\partial}{\partial z}\Bigg(\begin{bmatrix} \langle \mathbf{1}, \frac{1}{2}(u_{i,j}+u_{j,i})\,\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}) \rangle \\ \langle \mathbf{1}, \frac{1}{2}(u_{i,j}+u_{j,i})\,\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}) \rangle \end{bmatrix} \Bigg) = \mathbf{0} \tag{2.95}
$$

$$
\begin{aligned}
\langle (g_{zu}(u_i,z))^*, \delta u_i\rangle &= \left[\begin{array}{c} \langle \mathbf{1}, \frac{1}{2}(u_{i,j}+u_{j,i})\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k})\rangle \\ \langle \mathbf{1}, \frac{1}{2}(u_{i,j}+u_{j,i})\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k})\rangle \end{array}\right] \\
&= \left[\begin{array}{c} \langle \mathbf{1}, -u_i(\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}\rangle \\ +\langle \mathbf{1}, -u_i(\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j\rangle_{\partial\Omega_\tau} \\ \langle \mathbf{1}, -u_i(\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}\rangle \\ +\langle \mathbf{1}, u_i(\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j\rangle_{\partial\Omega_\tau} \end{array}\right] \\
&= \frac{\partial}{\partial u_i}\left(\left[\begin{array}{c} \langle -(\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}, u_i\rangle \\ \langle -(\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}, u_i\rangle \end{array}\right]\right) \\
&= \left[\begin{array}{c} \langle -(\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}, \delta u_i\rangle \\ \langle -(\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}, \delta u_i\rangle \end{array}\right]
\end{aligned} \tag{2.96}
$$

Discretization. We define finite-dimensional subspaces $\mathcal{U}_i^h \subset \mathcal{U}_i$ with basis $\mathcal{U}_i^h = \text{span}\{\phi_i^m\}_{m=1}^M$, $\mathcal{G}^h \subset \mathcal{G}$ with basis $\mathcal{G}^h = \text{span}\{\psi^n\}_{n=1}^N$, and $\mathcal{B}^h \subset \mathcal{B}$ with basis $\mathcal{B}^h = \text{span}\{\chi^o\}_{o=1}^O$. This leads to $u_i^h = \sum_{m=1}^M u_i^m\phi_i^m$, $\mu^h = \sum_{n=1}^N \mu^n\psi^n$, $\kappa^h = \sum_{o=1}^O \kappa^o\chi^o$,and $\lambda_i^h = \sum_{m=1}^M \lambda_i^m\phi_i^m$. Furthermore, the quantities δu_i^h, $\delta\mu^h$, $\delta\kappa^h$, and $\delta\lambda_i^h$ can be respectively approximated as follows: $\delta u_i^h = \sum_{m=1}^M \delta u_i^m\phi_i^m$, $\delta\mu^h = \sum_{n=1}^N \delta\mu^n\psi^n$, and $\delta\kappa^h = \sum_{o=1}^O \delta\kappa^o\chi^o$. Subsequently, the Galerkin approximation of the required operators to solve the parameter identification problem are defined as

$$
J(u_i^h, z^h) = \frac{\beta}{2}\langle u_i^h - \widehat{u}_i, u_i^h - \widehat{u}_i\rangle + R(\mu^h) + R(\kappa^h) \tag{2.97}
$$

$$
J_u(u_i^h, z^h) = \sum_{e=1}^{N_{elem}}\left(\sum_{a=1}^{M}\beta\langle\phi_i^a, u_i^h - \widehat{u}_i\rangle\right) \tag{2.98}
$$

$$
J_z(u_i^h, z^h) = \sum_{e=1}^{N_{elem}}\left(\left[\begin{array}{c} \sum_{a=1}^N \beta\langle\psi^a, R_\mu(\mu^h)\rangle \\ \sum_{a=1}^O \beta\langle\chi^a, R_\kappa(\kappa^h)\rangle \end{array}\right]\right) \tag{2.99}
$$

$$
J_{uu}(u_i^h, z^h) = \sum_{e=1}^{N_{elem}}\left(\sum_{a=1}^{M}\sum_{b=1}^{M}\beta\langle\phi^a, \phi^b\rangle\right) \tag{2.100}
$$

$$
J_{uz}(u_i^h, z^h) = \mathbf{0} \tag{2.101}
$$

$$
J_{zz}(u_i^h, z^h) = \sum_{e=1}^{N_{elem}}\left(\left[\begin{array}{c} \sum_{a=1}^N\sum_{b=1}^N \beta\langle\psi^a, R_{\mu\mu}(\mu^h)\psi^b\rangle \\ \sum_{a=1}^O\sum_{b=1}^O \beta\langle\chi^a, R_{\kappa\kappa}(\kappa^h)\chi^b\rangle \end{array}\right]\right) \tag{2.102}
$$

$$
J_{zu}(u_i^h, z^h) = \mathbf{0} \tag{2.103}
$$

$$
\begin{aligned}
g(u_i^h, z^h) = \sum_{e=1}^{N_{elem}}\Bigg(&\sum_{a=1}^{M}\langle\frac{1}{2}(\phi_{i,j}^a + \phi_{j,i}^a), \mathbf{C}_{ijkl}\frac{1}{2}(u_{k,l}^h + u_{l,k}^h)\rangle \\
&- \sum_{a=1}^{M}\langle\phi_i^a, \omega^2\rho\, u_i^h\rangle - \sum_{a=1}^{M}\langle\phi_i^a, \tau_i\rangle_{\partial\Omega_\tau}\Bigg)
\end{aligned} \tag{2.104}
$$

$$
g_u(u_i^h, z^h) = \sum_{e=1}^{N_{elem}}\left(\sum_{a=1}^{M}\langle\frac{1}{2}(\lambda_{i,j}^h + \lambda_{j,i}^h)\,\mathbf{C}_{ijkl}, \frac{1}{2}(\phi_{k,l}^a + \phi_{l,k}^a)\rangle - \sum_{a=1}^{M}\langle\omega^2\rho\,\lambda_i^h, \phi_i^a\rangle\right) \tag{2.105}
$$

$$g_z(u_i^h, z^h) = \sum_{e=1}^{N_{elem}} \left(\begin{bmatrix} \sum_{a=1}^{N} \langle \frac{1}{2}(\lambda_{i,j}^h + \lambda_{j,i}^h) \mathbf{C}_{ijkl}^{\mu} \, \frac{1}{2}(u_{k,l}^h + u_{l,k}^h, \psi_a) \rangle \\ \sum_{a=1}^{O} \langle \frac{1}{2}(\lambda_{i,j}^h + \lambda_{j,i}^h) \mathbf{C}_{ijkl}^{\kappa} \, \frac{1}{2}(u_{k,l}^h + u_{l,k}^h, \chi_a) \rangle \end{bmatrix} \right) \tag{2.106}$$

$$(g_u(u_i^h, z^h))^* = \sum_{e=1}^{N_{elem}} \left(\sum_{a=1}^{M} \langle \frac{1}{2}(\phi_{i,j}^a + \phi_{j,i}^a), \mathbf{C}_{ijkl} \, \frac{1}{2}(\lambda_{k,l}^h + \lambda_{l,k}^h) \rangle - \sum_{a=1}^{M} \langle \phi_i^a, \omega^2 \rho \, \lambda_i^h \rangle \right) \tag{2.107}$$

$$(g_z(u_i^h, z^h))^* = \sum_{e=1}^{N_{elem}} \left(\begin{bmatrix} \sum_{a=1}^{N} \langle \psi^a, \frac{1}{2}(u_{i,j}^h + u_{j,i}^h) \mathbf{C}_{ijkl}^{\mu} \, \frac{1}{2}(\lambda_{k,l}^h + \lambda_{l,k}^h) \rangle \\ \sum_{a=1}^{O} \langle \chi^a, \frac{1}{2}(u_{i,j}^h + u_{j,i}^h) \mathbf{C}_{ijkl}^{\kappa} \, \frac{1}{2}(\lambda_{k,l}^h + \lambda_{l,k}^h) \rangle \end{bmatrix} \right) \tag{2.108}$$

$$(g_{uu}(u_i^h, z^h))^* = \mathbf{0} \tag{2.109}$$

$$(g_{uz}(u_i^h, z^h))^* = \sum_{e=1}^{N_{elem}} \left(\begin{bmatrix} \sum_{a=1}^{M} \sum_{b=1}^{N} \langle \frac{1}{2}(\phi_{i,j}^a + \phi_{j,i}^a) \mathbf{C}_{ijkl}^{\mu} \, \frac{1}{2}(\lambda_{k,l}^h + \lambda_{l,k}^h), \psi^b \rangle \\ \sum_{a=1}^{M} \sum_{b=1}^{O} \langle \frac{1}{2}(\phi_{i,j}^a + \phi_{j,i}^a) \mathbf{C}_{ijkl}^{\kappa} \, \frac{1}{2}(\lambda_{k,l}^h + \lambda_{l,k}^h), \chi^b \rangle \end{bmatrix} \right) \tag{2.110}$$

$$(g_{zz}(u_i^h, z^h))^* = \mathbf{0} \tag{2.111}$$

$$(g_{zu}(u_i^h, z^h))^* = \sum_{e=1}^{N_{elem}} \left(\begin{bmatrix} \sum_{a=1}^{M} \sum_{b=1}^{N} \langle \frac{1}{2}(\phi_{i,j}^a + \phi_{j,i}^a), \mathbf{C}_{ijkl}^{\mu} \, \frac{1}{2}(\lambda_{k,l}^h + \lambda_{l,k}^h) \psi^b \rangle \\ \sum_{a=1}^{M} \sum_{b=1}^{O} \langle \frac{1}{2}(\phi_{i,j}^a + \phi_{j,i}^a), \mathbf{C}_{ijkl}^{\kappa} \frac{1}{2}(\lambda_{k,l}^h + \lambda_{l,k}^h) \, \chi^b \rangle \end{bmatrix} \right) \tag{2.112}$$

Parameter Identification in Nonlinear Elasticity

A parameter identification problem in nonlinear elastostatics is to estimate the shear modulus μ and bulk modulus κ from displacement measurements $\{\widehat{u}_i\}_{i=1,\ldots,d}$. We define $\Omega \subseteq \mathbb{R}^d$, $d \in \{1,2,3\}$, is the computational domain on the reference configuration with boundary $\partial\Omega = \partial\Omega_u \cup \partial\Omega_\tau$. The regions $\partial\Omega_u$ and $\partial\Omega_\tau$ are the boundaries where Dirichlet and Neumann conditions are respectively applied. Once more, we use standard tensor notation with Einstein summation. The indices i, j, k, and l take on the values $1,\ldots,d$. Partial differentiation is denoted by a comma.

The partial differential equations (PDEs) of nonlinear elastostatics are given by

$$\begin{aligned} -(F_{ik}S_{kj})_{,\,j} &= 0 \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega_u \\ (F_{ik}S_{kj})\, n_j &= \tau_i \quad \text{on} \quad \partial\Omega_\tau, \end{aligned} \tag{2.113}$$

where $\{u_i\}_{i=1,\ldots,d}$ denotes the displacement in the i-th direction and τ_i denotes the surface traction in the i-th direction. The unit outward normal component in the i-th direction on $\partial\Omega_\tau$ with respect to the region Ω is denoted by n_j.

The second Piola-Kirchhoff stress tensor for a Saint Venant-Kirchhoff material is defined as

$$S_{ij} = \mathbf{C}_{ijkl}\, E_{kl} \tag{2.114}$$

and the fourth-order tensor of elastic moduli is given by Eq. 2.37.

The Green strain tensor E_{ij} is defined as

$$E_{ij} = \frac{1}{2}\left(F_{ki}F_{kj} - \delta_{ij}\right), \tag{2.115}$$

where the deformation gradient F_{ij} is given by

$$F_{ij} = \frac{\partial u_i}{\partial x_j^0} + \delta_{ij}. \tag{2.116}$$

The Kronecker delta is denoted as δ_{ij} and a material point in the reference configuration as x^0. The Green strain tensor defined in Eq. 2.115 can be expressed in terms of the displacement field as

$$E_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right). \tag{2.117}$$

Lets consider the following NLP

$$\begin{aligned} &\underset{(u_i,\mu,\kappa)\,\in\,\mathcal{U}_i\times\mathcal{G}\times\mathcal{B}}{\text{minimize}} && \frac{\beta}{2}\langle u_i - \widehat{u}_i, u_i - \widehat{u}_i\rangle + R(\mu) + R(\kappa) \\ &\text{subject to} && \\ & && -(F_{ik}S_{kj})_{,j} = 0 \quad \text{in} \quad \Omega, \\ & && (F_{ik}S_{kj})\, n_j = \tau_i \quad \text{in} \quad \partial\Omega_\tau \end{aligned} \tag{2.118}$$

where $\mathcal{U} = \{u_i \colon u_i \in H^1(\Omega),\ u_i = 0 \text{ on } \partial\Omega_u\}$, $\mathcal{G} = \{\mu \colon \mu \in L^2(\Omega),\ \mu > 0\}$, and $\mathcal{B} = \{\kappa \colon \kappa \in L^2(\Omega),\ \kappa > 0\}$. $H^m(\Omega)$ are Sobolev spaces of square integrable functionals whose m-th derivatives are also square integrable. Whenever $m = 0$ we shall keep the notation with $L^2(\Omega)$. The L^2 inner product for the space $\mathcal{U}_i$ is given by Eq. 2.40 and the L^2 inner product for spaces $\mathcal{G}$ and $\mathcal{B}$ is given by 2.41.

First-order neccesary optimality conditions. The Lagrangian associated with the NLP defined in Eq. 2.118 is given by

$$\begin{aligned} \mathcal{L}(u_i, \mu, \kappa, \lambda_i) = &\frac{\beta}{2}\langle u_i - \widehat{u}_i, u_i - \widehat{u}_i\rangle + R(\mu) + R(\kappa) \\ &+ \langle \lambda_i, -(F_{ik}S_{kj})_{,j}\rangle + \langle \lambda_i, (F_{ik}S_{kj})\, n_j - \tau_i\rangle_{\partial\Omega_\tau} \end{aligned} \tag{2.119}$$

If $(\tilde{u}_i, \tilde{\mu}, \tilde{\kappa}) \in \mathcal{U}_i \times \mathcal{G} \times \mathcal{B}$ is a local solution to the NLP in Eq. 2.39, there exists Lagrange multipliers $\lambda_i \in \mathcal{U}_i$, such that the first-order necessary optimality conditions hold at $(\tilde{u}_i, \tilde{\mu}, \tilde{\kappa})$, i.e.

$$\begin{aligned} \nabla\mathcal{L}(u_i, \mu, \kappa, \lambda_i)(\delta u_i, \delta\mu, \delta\kappa, \delta\lambda_i) = &\mathcal{L}_u(u_i, \mu, \kappa, \lambda_i)\delta u_i + \mathcal{L}_\kappa(u_i, \mu, \kappa, \lambda_i)\delta\mu \\ &+ \mathcal{L}_\kappa(u_i, \kappa, \lambda_i)\delta\kappa + \mathcal{L}_\lambda(u_i, \mu, \kappa, \lambda_i)\delta\lambda_i = 0 \end{aligned}$$

Second-Order Sufficient Conditions. If $(\tilde{u}_i, \tilde{\mu}, \tilde{\kappa}, \lambda_i) \in \mathcal{U}_i \times \mathcal{G} \times \mathcal{B} \times \mathcal{U}_i$ satisfy the first-order necessary optimality conditions and the Hessian operator exist and is positive semidefinite, then $(\tilde{u}_i, \tilde{\mu}, \tilde{\kappa})$ is a strict local minimum.

Operators. Let $z \equiv \{\mu, \kappa\} \in \mathcal{G} \times \mathcal{B}$. Furthermore, the tensors $\lambda_{i,j}$ and $w_{i,j}$ are symmetric and are respectively defined as $\lambda_{i,j} = \frac{1}{2}(\lambda_{i,j} + \lambda_{j,i})$ and $w_{i,j} = \frac{1}{2}(w_{i,j} + w_{j,i})$. Thus, the operators required to solve the paramater identification problem are defined as

$$J(u_i, z) = \frac{\beta}{2}\langle u_i - \widehat{u}_i, u_i - \widehat{u}_i\rangle + R(\mu) + R(\kappa) \tag{2.120}$$

$$\begin{aligned}\langle J_u(u_i, z), \delta u_i\rangle &= \frac{\partial}{\partial u_i}\Big(\beta\langle u_i - \widehat{u}_i, u_i - \widehat{u}_i\rangle + R(\mu) + R(\kappa)\Big) \\ &= \beta\langle u_i - \widehat{u}_i, \delta u_i\rangle\end{aligned} \tag{2.121}$$

$$\begin{aligned}\langle J_z(u_i, z), \delta z\rangle &= \frac{\partial}{\partial z}\Big(\frac{\beta}{2}\langle u_i - \widehat{u}_i, u_i - \widehat{u}_i\rangle + R(\mu) + R(\kappa)\Big) \\ &= \begin{bmatrix} \langle R_\mu(\mu), \delta\mu\rangle \\ \langle R_\kappa(\kappa), \delta\kappa\rangle \end{bmatrix}\end{aligned} \tag{2.122}$$

$$\begin{aligned}\langle J_{uu}(u_i, z), \delta u_i\rangle &= \frac{\partial}{\partial u_i}\Big(\beta\langle \mathbf{1}, u_i - \widehat{u}_i\rangle\Big) \\ &= \beta\langle \mathbf{1}, \delta u_i\rangle\end{aligned} \tag{2.123}$$

$$\langle J_{uz}(u_i, z), \delta z\rangle = \frac{\partial}{\partial z}\Big(\beta\langle \mathbf{1}, u_i - \widehat{u}_i\rangle\Big) = \mathbf{0} \tag{2.124}$$

$$\begin{aligned}\langle J_{zz}(u_i, z), \delta z\rangle &= \frac{\partial}{\partial z}\left(\begin{bmatrix} \langle \mathbf{1}, R_\mu(\mu)\rangle \\ \langle \mathbf{1}, R_\kappa(\kappa)\rangle \end{bmatrix}\right) \\ &= \begin{bmatrix} \langle R_{\mu\mu}(\mu), \delta\mu\rangle \\ \langle R_{\kappa\kappa}(\kappa), \delta\kappa\rangle \end{bmatrix}\end{aligned} \tag{2.125}$$

$$\langle J_{zu}(u_i, z), \delta u_i\rangle = \frac{\partial}{\partial u_i}\left(\begin{bmatrix} \langle \mathbf{1}, R_\mu(\mu)\rangle \\ \langle \mathbf{1}, R_\kappa(\kappa)\rangle \end{bmatrix}\right) = \mathbf{0} \tag{2.126}$$

$$g(u_i, z) = \begin{bmatrix} -(F_{ik}S_{kj})_{,j} = 0 \quad \text{in} \quad \Omega \\ (F_{ik}S_{kj})\, n_j - \tau_i = 0 \quad \text{on} \quad \partial\Omega_\tau \end{bmatrix} \tag{2.127}$$

$$
\begin{aligned}
\langle g_u(u_i,z),\delta u_i\rangle &= \langle \lambda_i, -(F_{ik}S_{kj})_{,j}\rangle + \langle \lambda_i, (F_{ik}S_{kj})\,n_j - \tau_i\rangle_{\partial\Omega_\tau} = 0 \;\forall\; \lambda_i \in \mathcal{U}_i \\
&= \langle \lambda_{i,j}, F_{ik}S_{kj}\rangle - \langle \lambda_i, (F_{ik}S_{kj})\,n_j\rangle_{\partial\Omega_\tau} \\
&+ \langle \lambda_i, (F_{ik}S_{kj})\,n_j - \tau_i\rangle_{\partial\Omega_\tau} \\
&= \frac{\partial}{\partial u_i}\bigg(\langle \lambda_{i,j}, F_{ik}S_{kj}\rangle - \langle \lambda_i, \tau_i\rangle_{\partial\Omega_\tau}\bigg) \\
&= \langle \lambda_{i,j}, \delta u_{i,k}S_{kj}\rangle + \langle \lambda_{i,j}, F_{ip}\mathbf{C}_{pjkl}\,(\frac{1}{2}(\delta u_{q,k}F_{ql}))\\
&+ \langle \lambda_{i,j}, F_{ip}\mathbf{C}_{pjkl}\,(\frac{1}{2}(F_{qk}\delta u_{q,l}))\rangle \\
&= \langle -(\lambda_{i,k}S_{k,j})_{,j}, \delta u_i\rangle + \langle (\lambda_{i,k}S_{kj})\,n_j, \delta u_i\rangle_{\partial\Omega} \\
&+ \langle -(\lambda_{i,r}F_{rp}\mathbf{C}_{pjkl}\,(\frac{1}{2}F_{kl}))_{,j}, \delta u_i\rangle \\
&+ \langle (\lambda_{i,r}F_{rp}\mathbf{C}_{pjkl}\,(\frac{1}{2}F_{kl}))\,n_j, \delta u_i\rangle_{\partial\Omega} \\
&+ \langle -(\lambda_{i,r}F_{rp}\mathbf{C}_{pjkl}\,(\frac{1}{2}F_{lk}))_{,j}, \delta u_i\rangle \\
&+ \langle (\lambda_{i,r}F_{rp}\mathbf{C}_{pjkl}\,(\frac{1}{2}F_{lk})\,n_j, \delta u_i\rangle_{\partial\Omega} \\
&= \langle -(\lambda_{i,k}S_{kj})_{,j}, \delta u_i\rangle + \langle -(\lambda_{i,r}F_{rp}\mathbf{C}_{pjkl}\,(\frac{1}{2}F_{kl}))_{,j}, \delta u_i\rangle \\
&+ \langle -(\lambda_{i,r}F_{rp}\mathbf{C}_{pjkl}\,(\frac{1}{2}F_{lk}))_{,j}, \delta u_i\rangle
\end{aligned}
\tag{2.128}
$$

The boundary in Eq. 2.51 is defined as $\partial\Omega = \partial\Omega_\tau \cup \partial\Omega_u$. The application of the divergence theorem yields a set of homogeneous Dirichlet boundary condition on $\partial\Omega_u$ since $\lambda_i \in \mathcal{U}_i$ and $u_i \in \mathcal{U}_i$. Therefore, $u_i = \lambda_i = 0$ on $\partial\Omega_u$ by definition of the space $\mathcal{U}_i$. Finally, in order to ensure the validity of the set of equations defined in Eq. 2.51, proper Neumann boundary condition need to be defined. Therefore, a set of free Neumann boundary conditions is defined on $\partial\Omega_\tau$, i.e. $(\lambda_{i,k}S_{kj})n_j = 0$, $(\lambda_{i,r}F_{rp}\mathbf{C}_{pjkl}\,(\frac{1}{2}F_{kl}))n_j = 0$, and $(\lambda_{i,r}F_{rp}\mathbf{C}_{pjkl}\,(\frac{1}{2}F_{lk})\,n_j = 0$. These definitions will be used in the subsequent derivations.

$$
\begin{aligned}
\langle g_z(u_i,z),\delta z\rangle &= \langle \lambda_i, -(F_{ik}S_{kj})_{,j}\rangle + \langle \lambda_i, (F_{ik}S_{kj})\,n_j - \tau_i\rangle_{\partial\Omega_\tau} = 0 \;\forall\; \lambda_i \in \mathcal{U}_i \\
&= \langle \lambda_{i,j}, F_{ik}S_{kj}\rangle - \langle \lambda_i, (F_{ik}S_{kj})\,n_j\rangle_{\partial\Omega_\tau} \\
&+ \langle \lambda_i, (F_{ik}S_{kj})\,n_j - \tau_i\rangle_{\partial\Omega_\tau} \\
&= \frac{\partial}{\partial z}\bigg(\langle \lambda_{i,j}, F_{ik}S_{kj}\rangle - \langle \lambda_i, \tau_i\rangle_{\partial\Omega_\tau}\bigg) \\
&= \begin{bmatrix} \langle \lambda_{i,j}F_{ip}\mathbf{C}^{\mu}_{pjkl}E_{kl}, \delta\mu\rangle \\ \langle \lambda_{i,j}F_{ip}\mathbf{C}^{\kappa}_{pjkl}E_{kl}, \delta\kappa\rangle \end{bmatrix}
\end{aligned}
\tag{2.129}
$$

$$
\begin{aligned}
\langle \delta u_i, (g_u(u_i,z))^* \rangle &= \langle \lambda_i, -(F_{ik}S_{kj})_{,j} \rangle + \langle \lambda_i, (F_{ik}S_{kj})\, n_j - \tau_i \rangle_{\partial\Omega_\tau} = 0 \;\forall\; \lambda_i \in \mathcal{U}_i \\
&= \langle \lambda_{i,j}, F_{ik}S_{kj} \rangle - \langle \lambda_i, (F_{ik}S_{kj})\, n_j \rangle_{\partial\Omega_\tau} \\
&+ \langle \lambda_i, (F_{ik}S_{kj})\, n_j - \tau_i \rangle_{\partial\Omega_\tau} \\
&= \frac{\partial}{\partial u_i}\Big(\langle \lambda_{i,j}, F_{ik}S_{kj} \rangle - \langle \lambda_i, \tau_i \rangle_{\partial\Omega_\tau} \Big) \\
&= \langle \lambda_{i,j}, \delta u_{i,k} S_{kj} \rangle + \langle \lambda_{i,j}, F_{ip} \mathbf{C}_{pjkl}\, (\frac{1}{2}(\delta u_{q,k} F_{ql})) \\
&+ \langle \lambda_{i,j}, F_{ip} \mathbf{C}_{pjkl}\, (\frac{1}{2}(F_{qk} \delta u_{q,l})) \rangle \\
&= \langle \delta u_{i,j}, S_{ik} \lambda_{k,j} \rangle^* + \langle \delta u_{i,j}, (\frac{1}{2} F_{ip})\, \mathbf{C}_{pjkl}\, \lambda_{k,q} F_{ql} \rangle^* \\
&+ \langle \delta u_{i,j}, (\frac{1}{2} F_{ip}) \mathbf{C}_{pjkl}\, F_{qk} \lambda_{q,l} \rangle^* \\
&= \langle \delta u_i, -(S_{ik}\lambda_{k,j})_{,j} \rangle^* + \langle \delta u_i, (S_{ik}\lambda_{k,j})\, n_j \rangle^*_{\partial\Omega} \\
&+ \langle \delta u_i, -((\frac{1}{2} F_{ip})\, \mathbf{C}_{pjkl}\, \lambda_{k,q} F_{ql})_{,j} \rangle^* \\
&+ \langle \delta u_i, (\frac{1}{2} F_{ip})\, \mathbf{C}_{pjkl}\, \lambda_{k,q} F_{ql})\, n_j \rangle^*_{\partial\Omega} \\
&+ \langle \delta u_i, -(\frac{1}{2} F_{ip}) \mathbf{C}_{pjkl}\, F_{qk} \lambda_{q,l})_{,j} \rangle^* \\
&+ \langle \delta u_i, (\frac{1}{2} F_{ip}) \mathbf{C}_{pjkl}\, F_{qk} \lambda_{q,l})\, n_j \rangle^*_{\partial\Omega} \\
&= \langle \delta u_i, -(S_{ik}\lambda_{k,j})_{,j} \rangle^* + \langle \delta u_i, -((\frac{1}{2} F_{ip})\, \mathbf{C}_{pjkl}\, \lambda_{k,q} F_{ql})_{,j} \rangle^* \\
&+ \langle \delta u_i, -(\frac{1}{2} F_{ip}) \mathbf{C}_{pjkl}\, F_{qk} \lambda_{q,l})_{,j} \rangle^*
\end{aligned} \tag{2.130}
$$

The boundary in Eq. 2.130 is defined as $\partial\Omega = \partial\Omega_\tau \cup \partial\Omega_u$. The application of the divergence theorem yields a set of homogeneous Dirichlet boundary condition on $\partial\Omega_u$ since $\lambda_i \in \mathcal{U}_i$ and $u_i \in \mathcal{U}_i$. Therefore, $u_i = \lambda_i = 0$ on $\partial\Omega_u$ by definition of the space $\mathcal{U}_i$. Finally, in order to ensure the validity of the set of adjoint equations defined in Eq. 2.130, proper Neumann boundary condition are required. Therefore, a set of free Neumann boundary conditions is defined on $\partial\Omega_\tau$ and is given by $(S_{ik}\lambda_{k,j})n_j = 0$, $((\frac{1}{2}F_{ip})\mathbf{C}_{pjkl}\lambda_{k,q}F_{ql})n_j = 0$, and $((\frac{1}{2}F_{ip})\mathbf{C}_{pjkl}F_{kq}\lambda_{q,l})n_j = 0$ on $\partial\Omega_\tau$.

$$
\begin{aligned}
\langle \delta z, (g_z(u_i,z))^* \rangle &= \langle \lambda_i, -(F_{ik}S_{kj})_{,j} \rangle + \langle \lambda_i, (F_{ik}S_{kj})\, n_j - \tau_i \rangle_{\partial\Omega_\tau} = 0 \;\forall\; \lambda_i \in \mathcal{U}_i \\
&= \langle \lambda_{i,j}, F_{ik}S_{kj} \rangle - \langle \lambda_i, (F_{ik}S_{kj})\, n_j \rangle_{\partial\Omega_\tau} \\
&+ \langle \lambda_i, (F_{ik}S_{kj})\, n_j - \tau_i \rangle_{\partial\Omega_\tau} \\
&= \frac{\partial}{\partial z}\Big(\langle \lambda_{i,j}, F_{ik}S_{kj} \rangle - \langle \lambda_i, \tau_i \rangle_{\partial\Omega_\tau} \Big) \\
&= \begin{bmatrix} \langle \lambda_{i,j} F_{ip} \mathbf{C}^{\mu}_{pjkl} E_{kl}, \delta\mu \rangle \\ \langle \lambda_{i,j} F_{ip} \mathbf{C}^{\kappa}_{pjkl} E_{kl}, \delta\kappa \rangle \end{bmatrix} \\
&= \begin{bmatrix} \langle \delta\mu, E_{ij} \mathbf{C}^{\mu}_{ijkp} F_{lp} \lambda_{k,l} \rangle^* \\ \langle \delta\kappa, E_{ij} \mathbf{C}^{\kappa}_{ijkp} F_{lp} \lambda_{k,l} \rangle^* \end{bmatrix}
\end{aligned} \tag{2.131}
$$

$$
\begin{aligned}
\langle (g_{uu}(u_i,z))^*, \delta u_i \rangle &= \langle w_i, -(S_{ik}\lambda_{k,j})_{,j} \rangle + \langle w_i, -((\frac{1}{2}F_{ip})\,\mathbf{C}_{pjkl}\,\lambda_{k,q}F_{ql})_{,j} \rangle \\
&+ \langle w_i, -(\frac{1}{2}F_{ip})\mathbf{C}_{pjkl}\,F_{qk}\lambda_{q,l})_{,j} \rangle = 0 \quad \forall \quad w_i \in \mathcal{U}_i \\
&= \langle w_{i,j}, S_{ik}\lambda_{k,j} \rangle - \langle w_i, (S_{ik}\lambda_{k,j})\,n_j \rangle_{\partial\Omega} \\
&+ \langle w_{i,j}, (\frac{1}{2}F_{ip})\,\mathbf{C}_{pjkl}\,\lambda_{k,q}F_{ql} \rangle - \langle w_i, ((\frac{1}{2}F_{ip})\,\mathbf{C}_{pjkl}\,\lambda_{k,q}F_{ql})\,n_j \rangle_{\partial\Omega} \\
&+ \langle w_{i,j}, (\frac{1}{2}F_{ip})\,\mathbf{C}_{pjkl}\,F_{qk}\lambda_{q,l} \rangle - \langle w_i, ((\frac{1}{2}F_{ip})\,\mathbf{C}_{pjkl}\,F_{qk}\lambda_{q,l})\,n_j \rangle_{\partial\Omega} \\
&= \frac{\partial}{\partial u_i}\Bigg(\langle w_{i,j}, S_{ik}\lambda_{k,j} \rangle + \langle w_{i,j}, (\frac{1}{2}F_{ip})\,\mathbf{C}_{pjkl}\,\lambda_{k,q}F_{ql} \rangle \\
&+ \langle w_{i,j}, (\frac{1}{2}F_{ip})\,\mathbf{C}_{pjkl}\,F_{qk}\lambda_{q,l} \rangle \Bigg) \\
&= \langle w_{i,j}, \mathbf{C}_{ijkl}(\frac{1}{2}(\delta u_{q,k}F_{qp}))\lambda_{p,l} \rangle + \langle w_{i,j}, \mathbf{C}_{ijkl}(\frac{1}{2}(F_{q,k}\delta u_{q,p}))\lambda_{p,l} \rangle \\
&+ \langle w_{i,j}, (\frac{1}{2}\delta u_{i,p})\,\mathbf{C}_{pjkl}\,\lambda_{k,q}F_{ql} \rangle + \langle w_{i,j}, (\frac{1}{2}F_{ip})\,\mathbf{C}_{pjkl}\,\lambda_{k,q}\delta u_{q,l} \rangle \\
&+ \langle w_{i,j}, (\frac{1}{2}\delta u_{i,p})\,\mathbf{C}_{pjkl}\,F_{qk}\lambda_{q,l} \rangle + \langle w_{i,j}, (\frac{1}{2}F_{ip})\,\mathbf{C}_{pjkl}\,\delta u_{q,k}\lambda_{q,l} \rangle \\
&= \langle \delta u_{i,j}, \mathbf{C}_{ijkl}(\frac{1}{2}(w_{q,k}F_{qp}))\lambda_{p,l} \rangle + \langle \delta u_{i,j}, \mathbf{C}_{ijkl}(\frac{1}{2}(F_{q,k}w_{q,p}))\lambda_{p,l} \rangle \\
&+ \langle \delta u_{i,j}, (\frac{1}{2}w_{i,p})\,\mathbf{C}_{pjkl}\,\lambda_{k,q}F_{ql} \rangle + \langle \delta u_{i,j}, (\frac{1}{2}F_{ip})\,\mathbf{C}_{pjkl}\,\lambda_{k,q}w_{q,l} \rangle \\
&+ \langle \delta u_{i,j}, (\frac{1}{2}w_{i,p})\,\mathbf{C}_{pjkl}\,F_{qk}\lambda_{q,l} \rangle + \langle \delta u_{i,j}, (\frac{1}{2}F_{ip})\,\mathbf{C}_{pjkl}\,w_{q,k}\lambda_{q,l} \rangle \\
&= \langle -(\mathbf{C}_{ijkl}(\frac{1}{2}(w_{q,k}F_{qp}))\lambda_{p,l})_{,j}, \delta u_i \rangle \\
&+ \langle (\mathbf{C}_{ijkl}(\frac{1}{2}(w_{q,k}F_{qp}))\lambda_{p,l})\,n_j, \delta u_i \rangle_{\partial\Omega} \\
&+ \langle -(\mathbf{C}_{ijkl}(\frac{1}{2}(F_{q,k}w_{q,p}))\lambda_{p,l})_{,j}, \delta u_i \rangle \\
&+ \langle (\mathbf{C}_{ijkl}(\frac{1}{2}(F_{q,k}w_{q,p}))\lambda_{p,l})\,n_j, \delta u_i \rangle_{\partial\Omega} \\
&+ \langle -((\frac{1}{2}w_{i,p})\,\mathbf{C}_{pjkl}\,\lambda_{k,q}F_{ql})_{,j}, \delta u_i \rangle \\
&+ \langle ((\frac{1}{2}w_{i,p})\,\mathbf{C}_{pjkl}\,\lambda_{k,q}F_{ql})\,n_j, \delta u_i \rangle_{\partial\Omega} \\
&+ \langle -((\frac{1}{2}F_{ip})\,\mathbf{C}_{pjkl}\,\lambda_{k,q}w_{q,l})_{,j}, \delta u_i \rangle + \langle ((\frac{1}{2}F_{ip})\,\mathbf{C}_{pjkl}\,\lambda_{k,q}w_{q,l})\,n_j, \delta u_i \rangle_{\partial\Omega} \\
&+ \langle -((\frac{1}{2}w_{i,p})\,\mathbf{C}_{pjkl}\,F_{qk}\lambda_{q,l})_{,j}, \delta u_i \rangle + \langle ((\frac{1}{2}w_{i,p})\,\mathbf{C}_{pjkl}\,F_{qk}\lambda_{q,l})\,n_j, \delta u_i \rangle_{\partial\Omega} \\
&+ \langle -((\frac{1}{2}F_{ip})\,\mathbf{C}_{pjkl}\,w_{q,k}\lambda_{q,l})_{,j}, \delta u_i \rangle + \langle ((\frac{1}{2}F_{ip})\,\mathbf{C}_{pjkl}\,w_{q,k}\lambda_{q,l})\,n_j, \delta u_i \rangle_{\partial\Omega}
\end{aligned}
\tag{2.132}
$$

$$
\begin{aligned}
\langle (g_{uz}(u_i,z))^*, \delta z\rangle &= \langle w_i, -(S_{ik}\lambda_{k,j})_{,j}\rangle + \langle w_i, -((\frac{1}{2}F_{ip})\,\mathbf{C}_{pjkl}\,\lambda_{k,q}F_{ql})_{,j}\rangle \\
&\quad + \langle w_i, -(\frac{1}{2}F_{ip})\mathbf{C}_{pjkl}\,F_{qk}\lambda_{q,l})_{,j}\rangle = 0 \quad \forall \quad w_i \in \mathcal{U}_i \\
&= \langle w_{i,j}, S_{ik}\lambda_{k,j}\rangle - \langle w_i, (S_{ik}\lambda_{k,j})\, n_j\rangle_{\partial\Omega} \\
&\quad + \langle w_{i,j}, (\frac{1}{2}F_{ip})\,\mathbf{C}_{pjkl}\,\lambda_{k,q}F_{ql}\rangle - \langle w_i, ((\frac{1}{2}F_{ip})\,\mathbf{C}_{pjkl}\,\lambda_{k,q}F_{ql})\, n_j\rangle_{\partial\Omega} \\
&\quad + \langle w_{i,j}, (\frac{1}{2}F_{ip})\,\mathbf{C}_{pjkl}\,F_{qk}\lambda_{q,l}\rangle - \langle w_i, ((\frac{1}{2}F_{ip})\,\mathbf{C}_{pjkl}\,F_{qk}\lambda_{q,l})\, n_j\rangle_{\partial\Omega} \\
&= \frac{\partial}{\partial z}\Big(\langle w_{i,j}, S_{ik}\lambda_{k,j}\rangle + \langle w_{i,j}, (\frac{1}{2}F_{ip})\,\mathbf{C}_{pjkl}\,\lambda_{k,q}F_{ql}\rangle \Big) \\
&\quad + \frac{\partial}{\partial z}\Big(\langle w_{i,j}, (\frac{1}{2}F_{ip})\,\mathbf{C}_{pjkl}\,F_{qk}\lambda_{q,l}\rangle \Big) \\
&= \begin{bmatrix}
\langle w_{i,j}\mathbf{C}^{\mu}_{ijkl}S_{ik}\lambda_{k,j}, \delta\mu\rangle + \langle w_{i,j}(\frac{1}{2}F_{ip})\,\mathbf{C}^{\mu}_{pjkl}\,\lambda_{k,q}F_{ql}, \delta\mu\rangle \\
\langle w_{i,j}(\frac{1}{2}F_{ip})\,\mathbf{C}^{\mu}_{pjkl}\,F_{qk}\lambda_{q,l}, \delta\mu\rangle \\
\langle w_{i,j}\mathbf{C}^{\kappa}_{ijkl}S_{ik}\lambda_{k,j}, \delta\kappa\rangle + \langle w_{i,j}(\frac{1}{2}F_{ip})\,\mathbf{C}^{\kappa}_{pjkl}\,\lambda_{k,q}F_{ql}, \delta\kappa\rangle \\
\langle w_{i,j}(\frac{1}{2}F_{ip})\,\mathbf{C}^{\kappa}_{pjkl}\,F_{qk}\lambda_{q,l}, \delta\kappa\rangle
\end{bmatrix}
\end{aligned} \tag{2.133}
$$

$$
\begin{aligned}
\langle (g_{zu}(u_i,z))^*, \delta u_i\rangle &= \frac{\partial}{\partial u_i}\left(\begin{bmatrix} \langle \mathbf{1}, \lambda_{i,j}F_{ip}\mathbf{C}^{\mu}_{pjkl}E_{kl}\rangle \\ \langle \mathbf{1}, \lambda_{i,j}F_{ip}\mathbf{C}^{\kappa}_{pjkl}E_{kl}\rangle \end{bmatrix} \right) \\
&= \begin{bmatrix}
\langle \mathbf{1}, \lambda_{i,j}\delta u_{i,p}\mathbf{C}^{\mu}_{pjkl}E_{kl}\rangle + \langle v^{\mu}, \lambda_{i,j}F_{ip}\mathbf{C}^{\mu}_{pjkl}(\frac{1}{2}\delta u_{q,k}F_{ql})\rangle \\
\langle \mathbf{1}, \lambda_{i,j}F_{ip}\mathbf{C}^{\mu}_{pjkl}(\frac{1}{2}F_{qk}\delta u_{q,l})\rangle \\
\langle \mathbf{1}, \lambda_{i,j}\delta u_{i,p}\mathbf{C}^{\kappa}_{pjkl}E_{kl}\rangle + \langle v^{\kappa}, \lambda_{i,j}F_{ip}\mathbf{C}^{\kappa}_{pjkl}(\frac{1}{2}\delta u_{q,k}F_{ql})\rangle \\
\langle \mathbf{1}, \lambda_{i,j}F_{ip}\mathbf{C}^{\kappa}_{pjkl}(\frac{1}{2}F_{qk}\delta u_{q,l})\rangle
\end{bmatrix} \\
&= \begin{bmatrix}
\langle -(\lambda_{i,p}\mathbf{C}^{\mu}_{pjkl}E_{kl})_{,j}, \delta u_i\rangle + \langle (\lambda_{i,p}\mathbf{C}^{\mu}_{pjkl}E_{kl})\, n_j, \delta u_i\rangle_{\partial\Omega} \\
+\langle -(\lambda_{i,r}F_{rp}\mathbf{C}^{\mu}_{pjkl}(\frac{1}{2}F_{kl}))_{,j}, \delta u_i\rangle \\
+\langle (\lambda_{i,r}F_{rp}\mathbf{C}^{\mu}_{pjkl}(\frac{1}{2}F_{kl}))\, n_j, \delta u_i\rangle_{\partial\Omega} \\
+\langle -(\lambda_{i,r}F_{rp}\mathbf{C}^{\mu}_{pjkl}(\frac{1}{2}F_{lk}))_{,l}, \delta u_i\rangle \\
+\langle (\lambda_{i,r}F_{rp}\mathbf{C}^{\mu}_{pjkl}(\frac{1}{2}F_{lk}))\, n_j, \delta u_i\rangle_{\partial\Omega} \\
\langle -(\lambda_{i,p}\mathbf{C}^{\kappa}_{pjkl}E_{kl})_{,j}, \delta u_i\rangle + \langle (\lambda_{i,p}\mathbf{C}^{\kappa}_{pjkl}E_{kl})\, n_j, \delta u_i\rangle_{\partial\Omega} \\
+\langle -(\lambda_{i,r}F_{rp}\mathbf{C}^{\kappa}_{pjkl}(\frac{1}{2}F_{kl}))_{,j}, \delta u_i\rangle \\
+\langle (\lambda_{i,r}F_{rp}\mathbf{C}^{\kappa}_{pjkl}(\frac{1}{2}F_{kl}))\, n_j, \delta u_i\rangle_{\partial\Omega} \\
+\langle -(\lambda_{i,r}F_{rp}\mathbf{C}^{\kappa}_{pjkl}(\frac{1}{2}F_{lk}))_{,j}, \delta u_i\rangle \\
+\langle (\lambda_{i,r}F_{rp}\mathbf{C}^{\kappa}_{pjkl}(\frac{1}{2}F_{lk}))\, n_j, \delta u_i\rangle_{\partial\Omega}
\end{bmatrix}
\end{aligned} \tag{2.134}
$$

$$
\langle (g_{zz}(u_i,z))^*, \delta z\rangle = \frac{\partial}{\partial z}\left(\begin{bmatrix} \langle \mathbf{1}, \lambda_{i,j}F_{ip}\mathbf{C}^{\mu}_{pjkl}E_{kl}\rangle \\ \langle \mathbf{1}, \lambda_{i,j}F_{ip}\mathbf{C}^{\kappa}_{pjkl}E_{kl}\rangle \end{bmatrix} \right) = \mathbf{0} \tag{2.135}
$$

Discretization. We define finite-dimensional subspaces $\mathcal{U}_i^h \subset \mathcal{U}_i$ with basis $\mathcal{U}_i^h = \text{span}\{\phi_i^m\}_{m=1}^M$, $\mathcal{G}^h \subset \mathcal{G}$ with basis $\mathcal{G}^h = \text{span}\{\psi^n\}_{n=1}^N$, and $\mathcal{B}^h \subset \mathcal{B}$ with basis $\mathcal{B}^h = \text{span}\{\chi^o\}_{o=1}^O$. This leads to $u_i^h = \sum_{m=1}^M u_i^m\phi_i^m$, $\mu^h = \sum_{n=1}^N \mu^n\psi^n$, $\kappa^h = \sum_{o=1}^O \kappa^o\chi^o$,and $\lambda_i^h = \sum_{m=1}^M \lambda_i^m\phi_i^m$. Furthermore, the quantities δu_i^h, $\delta\mu^h$, $\delta\kappa^h$, and $\delta\lambda_i^h$ can be respectively approximated as follows: $\delta u_i^h = \sum_{m=1}^M \delta u_i^m\phi_i^m$, $\delta\mu^h = \sum_{n=1}^N \delta\mu^n\psi^n$, and $\delta\kappa^h = \sum_{o=1}^O \delta\kappa^o\chi^o$. Subsequently, the Galerkin approximation of the required operators to solve the parameter identification problem are defined as

$$J(u_i^h, z^h) = \frac{\beta}{2}\langle u_i^h - \widehat{u}_i, u_i^h - \widehat{u}_i\rangle + R(\mu^h) + R(\kappa^h) \tag{2.136}$$

$$J_u(u_i^h, z^h) = \sum_{e=1}^{N_{elem}} \left(\sum_{a=1}^{M} \beta\langle \phi_i^a, u_i^h - \widehat{u}_i\rangle \right) \tag{2.137}$$

$$J_z(u_i^h, z^h) = \sum_{e=1}^{N_{elem}} \left(\begin{bmatrix} \sum_{a=1}^{N} \beta\langle \psi^a, R_\mu(\mu^h)\rangle \\ \sum_{a=1}^{O} \beta\langle \chi^a, R_\kappa(\kappa^h)\rangle \end{bmatrix} \right) \tag{2.138}$$

$$J_{uu}(u_i^h, z^h) = \sum_{e=1}^{N_{elem}} \left(\sum_{a=1}^{M}\sum_{b=1}^{M} \beta\langle \phi^a, \phi^b\rangle \right) \tag{2.139}$$

$$J_{uz}(u_i^h, z^h) = \mathbf{0} \tag{2.140}$$

$$J_{zz}(u_i^h, z^h) = \sum_{e=1}^{N_{elem}} \left(\begin{bmatrix} \sum_{a=1}^{N}\sum_{b=1}^{N} \beta\langle \psi^a, R_{\mu\mu}(\mu^h)\psi^b\rangle \\ \sum_{a=1}^{O}\sum_{b=1}^{O} \beta\langle \chi^a, R_{\kappa\kappa}(\kappa^h)\chi^b\rangle \end{bmatrix} \right) \tag{2.141}$$

$$J_{zu}(u_i^h, z^h) = \mathbf{0} \tag{2.142}$$

$$\begin{aligned} g(u_i^h, z^h) = \sum_{e=1}^{N_{elem}} \Bigg(& \sum_{a=1}^{M} \langle \phi_{i,j}^a, F_{ip}(u^h)\, \mathbf{C}_{pjkl}\, (\frac{1}{2}\, (F_{qk}(u^h)\, \delta u_{q,l}^h + \delta u_{q,k}^h\, F_{ql}(u^h)))\rangle \\ & + \sum_{a=1}^{M} \langle \phi_{i,j}^a, S_{ik}\, \delta u_{k,j}^h\rangle + \sum_{a=1}^{M} \langle \phi_i^a, \tau_i\rangle_{\partial\Omega_\tau} \Bigg) \end{aligned} \tag{2.143}$$

Notice that Eq. 2.143 is the linearization of the nonlinear elastostatics system of equations. This system of equations is solved using an iterative solver, e.g. Generalized Minimum Residual (GMRES).

$$\begin{aligned} g_u(u_i^h, z^h) = \sum_{e=1}^{N_{elem}} \Bigg(& \sum_{a=1}^{M} \langle \lambda_{i,k}^h S_{kj}, \phi_{i,j}^a\rangle + \sum_{a=1}^{M} \langle \lambda_{i,r}^h F_{rp}(u^h)\mathbf{C}_{pjkl}(\frac{1}{2}\, F_{kl}(u^h)), \phi_{i,j}^a\rangle \\ & + \sum_{a=1}^{M} \langle \lambda_{i,r}^h F_{rp}(u^h)\mathbf{C}_{pjkl}(\frac{1}{2}\, F_{lk}(u^h)), \phi_{i,j}^a\rangle \Bigg) \end{aligned} \tag{2.144}$$

$$g_z(u_i^h, z^h) = \sum_{e=1}^{N_{elem}} \left(\begin{bmatrix} \sum_{a=1}^{N} \langle \lambda_{i,j}^h F_{ip}(u^h)\mathbf{C}_{pjkl}^{\mu} E_{kl}(u^h), \psi^a\rangle \\ \sum_{a=1}^{O} \langle \lambda_{i,j}^h F_{ip}(u^h)\mathbf{C}_{pjkl}^{\kappa} E_{kl}(u^h), \chi^a\rangle \end{bmatrix} \right) \tag{2.145}$$

$$\begin{aligned} (g_z(u_i^h, z^h))^* = \sum_{e=1}^{N_{elem}} \Bigg(& \sum_{a=1}^{M} \langle \phi_{i,j}^a, S_{ik}\lambda_{k,j}^h\rangle + \sum_{a=1}^{M} \langle \phi_{i,j}^a, (\frac{1}{2}F_{ip}(u^h))\mathbf{C}_{pjkl}\lambda_{k,q}^h F_{ql}(u^h)\rangle \\ & + \sum_{a=1}^{M} \langle \phi_{i,j}^a, (\frac{1}{2}F_{ip}(u^h))\mathbf{C}_{pjkl} F_{qk}(u^h)\lambda_{q,l}^h\rangle \Bigg) \end{aligned} \tag{2.146}$$

$$(g_u(u_i^h, z^h))^* = \sum_{e=1}^{N_{elem}} \left(\begin{bmatrix} \sum_{a=1}^{N} \langle \psi^a, E_{ij}(u^h) \mathbf{C}^{\mu}_{ijkl} F_{lp}(u^h) \lambda^h_{k,p} \rangle \\ \sum_{a=1}^{O} \langle \chi^a, E_{ij}(u^h) \mathbf{C}^{\kappa}_{ijkl} F_{lp}(u^h) \lambda^h_{k,p} \rangle \end{bmatrix} \right) \tag{2.147}$$

$$\begin{aligned} (g_{uu}(u_i^h, z^h))^* = \sum_{e=1}^{N_{elem}} \Bigg(& \sum_{a=1}^{M} \sum_{b=1}^{M} \langle \phi^a_{i,j}, \mathbf{C}_{ijkl}(\frac{1}{2}\phi^b_{q,k} F_{qp}(u^h)) \lambda^h_{p,l} \rangle \\ & + \sum_{a=1}^{M} \sum_{b=1}^{M} \langle \phi^a_{i,j}, \mathbf{C}_{ijkl}(\frac{1}{2} F_{q,k}(u^h) \phi^b_{q,p}) \lambda^h_{p,l} \rangle \\ & + \sum_{a=1}^{M} \sum_{b=1}^{M} \langle \phi^a_{i,j}, (\frac{1}{2}\phi^b_{i,p} \mathbf{C}_{pjkl} \lambda^h_{k,q} F_{ql}(u^h) \rangle \\ & + \sum_{a=1}^{M} \sum_{b=1}^{M} \langle \phi^a_{i,j}, (\frac{1}{2} F_{ip}(u^h) \mathbf{C}_{pjkl} \lambda^h_{k,q} \phi^b_{q,l} \rangle \\ & + \sum_{a=1}^{M} \sum_{b=1}^{M} \langle \phi^a_{i,j}, (\frac{1}{2}\phi^b_{i,p} \mathbf{C}_{pjkl} F_{qk}(u^h) \lambda^h_{q,l}) \rangle \\ & + \sum_{a=1}^{M} \sum_{b=1}^{M} \langle \phi^a_{i,j}, (\frac{1}{2} F_{ip}(u^h) \mathbf{C}_{pjkl} \phi^b_{q,k} \lambda^h_{q,l} \rangle \end{aligned} \tag{2.148}$$

$$(g_{uz}(u_i^h, z^h))^* = \sum_{e=1}^{N_{elem}} \left(\begin{bmatrix} \sum_{a=1}^{M} \sum_{b=1}^{N} \langle \phi^a_{i,j} \mathbf{C}^{\mu}_{ijkl} S_{ik} \lambda^h_{k,j}, \psi^b \rangle \\ + \sum_{a=1}^{M} \sum_{b=1}^{N} \langle \phi^a_{i,j} (\frac{1}{2} F_{ip}(u^h) \mathbf{C}^{\mu}_{pjkl} \lambda^h_{k,q} F_{ql}(u^h), \psi^b \rangle \\ + \sum_{a=1}^{M} \sum_{b=1}^{N} \langle \phi^a_{i,j} (\frac{1}{2} F_{ip}(u^h) \mathbf{C}^{\mu}_{pjkl} F_{qk}(u^h) \lambda^h_{q,l}, \psi^b \rangle \\ \sum_{a=1}^{M} \sum_{b=1}^{N} \langle \phi^a_{i,j} \mathbf{C}^{\kappa}_{ijkl} S_{ik} \lambda^h_{k,j}, \chi^b \rangle \\ + \sum_{a=1}^{M} \sum_{b=1}^{N} \langle \phi^a_{i,j} (\frac{1}{2} F_{ip}(u^h) \mathbf{C}^{\kappa}_{pjkl} \lambda^h_{k,q} F_{ql}(u^h), \chi^b \rangle \\ + \sum_{a=1}^{M} \sum_{b=1}^{O} \langle \phi^a_{i,j} (\frac{1}{2} F_{ip}(u^h) \mathbf{C}^{\kappa}_{pjkl} F_{qk}(u^h) \lambda^h_{q,l}, \chi^b \rangle \end{bmatrix} \right) \tag{2.149}$$

$$(g_{uz}(u_i^h, z^h))^* = \mathbf{0} \tag{2.150}$$

$$(g_{zu}(u_i^h, z^h))^* = \sum_{e=1}^{N_{elem}} \left(\begin{bmatrix} \sum_{a=1}^{N} \sum_{b=1}^{M} \langle \psi^a \lambda^h_{i,p} \mathbf{C}^{\mu}_{ijkl} E_{kl}(u^h), \phi^b_{i,j} \rangle \\ \sum_{a=1}^{N} \sum_{b=1}^{M} \langle \psi^a \lambda^h_{i,r} F_{rp}(u^h) \mathbf{C}^{\mu}_{pjkl} (\frac{1}{2} F_{kl}(u^h), \phi^b_{i,j} \rangle \\ \sum_{a=1}^{N} \sum_{b=1}^{M} \langle \psi^a \lambda^h_{i,r} F_{rp}(u^h) \mathbf{C}^{\mu}_{pjkl} (\frac{1}{2} F_{lk}(u^h), \phi^b_{i,j} \rangle \\ \sum_{a=1}^{N} \sum_{b=1}^{M} \langle \psi^a \lambda^h_{i,p} \mathbf{C}^{\kappa}_{ijkl} E_{kl}(u^h), \phi^b_{i,j} \rangle \\ \sum_{a=1}^{N} \sum_{b=1}^{M} \langle \psi^a \lambda^h_{i,r} F_{rp}(u^h) \mathbf{C}^{\kappa}_{pjkl} (\frac{1}{2} F_{kl}(u^h), \phi^b_{i,j} \rangle \\ \sum_{a=1}^{N} \sum_{b=1}^{M} \langle \psi^a \lambda^h_{i,r} F_{rp}(u^h) \mathbf{C}^{\kappa}_{pjkl} (\frac{1}{2} F_{lk}(u^h), \phi^b_{i,j} \rangle \end{bmatrix} \right) \tag{2.151}$$

Parameter Identification in Structural-Acoustics

Let $D \subseteq \mathbb{R}^d$, $d \in \{1, 2, 3\}$ be a bounded domain of interest with an interior region D_s denoting an elastic body and an exterior region Ω_a denoting an incompressible acoustic body. Lets designate $\partial\Omega_a$ as the outer boundary of region $\partial\Omega_a$ and $\partial\Omega_s$ as the boundary of region D_s, which is also an interior boundary for Ω_a. The structural-acosutics interactions take place on the $\partial\Omega_c \subset \partial\Omega_s$ boundary. Let u_i denote the complex-valued displacement in the

i-th direction and p denote the complex-valued pressure field. The elastic body and acoustic fluid mass densities are respectively denoted by ρ_s and ρ_a. The elastic body shear modulus is μ and the bulk modulus is κ. The acoustic wave number is given by $\mathrm{k} = \omega\sqrt{\varsigma/\rho_a}$, where ς is the acoustic fluid bulk modulus. Angular frequency is denoted by ω. In the subsequent derivations we use standard tensor notation with Einstein summation for the fluid-structure interaction (FSI) equations. The indices i, j, k and l take on the values $1, \dots, d$. Partial differentiation is denoted by a comma.

The PDEs of structural-acoustics are given by

$$\begin{aligned}
-(\mathbf{C}_{ijkl}\epsilon_{kl})_{,j} - \rho_s\,\omega^2\,u_i &= f_i && \text{in} && \Omega_s \\
-p_{,i} - \mathrm{k}^2 p &= 0 && \text{in} && \Omega_a \\
u_i &= 0 && \text{on} && \partial\Omega_u \\
(\mathbf{C}_{ijkl}\epsilon_{kl})\,n_j &= \tau_i && \text{on} && \partial\Omega_\tau \\
(\mathbf{C}_{ijkl}\epsilon_{kl})\,n_j &= -p\,n_i && \text{on} && \partial\Omega_c \\
p_i n_i &= f(\beta - j\gamma\mathrm{k})\,p && \text{on} && \partial\Omega_a \\
p_i n_i &= \rho_a\omega^2 u_i\,n_i && \text{on} && \partial\Omega_c
\end{aligned} \tag{2.152}$$

where the structural boubdary is defined as $\partial\Omega_s = \partial\Omega_\tau \cup \partial\Omega_u \cup \partial\Omega_c$. The Dirichlet and Neumann conditions are repectively applied on $\partial\Omega_u$ and $\partial\Omega_\tau$. The surface traction in the i-th direction is denoted by τ_i, while the unit outward normal component in the i-th direction on $\partial\Omega_s$ with respect to the region Ω_s is denoted by n_i. The fourth-order tensor of elastic moduli, $\mathbf{C}_{ijkl}$, is defined in Eq. 2.37 and the symmetric part of the displacement gradient, ϵ_{kl}, is defined in Eq. 2.38.

Lets consider the following NLP:

$$\begin{aligned}
&\underset{(u_i,p,\mu,\kappa)\in\, \mathcal{U}_i\times\mathcal{P}\times\mathcal{G}\times\mathcal{B}}{\text{minimize}} && \frac{\alpha}{2}\langle u_i - \widehat{u}_i, u_i - \widehat{u}_i\rangle + R(\mu) + R(\kappa) \\
&\text{subject to} \\
&-(\mathbf{C}_{ijkl}\epsilon_{kl})_{,j} - \rho_s\,\omega^2\,u_i = f_i && \text{in} \quad \Omega_s \\
&-p_{,i} - \mathrm{k}^2 p = 0 && \text{in} \quad \Omega_f \\
&u_i = 0 && \text{on} \quad \partial\Omega_u \\
&(\mathbf{C}_{ijkl}\epsilon_{kl})\,n_j = \tau_i && \text{on} \quad \partial\Omega_\tau \\
&(\mathbf{C}_{ijkl}\epsilon_{kl})\,n_j = -p\,n_i && \text{on} \quad \partial\Omega_c \\
&p_i n_i = f(\beta - j\gamma\mathrm{k})\,p && \text{on} \quad \partial\Omega_a \\
&p_i n_i = \rho_a\omega^2 u_i\,n_i && \text{on} \quad \partial\Omega_c
\end{aligned} \tag{2.153}$$

where $\mathcal{U}_i = \{u_i \colon u_i \in H^1(\Omega_s),\ u_i = 0 \text{ on } \partial\Omega_u\}$, $\mathcal{P} = \{p \colon p \in H^1(\Omega_a)\}$, $\mathcal{G} = \{\mu \colon \mu \in L^2(\Omega),\ \mu > 0\}$, ad $\mathcal{B} = \{\kappa \colon \kappa \in L^2(\Omega),\ \kappa > 0\}$. $H^m(\Omega)$ are Sobolev spaces of square integrable functionals whose m-th derivatives are also square integrable. Whenever $m = 0$ we shall keep the notation with $L^2(\Omega)$. The L^2 inner product for the space $\mathcal{P}$ is defined as

$$\langle u, v\rangle := \int_\Omega u(x)\,\overline{v}(x)dx, \tag{2.154}$$

where the overline denotes complex conjugate. The L^2 inner product for the space $\mathcal{U}_i$ is given by Eq. 2.79 and for the spaces $\mathcal{G}$ and $\mathcal{B}$ is given by Eq. 2.41.

First-order neccesary optimality conditions. The Lagrangian associated with the NLP defined in Eq. 2.153 is given by

$$\begin{aligned}
&\mathcal{L}(u_i,p,\mu,\kappa,\lambda_i,\upsilon)\\
&\quad= \frac{\alpha}{2}\langle u_i-\widehat{u}_i, u_i-\widehat{u}_i\rangle + R(\mu) + R(\kappa) + \Re\Big[\langle\lambda_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))_{,j}\rangle\\
&\quad- \langle\lambda_i, \rho_s\,\omega^2\,u_i\rangle - \langle\lambda_i, f_i\rangle + \langle\lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j - \tau_i\rangle_{\partial\Omega_\tau}\\
&\quad+ \langle\lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j + p\,n_i\rangle_{\partial\Omega_c} + \langle\upsilon, -p_{i,i} - \mathrm{k}^2\rho_a\,p\rangle\\
&\quad+ \langle\upsilon, p_i n_i - f(\beta - j\gamma\mathrm{k})\,p\rangle_{\partial\Omega_a} + \langle\upsilon, p_i n_i - \rho_a\,\omega^2\,u_i\,n_i\rangle_{\partial\Omega_c}\Big],
\end{aligned} \tag{2.155}$$

If $(\tilde{u}_i, \tilde{p}, \tilde{\mu}, \tilde{\kappa}) \in \mathcal{U}_i \times \mathcal{P} \times \mathcal{G} \times \mathcal{B}$ is a local solution to the NLP in Eq. 2.153, there exists Lagrange multipliers $(\lambda_i, \upsilon) \in \mathcal{U}_i \times \mathcal{P}$, such that the first-order necessary optimality conditions hold at $(\tilde{u}_i, \tilde{p}, \tilde{\mu}, \tilde{\kappa})$, i.e.

$$\begin{aligned}
&\nabla\mathcal{L}(u_i,p,\mu,\kappa,\lambda_i,\upsilon)(\delta u_i, \delta p, \delta\mu, \delta\kappa, \delta\lambda_i, \delta\upsilon)\\
&\qquad= \mathcal{L}_u(u_i,p,\mu,\kappa,\lambda_i,\upsilon)\delta u_i + \mathcal{L}_p(u_i,p,\mu,\kappa,\lambda_i,\upsilon)\delta p + \mathcal{L}_\mu(u_i,p,\mu,\kappa,\lambda_i,\upsilon)\delta\mu\\
&\qquad+ \mathcal{L}_\kappa(u_i,p,\mu,\kappa,\lambda_i,\upsilon)\delta\kappa + \mathcal{L}_\lambda(u_i,p,\mu,\kappa,\lambda_i,\upsilon)\delta\lambda_i + \mathcal{L}_\upsilon(u_i,p,\mu,\kappa,\lambda_i,\upsilon)\delta\upsilon = 0
\end{aligned}$$

Second-Order Sufficient Conditions. If $(\tilde{u}_i, \tilde{p}, \tilde{\mu}, \tilde{\kappa}, \lambda_i, \upsilon) \in \mathcal{U}_i \times \mathcal{P} \times \mathcal{G} \times \mathcal{B} \times \mathcal{U}_i \times \mathcal{P}$ satisfy the first-order necessary optimality conditions and the Hessian operator exist and is positive semidefinite, then $(\tilde{u}_i, p, \tilde{\mu}, \tilde{\kappa})$ is a strict local minimum.

Operators. Let $u \equiv \{u_i, p\} \in \mathcal{U}_i \times \mathcal{P}$ and $z \equiv \{\mu, \kappa\} \in \mathcal{G} \times \mathcal{B}$, then the operators required to solve the paramater identification problem are defined as

$$J(u,z) = \frac{\beta}{2}\langle u_i-\widehat{u}_i, u_i-\widehat{u}_i\rangle + R(\mu) + R(\kappa) \tag{2.156}$$

$$\begin{aligned}
\langle J_u(u,z), \delta u\rangle &= \frac{\partial}{\partial u}\Big(\beta\langle u_i-\widehat{u}_i, u_i-\widehat{u}_i\rangle + R(\mu) + R(\kappa)\Big)\\
&= \begin{bmatrix} \beta\langle u_i-\widehat{u}_i, \delta u_i\rangle \\ \mathbf{0} \end{bmatrix}
\end{aligned} \tag{2.157}$$

$$\begin{aligned}
\langle J_z(u,z), \delta z\rangle &= \frac{\partial}{\partial z}\Big(\frac{\beta}{2}\langle u_i-\widehat{u}_i, u_i-\widehat{u}_i\rangle + R(\mu) + R(\kappa)\Big)\\
&= \begin{bmatrix} \langle R_\mu(\mu), \delta\mu\rangle \\ \langle R_\kappa(\kappa), \delta\kappa\rangle \end{bmatrix}
\end{aligned} \tag{2.158}$$

$$\begin{aligned}
\langle J_{uu}(u,z), \delta u\rangle &= \frac{\partial}{\partial u}\left(\begin{bmatrix} \beta\langle\mathbf{1}, u_i-\widehat{u}_i\rangle \\ \mathbf{0} \end{bmatrix}\right)\\
&= \begin{bmatrix} \beta\langle\mathbf{1}, \delta u_i\rangle \\ \mathbf{0} \end{bmatrix}
\end{aligned} \tag{2.159}$$

$$\langle J_{uz}(u,z),\delta z\rangle = \frac{\partial}{\partial z}\left(\left[\begin{array}{c}\beta\langle\mathbf{1},u_i-\widehat{u}_i\rangle\\ \mathbf{0}\end{array}\right]\right) = \mathbf{0} \tag{2.160}$$

$$\begin{aligned}\langle J_{zz}(u,z),\delta z\rangle &= \frac{\partial}{\partial z}\left(\left[\begin{array}{c}\langle\mathbf{1},R_\mu(\mu)\rangle\\ \langle\mathbf{1},R_\kappa(\kappa)\rangle\end{array}\right]\right)\\ &= \left[\begin{array}{c}\langle R_{\mu\mu}(\mu),\delta\mu\rangle\\ \langle R_{\kappa\kappa}(\kappa),\delta\kappa\rangle\end{array}\right]\end{aligned} \tag{2.161}$$

$$\langle J_{zu}(u,z),\delta u\rangle = \frac{\partial}{\partial u}\left(\left[\begin{array}{c}\langle\mathbf{1},R_\mu(\mu)\rangle\\ \langle\mathbf{1},R_\kappa(\kappa)\rangle\end{array}\right]\right) = \mathbf{0} \tag{2.162}$$

$$g(u,z) = \left[\begin{array}{c} -(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))_{,j} = f_i+\omega^2\rho_s\,u_i \quad \text{in} \quad \Omega_s\\ -p_{i,i}-\mathrm{k}^2 p = 0 \quad \text{in} \quad \Omega_a\\ (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j = \tau_i \quad \text{on} \quad \partial\Omega_\tau\\ (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j = -p\,n_i \quad \text{on} \quad \partial\Omega_c\\ p_i n_i = f(\beta-j\gamma\mathrm{k})p \quad \text{on} \quad \partial\Omega_a\\ p_i n_i = \rho_a\omega^2 u_i\,n_i \quad \text{on} \quad \partial\Omega_c \end{array}\right] \tag{2.163}$$

$$
\begin{aligned}
\langle g_u(u,z),\delta u\rangle &= \langle \lambda_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))_{,j} - \omega^2\rho_s\,u_i - f_i\rangle \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j - \tau_i\rangle_{\partial\Omega_\tau} \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j + p\,n_i\rangle_{\partial\Omega_c} \\
&+ \langle \upsilon, -p_{i,i} - \mathrm{k}^2 p\rangle + \langle \upsilon, p_i n_i - f(\beta - j - \gamma\mathrm{k})p\rangle_{\partial\Omega_a} \\
&+ \langle \upsilon, p_i n_i - \rho_a\omega^2 u_i\,n_i\rangle_{\partial\Omega_c} = 0 \quad \forall(\lambda_i,\upsilon)\in\mathcal{U}_i,\times\mathcal{P} \\
&= \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k})\rangle \\
&- \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j\rangle_{\partial\Omega_s} \\
&+ \langle \lambda_i, -\omega^2\rho_s\,u_i - f_i\rangle + \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j - \tau_i\rangle_{\partial\Omega_\tau} \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j + p\,n_i\rangle_{\partial\Omega_c} + \langle \upsilon_i, p_i\rangle - \langle \upsilon, p_i n_i\rangle_{\partial\Omega_a} \\
&- \langle \upsilon, p_i n_i\rangle_{\partial\Omega_c} - \langle \upsilon, \mathrm{k}^2 p\rangle + \langle \upsilon, p_i n_i - f(\beta - j\gamma\mathrm{k})p\rangle_{\partial\Omega_a} \\
&+ \langle \upsilon, p_i n_i - \rho_a\omega^2 u_i\,n_i\rangle_{\partial\Omega_c} \\
&= \frac{\partial}{\partial u}\Bigg(\langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k})\rangle + \langle \lambda_i, -\omega^2\rho_s\,u_i - f_i\rangle \\
&- \langle \lambda_i, \tau_i\rangle_{\partial\Omega_\tau} + \langle \lambda_i, p\,n_i\rangle_{\partial\Omega_c} + \langle \upsilon_i, p_i\rangle - \langle \upsilon, \mathrm{k}^2 p\rangle \\
&- \langle \upsilon, f(\beta - j\gamma\mathrm{k})p\rangle_{\partial\Omega_a} - \langle \upsilon, \rho_a\omega^2 u_i\,n_i\rangle_{\partial\Omega_c}\Bigg) \\
&= \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(\delta u_{k,l}+\delta u_{l,k})\rangle + \langle \lambda_i, -\omega^2\rho_s\,\delta u_i\rangle \\
&+ \langle \lambda_i, \delta p\,n_i\rangle_{\partial\Omega_c} + \langle \upsilon_i, \delta p_i\rangle - \langle \upsilon, \mathrm{k}^2\delta p\rangle - \langle \upsilon, f(\beta - j\gamma\mathrm{k})\delta p\rangle_{\partial\Omega_a} \\
&- \langle \upsilon, \rho_a\omega^2\delta u_i\,n_i\rangle_{\partial\Omega_c} \\
&= \begin{bmatrix}
-\langle(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}, \delta u_i\rangle \\
+\langle(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j, \delta u_i\rangle_{\partial\Omega_c} \\
-\langle \omega^2\rho_s\,\lambda_i, \delta u_i\rangle - \langle \rho_a\omega^2\upsilon\,n_i, \delta u_i\rangle_{\partial\Omega_c} \\
-\langle \upsilon_{i,i}, \delta p\rangle + \langle \upsilon\,n_i, \delta p\rangle_{\partial\Omega_a} + \langle \upsilon\,n_i, \delta p\rangle_{\partial\Omega_c} \\
-\langle \mathrm{k}^2\upsilon, \delta p\rangle - \langle f(\beta - j\gamma\mathrm{k})\upsilon, \delta p\rangle_{\partial\Omega_a} + \langle \lambda_i n_i, \delta p\rangle_{\partial\Omega_c}
\end{bmatrix}
\end{aligned} \tag{2.164}
$$

The structural boundary in Eq. 2.164 is defined as $\partial\Omega_s = \partial\Omega_\tau \cup \partial\Omega_u \cup \partial\Omega_c$. Furthermore, the fields $\lambda_i = 0$ and $u_i = 0$ on $\partial\Omega_u$ since $\lambda_i \in \mathcal{U}_i$ and $u_i \in \mathcal{U}_i$. Hence, the application of the divergence theorem yields a set of homogeneous boundary condition on $\partial\Omega_u$. However, proper Neumann boundary conditions need to be defined to ensure the validity of the problem. Thus, a set of free Neumann boundary conditions is defined on $\partial\Omega_\tau$. Hence, $(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j = 0$ on $\partial\Omega_\tau$. Finally, the derivation of Eq. 2.164 yields a set of nonhomogeneous Neumann boundary conditions on $\partial\Omega_c$. This set of Neumann boundary conditions is defined as $(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j = \rho_a\omega^2\upsilon\,n_i$.

The acoustic boundary in Eq. 2.164 is defined as $\partial\Omega = \partial\Omega_a \cup \partial\Omega_c$. The fields υ and p

are respectively defined in spaces $\upsilon \in \mathcal{P}$ and $p \in \mathcal{P}$. Thus, the application of the divergence theorem results in a set of nonhomogeneous Neumann boundary conditions on $\partial\Omega_a$. $\partial\Omega_c$. Therefore, $\upsilon\, n_i = f(\beta - j\gamma\mathrm{k})\upsilon$ on $\partial\Omega_a$ and $\upsilon\, n_i = \lambda_i n_i$ on $\partial\Omega_c$. These sets of acoustic boundary conditions, coupled with the proper sets of structural boundary conditions, ensure the validity of the operator defined in Eq. 2.164.

$$
\begin{aligned}
\langle g_z(u,z), \delta z\rangle &= \langle \lambda_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))_{,j} - \omega^2\rho_s\, u_i - f_i\rangle \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j - \tau_i\rangle_{\partial\Omega_\tau} \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j + p\, n_i\rangle_{\partial\Omega_c} \\
&+ \langle \upsilon, -p_{i,i} - \mathrm{k}^2 p\rangle + \langle \upsilon, p_i n_i - f(\beta - j - \gamma\mathrm{k})p\rangle_{\partial\Omega_a} \\
&+ \langle \upsilon, p_i n_i - \rho_a\omega^2 u_i\, n_i\rangle_{\partial\Omega_c} = 0 \quad \forall(\lambda_i, \upsilon) \in \mathcal{U}_i, \times\mathcal{P} \\
&= \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k})\rangle \\
&- \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j\rangle_{\partial\Omega_s} \\
&+ \langle \lambda_i, -\omega^2\rho_s\, u_i - f_i\rangle + \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j - \tau_i\rangle_{\partial\Omega_\tau} \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j + p\, n_i\rangle_{\partial\Omega_c} + \langle \upsilon_i, p_i\rangle - \langle \upsilon, p_i n_i\rangle_{\partial\Omega_a} \\
&- \langle \upsilon, p_i n_i\rangle_{\partial\Omega_c} - \langle \upsilon, \mathrm{k}^2 p\rangle + \langle \upsilon, p_i n_i - f(\beta - j\gamma\mathrm{k})p\rangle_{\partial\Omega_a} \\
&+ \langle \upsilon, p_i n_i - \rho_a\omega^2 u_i\, n_i\rangle_{\partial\Omega_c} \\
&= \frac{\partial}{\partial z}\Bigg(\langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k})\rangle + \langle \lambda_i, -\omega^2\rho_s\, u_i - f_i\rangle \\
&- \langle \lambda_i, \tau_i\rangle_{\partial\Omega_\tau} + \langle \lambda_i, p\, n_i\rangle_{\partial\Omega_c} + \langle \upsilon_i, p_i\rangle - \langle \upsilon, \mathrm{k}^2 p\rangle \\
&- \langle \upsilon, f(\beta - j\gamma\mathrm{k})p\rangle_{\partial\Omega_a} - \langle \upsilon, \rho_a\omega^2 u_i\, n_i\rangle_{\partial\Omega_c} \Bigg) \\
&= \begin{bmatrix} \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}), \delta\mu\rangle \\ \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}), \delta\kappa\rangle \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}
\end{aligned}
\tag{2.165}
$$

$$
\begin{aligned}
\langle \delta u, (g_u(u,z))^* \rangle &= \langle \lambda_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))_{,j} - \omega^2\rho_s\, u_i - f_i \rangle \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j - \tau_i \rangle_{\partial\Omega_\tau} \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j + p\, n_i \rangle_{\partial\Omega_c} \\
&+ \langle \upsilon, -p_{,i} - \mathrm{k}^2 p \rangle + \langle \upsilon, p_i n_i - f(\beta - j - \gamma\mathrm{k})p \rangle_{\partial\Omega_a} \\
&+ \langle \upsilon, p_i n_i - \rho_a\omega^2 u_i\, n_i \rangle_{\partial\Omega_c} = 0 \quad \forall (\lambda_i, \upsilon) \in \mathcal{U}_i, \times \mathcal{P} \\
&= \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}) \rangle \\
&- \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j \rangle_{\partial\Omega_s} \\
&+ \langle \lambda_i, -\omega^2\rho_s\, u_i - f_i \rangle + \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j - \tau_i \rangle_{\partial\Omega_\tau} \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j + p\, n_i \rangle_{\partial\Omega_c} + \langle \upsilon_i, p_i \rangle - \langle \upsilon, p_i n_i \rangle_{\partial\Omega_a} \\
&- \langle \upsilon, p_i n_i \rangle_{\partial\Omega_c} - \langle \upsilon, \mathrm{k}^2 p \rangle + \langle \upsilon, p_i n_i - f(\beta - j\gamma\mathrm{k})p \rangle_{\partial\Omega_a} \\
&+ \langle \upsilon, p_i n_i - \rho_a\omega^2 u_i\, n_i \rangle_{\partial\Omega_c} \\
&= \frac{\partial}{\partial u}\Bigg(\langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}) \rangle + \langle \lambda_i, -\omega^2\rho_s\, u_i - f_i \rangle \\
&- \langle \lambda_i, \tau_i \rangle_{\partial\Omega_\tau} + \langle \lambda_i, p\, n_i \rangle_{\partial\Omega_c} + \langle \upsilon_i, p_i \rangle \\
&- \langle \upsilon, \mathrm{k}^2 p \rangle - \langle \upsilon, f(\beta - j\gamma\mathrm{k})p \rangle_{\partial\Omega_a} - \langle \upsilon, \rho_a\omega^2 u_i\, n_i \rangle_{\partial\Omega_c} \Bigg) \\
&= \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(\delta u_{k,l}+\delta u_{l,k}) \rangle + \langle \lambda_i, -\omega^2\rho_s\, \delta u_i \rangle \\
&+ \langle \lambda_i, \delta p\, n_i \rangle_{\partial\Omega_c} + \langle \upsilon_i, \delta p_i \rangle - \langle \upsilon, \mathrm{k}^2 \delta p \rangle - \langle \upsilon, f(\beta - j\gamma\mathrm{k})\delta p \rangle_{\partial\Omega_a} \\
&- \langle \upsilon, \rho_a\omega^2 \delta u_i\, n_i \rangle_{\partial\Omega_c} \\
&= \langle \frac{1}{2}(\delta u_{i,j}+\delta u_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}-\lambda_{l,k}) \rangle^* + \langle \delta u_i, \omega^2\rho_s\, \lambda_i \rangle^* \\
&+ \langle \delta p, \lambda_i\, n_i \rangle^*_{\partial\Omega_c} + \langle \delta p_i, \upsilon_i \rangle^* - \langle \delta p, \mathrm{k}^2 \upsilon \rangle^* - \langle \delta p, f(\beta + j\gamma\mathrm{k})\upsilon \rangle^*_{\partial\Omega_a} \\
&- \langle \delta u_i, \rho_a\omega^2 \upsilon\, n_i \rangle^*_{\partial\Omega_c} \\
&= \left[\begin{array}{c}
-\langle (\delta u_i, \mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}))_{,j} \rangle^* \\
+\langle (\delta u_i, \mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}))\, n_j \rangle^*_{\partial\Omega_c} \\
-\langle \delta u_i, \omega^2\rho_s\, \lambda_i \rangle^* - \langle \delta u_i, \rho_a\omega^2 \upsilon\, n_i \rangle^*_{\partial\Omega_c} \\
-\langle \delta p, \upsilon_{i,i} \rangle^* + \langle \delta p, \upsilon\, n_i \rangle^*_{\partial\Omega_a} + \langle \delta p, \upsilon\, n_i \rangle^*_{\partial\Omega_c} \\
-\langle \delta p, \mathrm{k}^2 \upsilon \rangle^* - \langle \delta p, f(\beta + j\gamma\mathrm{k})\upsilon \rangle^*_{\partial\Omega_a} + \langle \delta p, \lambda_i n_i \rangle^*_{\partial\Omega_c}
\end{array} \right]
\end{aligned} \tag{2.166}
$$

The structural boundary in Eq. 2.166 is defined as $\partial\Omega_s = \partial\Omega_\tau \cup \partial\Omega_u \cup \partial\Omega_c$. Furthermore, the fields $\lambda_i = 0$ and $u_i = 0$ on $\partial\Omega_u$ since $\lambda_i \in \mathcal{U}_i$ and $u_i \in \mathcal{U}_i$. Hence, the application of the divergence theorem yields a set of homogeneous boundary condition on $\partial\Omega_u$. However, proper Neumann boundary conditions need to be defined to ensure a self-adjoint operator. Thus, a set of free Neumann boundary conditions is defined on $\partial\Omega_\tau$. Hence, $(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l} + \lambda_{l,k}))\, n_j = 0$ on $\partial\Omega_\tau$. Finally, the derivation of Eq. 2.164 yields a set of nonhomogeneous

Neumann boundary conditions on $\partial\Omega_c$. This set of Neumann boundary conditions is defined as $(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j = \rho_a\omega^2\upsilon\,n_i$.

The acoustic boundary in Eq. 2.166 is defined as $\partial\Omega = \partial\Omega_a \cup \partial\Omega_c$. The fields υ and p are respectively defined in spaces $\upsilon \in \mathcal{P}$ and $p \in \mathcal{P}$. Thus, the application of the divergence theorem results in a set of nonhomogeneous Neumann boundary conditions on $\partial\Omega_a$. $\partial\Omega_c$. Therefore, $\upsilon\,n_i = f(\beta + j\gamma\mathrm{k})\upsilon$ on $\partial\Omega_a$ and $\upsilon\,n_i = \lambda_i n_i$ on $\partial\Omega_c$. Notice the change in sign of the Sommerfeld radiation boundary condition imaginary component. This change in sign is due to the adjoint operation. These sets of acoustic boundary conditions ensure the validity of the adjoint operator for this problem.

$$
\begin{aligned}
\langle \delta z, (g_z(u,z))^*\rangle &= \langle \lambda_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))_{,j} - \omega^2\rho_s\,u_i - f_i\rangle \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j - \tau_i\rangle_{\partial\Omega_\tau} \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j + p\,n_i\rangle_{\partial\Omega_c} \\
&+ \langle \upsilon, -p_{i,i} - \mathrm{k}^2 p\rangle + \langle \upsilon, p_i n_i - f(\beta - j - \gamma\mathrm{k})p\rangle_{\partial\Omega_a} \\
&+ \langle \upsilon, p_i n_i - \rho_a\omega^2 u_i\,n_i\rangle_{\partial\Omega_c} = 0 \quad \forall(\lambda_i,\upsilon) \in \mathcal{U}_i, \times\mathcal{P} \\
&= \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k})\rangle \\
&- \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j\rangle_{\partial\Omega_s} \\
&+ \langle \lambda_i, -\omega^2\rho_s\,u_i - f_i\rangle + \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j - \tau_i\rangle_{\partial\Omega_\tau} \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j + p\,n_i\rangle_{\partial\Omega_c} + \langle \upsilon_i, p_i\rangle - \langle \upsilon, p_i n_i\rangle_{\partial\Omega_a} \\
&- \langle \upsilon, p_i n_i\rangle_{\partial\Omega_c} - \langle \upsilon, \mathrm{k}^2 p\rangle + \langle \upsilon, p_i n_i - f(\beta - j\gamma\mathrm{k})p\rangle_{\partial\Omega_a} \\
&+ \langle \upsilon, p_i n_i - \rho_a\omega^2 u_i\,n_i\rangle_{\partial\Omega_c} \\
&= \frac{\partial}{\partial z}\Big(\langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k})\rangle + \langle \lambda_i, -\omega^2\rho_s\,u_i - f_i\rangle \\
&- \langle \lambda_i, \tau_i\rangle_{\partial\Omega_\tau} + \langle \lambda_i, p\,n_i\rangle_{\partial\Omega_c} + \langle \upsilon_i, p_i\rangle - \langle \upsilon, \mathrm{k}^2 p\rangle \\
&- \langle \upsilon, f(\beta - j\gamma\mathrm{k})p\rangle_{\partial\Omega_a} - \langle \upsilon, \rho_a\omega^2 u_i\,n_i\rangle_{\partial\Omega_c}\Big) \\
&= \begin{bmatrix} \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}), \delta\mu\rangle \\ \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}), \delta\kappa\rangle \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \\
&= \begin{bmatrix} \langle \delta\mu, \frac{1}{2}(u_{i,j}+u_{j,i})\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k})\rangle^* \\ \langle \delta\kappa, \frac{1}{2}(u_{i,j}+u_{j,i})\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k})\rangle^* \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}
\end{aligned}
\tag{2.167}
$$

$$\langle (g_{uu}(u,z))^*, \delta u\rangle = \frac{\partial}{\partial u}\left(\begin{bmatrix} -\langle \mathbf{1}, (\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})),_j\rangle \\ +\langle \mathbf{1}, (\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}))\,n_j\rangle_{\partial\Omega_c} \\ -\langle \mathbf{1}, \omega^2\rho_s\,\lambda_i, \mathbf{1}\rangle - \langle \mathbf{1}, \rho_a\omega^2\upsilon\,n_i\rangle_{\partial\Omega_c} \\ -\langle \mathbf{1}, \upsilon_{i,i}\rangle + \langle \mathbf{1}, \upsilon\,n_i\rangle_{\partial\Omega_a} + \langle \mathbf{1}, \upsilon\,n_i\rangle_{\partial\Omega_c} \\ -\langle \mathbf{1}, f(\beta + j\gamma\mathrm{k})\upsilon\rangle_{\partial\Omega_a} \\ -\langle \mathbf{1}, \mathrm{k}^2\upsilon\rangle + \langle \mathbf{1}, \lambda_i n_i\rangle_{\partial\Omega_c} \end{bmatrix}\right) = \mathbf{0} \tag{2.168}$$

$$\begin{aligned} \langle (g_{uz}(u,z))^*, \delta z\rangle &= \begin{bmatrix} -\langle w_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})),_j,\rangle \\ +\langle w_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}))\,n_j,\rangle_{\partial\Omega_c} \\ -\langle w_i, \omega^2\rho_s\,\lambda_i\rangle - \langle w_i, \rho_a\omega^2\upsilon\,n_i\rangle_{\partial\Omega_c} = 0 \\ -\langle y, \upsilon_{i,i}\rangle + \langle y, \upsilon\,n_i\rangle_{\partial\Omega_a} + \langle y, \upsilon\,n_i\rangle_{\partial\Omega_c} \\ -\langle y, f(\beta - j\gamma\mathrm{k})\upsilon\rangle_{\partial\Omega_a} \\ -\langle y, \mathrm{k}^2\upsilon\rangle + \langle y, \lambda_i n_i\rangle_{\partial\Omega_c} = 0 \end{bmatrix} \;\forall\; w_i \in \mathcal{U}_i,\; y \in \mathcal{P} \\ &= \frac{\partial}{\partial z}\left(\begin{bmatrix} \langle \frac{1}{2}(w_{i,j}+w_{j,i}),\, \mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k})\rangle \\ -\langle w_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}))\,n_j\rangle_{\partial\Omega_c} \\ +\langle w_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}))\,n_j\rangle_{\partial\Omega_c} \\ -\langle w_i, \omega^2\rho_s\,\lambda_i\rangle - \langle w_i, \rho_a\omega^2\upsilon\,n_i\rangle_{\partial\Omega_c} = 0 \\ \langle y_i, \upsilon_i\rangle - \langle y, \upsilon\,n_i\rangle_{\partial\Omega_a} - \langle y, \upsilon\,n_i\rangle_{\partial\Omega_c} \\ +\langle y, \upsilon\,n_i\rangle_{\partial\Omega_a} + \langle y, \upsilon\,n_i\rangle_{\partial\Omega_c} \\ -\langle y, f(\beta - j\gamma\mathrm{k})\upsilon\rangle_{\partial\Omega_a} \\ -\langle y, \mathrm{k}^2\upsilon\rangle + \langle y, \lambda_i n_i\rangle_{\partial\Omega_c} = 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} \langle \frac{1}{2}(w_{i,j}+w_{j,i})\,\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}), \delta\mu\rangle \\ \langle \frac{1}{2}(w_{i,j}+w_{j,i})\,\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}), \delta\kappa\rangle \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \end{aligned} \tag{2.169}$$

$$\langle (g_{zz}(u,z))^*, \delta z\rangle = \frac{\partial}{\partial z}\left(\begin{bmatrix} \langle \mathbf{1}, \frac{1}{2}(u_{i,j}+u_{j,i})\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k})\rangle \\ \langle \mathbf{1}, \frac{1}{2}(u_{i,j}+u_{j,i})\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k})\rangle \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}\right) = \mathbf{0} \tag{2.170}$$

$$\langle (g_{zu}(u,z))^*, \delta u\rangle = \begin{bmatrix} \langle \mathbf{1}, \frac{1}{2}(u_{i,j}+u_{j,i})\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k})\rangle \\ \langle \mathbf{1}, \frac{1}{2}(u_{i,j}+u_{j,i})\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k})\rangle \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

$$= \frac{\partial}{\partial u}\left(\begin{bmatrix} \langle \mathbf{1}, -u_i(\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}\rangle \\ +\langle \mathbf{1}, u_i(\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j\rangle_{\partial\Omega_s} \\ \langle \mathbf{1}, -u_i(\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}\rangle \\ +\langle \mathbf{1}, u_i(\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j\rangle_{\partial\Omega_s} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}\right) \tag{2.171}$$

$$= \begin{bmatrix} -\langle (\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}, \delta u_i\rangle \\ +\langle (\mathbf{C}^{\mu}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j, \delta u_i\rangle_{\partial\Omega_s} \\ -\langle (\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))_{,j}, \delta u_i\rangle \\ +\langle (\mathbf{C}^{\kappa}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j, \delta u_i\rangle_{\partial\Omega_s} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

Discretization. Lets define finite-dimensional subspaces $\mathcal{U}_i^h \subset \mathcal{U}_i$ with basis $\mathcal{U}_i^h = \text{span}\{\phi_i^m\}_{m=1}^M$, $\mathcal{P}^h \subset \mathcal{P}$ with basis $\mathcal{P}^h = \text{span}\{\psi^n\}_{n=1}^N$, $\mathcal{G}^h \subset \mathcal{G}$ with basis $\{\theta^o\}_{o=1}^O$, and $\mathcal{B}^h \subset \mathcal{B}$ with basis $\{\chi^p\}_{p=1}^P$. This leads to $u_i^h = \sum_{m=1}^M u_i^m \phi_i^m$, $p^h = \sum_{n=1}^N p^n \psi^n$, $\mu^h = \sum_{o=1}^O \mu^o \theta^o$, $\kappa^h = \sum_{p=1}^P \kappa^p \chi^p$, $\lambda_i^h = \sum_{m=1}^M \lambda_i^m \phi_i^m$, and $\upsilon^h = \sum_{n=1}^N \upsilon^n \psi^n$. Furthermore, the quantities δu_i^h, δp^h, $\delta\mu^h$, $\delta\kappa^h$, $\delta\lambda_i$, and $\delta\upsilon^h$ can be respectively approximated as follow: $\delta u_i^h = \sum_{m=1}^M \delta u_i^m \phi_i^m$, $\delta p^h = \sum_{n=1}^N p^n \psi^n$, $\delta\mu^h = \sum_{o=1}^O \mu^o \theta^o$, $\delta\kappa^h = \sum_{p=1}^O \kappa^p \chi^p$, $\delta\lambda_i^h = \sum_{m=1}^M \delta\lambda_i^m \phi_i^m$, and $\delta\upsilon^h = \sum_{n=1}^N \delta\upsilon^n \psi^n$. Subsequently, the Galerkin approximation of the required operators to solve the parameter identification problem are defined as

$$J(u^h, z^h) = \frac{\beta}{2}\langle u_i^h - \widehat{u}_i, u_i^h - \widehat{u}_i\rangle + R(\mu^h) + R(\kappa^h) \tag{2.172}$$

$$J_u(u^h, z^h) = \sum_{e=1}^{N_{elem}} \left(\sum_{a=1}^{M} \beta\langle \phi_i^a, u_i^h - \widehat{u}_i\rangle \right) \tag{2.173}$$

$$J_z(u^h, z^h) = \sum_{e=1}^{N_{elem}} \left(\begin{bmatrix} \sum_{a=1}^O \beta\langle \theta^a, R_\mu(\mu^h)\rangle \\ \sum_{a=1}^P \beta\langle \chi^a, R_\kappa(\kappa^h)\rangle \end{bmatrix} \right) \tag{2.174}$$

$$J_{uu}(u^h, z^h) = \sum_{e=1}^{N_{elem}} \left(\sum_{a=1}^{M}\sum_{b=1}^{M} \beta\langle \phi^a, \phi^b\rangle \right) \tag{2.175}$$

$$J_{uz}(u^h, z^h) = \mathbf{0} \tag{2.176}$$

$$J_{zz}(u^h, z^h) = \sum_{e=1}^{N_{elem}} \left(\begin{bmatrix} \sum_{a=1}^O \sum_{b=1}^O \beta\langle \theta^a, R_{\mu\mu}(\mu^h)\theta^b\rangle \\ \sum_{a=1}^P \sum_{b=1}^P \beta\langle \chi^a, R_{\kappa\kappa}(\kappa^h)\chi^b\rangle \end{bmatrix} \right) \tag{2.177}$$

$$J_{zu}(u^h, z^h) = \mathbf{0} \tag{2.178}$$

$$g(u^h, z^h) = \sum_{e=1}^{N_{elem}} \left(\begin{bmatrix} \sum_{a=1}^{M} \langle \frac{1}{2}(\phi^a_{i,j} + \phi^a_{j,i}), \mathbf{C}_{ijkl}\, \frac{1}{2}(u^h_{k,l} + u^h_{l,k}) \rangle \\ - \sum_{a=1}^{M} \langle \phi^a_i, \rho_s\, \omega^2\, u^h_i + f_i \rangle \\ - \sum_{a=1}^{M} \langle \phi^a_i, \tau_i \rangle_{\partial\Omega_\tau} + \sum_{a=1}^{M} \langle \phi^a_i, p^h\, n_i \rangle_{\partial\Omega_c} = 0 \\ \sum_{b=1}^{N} \langle \psi^b_i, p^h_i \rangle - \sum_{b=1}^{N} \langle \psi^b_i, \mathrm{k}^2\, \rho_a\, p^h \rangle \\ - \sum_{b=1}^{N} \langle \psi^b, f(\beta - j\gamma \mathrm{k})\, p^h \rangle_{\partial\Omega_a} \\ - \sum_{b=1}^{N} \langle \psi^b, \rho_a\, \omega^2\, u^h_i\, n_i \rangle_{\partial\Omega_c} = 0 \end{bmatrix} \right) \tag{2.179}$$

$$g_u(u^h, z^h) = \sum_{e=1}^{N_{elem}} \left(\begin{bmatrix} \sum_{a=1}^{M} \langle \frac{1}{2}(\phi^a_{i,j} + \phi^a_{j,i}), \mathbf{C}_{ijkl}\, \frac{1}{2}(\lambda^h_{k,l} + \lambda^h_{l,k}) \rangle \\ - \sum_{a=1}^{M} \langle \phi^a_i, \rho_s\, \omega^2\, \lambda^h_i \rangle + \sum_{a=1}^{M} \langle \phi^a_i, \rho_a\, \omega^2\, \upsilon^h\, n_i \rangle_{\partial\Omega_c} = 0 \\ \sum_{b=1}^{N} \langle \psi^b_i, \upsilon^h_i \rangle - \sum_{b=1}^{N} \langle \psi^b, \mathrm{k}^2\, \rho_a\, \upsilon^h \rangle \\ - \sum_{b=1}^{N} \langle \psi^b, f(\beta - j\gamma \mathrm{k})\, \upsilon^h \rangle_{\partial\Omega_a} \\ - \sum_{b=1}^{N} \langle \psi^b, \lambda^h_i\, n_i \rangle_{\partial\Omega_c} = 0 \end{bmatrix} \right) \tag{2.180}$$

$$g_z(u^h, z^h) = \sum_{e=1}^{N_{elem}} \left(\begin{bmatrix} \sum_{a=1}^{O} \langle \theta^a, \frac{1}{2}(\lambda^h_{i,j} + \lambda^h_{j,i}) \mathbf{C}^{\mu}_{ijkl}\, \frac{1}{2}(u^h_{k,l} + u^h_{l,k}) \rangle \\ \sum_{a=1}^{P} \langle \chi^a, \frac{1}{2}(\lambda^h_{i,j} + \lambda^h_{j,i}) \mathbf{C}^{\kappa}_{ijkl}\, \frac{1}{2}(u^h_{k,l} + u^h_{l,k}) \rangle \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \right) \tag{2.181}$$

$$(g_u(u^h, z^h))^* = \sum_{e=1}^{N_{elem}} \left(\begin{bmatrix} \sum_{a=1}^{M} \langle \frac{1}{2}(\phi^a_{i,j} + \phi^a_{j,i}), \mathbf{C}_{ijkl}\, \frac{1}{2}(\lambda^h_{k,l} + \lambda^h_{l,k}) \rangle \\ - \sum_{a=1}^{M} \langle \phi^a_i, \rho_s\, \omega^2\, \lambda^h_i \rangle \\ + \sum_{a=1}^{M} \langle \phi^a_i, \rho_a\, \omega^2\, \upsilon^h\, n_i \rangle_{\partial\Omega_c} = 0 \\ \sum_{b=1}^{N} \langle \psi^b_i, \upsilon^h_i \rangle - \sum_{b=1}^{N} \langle \psi^b, \mathrm{k}^2\, \rho_a\, \upsilon^h \rangle \\ - \sum_{b=1}^{N} \langle \psi^b, f(\beta - j\gamma \mathrm{k})\, \upsilon^h \rangle_{\partial\Omega_a} \\ - \sum_{b=1}^{N} \langle \psi^b, \lambda^h_i\, n_i \rangle_{\partial\Omega_c} = 0 \end{bmatrix} \right) \tag{2.182}$$

$$(g_z(u^h, z^h))^* = \sum_{e=1}^{N_{elem}} \left(\begin{bmatrix} \sum_{a=1}^{O} \langle \theta^a, \frac{1}{2}(u^h_{i,j} + u^h_{j,i}) \mathbf{C}^{\mu}_{ijkl}\, \frac{1}{2}(\lambda^h_{k,l} + \lambda^h_{l,k}) \rangle \\ \sum_{a=1}^{P} \langle \chi^a, \frac{1}{2}(u^h_{i,j} + u^h_{j,i}) \mathbf{C}^{\kappa}_{ijkl}\, \frac{1}{2}(\lambda^h_{k,l} + \lambda^h_{l,k}) \rangle \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \right) \tag{2.183}$$

$$(g_{uu}(u^h, z^h))^* = \mathbf{0} \tag{2.184}$$

$$(g_{uz}(u^h, z^h))^* = \sum_{e=1}^{N_{elem}} \left(\begin{bmatrix} \sum_{a=1}^{M} \sum_{b=1}^{O} \langle \frac{1}{2}(\phi^a_{i,j} + \phi^a_{j,i}) \mathbf{C}^{\mu}_{ijkl}\, \frac{1}{2}(\lambda^h_{k,l} + \lambda^h_{l,k}), \theta^b \rangle \\ \sum_{a=1}^{M} \sum_{b=1}^{O} \langle \frac{1}{2}(\phi^a_{i,j} + \phi^a_{j,i}) \mathbf{C}^{\kappa}_{ijkl}\, \frac{1}{2}(\lambda^h_{k,l} + \lambda^h_{l,k}), \chi^b \rangle \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \right) \tag{2.185}$$

$$(g_{zz}(u^h, z^h))^* = \mathbf{0} \tag{2.186}$$

$$(g_{zu}(u^h, z^h))^* = \sum_{e=1}^{N_{elem}} \left(\begin{bmatrix} \sum_{a=1}^{M} \sum_{b=1}^{O} \langle \frac{1}{2}(\phi^a_{i,j} + \phi^a_{j,i}) \mathbf{C}^{\mu}_{ijkl}\, \frac{1}{2}(\lambda^h_{k,l} + \lambda^h_{l,k}), \theta^b \rangle \\ \sum_{a=1}^{M} \sum_{b=1}^{O} \langle \frac{1}{2}(\phi^a_{i,j} + \phi^a_{j,i}) \mathbf{C}^{\kappa}_{ijkl}\, \frac{1}{2}(\lambda^h_{k,l} + \lambda^h_{l,k}), \chi^b \rangle \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \right) \tag{2.187}$$

Optimal Control in Structural-Acoustics

Lets consider the structural-acoustics partial differential equations defined in Eq. 2.152. An optimal control problem in structural-acoustics can be defined as: estimate the surface traction forces z_i that yield a desired displacement field, i.e.

$$\underset{(u_i,p,z_i)\in\, \mathcal{U}_i\times\mathcal{P}\times\mathcal{Z}_i}{\text{minimize}} \qquad \frac{\alpha}{2}\langle u_i-\widehat{u}_i, u_i-\widehat{u}_i\rangle + R(z_i)$$

subject to

$$\begin{aligned}
-(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))_{,j} &= \rho_s\,\omega^2\,u_i + f_i && \text{in} && \Omega_s \\
-p_{,ii} - \mathrm{k}^2 p &= 0 && \text{in} && \Omega_a \\
u_i &= 0 && \text{on} && \partial\Omega_u \\
(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j &= z_i && \text{on} && \partial\Omega_z \\
(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j &= -p\,n_i && \text{on} && \partial\Omega_c \\
pn_i &= f(\beta - i\gamma\mathrm{k})\,p && \text{on} && \partial\Omega_a \\
pn_i &= \rho_f\omega^2 u_i\,n_i && \text{on} && \partial\Omega_c
\end{aligned} \tag{2.188}$$

where $\mathcal{U}_i = \{u_i\colon u_i \in H^1(\Omega_s),\ u_i = 0 \text{ on } \partial\Omega_u\}$, $\mathcal{P} = \{p\colon p \in H^1(\Omega_a)\}$, and $\mathcal{Z}_i = \{z_i\colon z_i \in L^2(\partial\Omega_z)\}$. The L^2 inner products for spaces $\mathcal{U}_i$ and $\mathcal{P}$ are respectively defined in Eqs. 2.79 and 2.41.

First-order neccesary optimality conditions. The Lagrangian associated with the optimal control problem defined in Eq. 2.188 is given by

$$\begin{aligned}
\mathcal{L}(u_i,p,z_i,\lambda_i,\upsilon) = \frac{\alpha}{2}\langle u_i-\widehat{u}_i, u_i-\widehat{u}_i\rangle + R(z_i) + \Re\Big[&\langle\lambda_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))_{,j}\rangle \\
&- \langle\lambda_i, \rho_s\,\omega^2\,u_i\rangle - \langle\lambda_i, f_i\rangle + \langle\lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j - z_i\rangle_{\partial\Omega_z} \\
&+ \langle\lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j + p\,n_i\rangle_{\partial\Omega_c} + \langle\upsilon, -p_{i,i} - \mathrm{k}^2\rho_a\,p\rangle \\
&+ \langle\upsilon, p_i n_i - f(\beta - j\gamma\mathrm{k})\,p\rangle_{\partial\Omega_a} + \langle\upsilon, p_i n_i - \rho_a\,\omega^2\,u_i\,n_i\rangle_{\partial\Omega_c}\Big],
\end{aligned} \tag{2.189}$$

If $(\tilde{u}_i,\tilde{p},\tilde{z}_i)\in\mathcal{U}_i\times\mathcal{P}\times\mathcal{Z}_i$ is a local solution to the optimal control problems in Eq. 2.189, there exists Lagrange multipliers $(\lambda_i,\upsilon)\in\mathcal{U}_i\times\mathcal{P}$, such that the first-order necessary optimality conditions hold at $(\tilde{u}_i,\tilde{p},\tilde{z}_i)$, i.e.

$$\begin{aligned}
\nabla\mathcal{L}(u_i,p,z_i,\lambda_i,\upsilon)(\delta u_i,\delta p,\delta z_i,\delta\lambda_i,\delta\upsilon) &= \mathcal{L}_u(u_i,p,z_i,\lambda_i,\upsilon)\delta u_i + \mathcal{L}_p(u_i,p,z_i,\lambda_i,\upsilon)\delta p \\
&+ \mathcal{L}_\mu(u_i,p,z_i,\lambda_i,\upsilon)\delta z_i + \mathcal{L}_\kappa(u_i,p,z_i,\lambda_i,\upsilon)\delta\lambda_i + \mathcal{L}_\upsilon(u_i,p,z_i,\lambda_i,\upsilon)\delta\upsilon = 0
\end{aligned}$$

Second-Order Sufficient Conditions. If $(\tilde{u}_i,\tilde{p},\tilde{z}_i,\lambda_i,\upsilon)\in\mathcal{U}_i\times\mathcal{P}\times\mathcal{Z}_i\times\mathcal{U}_i\times\mathcal{P}$ satisfy the first-order necessary optimality conditions and the Hessian operator exist and is positive semidefinite, then $(\tilde{u}_i,p,\tilde{z}_i)$ is a strict local minimum.

Operators. Let $u \equiv \{u_i, p\} \in \mathcal{U}_i \times \mathcal{P}$, then the operators required to solve the optimal control problem are defined as

$$J(u, z_i) = \frac{\alpha}{2}\langle u_i - \widehat{u}_i, u_i - \widehat{u}_i\rangle + R(z_i) \tag{2.190}$$

$$\begin{aligned}\langle J_u(u, z_i), \delta u\rangle &= \frac{\partial}{\partial u}\left(\frac{\alpha}{2}\langle u_i - \widehat{u}_i, u_i - \widehat{u}_i\rangle + R(z_i)\right)\\ &= \begin{bmatrix} = \alpha\langle u_i - \widehat{u}_i, \delta u_i\rangle \\ \mathbf{0} \end{bmatrix}\end{aligned} \tag{2.191}$$

$$\begin{aligned}\langle J_z(u, z_i), \delta z_i\rangle &= \frac{\partial}{\partial z_i}\left(\frac{\alpha}{2}\langle u_i - \widehat{u}_i, u_i - \widehat{u}_i\rangle + R(z_i)\right)\\ &= \langle R_z(z_i), \delta z_i\rangle\end{aligned} \tag{2.192}$$

$$\begin{aligned}\langle J_{uu}(u, z_i), \delta u\rangle &= \frac{\partial}{\partial u}\left(\begin{bmatrix} \alpha\langle \mathbf{1}, u_i - \widehat{u}_i\rangle \\ \mathbf{0} \end{bmatrix}\right)\\ &= \begin{bmatrix} \alpha\langle \mathbf{1}, \delta u_i\rangle \\ \mathbf{0} \end{bmatrix}\end{aligned} \tag{2.193}$$

$$\langle J_{uz}(u, z_i), \delta z_i\rangle = \frac{\partial}{\partial z_i}\left(\begin{bmatrix} \alpha\langle \mathbf{1}, u_i - \widehat{u}_i\rangle \\ \mathbf{0} \end{bmatrix}\right) = \mathbf{0} \tag{2.194}$$

$$\begin{aligned}\langle J_{zz}(u, z_i), \delta z_i\rangle &= \frac{\partial}{\partial z_i}\Big(\langle \mathbf{1}, R_z(z_i)\rangle\Big)\\ &= \langle R_{zz}(z_i), \delta z_i\rangle\end{aligned} \tag{2.195}$$

$$\langle J_{zu}(u, z_i), \delta u\rangle = \frac{\partial}{\partial u}\Big(\langle \mathbf{1}, R_z(z_i)\rangle\Big) = \mathbf{0} \tag{2.196}$$

$$g(u, z_i) = \begin{bmatrix} -(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l} + u_{l,k}))_{,j} = f_i + \omega^2\rho_s\,u_i \quad \text{in} \quad \Omega_s \\ -p_{,i} - \mathrm{k}^2 p = 0 \quad \text{in} \quad \Omega_a \\ (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l} + u_{l,k}))\,n_j = z_i \quad \text{on} \quad \partial\Omega_z \\ (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l} + u_{l,k}))\,n_j = -p\,n_i \quad \text{on} \quad \partial\Omega_c \\ p_i n_i = f(\beta - j\gamma\mathrm{k})p \quad \text{on} \quad \partial\Omega_a \\ p_i n_i = \rho_a\omega^2 u_i\,n_i \quad \text{on} \quad \partial\Omega_c \end{bmatrix} \tag{2.197}$$

$$
\begin{aligned}
\langle g_u(u,z_i),\delta u\rangle &= \langle \lambda_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k})),_j - \omega^2\rho_s\,u_i - f_i\rangle \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j - z_i\rangle_{\partial\Omega_z} \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j + p\,n_i\rangle_{\partial\Omega_c} \\
&+ \langle \upsilon, -p_{,i} - \mathrm{k}^2 p\rangle + \langle \upsilon, p_i n_i - f(\beta - j - \gamma\mathrm{k})p\rangle_{\partial\Omega_a} \\
&+ \langle \upsilon, p_i n_i - \rho_a\omega^2 u_i\,n_i\rangle_{\partial\Omega_c} = 0 \quad \forall(\lambda_i,\upsilon)\in\mathcal{U}_i,\times\mathcal{P} \\
&= \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}), \mathbf{C}_{ijkl}\frac{1}{2}(u_{k,l}+u_{l,k})\rangle - \langle \lambda_i, (\mathbf{C}_{ijkl}\frac{1}{2}(u_{k,l}+u_{l,k}))n_j\rangle_{\partial\Omega_s} \\
&+ \langle \lambda_i, -\omega^2\rho_s\,u_i - f_i\rangle + \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j - z_i\rangle_{\partial\Omega_z} \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\,n_j + p\,n_i\rangle_{\partial\Omega_c} + \langle \upsilon_i, p_i\rangle - \langle \upsilon, p_i n_i\rangle_{\partial\Omega_a} \\
&- \langle \upsilon, p_i n_i\rangle_{\partial\Omega_c} - \langle \upsilon, \mathrm{k}^2 p\rangle + \langle \upsilon, p_i n_i - f(\beta - j\gamma\mathrm{k})p\rangle_{\partial\Omega_a} \\
&+ \langle \upsilon, p_i n_i - \rho_a\omega^2 u_i\,n_i\rangle_{\partial\Omega_c} \qquad\qquad (2.198) \\
&= \frac{\partial}{\partial u}\Bigg(\langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k})\rangle + \langle \lambda_i, -\omega^2\rho_s\,u_i - f_i\rangle \\
&- \langle \lambda_i, z_i\rangle_{\partial\Omega_z} + \langle \lambda_i, p\,n_i\rangle_{\partial\Omega_c} + \langle \upsilon_i, p_i\rangle - \langle \upsilon, \mathrm{k}^2 p\rangle \\
&- \langle \upsilon, f(\beta - j\gamma\mathrm{k})p\rangle_{\partial\Omega_a} - \langle \upsilon, \rho_a\omega^2 u_i\,n_i\rangle_{\partial\Omega_c}\Bigg) \\
&= \langle \frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(\delta u_{k,l}+\delta u_{l,k})\rangle + \langle \lambda_i, -\omega^2\rho_s\,\delta u_i\rangle \\
&+ \langle \lambda_i, \delta p\,n_i\rangle_{\partial\Omega_c} + \langle \upsilon_i, \delta p_i\rangle - \langle \upsilon, \mathrm{k}^2\delta p\rangle - \langle \upsilon, f(\beta - j\gamma\mathrm{k})\delta p\rangle_{\partial\Omega_a} \\
&- \langle \upsilon, \rho_a\omega^2\delta u_i\,n_i\rangle_{\partial\Omega_c} \\
&= \left[\begin{array}{c}
-\langle(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k})),_j, \delta u_i\rangle \\
+\langle(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j, \delta u_i\rangle_{\partial\Omega_c} \\
-\langle \omega^2\rho_s\,\lambda_i, \delta u_i\rangle - \langle \rho_a\omega^2\upsilon\,n_i, \delta u_i\rangle_{\partial\Omega_c} \\
-\langle \upsilon_{i,i}, \delta p\rangle + \langle \upsilon\,n_i, \delta p\rangle_{\partial\Omega_a} + \langle \upsilon\,n_i, \delta p\rangle_{\partial\Omega_c} \\
-\langle \mathrm{k}^2\upsilon, \delta p\rangle - \langle f(\beta - j\gamma\mathrm{k})\upsilon, \delta p\rangle_{\partial\Omega_a} + \langle \lambda_i n_i, \delta p\rangle_{\partial\Omega_c}
\end{array}\right]
\end{aligned}
$$

The structural boundary in Eq. 2.198 is defined as $\partial\Omega_s = \partial\Omega_z \cup \partial\Omega_u \cup \partial\Omega_c$. The fields $\lambda_i = 0$ and $u_i = 0$ on $\partial\Omega_u$ since $\lambda_i \in \mathcal{U}_i$ and $u_i \in \mathcal{U}_i$. Hence, the application of the divergence theorem yields a set of homogeneous boundary conditions on $\partial\Omega_u$. Furthermore, in order to ensure the validity of the operators derived in Eq. 2.198, proper Neumann boundary condition are required. Thus, a set of free Neumann boundary conditions is defined on $\partial\Omega_z$, i.e. $(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j = 0$ on $\partial\Omega_z$. Finally, notice that the derivation shown in Eq. 2.198, yields a set of nonhomogeneous Neumann boundary condition on $\partial\Omega_c$ defined as $(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j = \rho_a\omega^2\upsilon\,n_i$.

The acoustic boundary is defined as $\partial\Omega = \partial\Omega_a \cup \partial\Omega_c$. The application of the divergence theorem yields nonhomogeneous Neumann boundary conditions on $\partial\Omega_a$ and $\partial\Omega_c$. Therefore, these sets of boundary conditions are defined as $\upsilon\,n_i = f(\beta - j\gamma\mathrm{k})\upsilon$ on $\partial\Omega_a$ and $\upsilon\,n_i =$

$\lambda_i n_i$ on $\partial\Omega_c$. These sets of structural and acoustic boundary conditions ensure the validity of the equations defined in Eq. 2.198.

$$
\begin{aligned}
\langle g_z(u, z_i), \delta z_i\rangle = \frac{\partial}{\partial z_i}\Bigg(& \langle \lambda_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l} + u_{l,k}))_{,\,j} - \omega^2 \rho_s\, u_i - f_i\rangle \\
& + \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l} + u_{l,k}))\, n_j - z_i\rangle_{\partial\Omega_z} \\
& + \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l} + u_{l,k}))\, n_j + p\, n_i\rangle_{\partial\Omega_c} \\
& + \langle \upsilon, -p_{i,i} - \mathrm{k}^2 p\rangle + \langle \upsilon, p_i n_i - f(\beta - j - \gamma \mathrm{k})p\rangle_{\partial\Omega_a} \\
& + \langle \upsilon, p_i n_i - \rho_a \omega^2 u_i\, n_i\rangle_{\partial\Omega_c} = 0\Bigg) \quad \forall(\lambda_i, \upsilon) \in \mathcal{U}_i, \times\mathcal{P} \\
& = \langle \lambda_i, \delta z_i\rangle_{\partial\Omega_z}
\end{aligned}
\tag{2.199}
$$

$$
\begin{aligned}
\langle \delta z_i, (g_z(u, z_i))^*\rangle = \frac{\partial}{\partial z_i}\Bigg(& \langle \lambda_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l} + u_{l,k}))_{,\,j} - \omega^2 \rho_s\, u_i - f_i\rangle \\
& + \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l} + u_{l,k}))\, n_j - z_i\rangle_{\partial\Omega_z} \\
& + \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l} + u_{l,k}))\, n_j + p\, n_i\rangle_{\partial\Omega_c} \\
& + \langle \upsilon, -p_{i,i} - \mathrm{k}^2 p\rangle + \langle \upsilon, p_i n_i - f(\beta - j - \gamma \mathrm{k})p\rangle_{\partial\Omega_a} \\
& + \langle \upsilon, p_i n_i - \rho_a \omega^2 u_i\, n_i\rangle_{\partial\Omega_c} = 0\Bigg) \quad \forall(\lambda_i, \upsilon) \in \mathcal{U}_i, \times\mathcal{P} \\
& = \langle \lambda_i, \delta z_i\rangle_{\partial\Omega_z} \\
& = \langle \delta z_i, \lambda_i\rangle^*_{\partial\Omega_z}
\end{aligned}
\tag{2.200}
$$

$$
\langle (g_{zz}(u, z_i))^*, \delta z_i\rangle = \frac{\partial}{\partial z_i}\Bigg(\langle \mathbf{1}, \lambda_i\rangle_{\partial\Omega_z}\Bigg) = \mathbf{0} \tag{2.201}
$$

$$
\langle (g_{zu}(u, z_i))^*, \delta u\rangle = \frac{\partial}{\partial u}\Bigg(\langle \mathbf{1}, \lambda_i\rangle_{\partial\Omega_z}\Bigg) = \mathbf{0} \tag{2.202}
$$

$$
\begin{aligned}
\langle \delta u, (g_u(u,z_i))^* \rangle &= \langle \lambda_i, -(\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))_{,j} - \omega^2 \rho_s\, u_i - f_i \rangle \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j - z_i \rangle_{\partial\Omega_z} \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j + p\, n_i \rangle_{\partial\Omega_c} \\
&+ \langle \upsilon, -p_{i,i} - \mathrm{k}^2 p \rangle + \langle \upsilon, p_i n_i - f(\beta - j - \gamma \mathrm{k}) p \rangle_{\partial\Omega_a} \\
&+ \langle \upsilon, p_i n_i - \rho_a \omega^2 u_i\, n_i \rangle_{\partial\Omega_c} = 0 \quad \forall (\lambda_i, \upsilon) \in \mathcal{U}_i, \times \mathcal{P} \\
&= \langle \frac{1}{2}(\lambda_{i,j} + \lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}) \rangle \\
&- \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j \rangle_{\partial\Omega_s} \\
&+ \langle \lambda_i, -\omega^2 \rho_s\, u_i - f_i \rangle + \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j - z_i \rangle_{\partial\Omega_z} \\
&+ \langle \lambda_i, (\mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}))\, n_j + p\, n_i \rangle_{\partial\Omega_c} + \langle \upsilon_i, p_i \rangle - \langle \upsilon, p_i n_i \rangle_{\partial\Omega_a} \\
&- \langle \upsilon, p_i n_i \rangle_{\partial\Omega_c} - \langle \upsilon, \mathrm{k}^2 p \rangle + \langle \upsilon, p_i n_i - f(\beta - j\gamma \mathrm{k}) p \rangle_{\partial\Omega_a} \\
&+ \langle \upsilon, p_i n_i - \rho_a \omega^2 u_i\, n_i \rangle_{\partial\Omega_c} \\
&= \frac{\partial}{\partial u}\Bigg(\langle \frac{1}{2}(\lambda_{i,j} + \lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(u_{k,l}+u_{l,k}) \rangle + \langle \lambda_i, -\omega^2 \rho_s\, u_i - f_i \rangle \\
&- \langle \lambda_i, z_i \rangle_{\partial\Omega_z} + \langle \lambda_i, p\, n_i \rangle_{\partial\Omega_c} + \langle \upsilon_i, p_i \rangle \\
&- \langle \upsilon, \mathrm{k}^2 p \rangle - \langle \upsilon, f(\beta - j\gamma \mathrm{k}) p \rangle_{\partial\Omega_a} - \langle \upsilon, \rho_a \omega^2 u_i\, n_i \rangle_{\partial\Omega_c} \Bigg) \\
&= \langle \frac{1}{2}(\lambda_{i,j} + \lambda_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(\delta u_{k,l}+\delta u_{l,k}) \rangle + \langle \lambda_i, -\omega^2 \rho_s\, \delta u_i \rangle \\
&+ \langle \lambda_i, \delta p\, n_i \rangle_{\partial\Omega_c} + \langle \upsilon_i, \delta p_i \rangle - \langle \upsilon, \mathrm{k}^2 \delta p \rangle - \langle \upsilon, f(\beta - j\gamma \mathrm{k}) \delta p \rangle_{\partial\Omega_a} \\
&- \langle \upsilon, \rho_a \omega^2 \delta u_i\, n_i \rangle_{\partial\Omega_c} \\
&= \langle \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i}), \mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l} - \lambda_{l,k}) \rangle^* + \langle \delta u_i, \omega^2 \rho_s\, \lambda_i \rangle^* \\
&+ \langle \delta p, \lambda_i\, n_i \rangle^*_{\partial\Omega_c} + \langle \delta p_i, \upsilon_i \rangle^* - \langle \delta p, \mathrm{k}^2 \upsilon \rangle^* - \langle \delta p, f(\beta + j\gamma \mathrm{k}) \upsilon \rangle^*_{\partial\Omega_a} \\
&- \langle \delta u_i, \rho_a \omega^2 \upsilon\, n_i \rangle^*_{\partial\Omega_c} \\
&= \begin{bmatrix}
-\langle (\delta u_i, \mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{i,j} + \lambda_{j,i}))_{,j} \rangle^* \\
+\langle (\delta u_i, \mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{i,j} + \lambda_{j,i}))\, n_j \rangle^*_{\partial\Omega_c} \\
-\langle \delta u_i, \omega^2 \rho_s\, \lambda_i \rangle^* - \langle \delta u_i, \rho_a \omega^2 \upsilon\, n_i \rangle^*_{\partial\Omega_c} \\
-\langle \delta p, \upsilon_{i,i} \rangle^* + \langle \delta p, \upsilon\, n_i \rangle^*_{\partial\Omega_a} + \langle \delta p, \upsilon\, n_i \rangle^*_{\partial\Omega_c} \\
-\langle \delta p, \mathrm{k}^2 \upsilon \rangle^* - \langle \delta p, f(\beta - j\gamma \mathrm{k}) \upsilon \rangle^*_{\partial\Omega_a} + \langle \delta p, \lambda_i n_i \rangle^*_{\partial\Omega_c}
\end{bmatrix}
\end{aligned}
\tag{2.203}
$$

The structural boundary in Eq. 2.203 is defined as $\partial\Omega_s = \partial\Omega_z \cup \partial\Omega_u \cup \partial\Omega_c$. The fields $\lambda_i = 0$ and $u_i = 0$ on $\partial\Omega_u$ since $\lambda_i \in \mathcal{U}_i$ and $u_i \in \mathcal{U}_i$. Hence, the application of the divergence theorem yields a set of homogeneous boundary conditions on $\partial\Omega_u$. Furthermore, in order to ensure a self-adjoint operator, proper Neumann boundary condition are required. Thus, a set of free Neumann boundary conditions is defined on $\partial\Omega_z$, i.e. $(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l} + \lambda_{l,k}))\, n_j = 0$ on $\partial\Omega_z$. Finally, a set of nonhomogeneous Neumann boundary condition is defined on

$\partial\Omega_c$. This set of nonhomogeneous Neumann boundary condition is a consequence of the derivations shown in Eq. 2.203 and is defined as $(\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{k,l}+\lambda_{l,k}))\,n_j = \rho_a\omega^2\upsilon\,n_i$.

The acoustic boundary is defined as $\partial\Omega = \partial\Omega_a \cup \partial\Omega_c$. The application of the divergence theorem yields nonhomogeneous Neumann boundary conditions on $\partial\Omega_a$ and $\partial\Omega_c$. Therefore, these sets of boundary conditions are defined as $\upsilon\,n_i = f(\beta + j\gamma\mathrm{k})\upsilon$ on $\partial\Omega_a$ and $\upsilon\,n_i = \lambda_i n_i$ on $\partial\Omega_c$. Notice the change in sign of the Sommerfeld radiation boundary condition imaginary component. This change in sign is due to the adjoint operation. These sets of structural and acoustic boundary conditions ensure the validity of the adjoint equations defined in Eq. 2.203.

$$\langle (g_{uu}(u,z_i))^*, \delta u\rangle = \frac{\partial}{\partial u}\left(\left[\begin{array}{c} -\langle(\mathbf{1}, \mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})),_{j}\rangle \\ +\langle \mathbf{1}, (\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}))\,n_j\rangle_{\partial\Omega_c} \\ -\langle \mathbf{1}, \omega^2\rho_s\,\lambda_i\rangle - \langle \mathbf{1}, \rho_a\omega^2\upsilon\,n_i\rangle_{\partial\Omega_c} \\ -\langle \mathbf{1}, \upsilon_{i,i}\rangle + \langle \mathbf{1}, \upsilon\,n_i\rangle_{\partial\Omega_a} + \langle \mathbf{1}, \upsilon\,n_i\rangle_{\partial\Omega_c} \\ -\langle \mathbf{1}, f(\beta - j\gamma\mathrm{k})\upsilon\rangle_{\partial\Omega_a} \\ -\langle \mathbf{1}, \mathrm{k}^2\upsilon\rangle + \langle \mathbf{1}, \lambda_i n_i\rangle_{\partial\Omega_c} \end{array}\right]\right) = \mathbf{0} \tag{2.204}$$

$$\langle (g_{uz}(u,z_i))^*, \delta z_i\rangle = \frac{\partial}{\partial z_i}\left(\left[\begin{array}{c} -\langle \mathbf{1}, (\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{i,j}+\lambda_{j,i})),_{j}\rangle \\ +\langle \mathbf{1}, (\mathbf{C}_{ijkl}\,\frac{1}{2}(\lambda_{i,j}+\lambda_{j,i}))\,n_j\rangle_{\partial\Omega_c} \\ -\langle \mathbf{1}, \omega^2\rho_s\,\lambda_i\rangle - \langle \mathbf{1}, \rho_a\omega^2\upsilon\,n_i\rangle_{\partial\Omega_c} \\ -\langle \mathbf{1}, \upsilon_{i,i}\rangle + \langle \mathbf{1}, \upsilon\,n_i\rangle_{\partial\Omega_a} + \langle \mathbf{1}, \upsilon\,n_i\rangle_{\partial\Omega_c} \\ -\langle \mathbf{1}, f(\beta - j\gamma\mathrm{k})\upsilon\rangle_{\partial\Omega_a} \\ -\langle \mathbf{1}, \mathrm{k}^2\upsilon\rangle + \langle \mathbf{1}, \lambda_i n_i\rangle_{\partial\Omega_c} \end{array}\right]\right) = \mathbf{0} \tag{2.205}$$

Discretization. Lets define finite-dimensional subspaces $\mathcal{U}_i^h \subset \mathcal{U}_i$ with basis $\mathcal{U}_i^h = \mathrm{span}\{\phi_i^m\}_{m=1}^M$, $\mathcal{P}^h \subset \mathcal{P}$ with basis $\mathcal{P}^h = \mathrm{span}\{\psi^n\}_{n=1}^N$, and $\mathcal{Z}_i^h \subset \mathcal{Z}_i$ with basis $\{\theta_i^o\}_{o=1}^O$. This leads to $u_i^h = \sum_{m=1}^M u_i^m\phi_i^m$, $p^h = \sum_{n=1}^N p^n\psi^n$, $z_i^h = \sum_{o=1}^O z_i^o\theta_i^o$, $\lambda_i^h = \sum_{m=1}^M \lambda_i^m\phi_i^m$, and $\upsilon^h = \sum_{n=1}^N \upsilon^n\psi^n$. Furthermore, the quantities δu_i^h, δp^h, δz_i^h, $\delta\lambda_i$, and $\delta\upsilon^h$ can be respectively approximated as follow: $\delta u_i^h = \sum_{m=1}^M \delta u_i^m\phi_i^m$, $\delta p^h = \sum_{n=1}^N p^n\psi^n$, $\delta z_i^h = \sum_{o=1}^O z_i^o\theta_i^o$, $\delta\lambda_i^h = \sum_{m=1}^M \delta\lambda_i^m\phi_i^m$, and $\delta\upsilon^h = \sum_{n=1}^N \delta\upsilon^n\psi^n$. Subsequently, the Galerkin approximation of the required operators to solve the optimal control problem are defined as

$$J(u^h, z_i^h) = \frac{\alpha}{2}\langle u_i^h - \widehat{u}_i, u_i^h - \widehat{u}_i\rangle + R(z_i^h) \tag{2.206}$$

$$J_u(u^h, z_i^h) = \sum_{e=1}^{N_{elem}}\left(\sum_{a=1}^{M}\alpha\langle\phi^a, u_i^h - \widehat{u}_i\rangle\right) \tag{2.207}$$

$$J_z(u^h, z_i^h) = \sum_{e=1}^{N_{elem}}\left(\sum_{a=1}^{O}\langle\theta^a, R_z(z_i^h)\right) \tag{2.208}$$

$$J_{uu}(u^h, z_i^h) = \sum_{e=1}^{N_{elem}}\left(\sum_{a=1}^{M}\sum_{b=1}^{M}\alpha\langle\phi^a, \phi^b\rangle\right) \tag{2.209}$$

$$J_{uz}(u^h, z_i^h) = \mathbf{0} \tag{2.210}$$

$$J_{zz}(u^h, z_i^h) = \sum_{e=1}^{N_{elem}} \left(\sum_{a=1}^{O} \sum_{b=1}^{O} \langle \theta^a, R_{zz}(z_i^h) \theta^b \rangle \right) \tag{2.211}$$

$$J_{zu}(u^h, z_i^h) = \mathbf{0} \tag{2.212}$$

$$g(u^h, z_i^h) = \sum_{e=1}^{N_{elem}} \left(\begin{bmatrix} \sum_{a=1}^{M} \langle \frac{1}{2}(\phi^a_{i,j} + \phi^a_{j,i}), \mathbf{C}_{ijkl}\, \frac{1}{2}(u^h_{k,l} + u^h_{l,k}) \rangle \\ - \sum_{a=1}^{M} \langle \phi^a_i, \rho_s\, \omega^2\, u^h_i + f_i \rangle \\ - \sum_{a=1}^{M} \langle \phi^a_i, z_i \rangle_{\partial\Omega_z} + \sum_{a=1}^{M} \langle \phi^a_i, p^h\, n_i \rangle_{\partial\Omega_c} = 0 \\ \sum_{b=1}^{N} \langle \psi^b_i, p^h_i \rangle - \sum_{b=1}^{N} \langle \psi^b_i, \mathrm{k}^2\, \rho_a\, p^h \rangle \\ - \sum_{b=1}^{N} \langle \psi^b, f(\beta - j\gamma \mathrm{k})\, p^h \rangle_{\partial\Omega_a} \\ - \sum_{b=1}^{N} \langle \psi^b, \rho_a\, \omega^2\, u^h_i\, n_i \rangle_{\partial\Omega_c} = 0 \end{bmatrix} \right) \tag{2.213}$$

$$g_u(u^h, z_i^h) = \sum_{e=1}^{N_{elem}} \left(\begin{bmatrix} \sum_{a=1}^{M} \langle \frac{1}{2}(\phi^a_{i,j} + \phi^a_{j,i}), \mathbf{C}_{ijkl}\, \frac{1}{2}(\lambda^h_{k,l} + \lambda^h_{l,k}) \rangle \\ - \sum_{a=1}^{M} \langle \phi^a_i, \rho_s\, \omega^2\, \lambda^h_i \rangle + \sum_{a=1}^{M} \langle \phi^a_i, \rho_a\, \omega^2\, \upsilon^h\, n_i \rangle_{\partial\Omega_c} = 0 \\ \sum_{b=1}^{N} \langle \psi^b_i, \upsilon^h_i \rangle - \sum_{b=1}^{N} \langle \psi^b, \mathrm{k}^2\, \rho_a\, \upsilon^h \rangle \\ - \sum_{b=1}^{N} \langle \psi^b, f(\beta - j\gamma \mathrm{k})\, \upsilon^h \rangle_{\partial\Omega_a} \\ - \sum_{b=1}^{N} \langle \psi^b, \lambda^h_i\, n_i \rangle_{\partial\Omega_c} = 0 \end{bmatrix} \right) \tag{2.214}$$

$$g_z(u^h, z_i^h) = \sum_{e=1}^{N_{elem}} \left(\sum_{a=1}^{O} \langle \lambda^h_i, \theta^a_i \rangle \right) \tag{2.215}$$

$$(g_u(u^h, z_i^h))^* = \sum_{e=1}^{N_{elem}} \left(\begin{bmatrix} \sum_{a=1}^{M} \langle \frac{1}{2}(\phi^a_{i,j} + \phi^a_{j,i}), \mathbf{C}_{ijkl}\, \frac{1}{2}(\lambda^h_{k,l} + \lambda^h_{l,k}) \rangle \\ - \sum_{a=1}^{M} \langle \phi^a_i, \rho_s\, \omega^2\, \lambda^h_i \rangle \\ + \sum_{a=1}^{M} \langle \phi^a_i, \rho_a\, \omega^2\, \upsilon^h\, n_i \rangle_{\partial\Omega_c} = 0 \\ \sum_{b=1}^{N} \langle \psi^b_i, \upsilon^h_i \rangle - \sum_{b=1}^{N} \langle \psi^b, \mathrm{k}^2\, \rho_a\, \upsilon^h \rangle \\ - \sum_{b=1}^{N} \langle \psi^b, f(\beta - j\gamma \mathrm{k})\, \upsilon^h \rangle_{\partial\Omega_a} \\ - \sum_{b=1}^{N} \langle \psi^b, \lambda^h_i\, n_i \rangle_{\partial\Omega_c} = 0 \end{bmatrix} \right) \tag{2.216}$$

$$(g_z(u^h, z_i^h))^* = \sum_{e=1}^{N_{elem}} \left(\sum_{a=1}^{O} \langle \theta^a_i, \lambda^h_i \rangle \right) \tag{2.217}$$

$$(g_{uu}(u^h, z_i^h))^* = \mathbf{0} \tag{2.218}$$

$$(g_{uz}(u^h, z_i^h))^* = \mathbf{0} \tag{2.219}$$

$$(g_{zz}(u^h, z_i^h))^* = \mathbf{0} \tag{2.220}$$

$$(g_{zu}(u^h, z_i^h))^* = \mathbf{0} \tag{2.221}$$

Optimal Control in Steady-State Acoustics

An optimal control problem in steady-state acoustics is to estimate the source term g_i to obtain a desired pressure field $\widehat{u}$. We define $\Omega \subseteq \mathbb{R}^d$, $d \in \{1, 2, 3\}$, as the computational domain with boundary $\partial\Omega = \partial\Omega_u \cup \partial\Omega_r$. The regions $\partial\Omega_u$ and $\partial\Omega_r$ are the boundaries where Dirichlet and Neumann conditions are applied, respectively. Once more, we use standard tensor notation with Einstein summation. The index i takes on the values $1, \ldots, d$. Partial differentiation is denoted by a comma.

The PDEs of steady-state acoustics are given by

$$\begin{aligned} -u_{i,i} - \mathrm{k}^2 u = z \qquad & \text{in} \quad \Omega \\ u = 0 \qquad & \text{on} \quad \partial\Omega_u \\ u_i n_i = f\,(\beta - j\gamma \mathrm{k})\,u \qquad & \text{on} \quad \partial\Omega_r, \end{aligned} \tag{2.222}$$

where u denotes the complex-valued pressure field. The fluid mass density is denoted by ρ_f. The acoustic wave number is given by $\mathrm{k} = \omega\sqrt{\kappa_f \,/\, \rho_f}$, where κ_f is the fluid bulk modulus. Angular frequency is denoted by ω.

Impedance boundary conditions are imposed on $\partial\Omega_r$ to allow the acoustic wave to pass through and not reflect back into the computational domain. The geometric factor related to the metric factors of the curvilinear coordinate system used on the boundary is denoted by γ and the spreading loss term is denoted by β.

Lets consider the following optimal control problem

$$\begin{aligned} &\underset{(u,z)\,\in\,\mathcal{U}\times\mathcal{Z}}{\text{minimize}} \qquad \frac{\alpha}{2}\langle u - \widehat{u}, u - \widehat{u}\rangle + R(z) \\ &\text{subject to} \\ &\qquad -u_{i,i} - \mathrm{k}^2 u = z \qquad \text{in} \quad \Omega, \\ &\qquad u_i n_i = f\,(\beta - j\gamma \mathrm{k})\,u \qquad \text{in} \quad \partial\Omega_r \end{aligned} \tag{2.223}$$

where $\mathcal{U} = \{u \colon u \in H^1(\Omega)\,,\ u = 0 \text{ on } \partial\Omega_u\}$, $\mathcal{Z} = \{z \colon z \in L^2(\Omega)\}$. $H^m(\Omega)$ are Sobolev spaces of square integrable functionals whose m-th derivatives are also square integrable. Whenever $m = 0$ we shall keep the notation with $L^2(\Omega)$.

First-order neccesary optimality conditions. The Lagrangian associated with the optimal control problem defined in Eq. 2.223 is given by

$$\begin{aligned} \mathcal{L}(u, g, \lambda) = &\frac{\alpha}{2}\langle u - \widehat{u}, u - \widehat{u}\rangle + R(z) \\ &+ \Re[\langle \lambda, -u_{i,i} - \mathrm{k}^2 u - z\rangle + \langle \lambda, u_i n_i - f(\beta - j\gamma\mathrm{k})u)\rangle] \end{aligned} \tag{2.224}$$

If $\{\tilde{u}, \tilde{z}\} \in \mathcal{U} \times \mathcal{Z}$ is a local solution to the optimal control problem in Eq. 2.2, there exists Lagrange multipliers $\lambda \in \mathcal{U}$, such that the first-order necessary optimality conditions hold at $\{\tilde{u}, \tilde{z}\}$, i.e.

$$\nabla\mathcal{L}(u, z, \lambda)(\delta u, \delta z, \delta\lambda) = \mathcal{L}_u(u, z, \lambda)\delta u + \mathcal{L}_z(u, z, \lambda)\delta z + \mathcal{L}_\lambda(u, z, \lambda)\delta\lambda = 0 \tag{2.225}$$

Second-Order Sufficient Conditions. If $(\tilde{u}, \tilde{z}, \lambda) \in \mathcal{U} \times \mathcal{Z} \times \mathcal{U}$ satisfy the first-order necessary optimality conditions and the Hessian operator exist and is positive semidefinite, then $(\tilde{u}, \tilde{z})$ is a strict local minimum.

Operators. The operators required to solve the optimal control problem are defined as

$$J(u, z) = \frac{\alpha}{2}\langle u - \widehat{u}, u - \widehat{u}\rangle + R(z) \tag{2.226}$$

$$\begin{aligned}\langle J_u(u, z), \delta u\rangle &= \frac{\partial}{\partial u}\left(\frac{\alpha}{2}\langle u - \widehat{u}, u - \widehat{u}\rangle + R(z)\right) \\ &= \alpha\langle u - \widehat{u}, \delta u\rangle\end{aligned} \tag{2.227}$$

$$\begin{aligned}\langle J_z(u, z), \delta z\rangle &= \frac{\partial}{\partial z}\left(\frac{\alpha}{2}\langle u - \widehat{u}, u - \widehat{u}\rangle + R(z)\right) \\ &= \langle R_z(z), \delta z\rangle\end{aligned} \tag{2.228}$$

$$\begin{aligned}\langle J_{uu}(u, z), \delta u\rangle &= \frac{\partial}{\partial u}\Big(\alpha\langle \mathbf{1}, u - \widehat{u}\rangle\Big) \\ &= \alpha\langle \mathbf{1}, \delta u\rangle\end{aligned} \tag{2.229}$$

$$\langle J_{uz}(u, z), \delta z\rangle = \frac{\partial}{\partial z}\Big(\alpha\langle \mathbf{1}, u - \widehat{u}\rangle\Big) = \mathbf{0} \tag{2.230}$$

$$\begin{aligned}\langle J_{zz}(u, z), \delta z\rangle &= \frac{\partial}{\partial z}\Big(\langle \mathbf{1}, R_z(z)\rangle\Big) \\ &= \langle R_{zz}(z), \delta z\rangle\end{aligned} \tag{2.231}$$

$$\langle J_{zu}(u, z), \delta u\rangle = \frac{\partial}{\partial u}\Big(\langle \mathbf{1}, R_z(z)\rangle\Big) = \mathbf{0} \tag{2.232}$$

$$g(u, z) = \left[\begin{array}{ccc} -u_{i,i} - \mathrm{k}^2 u = z & \text{in} & \Omega \\ u_i n_i = f(\beta - j\gamma\mathrm{k})u & \text{on} & \partial\Omega_r \end{array}\right] \tag{2.233}$$

$$\begin{aligned}\langle g_u(u, z), \delta u\rangle &= \langle \lambda, -u_{i,i} - \mathrm{k}^2 u - z\rangle + \langle \lambda, u_i n_i - f(\beta - j\gamma\mathrm{k})u\rangle_{\partial\Omega_r} = 0 \ \forall\ \lambda \in \mathcal{U} \\ &= \langle \lambda_i, u_i - \mathrm{k}^2 u - z\rangle - \langle \lambda, u_i n_i\rangle_{\partial\Omega} + \langle \lambda, u_i n_i - f(\beta - j\gamma\mathrm{k})u\rangle_{\partial\Omega_r} \\ &= \langle \lambda_i, u_i - \mathrm{k}^2 u - z\rangle - \langle \lambda, f(\beta - j\gamma\mathrm{k})u\rangle_{\partial\Omega_r} \\ &= \langle -\lambda_{i,i}, u\rangle + \langle \lambda_i n_i, u\rangle_{\partial\Omega} - \langle \lambda, \mathrm{k}^2 u + z\rangle - \langle \lambda, f(\beta - j\gamma\mathrm{k})u\rangle_{\partial\Omega_r} \\ &= \frac{\partial}{\partial u}\Big(\langle -\lambda_{i,i}, u\rangle - \langle \lambda, \mathrm{k}^2 u + z\rangle + \langle \lambda_i n_i, u\rangle_{\partial\Omega_r} - \langle \lambda, f(\beta - j\gamma\mathrm{k})u\rangle_{\partial\Omega_r}\Big) \\ &= \langle -\lambda_{i,i}, \delta u\rangle - \langle \mathrm{k}^2\lambda, \delta u\rangle + \langle \lambda_i n_i, \delta u\rangle_{\partial\Omega_r} - \langle f(\beta - j\gamma\mathrm{k})\lambda, \delta u\rangle_{\partial\Omega_r}\end{aligned} \tag{2.234}$$

In Eq. 2.234 the boundary is defined as $\partial\Omega = \partial\Omega_r \cup \partial\Omega_u$. Since $\lambda \in \mathcal{U}$ and $u \in \mathcal{U}$, then $\lambda = u = 0$ on $\partial\Omega_u$. Hence, the application of the divergence theorem results in a set of homogeneous boundary conditions on $\partial\Omega_u$. However, the Sommerfeld radiation boundary condition are nonhomogeneous on $\partial\Omega_r$, i.e. $u \neq 0$. Resulting in a set of nonhomogeneous boundary condition defined as $\lambda_i n_i = f(\beta - j\gamma \mathrm{k})\lambda$ on $\partial\Omega_r$.

$$\begin{aligned}
\langle g_z(u,z), \delta z\rangle &= \langle \lambda, -u_{i,i} - \mathrm{k}^2 u - z\rangle + \langle \lambda, u_i n_i - f(\beta - j\gamma\mathrm{k})u\rangle_{\partial\Omega_r} = 0 \ \forall\ \lambda \in \mathcal{U} \\
&= \langle \lambda_i, u_i - \mathrm{k}^2 u - z\rangle - \langle \lambda, u_i n_i\rangle_{\partial\Omega} + \langle \lambda, u_i n_i - f(\beta - j\gamma\mathrm{k})u\rangle_{\partial\Omega_r} \\
&= \langle \lambda_i, u_i - \mathrm{k}^2 u - z\rangle - \langle \lambda, f(\beta - j\gamma\mathrm{k})u\rangle_{\partial\Omega_r} \\
&= \langle -\lambda_{i,i}, u\rangle + \langle \lambda_i n_i, u\rangle_{\partial\Omega} - \langle \lambda, \mathrm{k}^2 u + z\rangle - \langle \lambda, f(\beta - j\gamma\mathrm{k})u\rangle_{\partial\Omega_r} \\
&= \frac{\partial}{\partial z}\Bigg(\langle -\lambda_{i,i}, u\rangle - \langle \lambda, \mathrm{k}^2 u + z\rangle + \langle \lambda_i n_i, u\rangle_{\partial\Omega_r} - \langle \lambda, f(\beta - j\gamma\mathrm{k})u\rangle_{\partial\Omega_r}\Bigg) \\
&= -\langle \lambda, \delta z\rangle
\end{aligned} \tag{2.235}$$

$$\begin{aligned}
\langle \delta u, (g_u(u,z))^*\rangle &= \langle \lambda, -u_{i,i} - \mathrm{k}^2 u - z\rangle + \langle \lambda, u_i n_i - f(\beta - j\gamma\mathrm{k})u\rangle_{\partial\Omega_r} = 0 \ \forall\ \lambda \in \mathcal{U} \\
&= \langle \lambda_i, u_i - \mathrm{k}^2 u - z\rangle - \langle \lambda, u_i n_i\rangle_{\partial\Omega} + \langle \lambda, u_i n_i - f(\beta - j\gamma\mathrm{k})u\rangle_{\partial\Omega_r} \\
&= \langle \lambda_i, u_i - \mathrm{k}^2 u - z\rangle - \langle \lambda, f(\beta - j\gamma\mathrm{k})u\rangle_{\partial\Omega_r} \\
&= \langle -\lambda_{i,i}, u\rangle + \langle \lambda_i n_i, u\rangle_{\partial\Omega} - \langle \lambda, \mathrm{k}^2 u + z\rangle - \langle \lambda, f(\beta - j\gamma\mathrm{k})u\rangle_{\partial\Omega_r} \\
&= \langle u, -\lambda_{i,i}\rangle^* + \langle u, \lambda_i n_i\rangle^*_{\partial\Omega} - \langle \mathrm{k}^2 u + z, \lambda\rangle^* - \langle u, f(\beta + j\gamma\mathrm{k})\lambda\rangle_{\partial\Omega_r} \\
&= \frac{\partial}{\partial u}\Bigg(\langle u, -\lambda_{i,i}\rangle^* - \langle \mathrm{k}^2 u + z, \lambda\rangle^* + \langle u, \lambda_i n_i\rangle^*_{\partial\Omega_r} - \langle u, f(\beta + j\gamma\mathrm{k})\lambda\rangle^*_{\partial\Omega_r}\Bigg) \\
&= \langle \delta u, -\lambda_{i,i}\rangle^* - \langle \delta u, \mathrm{k}^2\lambda\rangle^* + \langle \delta u, \lambda_i n_i\rangle^*_{\partial\Omega_r} - \langle \delta u, f(\beta + j\gamma\mathrm{k})\lambda\rangle^*_{\partial\Omega_r}
\end{aligned} \tag{2.236}$$

In Eq. 2.236 the boundary is defined as $\partial\Omega = \partial\Omega_r \cup \partial\Omega_u$. Since $\lambda \in \mathcal{U}$ and $u \in \mathcal{U}$, then $\lambda = u = 0$ on $\partial\Omega_u$. Hence, the application of the divergence theorem results in a set of homogeneous boundary conditions on $\partial\Omega_u$. However, once again, the Sommerfeld radiation boundary condition are nonhomogeneous on $\partial\Omega_r$. This yields a set of nonhomogeneous boundary condition defined as $\lambda_i n_i = f(\beta + j\gamma\mathrm{k})\lambda$ on $\partial\Omega_r$. Notice the change in sign of the Sommerfeld radiation boundary condition imaginary component. This change in sign is due to the adjoint operation.

$$\begin{aligned}
\langle \delta z, (g_z(u,z))^*\rangle &= \langle \lambda, -u_{i,i} - \mathrm{k}^2 u - z\rangle + \langle \lambda, u_i n_i - f(\beta - j\gamma\mathrm{k})u\rangle_{\partial\Omega_r} = 0 \ \forall\ \lambda \in \mathcal{U} \\
&= \langle \lambda_i, u_i - \mathrm{k}^2 u - z\rangle - \langle \lambda, u_i n_i\rangle_{\partial\Omega} + \langle \lambda, u_i n_i - f(\beta - j\gamma\mathrm{k})u\rangle_{\partial\Omega_r} \\
&= \langle \lambda_i, u_i - \mathrm{k}^2 u - z\rangle - \langle \lambda, f(\beta - j\gamma\mathrm{k})u\rangle_{\partial\Omega_r} \\
&= \langle u, -\lambda_{i,i}\rangle^* + \langle u, \lambda_i n_i\rangle^*_{\partial\Omega} - \langle \mathrm{k}^2 u + z, \lambda\rangle^* - \langle u, f(\beta + j\gamma\mathrm{k})\lambda\rangle_{\partial\Omega_r} \\
&= \frac{\partial}{\partial z}\Bigg(\langle u, -\lambda_{i,i}\rangle^* - \langle \mathrm{k}^2 u + z, \lambda\rangle^* + \langle u, \lambda_i n_i\rangle^*_{\partial\Omega_r} - \langle u, f(\beta + j\gamma\mathrm{k})\lambda\rangle^*_{\partial\Omega_r}\Bigg) \\
&= -\langle \delta z, \lambda\rangle^*
\end{aligned} \tag{2.237}$$

$$\langle (g_{uu}(u,z))^*, \delta u\rangle = \frac{\partial}{\partial u}\Bigg(\langle \mathbf{1}, -\lambda_{i,i}\rangle - \langle \mathbf{1}, \mathrm{k}^2\lambda\rangle + \langle \mathbf{1}, \lambda_i n_i\rangle_{\partial\Omega_r} - \langle \mathbf{1}, f(\beta + j\gamma\mathrm{k})\lambda\rangle_{\partial\Omega_r}\Bigg) = \mathbf{0} \tag{2.238}$$

$$\langle (g_{uz}(u,z))^*, \delta z\rangle = \frac{\partial}{\partial z}\Bigg(\langle \mathbf{1}, -\lambda_{i,i}\rangle - \langle \mathbf{1}, \mathrm{k}^2\lambda\rangle + \langle \mathbf{1}, \lambda_i n_i\rangle_{\partial\Omega_r} - \langle \mathbf{1}, f(\beta + j\gamma\mathrm{k})\lambda\rangle_{\partial\Omega_r}\Bigg) = \mathbf{0} \tag{2.239}$$

$$\langle (g_{zu}(u,z))^*, \delta u\rangle = \frac{\partial}{\partial u}\Bigg(\langle \mathbf{1}, \lambda\rangle\Bigg) = \mathbf{0} \tag{2.240}$$

$$\langle (g_{zz}(u,z))^*, \delta z\rangle = \frac{\partial}{\partial z}\Bigg(\langle \mathbf{1}, \lambda\rangle\Bigg) = \mathbf{0} \tag{2.241}$$

Discretization. We define finite-dimensional subspaces $\mathcal{U}^h \subset \mathcal{U}$ with basis $\mathcal{U}^h = \mathrm{span}\{\phi^m\}_{m=1}^M$ and $\mathcal{Z}^h \subset \mathcal{Z}$ with basis $\mathcal{Z}^h = \mathrm{span}\{\psi^n\}_{n=1}^N$. This leads to $u^h = \sum_{m=1}^M u^m\phi^m$, $z^h = \sum_{n=1}^N z^n\psi^n$, and $\lambda^h = \sum_{m=1}^M \lambda^m\phi^m$. Furthermore, the quantities δu^h and δz^h can be respectively approximated as follow: $\delta u^h = \sum_{m=1}^M \delta u^m\phi^m$ and $\delta z^h = \sum_{n=1}^N \delta z^n\psi^n$. Subsequently, the Galerkin approximation of the required operators to solve the optimal control problem are defined as

$$J(u^h, z^h) = \frac{\alpha}{2}\langle u^h - \widehat{u}, u^h - \widehat{u}\rangle + R(z^h) \tag{2.242}$$

$$J_u(u^h, z^h) = \sum_{e=1}^{N_{elem}}\Bigg(\sum_{a=1}^{M}\alpha\langle \phi^a, u^h - \widehat{u}\rangle\Bigg) \tag{2.243}$$

$$J_z(u^h, z^h) = \sum_{e=1}^{N_{elem}}\Bigg(\sum_{a=1}^{N}\langle \psi^a, R_z(z^h)\Bigg) \tag{2.244}$$

$$J_{uu}(u^h, z^h) = \sum_{e=1}^{N_{elem}}\Bigg(\sum_{a=1}^{M}\sum_{b=1}^{M}\alpha\langle \phi^a, \phi^b\rangle\Bigg) \tag{2.245}$$

$$J_{uz}(u^h, z^h) = \mathbf{0} \tag{2.246}$$

$$J_{zz}(u^h, z^h) = \sum_{e=1}^{N_{elem}}\Bigg(\sum_{a=1}^{N}\sum_{b=1}^{N}\langle \psi^a, R_{zz}(z^h)\psi^b\rangle\Bigg) \tag{2.247}$$

$$J_{zu}(u^h, z^h) = \mathbf{0} \tag{2.248}$$

$$g(u,z) = \sum_{e=1}^{N_{elem}}\Bigg(\sum_{a=1}^{M}\langle \phi^a_i, u^h_i\rangle - \sum_{a=1}^{M}\langle \phi^a, \mathrm{k}^2 u^h\rangle + \sum_{a=1}^{M}\langle \phi^a, f(\beta - j\gamma\mathrm{k})u^h\rangle_{\partial\Omega_r}\Bigg) \tag{2.249}$$

$$g_u(u,z) = \sum_{e=1}^{N_{elem}}\Bigg(\sum_{a=1}^{M}\langle \phi^a_i, \lambda^h_i\rangle - \sum_{a=1}^{M}\langle \phi^a, \mathrm{k}^2 \lambda^h\rangle + \sum_{a=1}^{M}\langle \phi^a, f(\beta - j\gamma\mathrm{k})\lambda^h\rangle_{\partial\Omega_r}\Bigg) \tag{2.250}$$

$$g_z(u,z) = \sum_{e=1}^{N_{elem}}\Bigg(\sum_{a=1}^{N}\langle \lambda^h, \psi^a\rangle\Bigg) \tag{2.251}$$

$$(g_u(u,z))^* = \sum_{e=1}^{N_{elem}}\Bigg(\sum_{a=1}^{M}\langle \phi^a_i, \lambda^h_i\rangle - \sum_{a=1}^{M}\langle \phi^a, \mathrm{k}^2 \lambda^h\rangle + \sum_{a=1}^{M}\langle \phi^a, f(\beta + j\gamma\mathrm{k})\lambda^h\rangle_{\partial\Omega_r}\Bigg) \tag{2.252}$$

$$(g_z(u,z))^* = \sum_{e=1}^{N_{elem}}\Bigg(\sum_{a=1}^{N}\langle \psi^a, \lambda^h\rangle\Bigg) \tag{2.253}$$

$$(g_{uu}(u,z))^* = \mathbf{0} \tag{2.254}$$

$$(g_{uz}(u,z))^* = \mathbf{0} \tag{2.255}$$

$$(g_{zz}(u,z))^* = \mathbf{0} \tag{2.256}$$

$$(g_{zu}(u,z))^* = \mathbf{0} \tag{2.257}$$

References

[1] D.P. Bertsekas. *Nonlinear Programming.* Athena Scientific, 1999.

[2] J. Nocedal and S. J. Wright. *Numerical Optimization.* Springer, 2006.

[3] P.J. Olver and C. Shakiban. *Applied Linear Algebra.* Prentice Hall, 2005.

[4] R. T. Rockafellar. Lagrange multipliers and optimality. *SIAM Review*, 35:183–238, 1993.

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