Single-Axis Three-Beam Amplitude Monopulse Antenna – Signal Processing Issues

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Abstract

Typically, when three or more antenna beams along a single axis are required, the answer has been multiple antenna phase-centers, essentially a phase-monopulse system. Such systems and their design parameters are well-reported in the literature. Less appreciated is that three or more antenna beams can also be generated in an amplitude-monopulse fashion. Consequently, design guidelines and performance analysis of such antennas is somewhat under-reported in the literature. We provide discussion herein of three beams arrayed in a single axis with an amplitude-monopulse configuration.

Acknowledgements

Contents

Foreword

This report details the results of an academic study. It does not presently exemplify any modes, methodologies, or techniques employed by any operational system known to the authors.

Classification

The specific mathematics and algorithms presented herein do not bear any release restrictions or distribution limitations.

This distribution limitations of this report are in accordance with the classification guidance detailed in the memorandum "Classification Guidance Recommendations for Sandia Radar Testbed Research and Development", DRAFT memorandum from Brett Remund (Deputy Director, RF Remote Sensing Systems, Electronic Systems Center) to Randy Bell (US Department of Energy, NA-22), February 23, 2004. Sandia has adopted this guidance where otherwise none has been given.

This report formalizes preexisting informal notes and other documentation on the subject matter herein.

1 Introduction & Background

It has been appreciated since the early days of radio that direction finding was easiest to achieve by directing a null in an antenna pattern towards a signal source that otherwise exhibits good Signal-to-Noise Ratio (SNR). Nulls are typically sharper than the antenna mainlobe peak, ideally being singular points with zero response in a particular direction, and sharp sidewall (to the notch surrounding the null) responses.

In the classical monopulse tracking radar, this means pointing the antenna such that a null in some 'difference' beam is directed towards the target being tracked. Classically this means a difference signal between two otherwise identical beams, with a characteristic allowing knowledge of which side of the null a target signal manifests to allow steering the null in the proper direction.

A definitive text on dual-beam (in any one direction or axis) monopulse antennas is by Sherman and Barton.[1] We note that although their text analyzes antennas with more than two beams, they are generally referring to dual-axis monopulse, where nevertheless any one axis is limited to only two beams.

In a classical monopulse tracking radar, we are generally concerned with but a single target echo signal, presumed to be unperturbed by any other neighboring target echo signals.

However, for airborne Ground Moving Target Indicator (GMTI) radar, and its derivative Dismount Moving Target Indicator (DMTI) radar, the single target echo signal paradigm fails, and a target must be detected and located in the presence of clutter, i.e. other undesired target-like signals. The overwhelmingly popular antenna architecture for this condition is a multi-aperture (a.k.a. multi-subaperture, multi-phase-center) antenna. These antennas fall within the definition of a phase-monopulse antenna, albeit with more than two phase centers, but phase-monopulse antenna nevertheless.

Among others, Doerry and Bickel discuss Direction of Arrival (DOA) measurements from multiple phase centers in an earlier report.[2] Doerry and Bickel also discuss limits to clutter cancellation using multiple phase centers in another report.[3]

Equivalent to a two-aperture phase-monopulse antenna, the dual-beam amplitude monopulse antenna is generally well understood, and in fact a popular architecture, especially with dish reflector antenna configurations. However, less well appreciated is that just as N phase centers can steer $N-1$ nulls, so too can more nulls be generated by more simultaneous beams along a common axis in a general amplitude monopulse antenna. Amplitude monopulse antennas with more than two beams in any one axis are not well-reported in the literature.

In this report we will analyze and characterize a 3-beam amplitude monopulse antenna, concerning ourselves with all beams in a single axis as might be useful for GMTI/DMTI

applications. We will generally assume a dish reflector antenna with feeds offset from the focal point to yield squinted beams.

Our interest is in the characteristics of the antenna that allow useful processing for the GMTI/DMTI applications. We will not concern ourselves with the details of the physical design of the electro-mechanical antenna instrument itself. In that sense, this report is not about "how to build the antenna right," but rather this report is about "building the right antenna." Our Figure-of-Merit (FOM) will be the noise in the angle estimate for a target signal within the principal, or main reference beam of the antenna. This is consistent with the dual-beam analysis in the Sherman and Barton text.

However, before we examine three beams, we will first review the dual-beam amplitude monopulse antenna characteristics.
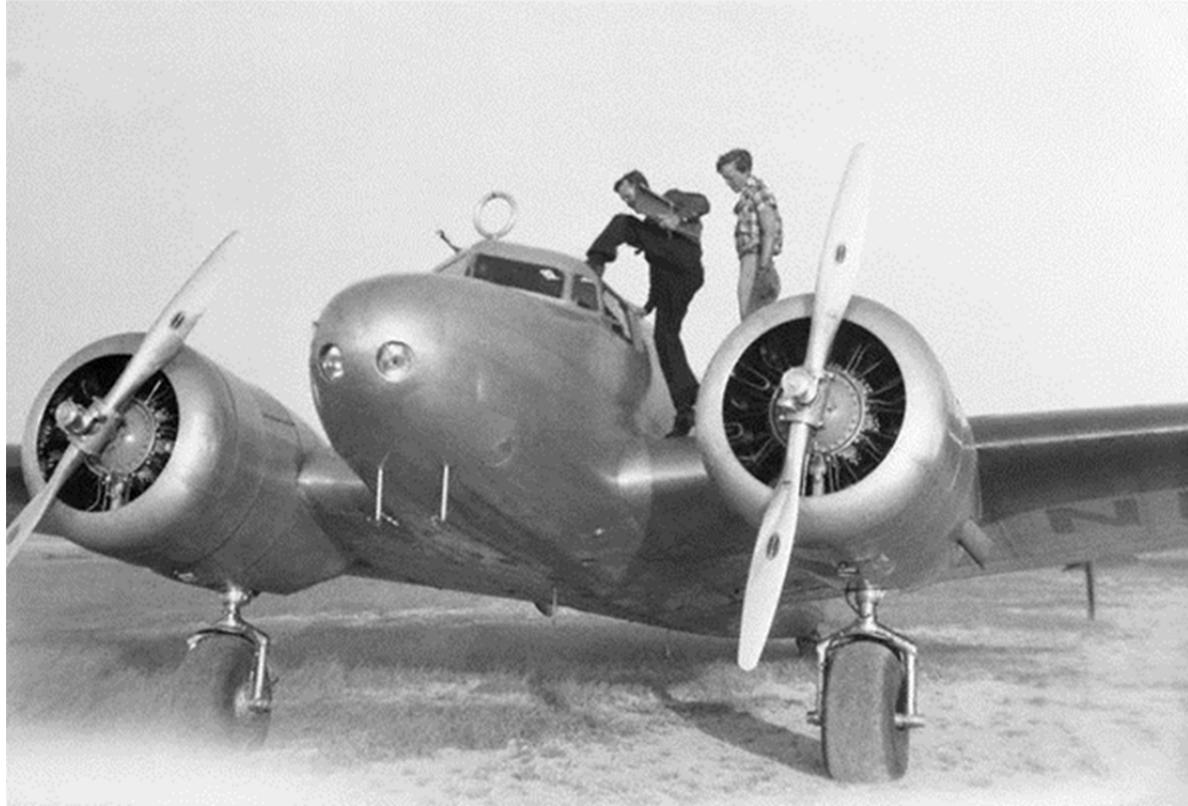
Figure 1. Fred Noonan and Amelia Earhart board their Lockheed Electra aircraft at San Juan, Puerto Rico, on 2 June 1937. Note the Radio Direction Finding (RDF) loop antenna just above the cockpit windshield. The loop antenna has a figure-8 gain pattern, and was employed by steering a null towards a radio signal source at a known location, giving the aircraft a bearing towards the source. Null-steering remains an essential direction finding technique. (Bettmann/CORBIS)

2 Dual-Beam Amplitude Monopulse

Let us assume that the antenna is capable of a beam-shape defined as

$$g_0(\theta) = \text{generalized beam shape,} \tag{1}$$

where

$$\theta = \text{off-boresight angle.} \tag{2}$$

We will for convenience assume that this is symmetric, i.e. an even function, that is

$$g_0(\theta) = g_0(-\theta). \tag{3}$$

For further convenience, we will define the beamwidth of $g_0(\theta)$ as

$$\theta_g = \text{constituent beamwidth, nominally at the } -3 \text{ dBc level.} \tag{4}$$

We now assume that the monopulse antenna pattern is composed of two beams of identical shape, but equally squinted in opposite directions, that is

$$g_1(\theta) = g_0(\theta + \theta_{sq}) = \text{beam \#1, and}$$
$$g_2(\theta) = g_0(\theta - \theta_{sq}) = \text{beam \#2,} \tag{5}$$

where we will assume these patterns to be real-valued, with

$$\pm\theta_{sq} = \text{squint angle offset of the two beams from boresight.} \tag{6}$$

A target echo will be received by both of these beams, albeit generally with different amplitudes. Accordingly, we define voltage measurements from the antennas as

$$m_1 = A_m \, g_1(\theta_m) = \text{signal at first beam, and}$$
$$m_2 = A_m \, g_2(\theta_m) = \text{signal at second beam,} \tag{7}$$

where these quantities are real-valued, and

$$\theta_m = \text{the direction of arrival of a received signal, and}$$
$$A_m = \text{some real scale factor with respect to the beam patterns.} \tag{8}$$

Now we wish to steer a null in the direction of the received signal by linearly combining these signals. Accordingly we can set up the matrix equation

$$\begin{bmatrix} 1 & 0 \\ m_1 & m_2 \end{bmatrix} \mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{9}$$

where

$$\mathbf{w} = \text{weight vector,} \tag{10}$$

with real-valued weights. The purpose of the top row in the matrix of Eq. (9) is to guarantee the non-trivial solution (other than all zero weights).

2.1 Optimal Weight Solution

The matrix equation can be solved for the optimum weight vector as

$$\mathbf{w}_{opt} = \begin{bmatrix} 1 & 0 \\ m_1 & m_2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{11}$$

which can be reduced to

$$\mathbf{w}_{opt} = \begin{bmatrix} 1 & -\dfrac{m_1}{m_2} \end{bmatrix}^{T}, \tag{12}$$

where the superscript 'T' denotes transpose. Note that we can equate

$$\frac{m_1}{m_2} = \frac{g_1(\theta_m)}{g_2(\theta_m)}. \tag{13}$$

Consequently, our optimum weight solution reduces to

$$\mathbf{w}_{opt} = \begin{bmatrix} 1 & -\dfrac{g_1(\theta_m)}{g_2(\theta_m)} \end{bmatrix}^{T}. \tag{14}$$

Note that in the boresight direction, where $\theta_m = 0$, and $m_1 = m_2$, we calculate

$$\mathbf{w}_{opt} = \begin{bmatrix} 1 & -1 \end{bmatrix}^{T} = \text{optimum weight vector for a boresight null.} \tag{15}$$

This simply says that if we wish to steer a null in the direction of boresight, we do so by taking the difference of the voltages from the two beams, as we might have intuitively already guessed.

2.2 Difference Signal and Monopulse Slope

Now consider the case where we wish to filter (use weights $\mathbf{w}_{opt}$) for the boresight case $\theta_m = 0$, but in fact have m_i for something slightly offset from this angle. We will of course generate an error due to this mismatch, which we calculate as

$$\varepsilon = \begin{bmatrix} m_1 & m_2 \end{bmatrix} \mathbf{w}_{opt} = (m_1 - m_2) = -A_m \left(g_2(\theta_m) - g_1(\theta_m) \right). \tag{16}$$

It is convenient for us to define a difference pattern such that

$$d(\theta) = \frac{g_2(\theta) - g_1(\theta)}{\sqrt{2}} = \text{difference signal.} \tag{17}$$

The $\sqrt{2}$ in the denominator is somewhat traditional for dual-beam monopulse, and is meant to address a conservation of power concern with difference and sum patterns. We discuss sum patterns later. Clearly, in the boresight direction,

$$d(0) = 0 = \text{difference signal in boresight direction.} \tag{18}$$

Let us use a first-order Taylor series expansion of the difference signal about the boresight direction. We can write this as

$$d(\theta) \approx \left(d(\theta)\big|_{\theta=0} \right) + \left(\frac{d}{d\theta} d(\theta) \bigg|_{\theta=0} \right) \theta = \text{difference signal approximation.} \tag{19}$$

We calculate the linear term as, and identify it as,

$$k_d = \frac{d}{d\theta} d(\theta) \bigg|_{\theta=0} = \text{monopulse slope.} \tag{20}$$

This lets us write the difference signal in the neighborhood of the null as approximately

$$d(\theta) \approx k_d \theta = \text{difference signal approximation.} \tag{21}$$

This relationship lets us estimate the offset angle for a received signal by calculating

$$\hat{\theta}_m = \frac{1}{k_d} d(\theta_m) = \frac{1}{k_d \sqrt{2}} \left(g_2(\theta_m) - g_1(\theta_m) \right), \tag{22}$$

where here $d(\theta_m)$ is essentially derived from the data. The offset angle estimate can also be written in terms of signal voltages as

$$\hat{\theta}_m = \frac{1}{A_m k_d \sqrt{2}} (m_2 - m_1). \tag{23}$$

It is often convenient to define the complex monopulse ratio (or simply "monopulse ratio") as the normalized function

$$r(\theta) = \frac{d(\theta)}{s(\theta)} = \text{monopulse ratio}, \tag{24}$$

where

$$s(\theta) = \text{reference beam, which we keep generic for the moment.} \tag{25}$$

The convenience is in the fact that $r(\theta)$ can be calculated from measurements directly, with any influence from scale factor A_m essentially dividing out, and therefore mitigated. Furthermore, the monopulse slope is likewise often scaled to a reference signal level, and written relative to a reference beamwidth, to yield a normalized quantity

$$k_m = \theta_{ref} \left. \frac{d}{d\theta} r(\theta) \right|_{\theta=0} = \frac{\theta_{ref}}{s_0} k_d = \text{normalized monopulse slope}, \tag{26}$$

where from the reference beam $s(\theta)$, we identify

$$\theta_{ref} = \text{reference antenna beamwidth, and}$$

$$s_0 = s(0) = \text{reference signal level.} \tag{27}$$

2.3 Sum Signal

For a dual-feed amplitude monopulse system, the reference beam typically refers to the sum of the constituent beam patterns, calculated as

$$s(\theta) = \frac{g_2(\theta) + g_1(\theta)}{\sqrt{2}} = \text{sum signal.} \tag{28}$$

In this case, we identify

$$\theta_{ref} = -3 \text{ dB beamwidth of the sum beam, and}$$

$$s_0 = s(0) = \text{sum signal in the boresight direction.} \tag{29}$$

We emphasize that other feed structures exist, for which Eq. (28) and Eq. (29) are not true. Some of these are illustrated in Figure 2, particularly (b) and (c).
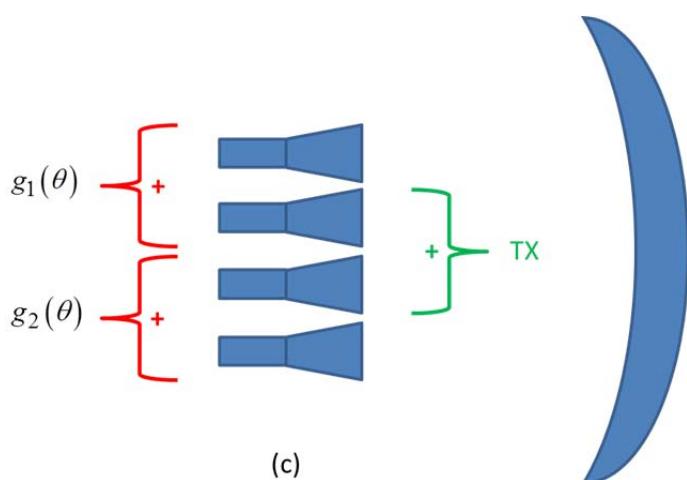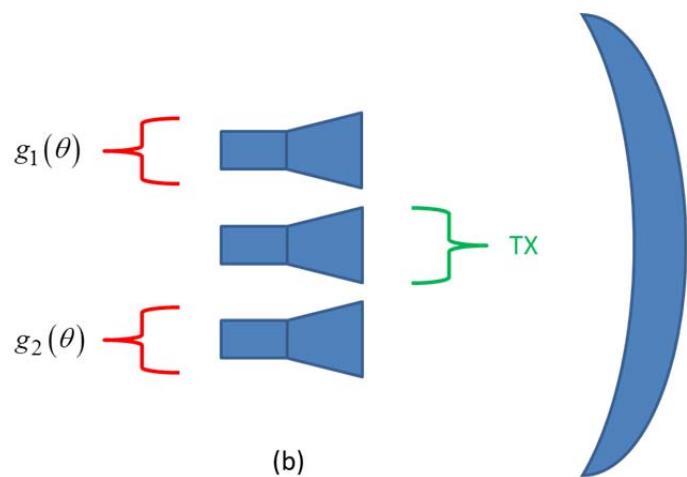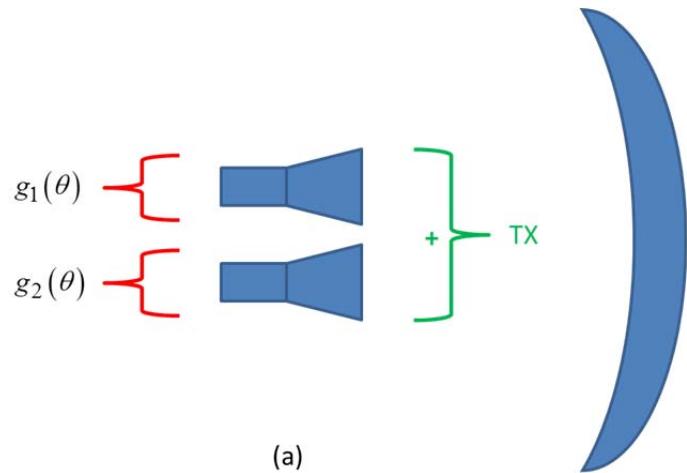
Figure 2. Dual-beam amplitude monopulse feed configurations.

2.4 Noise in DOA Angle Estimate

Now let us assume that we have noisy measurements, where

$$x_1 = m_1 + n_1 \text{, and}$$
$$x_2 = m_2 + n_2 \text{,} \tag{30}$$

where

$$n_1 = \text{noise voltage perturbing the first beam signal, and}$$
$$n_2 = \text{noise voltage perturbing the second beam signal,} \tag{31}$$

where the noise voltages are independent zero-mean Gaussian, with variance

$$\sigma_n^2 = E\left\{|n_1|^2\right\} = E\left\{|n_2|^2\right\}. \tag{32}$$

In this case, the estimate of the received signal's DOA is

$$\hat{\theta}_m = \frac{\left(m_2 + n_2 - m_1 - n_1\right)}{A_m k_d \sqrt{2}} = \frac{\left(m_2 - m_1\right)}{A_m k_d \sqrt{2}} + \frac{\left(n_2 - n_1\right)}{A_m k_d \sqrt{2}} \text{,} \tag{33}$$

which of course will be in error with variance

$$\sigma_\theta^2 = \frac{\sigma_n^2}{A_m^2 k_d^2}. \tag{34}$$

This can be transmogrified to the expression for RMS DOA angle noise as

$$\sigma_\theta = \frac{\theta_{ref}}{k_m \sqrt{A_m^2 \, s_0^2 / \sigma_n^2}} = \frac{\theta_{ref}}{k_m \sqrt{2SNR}} \text{,} \tag{35}$$

where, because we are dealing with the real-valued components of signal and noise, we identify

$$SNR = \frac{A_m^2 s_0^2}{2\sigma_n^2} = \text{Signal-to-Noise power ratio.} \tag{36}$$

Note that the SNR is a ratio of the received power in the reference beam to the noise power in the squinted beams, evaluated in the boresight direction. Any improvement to the transmitted signal, such as the gain of a transmit antenna, are embodied in the scale factor A_m. Eq. (35) is the same expression given in Sherman and Barton.

2.5 Figure of Merit for Optimum Squint Angle

We desire to minimize the variance of the DOA angle estimate σ_θ^2, specifically in the boresight direction. The individual measurement noise variance σ_n^2 is presumed to be constant. Consequently, we desire a monopulse antenna design that maximizes $A_m k_d$.

The factor A_m embodies the signal gain factors that include transmitter power, transmit antenna gain, range, and other miscellaneous gains and losses. We shall concern ourselves here only with the transmit antenna gain component. Accordingly, we define

$$s_T = \text{transmit antenna signal in the receive antenna boresight direction.} \qquad (37)$$

For a monostatic antenna, which we will assume hereafter, we typically equate

$$s_T = s_0 . \qquad (38)$$

Consequently, we define the figure of merit for our antenna as

$$FOM = s_0 k_d . \qquad (39)$$

Our intent now is to select a squint angle θ_{sq} that maximizes the FOM. We stipulate that this will depend on the individual beam shapes. Nevertheless, we emphasize a key point that anything that reduces the transmitted signal strength s_T without a corresponding increase in monopulse slope k_d will reduce the FOM and increase DOA noise. This is undesirable.

We examine two cases. For both, we shall assume a beam-shape from Sherman and Barton, namely

$$g_0(\theta) = \frac{\cos(\pi K_\theta \theta)}{1 - (2K_\theta \theta)^2} , \qquad (40)$$

where

$$K_\theta = 1.189/\theta_{ref} . \qquad (41)$$

This value for K_θ guarantees that the reference beamwidth θ_{ref} is in fact the one-way -3 dBc beamwidth for a constituent beam.

Case 1.

We duplicate here the case presented in Sherman and Barton where for this case our reference beam is the sum beam, namely

$$s(\theta) = \frac{g_2(\theta) + g_1(\theta)}{\sqrt{2}} = \text{sum signal.} \tag{42}$$

This corresponds to a feed configuration illustrated in Figure 2(a).

Numerical analysis indicated a peak FOM of approximately 1.4749 at a squint angle of approximately 0.453 constituent beamwidths. That is

$$\theta_{sq,optimum} \approx 0.453 \, \theta_g = \text{optimum squint angle.} \tag{43}$$

Note that this means that the two squinted beams are equal at a point where their relative gain is −2.45 dBc. The corresponding squinted beams are plotted in Figure 6. Their monopulse ratio is plotted in Figure 7, and the two-way patterns are plotted in Figure 8.

Case 2.

We present here the case where our reference beam is the constituent beam shape itself, aligned with the boresight direction, namely

$$s(\theta) = \sqrt{2} g_0(\theta) = \text{reference signal.} \tag{44}$$

This is equivalent to a sum of beams with zero squint angle, and accounts for the $\sqrt{2}$ factor. This corresponds to a feed configuration illustrated in Figure 2(c).

Numerical analysis indicated a peak FOM of approximately 2.245 at a squint angle of approximately 0.681 constituent beamwidths. That is

$$\theta_{sq,optimum} \approx 0.681 \, \theta_g = \text{optimum squint angle.} \tag{45}$$

Note that in this case s_0 is constant, and the FOM depends only on k_d. Furthermore, since θ_{ref} is also constant, the peak in the FOM is also a peak in normalized monopulse slope k_m.

Note that this means that the two squinted beams are equal at a point where their relative gain is about −5.9 dBc. With respect to Case 1, these beams are squinted a little wider. The corresponding squinted beams are plotted in Figure 12. Their monopulse ratio is plotted in Figure 13, and the two-way patterns are plotted in Figure 14.
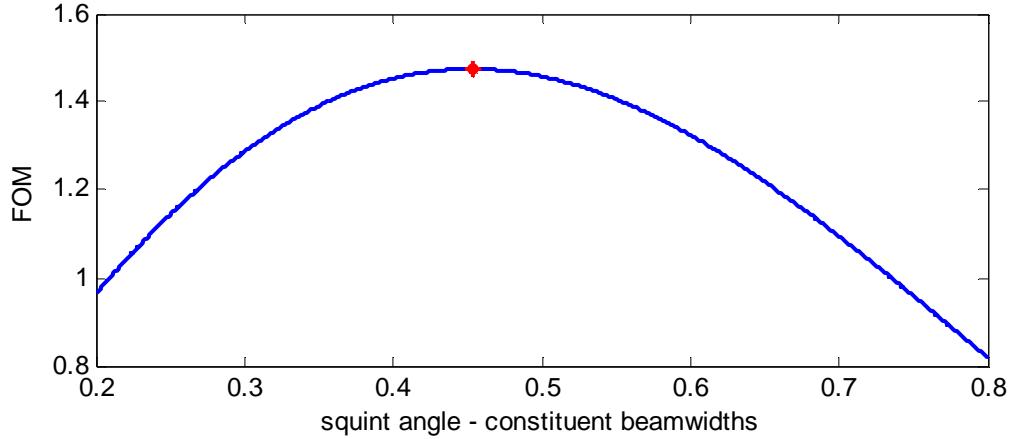
Figure 3. Figure of Merit for Case 1. The peak is indicated with the red '*'.
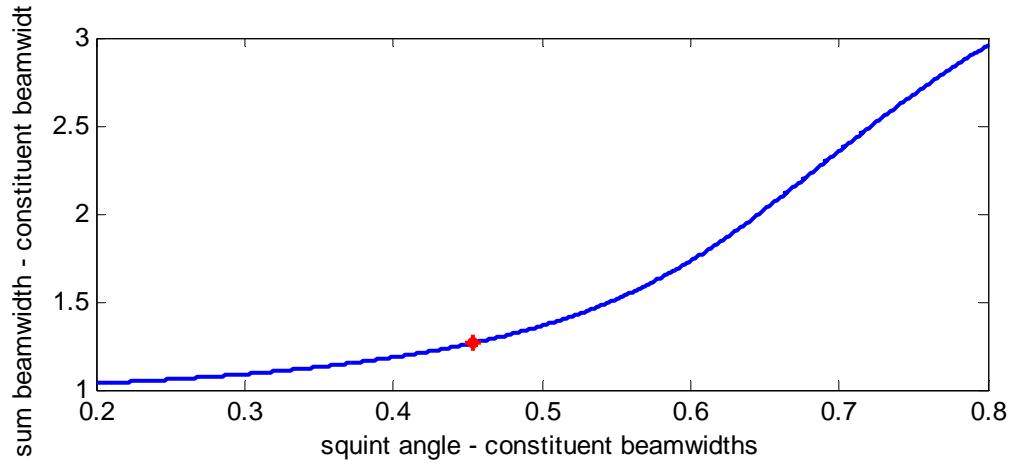
Figure 4. Reference (Sum beam) beamwidth for Case 1. The red '*' corresponds to the squint angle with maximum FOM.
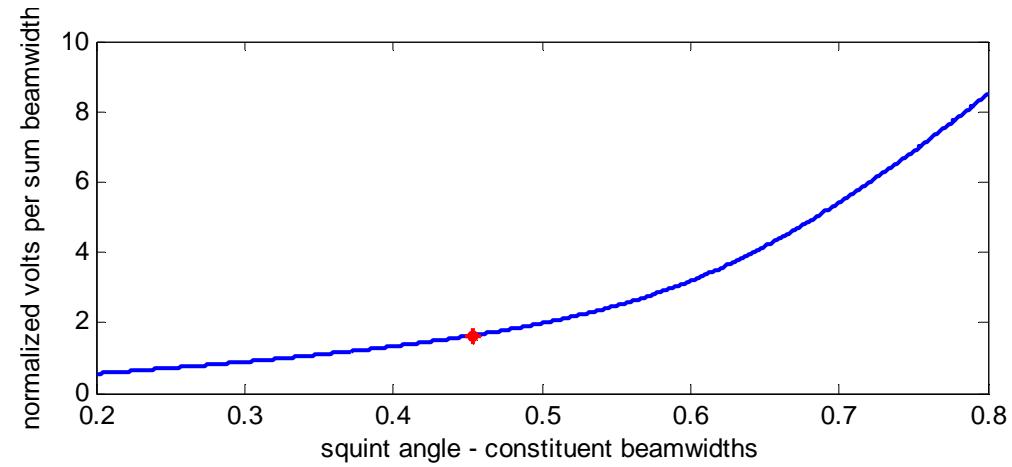
Figure 5. Normalized monopulse slope for Case 1. The red '*' corresponds to the squint angle with maximum FOM.
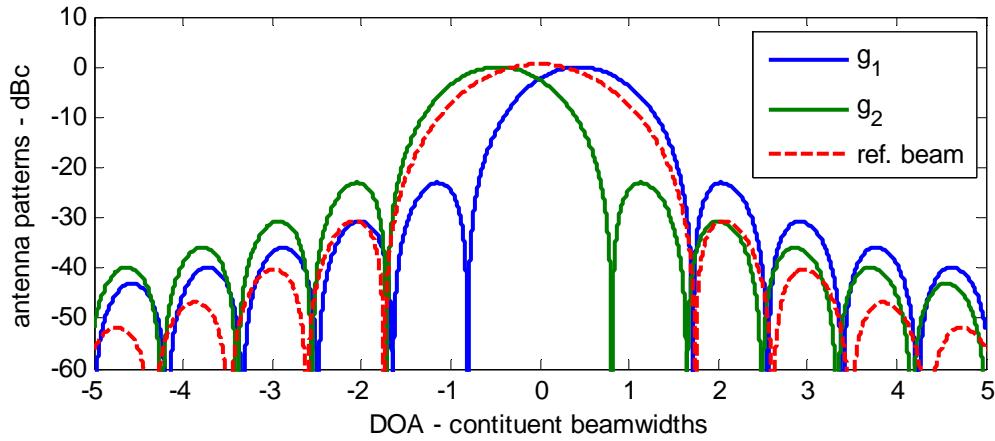
Figure 6. Beam patterns with maximum FOM for Case 1.
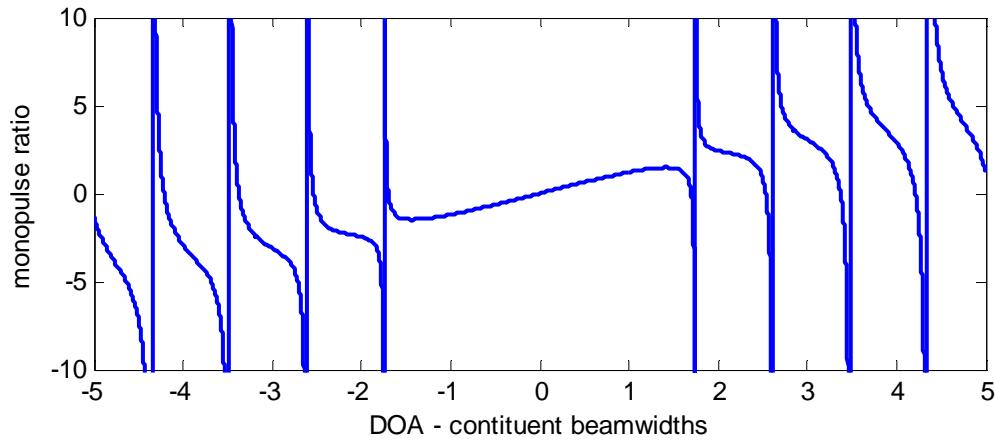
Figure 7. Monopulse ratio for Case 1.
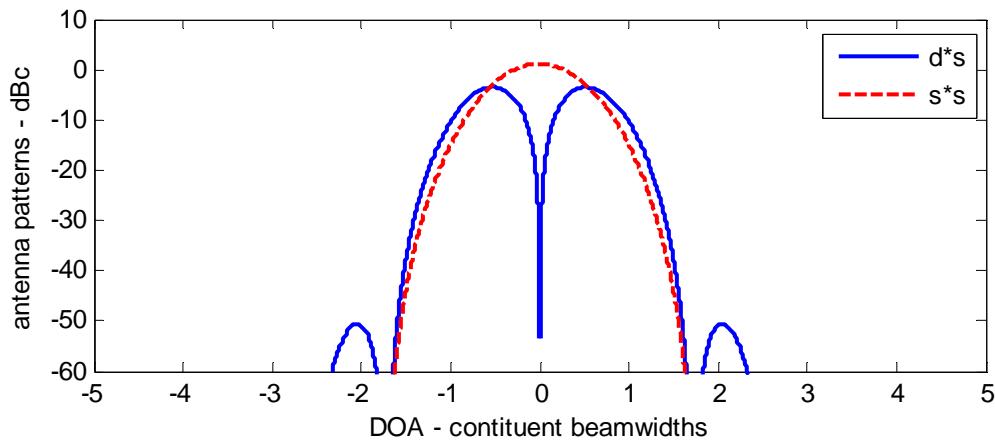
Figure 8. Two-way antenna patterns for Case 1.
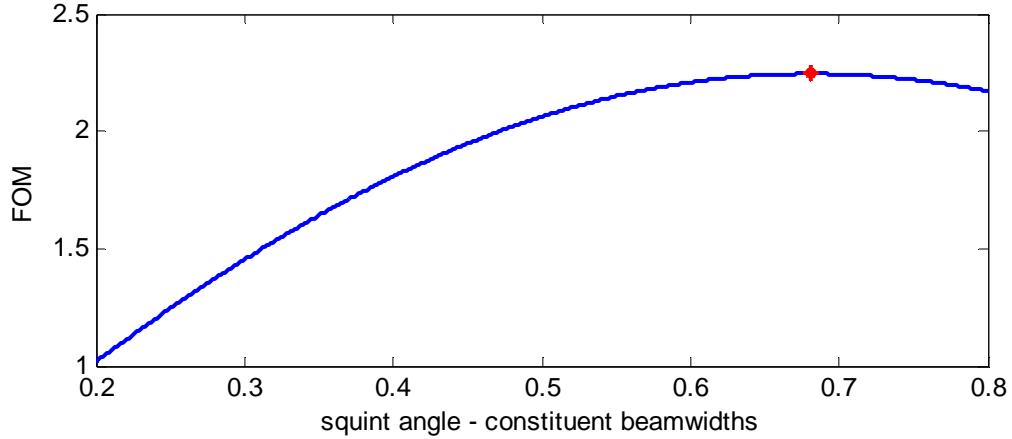
Figure 9. Figure of Merit for Case 2. The peak is indicated with the red '*'.
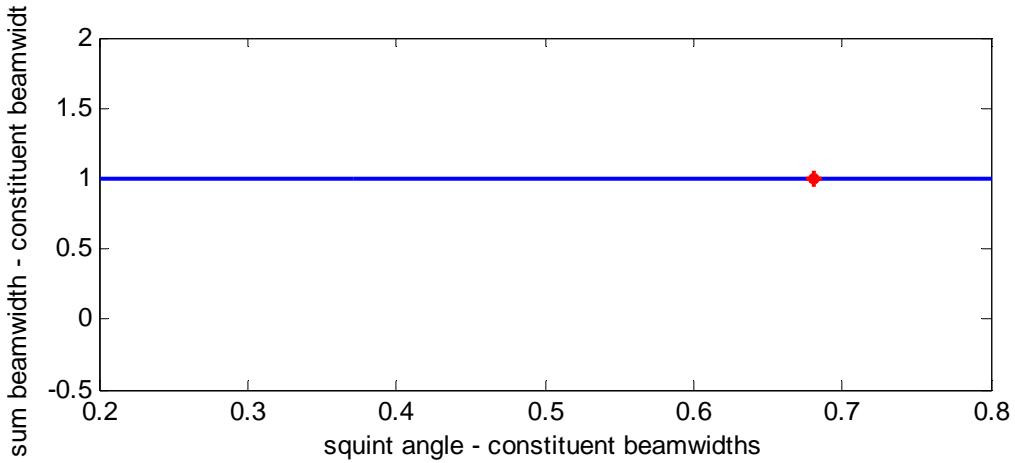
Figure 10. Reference beamwidth for Case 2. The red '*' corresponds to the squint angle with maximum FOM. Clearly, the reference beamwidth is constant as stipulated.
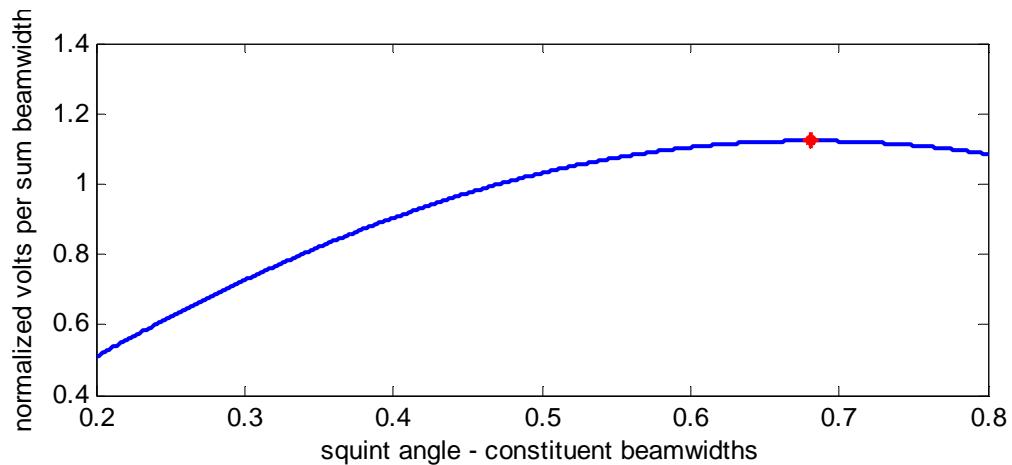
Figure 11. Normalized monopulse slope for Case 2. The red '*' corresponds to the squint angle with maximum FOM.
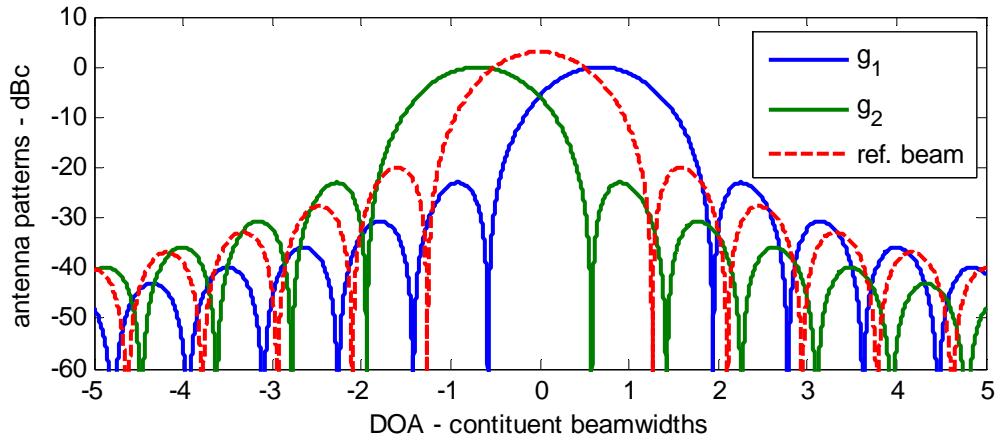
Figure 12. Beam patterns with maximum FOM for Case 2.
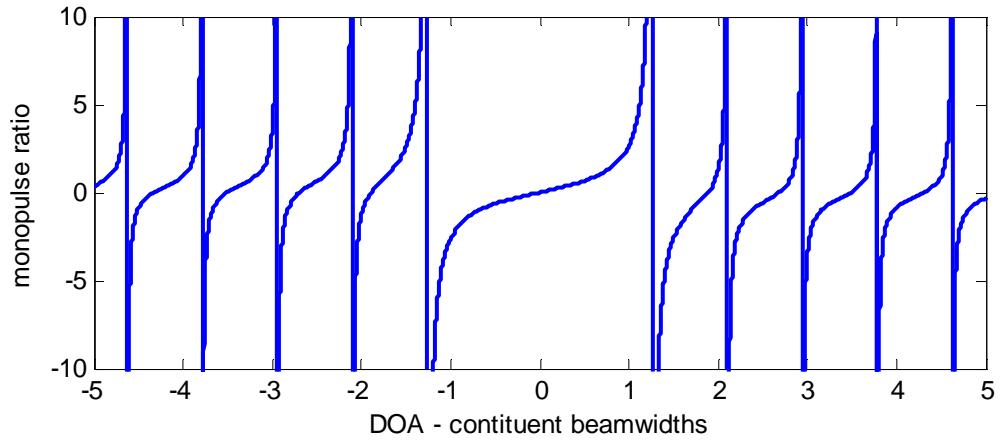
Figure 13. Monopulse ratio for Case 2.
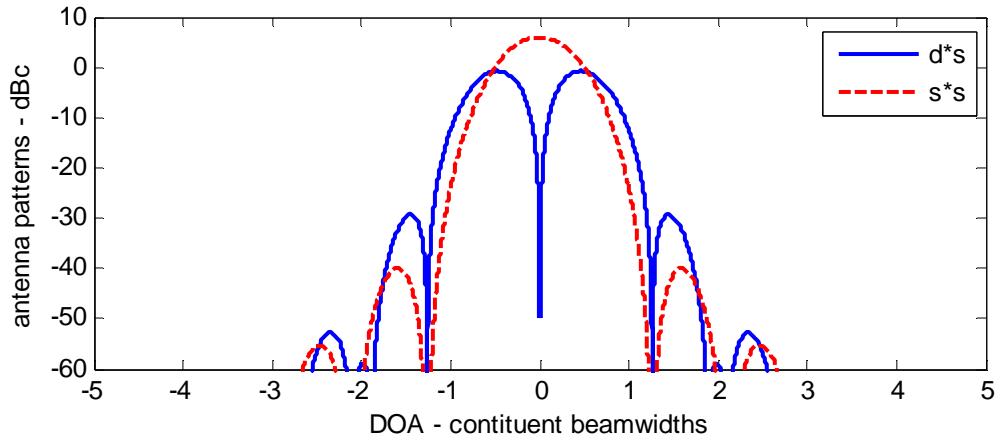
Figure 14. Two-way antenna patterns for Case 2.

2.6 Comments and Notes

We make several comments in no particular order.

- The useable range of angles for estimating DOA is some fraction of the null-to-null beamwidth of the reference beam.

- The monopulse slope is fairly linear over a range of DOA angles within the −3 dB beamwidth of the reference beam.

- If also used as a transmit beam, a narrower reference beam will provide more signal gain and a more accurate/precise estimate DOA angle, but will also limit the useable range of DOA angles to something less than a wider reference beam. Nevertheless, the corresponding less noise in the DOA estimate with a narrower reference beam is generally preferred.

- The two-way patterns also show a difference in sidelobe levels for the difference channel. Clearly we desire low sidelobes. However an analysis of "optimum" with respect to constraining sidelobe levels is beyond the scope of this report.

- We have made the tacit assumption that there are no issues with cross-polarization response. It is well-known that offset-fed dish reflectors are susceptible to cross-polarization error signals. Furthermore, we are treating the antenna in a free-space environment, without contamination from multipath signals. Techniques are available to assist in mitigating these error sources.[4] Specifics are beyond the scope of this report.

- Although also beyond the scope of this report, we do mention that various error sources do exist in practical monopulse systems. Some of these are addressed in a report by Bickel.[5] Mitigating these errors is typically a calibration function, although signal processing techniques can be employed for some of them.[6]

"The difference between fiction and reality? Fiction has to make sense."
-- Tom Clancy

3 Three-Beam Amplitude Monopulse

As with the dual-beam analysis, let us again assume that the antenna is capable of a symmetric beam-shape defined as

$$g_0(\theta) = \text{generalized beam shape,} \tag{46}$$

where

$$\theta = \text{off-boresight angle,} \tag{47}$$

and again for convenience,

$$\theta_g = \text{constituent beamwidth of } g_0(\theta), \text{ nominally at the } -3 \text{ dBc level.} \tag{48}$$

Now, however, we assume that the monopulse antenna pattern is composed of three beams, with a center beam and two more equally squinted in opposite directions, that is

$$
\begin{aligned}
g_1(\theta) &= g_0(\theta + \theta_{sq}) = \text{beam \#1,} \\
g_2(\theta) &= g_0(\theta) = \text{beam \#2, and} \\
g_3(\theta) &= g_0(\theta - \theta_{sq}) = \text{beam \#3,}
\end{aligned}
\tag{49}
$$

where

$$\pm\theta_{sq} = \text{squint angle offset of outer beams from boresight.} \tag{50}$$

A single target's echo will be received by each of these beams, albeit generally with different amplitudes. Accordingly, we define voltage measurements from the antennas as

$$
\begin{aligned}
m_1 &= A_m \, g_1(\theta_m) = \text{target signal at first beam,} \\
m_2 &= A_m \, g_2(\theta_m) = \text{target signal at second beam, and} \\
m_3 &= A_m \, g_3(\theta_m) = \text{target signal at third beam,}
\end{aligned}
\tag{51}
$$

where these are all real-valued, and

$$
\begin{aligned}
\theta_m &= \text{the direction of arrival of a received signal, and} \\
A_m &= \text{some scale factor with respect to the beam patterns.}
\end{aligned}
\tag{52}
$$

Based on these responses, we define the vector

$$\mathbf{m} = \begin{bmatrix} m_1 & m_2 & m_3 \end{bmatrix}^T = \text{real-valued signal vector.} \tag{53}$$

We furthermore define a weight scaling vector to avoid a trivial solution as

$$\mathbf{v}_s = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T = \text{scaling constraint vector.} \tag{54}$$

Our constraint to a solution for an optimal weight vector is then

$$\mathbf{v}_s^T \mathbf{w} = 1 = \text{the scale constraint for the weights,} \tag{55}$$

where

$$\mathbf{w} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}^T = \text{weight vector.} \tag{56}$$

Now we wish to steer a null in the direction of the received signal by linearly combining these signals. Accordingly we can set up the matrix equation

$$\begin{bmatrix} \mathbf{v}_s^T \\ \mathbf{m}^T \end{bmatrix} \mathbf{w} = \begin{bmatrix} 0 & 1 & 0 \\ m_1 & m_2 & m_3 \end{bmatrix} \mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{57}$$

We observe that since the weight vector is combining only real-valued quantities, it may also be limited to real-valued elements.

Note that we effectively have two equations with three unknowns. Therefore, no unique solution exists for Eq. (57). We will next proceed by stipulating that we seek the weight vector that provides a null with minimum noise.

3.1 Minimum-Noise Weight Solution

We now identify the following vectors, namely

$$\mathbf{n} = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix}^T = \text{real-valued noise vector,}$$
$$\mathbf{x} = \mathbf{m} + \mathbf{n} = \text{noisy measurement vector,} \tag{58}$$

where the noise voltages are zero-mean Gaussian, independent of each other, but identically distributed with variance

$$\sigma_n^2 = E\left\{|n_1|^2\right\} = E\left\{|n_2|^2\right\} = E\left\{|n_3|^2\right\}. \tag{59}$$

We desire to select weights such that

$$\mathbf{x}^T \mathbf{w} = 0, \text{ or at least minimum in the statistical sense,} \tag{60}$$

but subject to the aforementioned constraints. This is embodied in the matrix equation

$$\begin{bmatrix} \mathbf{v}_s^T \\ \mathbf{x}^T \end{bmatrix} \mathbf{w} = \begin{bmatrix} 1 \\ \varepsilon \end{bmatrix},$$ (61)

which may also be written as

$$\mathbf{w}^T \begin{bmatrix} \mathbf{v}_s & \mathbf{x} \end{bmatrix} = \begin{bmatrix} 1 & \varepsilon \end{bmatrix},$$ (62)

where we define the error term

$$\varepsilon = \mathbf{w}^T \mathbf{x} = \text{error that we wish to minimize in the statistical sense.}$$ (63)

We stipulate that the optimal weights occur when $|\varepsilon|^2$ is minimized in the statistical sense. To proceed, we recognize that

$$E\left\langle \left| \mathbf{w}^T \mathbf{x} \right|^2 \right\rangle = \mathbf{w}^T E\left\langle \mathbf{x} \mathbf{x}^T \right\rangle \mathbf{w} = \mathbf{w}^T \mathbf{R_{xx}} \mathbf{w},$$ (64)

where

$$E\langle y \rangle = \text{is the expected value of } y, \text{ and}$$
$$\mathbf{R_{xx}} = E\left\langle \mathbf{x}\, \mathbf{x}^T \right\rangle = \text{the covariance matrix of } \mathbf{x}.$$ (65)

We employ the method of Lagrange multipliers and formulate the Lagrange function with the single constraint as

$$\Lambda = \mathbf{w}^T \mathbf{R_{xx}} \mathbf{w} - 2\alpha_s \left(\mathbf{w}^T \mathbf{v}_s - 1 \right),$$ (66)

where

$$\alpha_s = \text{Lagrange multiplier associated with the scaling constraint.}$$ (67)

We may then find the optimum weight vector by taking the derivative of the Lagrange function with respect to $\mathbf{w}^T$ and setting it to zero. Doing so yields

$$2\mathbf{R_{xx}} \mathbf{w}_{opt} - 2\alpha_s \mathbf{v}_s = 0.$$ (68)

This may be solved for the optimal weights as

$$\mathbf{w}_{opt} = \mathbf{R_{xx}}^{-1} \left(\alpha_s \mathbf{v}_s \right) = \alpha_s \mathbf{R_{xx}}^{-1} \mathbf{v}_s.$$ (69)

The earlier constraint may now be employed to derive the equation

$$\mathbf{v}_s{}^T \left(\alpha_s \mathbf{R_{xx}}^{-1} \mathbf{v}_s \right) = \alpha_s \left(\mathbf{v}_s{}^T \mathbf{R_{xx}}^{-1} \mathbf{v}_s \right) = 1 , \tag{70}$$

which can be solved to yield

$$\alpha_s = \frac{1}{\left(\mathbf{v}_s{}^T \mathbf{R_{xx}}^{-1} \mathbf{v}_s \right)} . \tag{71}$$

This in turn can be used to solve for the optimal weight vector to be

$$\mathbf{w}_{opt} = \frac{\mathbf{R_{xx}}^{-1} \mathbf{v}_s}{\left(\mathbf{v}_s{}^T \mathbf{R_{xx}}^{-1} \mathbf{v}_s \right)} = \text{optimal weight vector.} \tag{72}$$

At this point, we have the optimal weight vector to minimize the expected value of $|\varepsilon|^2$. We still have an issue in determining the covariance matrix $\mathbf{R_{xx}}$ and finding its inverse.

Evaluating the Covariance Matrix Inverse

First we examine the covariance matrix, which we recall and expand to

$$\mathbf{R_{xx}} = E\left\langle \mathbf{x}\,\mathbf{x}^T \right\rangle = E\left\langle \mathbf{m}\,\mathbf{m}^T + \mathbf{n}\,\mathbf{n}^T \right\rangle . \tag{73}$$

We expand this in turn and evaluate it to

$$\mathbf{R_{xx}} = \mathbf{m}\,\mathbf{m}^T + \sigma_n^2 \mathbf{I} = \begin{bmatrix} m_1 m_1 & m_1 m_2 & m_1 m_3 \\ m_2 m_1 & m_2 m_2 & m_2 m_3 \\ m_3 m_1 & m_3 m_2 & m_3 m_3 \end{bmatrix} + \sigma_n^2 \mathbf{I} . \tag{74}$$

Note that it is the noise that makes this covariance matrix invertible. We may rearrange some things to equate

$$\mathbf{R_{xx}} = \sigma_n^2 \left(\mathbf{I} + \frac{1}{\sigma_n^2} \mathbf{m}\,\mathbf{m}^T \right) . \tag{75}$$

The inverse can be calculated using the Sherman–Morrison formula to be identically

$$\mathbf{R}_{\mathbf{xx}}^{-1} = \frac{1}{\sigma_n^2}\left(\mathbf{I} - \frac{\mathbf{m}\mathbf{m}^T}{\left(\sigma_n^2 + \mathbf{m}^T\mathbf{m}\right)}\right) = \frac{1}{\sigma_n^2}\left(\mathbf{I} - \frac{\mathbf{m}\mathbf{m}^T}{\left(\sigma_n^2 + \sum_i |m_i|^2\right)}\right). \tag{76}$$

Back to the Optimal Weight Vector

We plug this covariance matrix inverse into Eq. (72) and manifest the optimal weights as

$$\mathbf{w}_{opt} = \frac{\frac{1}{\sigma_n^2}\left(\mathbf{I} - \frac{\mathbf{m}\mathbf{m}^T}{\left(\sigma_n^2 + \sum_i |m_i|^2\right)}\right)\mathbf{v}_s}{\mathbf{v}_s^T\frac{1}{\sigma_n^2}\left(\mathbf{I} - \frac{\mathbf{m}\mathbf{m}^T}{\left(\sigma_n^2 + \sum_i |m_i|^2\right)}\right)\mathbf{v}_s} = \frac{\left(\mathbf{v}_s - \frac{\mathbf{m}\mathbf{m}^T\mathbf{v}_s}{\left(\sigma_n^2 + \sum_i |m_i|^2\right)}\right)}{\left(1 - \frac{\left|\mathbf{v}_s^T\mathbf{m}\right|^2}{\left(\sigma_n^2 + \sum_i |m_i|^2\right)}\right)}. \tag{77}$$

For the specific defined vector $\mathbf{v}_s$, this can be simplified to

$$\mathbf{w}_{opt} = \frac{\left(\left(\sigma_n^2 + \sum_i |m_i|^2\right)\mathbf{v}_s - m_2\mathbf{m}\right)}{\left(\sigma_n^2 + \sum_i |m_i|^2 - |m_2|^2\right)}, \tag{78}$$

and simplified some more to

$$\mathbf{w}_{opt} = \left[\left(\frac{-m_2 m_1}{\sigma_n^2 + |m_1|^2 + |m_3|^2}\right) \quad 1 \quad \left(\frac{-m_2 m_3}{\sigma_n^2 + |m_1|^2 + |m_3|^2}\right)\right]^T. \tag{79}$$

Note that the outside beams are weighted inversely proportional to the noise level present in their signals, and proportional to their beam gains. For noisier signals, the outside beams are discounted more. This suggests that for the outside beams to contribute meaningfully to generating a null, we need to maintain a good SNR in those beams. Since noise is fixed, this means we need a significant contribution from their signals.

Another interesting fact is that with noise, the optimum weight vector does not in fact create an absolute null. That is, if we account for noise, and end up with a noise-free input, then no absolute null is created. This is because as we recall the goal was to minimize the expected error for an expected amount of noise, not necessarily to guarantee no signal, i.e. an absolute null. Said another way, the optimal weights attempt to only suppress the signal down to the noise level. For large SNR, corresponding to $\sigma_n^2 \ll 1$, the optimum weight vector approaches

$$\mathbf{w}_{opt} = \left[\left(\frac{-m_2 m_1}{|m_1|^2 + |m_3|^2} \right) \quad 1 \quad \left(\frac{-m_2 m_3}{|m_1|^2 + |m_3|^2} \right) \right]^T . \tag{80}$$

In the boresight direction we have real-valued responses with $m_3 = m_1$, and the optimal weight vector for large SNR approaches

$$\mathbf{w}_{opt} = \left[\left(\frac{-m_2}{2m_1} \right) \quad 1 \quad \left(\frac{-m_2}{2m_1} \right) \right]^T . \tag{81}$$

In terms of the beam patterns, this reduces to

$$\mathbf{w}_{opt} = \left[\left(\frac{-g_0(0)}{2g_0(\theta_{sq})} \right) \quad 1 \quad \left(\frac{-g_0(0)}{2g_0(\theta_{sq})} \right) \right]^T . \tag{82}$$

This solution does guarantee an absolute null for the signal, but not necessarily a minimum noise solution.

3.1.1 Monopulse Slope for Minimum-Noise Weight Solution

Now consider the case where we wish to filter (use weights $\mathbf{w}_{opt}$) for the case $\theta_m = 0$, but in fact have actual m_i for something slightly offset from this angle, namely $\theta_m \neq 0$. We will of course then have a mismatch, and therewith generate an error, which we calculate as

$$\varepsilon_m = \mathbf{w}_{opt}^T \mathbf{m} = \left(w_1 m_1 + m_2 + w_1 m_3 \right) = A_m \left(w_1 g_1(\theta_m) + g_2(\theta_m) + w_1 g_3(\theta_m) \right), \tag{83}$$

where we recall that the high SNR weight solution approaches

$$w_1 = \frac{-g_0(0)}{2g_0(\theta_{sq})} . \tag{84}$$

Note that this error is due simply to the mismatched weights, and contains no contribution due to measurement noise. The error contribution due to noise is calculated as

$$\varepsilon_n = \mathbf{w}_{opt}^T \mathbf{n} = \left(w_1 n_1 + n_2 + w_1 n_3 \right).$$
(85)

The question we now have is "What offset θ_m will statistically yield the same error as noise with error ε_n?" We now define a function related to the mismatch error as

$$d(\theta) = w_1 g_1(\theta) + g_2(\theta) + w_1 g_3(\theta).$$
(86)

This function is analogous to the difference signal of a dual-beam monopulse configuration, but will hereafter be referred to as the "DOA function." We expand this into a second-order Taylor series as

$$d(\theta) = d(0) + \left[\frac{d}{d\theta} d(\theta) \Big|_{\theta=0} \right] \theta + \left[\frac{1}{2} \frac{d^2}{d\theta^2} d(\theta) \Big|_{\theta=0} \right] \theta^2,$$
(87)

which we expand to

$$d(\theta) \approx \left\{ \begin{array}{l} w_1 g_1(0) + g_2(0) + w_1 g_3(0) \\ + \left[w_1 \frac{d}{d\theta} g_1(\theta) + \frac{d}{d\theta} g_2(\theta) + w_1 \frac{d}{d\theta} g_3(\theta) \right]_{\theta=0} \theta \\ + \frac{1}{2} \left[w_1 \frac{d^2}{d\theta^2} g_1(\theta) + \frac{d^2}{d\theta^2} g_2(\theta) + w_1 \frac{d^2}{d\theta^2} g_3(\theta) \right]_{\theta=0} \theta^2 \end{array} \right\},$$
(88)

and then simplify to only the non-zero terms as

$$d(\theta) \approx \frac{1}{2} \left[w_1 \frac{d^2}{d\theta^2} g_1(\theta) + \frac{d^2}{d\theta^2} g_2(\theta) + w_1 \frac{d^2}{d\theta^2} g_3(\theta) \right]_{\theta=0} \theta^2.$$
(89)

We now define a quadratic component factor

$$\mu_d \approx \left[w_1 \frac{d^2}{d\theta^2} g_1(\theta) + \frac{d^2}{d\theta^2} g_2(\theta) + w_1 \frac{d^2}{d\theta^2} g_3(\theta) \right]_{\theta=0},$$
(90)

such that the DOA function can be written as

$$d(\theta) \approx \frac{\mu_d}{2}\theta^2.$$ (91)

The mismatch error itself can then be written as

$$\varepsilon_m = A_m d(\theta) \approx A_m \frac{\mu_d}{2}\theta^2.$$ (92)

The net result is that $d(\theta)$ is an even function, both in its expansion, and more generally by definition. Consequently there is no way to distinguish negative phase offsets from positive phase offsets based on a value for $d(\theta)$. This furthermore means that the monopulse slope is zero in the boresight direction, and hence unusable for estimating DOA.

That is not to say that this particular weight solution isn't useful for 'detecting' target energy. It just isn't particularly useful for 'locating' a target, i.e. determining DOA.

3.1.2 Multiple-Lobing of Minimum Noise Weight Solutions

In the previous section we determined that filtering to the beam center provided us with a mismatch error that was an even function of DOA angle, which proved ambiguous to determine the DOA angle.

We do note that if a target signal is offset from boresight, then we should be able to achieve a better null by steering the weight vector to one side of broadside than the other. Consequently, we ask ourselves now "can we get direction information by using two weight vectors, each steered in opposite directions from broadside, and combining the results in some fashion?" Accordingly, we define two "test" angles as

$$\pm\theta_t = \text{the test angle offsets from boresight,}$$ (93)

and we define corresponding responses from the two different test directions, or lobes, as

$$\mathbf{m}_a = \begin{bmatrix} g_1(\theta_t) & g_2(\theta_t) & g_3(\theta_t) \end{bmatrix}^T \text{, and}$$
$$\mathbf{m}_b = \begin{bmatrix} g_1(-\theta_t) & g_2(-\theta_t) & g_3(-\theta_t) \end{bmatrix}^T.$$ (94)

Without loss of generality, since these are simply test angles, we will assume a unit amplitude scaling for these specific vectors.

The two weight vectors to be employed are then correspondingly calculated as

$$\mathbf{w}_a = \left[\left(\frac{-g_2(\theta_t)\,g_1(\theta_t)}{\left|g_1(\theta_t)\right|^2 + \left|g_3(\theta_t)\right|^2} \right) \quad 1 \quad \left(\frac{-g_2(\theta_t)\,g_3(\theta_t)}{\left|g_1(\theta_t)\right|^2 + \left|g_3(\theta_t)\right|^2} \right) \right]^T, \text{ and}$$

$$\mathbf{w}_b = \left[\left(\frac{-g_2(-\theta_t)\,g_1(-\theta_t)}{\left|g_1(-\theta_t)\right|^2 + \left|g_3(-\theta_t)\right|^2} \right) \quad 1 \quad \left(\frac{-g_2(-\theta_t)\,g_3(-\theta_t)}{\left|g_1(-\theta_t)\right|^2 + \left|g_3(-\theta_t)\right|^2} \right) \right]^T. \tag{95}$$

From symmetry, we identify

$$g_2(-\theta_t) = g_2(\theta_t), \text{ and}$$
$$g_3(-\theta_t) = g_1(\theta_t). \tag{96}$$

This lets us simplify our weight vectors to

$$\mathbf{w}_a = \left[\left(\frac{-g_2(\theta_t)\,g_1(\theta_t)}{\left|g_1(\theta_t)\right|^2 + \left|g_1(-\theta_t)\right|^2} \right) \quad 1 \quad \left(\frac{-g_2(\theta_t)\,g_1(-\theta_t)}{\left|g_1(\theta_t)\right|^2 + \left|g_1(-\theta_t)\right|^2} \right) \right]^T, \text{ and}$$

$$\mathbf{w}_b = \left[\left(\frac{-g_2(\theta_t)\,g_1(-\theta_t)}{\left|g_1(\theta_t)\right|^2 + \left|g_1(-\theta_t)\right|^2} \right) \quad 1 \quad \left(\frac{-g_2(\theta_t)\,g_1(\theta_t)}{\left|g_1(\theta_t)\right|^2 + \left|g_1(-\theta_t)\right|^2} \right) \right]^T. \tag{97}$$

Clearly, the two weight vectors are just reversed sequences from each other, which is as we might expect from symmetry arguments.

Now consider a new signal with unknown direction θ_m and unknown amplitude A_m. The two weight vectors will yield two different errors for this signal given by

$$\varepsilon_a = \mathbf{w}_a^T \mathbf{m}, \text{ and}$$
$$\varepsilon_b = \mathbf{w}_b^T \mathbf{m}. \tag{98}$$

An offset in test angles $\pm\theta_t$ will generate a difference in these measures. Consequently, we are interested in how the difference in these errors changes with test angles. That is, we wish to identify

$$\xi = \frac{\varepsilon_b - \varepsilon_a}{2\theta_t} = \frac{\mathbf{w}_b^T \mathbf{m} - \mathbf{w}_a^T \mathbf{m}}{2\theta_t} = \frac{1}{2\theta_t}(\mathbf{w}_b - \mathbf{w}_a)^T \mathbf{m}, \tag{99}$$

which can be expanded and then reduced to

$$\xi = \frac{1}{2\theta_t} \begin{bmatrix} g_2(\theta_t)\left(\dfrac{g_1(\theta_t)-g_1(-\theta_t)}{|g_1(\theta_t)|^2+|g_1(-\theta_t)|^2}\right) \\ 0 \\ -g_2(\theta_t)\left(\dfrac{g_1(\theta_t)-g_1(-\theta_t)}{|g_1(\theta_t)|^2+|g_1(-\theta_t)|^2}\right) \end{bmatrix}^T \mathbf{m}. \tag{100}$$

If we take the limit as the test angle goes to zero, we calculate

$$\lim_{\theta_t \to 0} \xi = \begin{bmatrix} \dfrac{g_2(0)}{2|g_1(0)|^2}\left(\dfrac{d}{d\theta}g_1(\theta)\Big|_{\theta=0}\right) \\ 0 \\ -\dfrac{g_2(0)}{2|g_1(0)|^2}\left(\dfrac{d}{d\theta}g_1(\theta)\Big|_{\theta=0}\right) \end{bmatrix}^T \mathbf{m} = \mathbf{w}_{net}^T \mathbf{m}, \tag{101}$$

where

$$\mathbf{w}_{net} = \begin{bmatrix} w_{net,1} & 0 & -w_{net,1} \end{bmatrix}^T, \text{ and}$$

$$w_{net,1} = \frac{g_2(0)}{2|g_1(0)|^2}\left(\frac{d}{d\theta}g_1(\theta)\Big|_{\theta=0}\right). \tag{102}$$

From this we make several important observations.

- This analysis is for a single unknown signal in the vicinity of the boresight angle.

- The difference in the errors does not depend on the received signal from the center beam. Only the outer beams contribute a sensitivity to the DOA angle for an unknown signal.

- The important quantity of the outer beams is the slope of their beam patterns in the vicinity of the target direction.

- The 'net' weight vector in Eq. (101) clearly exhibits an 'odd' symmetry.

This suggests that a minimum-noise criterion for choosing a weight vector is by itself inadequate. Improved criteria would include forcing an odd symmetry to the weight vector. Having learned what we needed here, we examine the odd-symmetry weight solution in the next section.

3.2 Odd-Symmetry Weight Solution

We now limit the DOA function $d(\theta)$ to be an odd function. This means that we want to discount, in fact even zero, any even constituents in $d(\theta)$, where more generally

$$d(\theta) = w_1 g_1(\theta) + w_2 g_2(\theta) + w_3 g_3(\theta). \tag{103}$$

Since $g_2(\theta) = g_0(\theta)$ is inherently even, this means that we need to constrain $w_2 = 0$. Consequently, we define a new constraint to our weight solution embodied in a zero response with the vector

$$\mathbf{v}_o = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T = \text{odd-symmetry constraint vector}, \tag{104}$$

while our scaling constraint vector now becomes

$$\mathbf{v}_s = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T = \text{odd-symmetry constraint vector}. \tag{105}$$

Our matrix equation now becomes

$$\begin{bmatrix} \mathbf{v}_o^T \\ \mathbf{v}_s^T \\ \mathbf{m}^T \end{bmatrix} \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{106}$$

This matrix is generally of full rank, meaning it can be inverted. Furthermore, we have constrained $w_1 = 1$, and $w_2 = 0$. Full rank of course requires that we have a non-zero m_3 response.

Note that since m_2 is irrelevant due to $w_2 = 0$, the matrix equation can be reduced to the form

$$\begin{bmatrix} 1 & 0 \\ m_1 & m_3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{107}$$

Note further that this is essentially Eq. (9) for the two outside beams. This means that the analysis for the dual-beam configuration applies. The optimum weights are therefore identified as

$$\mathbf{w}_{opt} = \begin{bmatrix} 1 & 0 & -\dfrac{g_1(\theta_m)}{g_3(\theta_m)} \end{bmatrix}^T. \tag{108}$$

Note further yet that in the boresight direction, where $\theta_m = 0$, and $m_1 = m_3$, we calculate

$$\mathbf{w}_{opt} = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T = \text{weight vector for a boresight null.} \tag{109}$$

3.2.1 Monopulse Slope for Odd-Symmetry Weight Solution

The analysis of sections 2.2 and 2.4 apply here.

As with the dual-beam monopulse analysis, we allow here that the DOA function becomes

$$d(\theta) = \frac{g_3(\theta) - g_1(\theta)}{\sqrt{2}} = \text{difference signal.} \tag{110}$$

As before, in the boresight direction,

$$d(0) = 0 = \text{difference signal in boresight direction.} \tag{111}$$

We again write this signal in terms of a first-order Taylor series expansion and calculate the linear term as

$$k_d = \left. \frac{d}{d\theta} d(\theta) \right|_{\theta=0} = \text{monopulse slope.} \tag{112}$$

This lets us again linearize the difference signal as

$$d(\theta) \approx k_d \theta = \text{difference signal.} \tag{113}$$

This relationship lets us estimate the offset angle for a received signal by calculating

$$\hat{\theta}_m = \frac{1}{k_d} d(\theta_m) = \frac{1}{k_d \sqrt{2}} \left(g_3(\theta_m) - g_1(\theta_m) \right), \tag{114}$$

where here $d(\theta_m)$ is derived from the data. This can also be written as

$$\hat{\theta}_m = \frac{1}{A_m k_d \sqrt{2}} (m_3 - m_1). \tag{115}$$

We repeat that the monopulse slope is often scaled to be relative to a reference beamwidth and reference signal level to yield a normalized quantity

$$k_m = \frac{\theta_{ref}}{s_0} k_d = \text{normalized monopulse slope,} \tag{116}$$

where

$$\theta_{ref} = \text{reference antenna beamwidth, and}$$

$$s_0 = \text{reference signal level.} \tag{117}$$

Recall that we are dealing with the real-valued components of signal and noise, and we still identify

$$SNR = \frac{A_m^2 s_0^2}{2\sigma_n^2} = \text{Signal-to-Noise power ratio.} \tag{118}$$

Note that the SNR is a ratio of the received power in the reference beam to the noise power in the squinted beams, evaluated in the boresight direction. Any improvement to the transmitted signal, such as the gain of a transmit antenna, are embodied in the scale factor A_m.

The DOA noise can then still be written as

$$\sigma_\theta = \frac{\theta_{ref}}{k_m \sqrt{2SNR}} \, . \tag{119}$$

3.2.2 Figure of Merit for Optimum Squint Angle

The analysis in section 2.5 applies here.

We again desire to minimize the variance of the angle estimate σ_θ^2. The individual measurement noise variance σ_n^2 is still presumed to be constant. Consequently, we desire a monopulse antenna design that maximizes $A_m k_d$.

The factor A_m embodies the signal gain factors that include transmitter power, transmit antenna gain, range, and other miscellaneous gains and losses. We shall concern ourselves here only with the transmit antenna gain component. Accordingly, we define

$$s_T = \text{transmit antenna signal in the receive antenna boresight direction.} \tag{120}$$

For a monostatic antenna, which we will assume hereafter, we typically equate

$$s_T = s_0 \, . \tag{121}$$

Consequently, we define the figure of merit for our antenna as

$$FOM = s_0 k_d . \tag{122}$$

Our intent now is to select a squint angle θ_{sq} that maximizes the FOM. We stipulate that this will depend on the individual beam shapes. Nevertheless, we emphasize a key point that anything that reduces the transmitted signal strength s_T without a corresponding increase in monopulse slope k_d will reduce the FOM and increase DOA noise.

We shall again assume a beam-shape from Sherman and Barton, namely

$$g_0(\theta) = \frac{\cos(\pi K_\theta \theta)}{1 - (2K_\theta \theta)^2} , \tag{123}$$

where

$$K_\theta = 1.189/\theta_{ref} . \tag{124}$$

Since we have the additional beam $g_2(\theta)$, we can readily adopt case 2 in section 2.5 as our monopulse architecture, with $g_2(\theta)$ as our reference beam.

Numerical analysis indicates a peak FOM at a squint angle of approximately 0.681 constituent beamwidths. That is

$$\theta_{sq,optimum} \approx 0.681\,\theta_g = \text{optimum squint angle.} \tag{125}$$

Note that in this case s_0 is constant, and the FOM depends only on k_d.

This also means that the two squinted beams are equal at a point where their relative gain is about -5.9 dBc. This means that the 'outside' beams need to have their gains intersect at a relative gain of about -5.9 dBc. These beams are plotted in Figure 15.

Furthermore, since the factor A_m needs to be maximized, this means that we need maximum transmit power directed in the boresight direction. This implies that given the choice of spreading the transmit power among the several individual beams, we achieve optimum by applying the transmit power all to the single center beam $g_2(\theta)$. Spreading the power evenly to all beams is clearly not at all optimal, and in fact hurts performance.
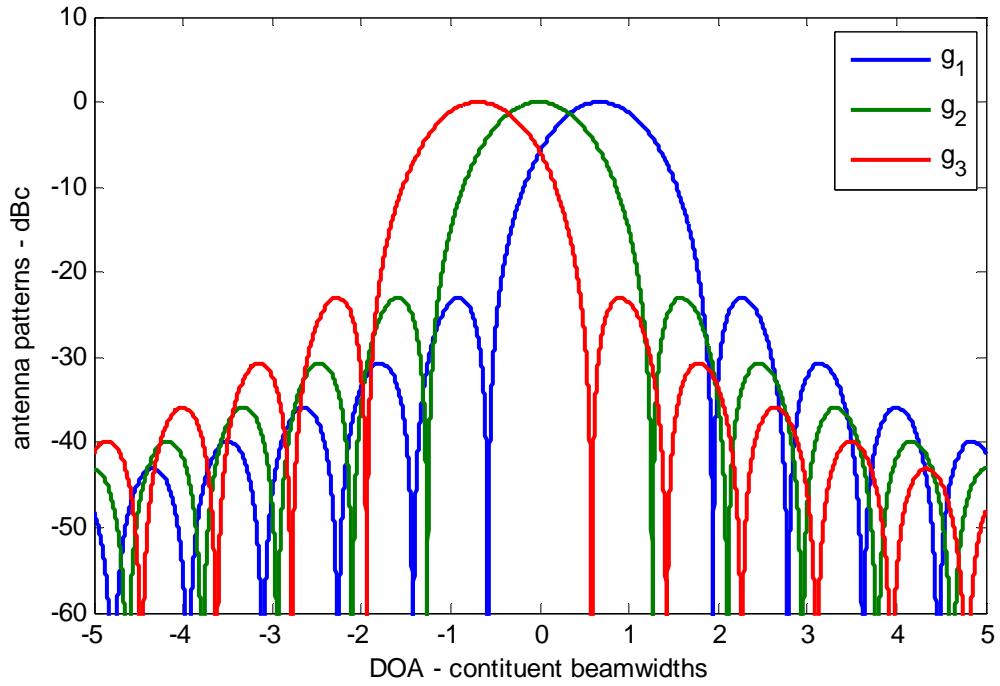
**Figure 15. Optimal beams for maximum FOM for a single target DOA estimation.**

The preferred architecture is illustrated in Figure 16. Other architectures also exist.
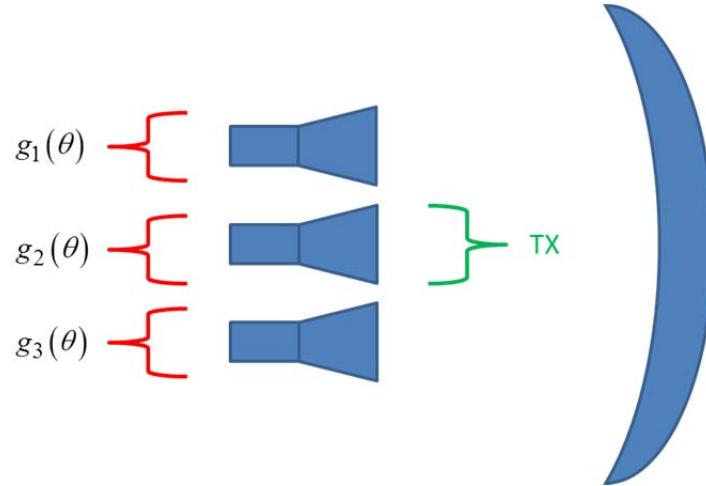

Figure 16. Three-beam amplitude monopulse feed configuration.

3.3 Constrained-Null Weight Solution

The purpose of three beams is after all to provide essentially two nulls. We now seek a solution that provides two distinct nulls at specific test angles. With this solution, we ask "What is the sensitivity of the second null's position to noise given the first null's known position?"

Accordingly, we define two "test" angles as

θ_{t1} = first test angle offset from boresight, and

θ_{t2} = second test angle offset from boresight, (126)

with the constraint that these angles are different, that is

$$\theta_{t2} \neq \theta_{t1}. \tag{127}$$

Therewith, we define responses from the two different test directions as

$$\mathbf{m}_{t1} = \begin{bmatrix} g_1(\theta_{t1}) & g_2(\theta_{t1}) & g_3(\theta_{t1}) \end{bmatrix}^T , \text{ and}$$
$$\mathbf{m}_{t2} = \begin{bmatrix} g_1(\theta_{t2}) & g_2(\theta_{t2}) & g_3(\theta_{t2}) \end{bmatrix}^T. \tag{128}$$

Without loss of generality, since these are simply test angles, we will assume a unit amplitude scaling for both of them. Furthermore, we define

$$\mathbf{v}_s = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T = \text{scaling constraint vector.} \tag{129}$$

We shall furthermore assume that both angles are located within the center beam, i.e. both $g_2(\theta_{t1})$ and $g_2(\theta_{t2})$ offer a significant response. More on this later. Consequently, we may construct the matrix equation

$$\begin{bmatrix} \mathbf{v}_s^T \\ \mathbf{m}_{t1}^T \\ \mathbf{m}_{t2}^T \end{bmatrix} \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{130}$$

We will constrain ourselves to the condition that this matrix is invertible, which means

$$\det\left(\begin{bmatrix} \mathbf{v}_s & \mathbf{m}_{t1} & \mathbf{m}_{t2} \end{bmatrix}^T\right) \neq 0 , \tag{131}$$

in which case the optimum weight vector is calculated as

$$\mathbf{w}_{opt} = \begin{bmatrix} \mathbf{v}_s^T \\ \mathbf{m}_{t1}^T \\ \mathbf{m}_{t2}^T \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \left(\dfrac{g_2(\theta_{t1})g_3(\theta_{t2}) - g_3(\theta_{t1})g_2(\theta_{t2})}{g_3(\theta_{t1})g_1(\theta_{t2}) - g_1(\theta_{t1})g_3(\theta_{t2})} \right) \\ 1 \\ \left(\dfrac{g_1(\theta_{t1})g_2(\theta_{t2}) - g_2(\theta_{t1})g_1(\theta_{t2})}{g_3(\theta_{t1})g_1(\theta_{t2}) - g_1(\theta_{t1})g_3(\theta_{t2})} \right) \end{bmatrix}. \tag{132}$$

Note that if this filter is applied to a noise vector given by

$$\mathbf{n} = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix}^T = \text{real-valued noise vector}, \tag{133}$$

where the noise voltages are zero-mean Gaussian, independent of each other, but identically distributed with variance

$$\sigma_n^2 = E\left\{|n_1|^2\right\} = E\left\{|n_2|^2\right\} = E\left\{|n_3|^2\right\}, \tag{134}$$

then the output noise power is calculated as

$$\sigma_{n,total}^2 = \sigma_n^2 \left(\mathbf{w}_{opt}^T \mathbf{w}_{opt} \right) = \sigma_n^2 \sum_i w_i^2. \tag{135}$$

3.3.1 Monopulse Slope for Constrained-Null Weight Solution

We now define the DOA function in the neighborhood of θ_{t2} as

$$d(\theta) = \mathbf{w}_{opt}^T \begin{bmatrix} g_1(\theta_{t2}+\theta) & g_2(\theta_{t2}+\theta) & g_3(\theta_{t2}+\theta) \end{bmatrix}^T. \tag{136}$$

We define a monopulse slope as the slope of $d(\theta)$ at θ_{t2} given a null at θ_{t1}. The monopulse slope will then be a function of both test angles. Accordingly, we define

$$k_d(\theta_{t2}|\theta_{t1}) = \frac{d}{d\theta}d(\theta)\bigg|_{\theta=0}. \tag{137}$$

We may expand this monopulse slope slightly as

$$k_d\left(\theta_{t2}\big|\theta_{t1}\right) = \mathbf{w}_{opt}^T \begin{bmatrix} \dfrac{d}{d\theta}g_1\left(\theta_{t2}+\theta\right)\Big|_{\theta=0} \\[1.5ex] \dfrac{d}{d\theta}g_2\left(\theta_{t2}+\theta\right)\Big|_{\theta=0} \\[1.5ex] \dfrac{d}{d\theta}g_3\left(\theta_{t2}+\theta\right)\Big|_{\theta=0} \end{bmatrix} = \mathbf{w}_{opt}^T \begin{bmatrix} \dfrac{d}{d\theta}g_1\left(\theta\right)\Big|_{\theta=\theta_{t2}} \\[1.5ex] \dfrac{d}{d\theta}g_2\left(\theta\right)\Big|_{\theta=\theta_{t2}} \\[1.5ex] \dfrac{d}{d\theta}g_3\left(\theta\right)\Big|_{\theta=\theta_{t2}} \end{bmatrix}. \tag{138}$$

From this we observe

- The answer is a weighted sum of the slopes of the antenna beams at the null angle of interest.
- The weight vector is a function of both null angles.
- This is clearly a 2-dimensional function.

Nevertheless, our estimate of the DOA function in the vicinity of θ_{t2} is calculated as

$$d\left(\theta\right) \approx d\left(0\right) + \left[k_d\left(\theta_{t2}\big|\theta_{t1}\right)\right]\theta = \left[k_d\left(\theta_{t2}\big|\theta_{t1}\right)\right]\theta \, , \tag{139}$$

where θ is an offset from θ_{t2}.

The monopulse slope can still be scaled to be relative to a reference beamwidth and reference signal level to yield a normalized quantity

$$k_m\left(\theta_{t2}\big|\theta_{t1}\right) = \frac{\theta_{ref}\, k_d\left(\theta_{t2}\big|\theta_{t1}\right)}{s_0\sqrt{\mathbf{w}_{opt}^T\mathbf{w}_{opt}}} = \text{normalized monopulse slope,} \tag{140}$$

from which we identify

$$s_0 = 1 = \text{reference beam gain with respect to constituent beam.} \tag{141}$$

We elaborate on this later.

3.3.2 Noise in the Angle Estimate

Consider a received signal in noise

$$\mathbf{x} = \mathbf{m}_1 + \mathbf{m}_2 + \mathbf{n} \, , \tag{142}$$

where

$$\mathbf{m}_1 = A_1 \begin{bmatrix} g_1(\theta_1) & g_2(\theta_1) & g_3(\theta_1) \end{bmatrix}^T, \text{ and}$$

$$\mathbf{m}_2 = A_2 \begin{bmatrix} g_1(\theta_2 + \theta_m) & g_2(\theta_2 + \theta_m) & g_3(\theta_2 + \theta_m) \end{bmatrix}^T. \tag{143}$$

where

θ_1 = known first signal DOA angle,

A_1 = unknown scale factor of first signal,

θ_2 = presumed second signal DOA angle,

θ_m = actual offset of second signal DOA angle from θ_2, and

A_2 = unknown scale factor of second signal. $\tag{144}$

We shall presume that $\mathbf{w}_{opt}^T$ is calculated based on θ_1 and θ_2, but without knowledge of θ_m. Note that since θ_1 is precisely known, it can be precisely nulled, leaving

$$\mathbf{w}_{opt}^T \mathbf{x} = \mathbf{w}_{opt}^T \mathbf{m}_2 + \mathbf{w}_{opt}^T \mathbf{n}. \tag{145}$$

The error contribution due to noise is calculated as

$$\varepsilon_n = \mathbf{w}_{opt}^T \mathbf{n}. \tag{146}$$

The error contribution due to angle offset θ_m is

$$\varepsilon_m = \mathbf{w}_{opt}^T \mathbf{m}_2 = A_2 \left[k_d \left(\theta_2 | \theta_1 \right) \right] \theta_m. \tag{147}$$

Setting these errors equal to each other yields the equation to estimate the equivalent offset DOA due to the noise as

$$\hat{\theta}_m = \frac{\mathbf{w}_{opt}^T \mathbf{n}}{A_2 \left[k_d \left(\theta_2 | \theta_1 \right) \right]}. \tag{148}$$

If we estimate θ_m in this manner, we will do so with a statistical variance of the DOA noise calculated to be

$$\sigma_\theta^2 = \frac{\sigma_{n,total}^2}{A_2^2 \left[k_d \left(\theta_2 | \theta_1 \right) \right]^2} = \frac{\sigma_n^2 \mathbf{w}_{opt}^T \mathbf{w}_{opt}}{A_2^2 \left[k_d \left(\theta_2 | \theta_1 \right) \right]^2}. \tag{149}$$

Some substitutions allow us to write this as

$$\sigma_\theta^2 = \frac{\theta_{ref}^2}{\left(\dfrac{\theta_{ref}^2 \left[k_d \left(\theta_2 | \theta_1 \right) \right]^2}{\left(\mathbf{w}_{opt}^{\mathrm{T}} \mathbf{w}_{opt} \right) s_0^2} \right) \left(\dfrac{A_2^2 s_0^2}{\sigma_n^2} \right)} \,, \tag{150}$$

from which we recall and otherwise identify

$$k_m \left(\theta_2 | \theta_1 \right) = \frac{\theta_{ref} k_d \left(\theta_2 | \theta_1 \right)}{s_0 \sqrt{\mathbf{w}_{opt}^{\mathrm{T}} \mathbf{w}_{opt}}} = \text{normalized monopulse slope,}$$

$$SNR = \left(\frac{A_2^2 s_0^2}{2 \sigma_n^2} \right), \text{ and}$$

$$s_0 = 1 = \text{reference beam gain with respect to constituent beam,} \tag{151}$$

and then write the DOA angle RMS noise in the familiar form

$$\sigma_\theta = \frac{\theta_{ref}}{\left| k_m \left(\theta_2 | \theta_1 \right) \right| \sqrt{2 SNR}} \,. \tag{152}$$

The SNR is the reference beam signal power to constituent beam noise. The absolute value is used to acknowledge that the normalized monopulse slope may be of either sign, depending on individual null positions.

3.3.3 *Figure of Merit for Optimum Squint Angle*

Recall that the signal amplitude is a function of the transmit beam gain, which means it includes its pattern for off-boresight angles. Consequently, our figure of merit for θ_2 given θ_1 which we wish to maximize is based on minimizing the RMS DOA noise, and becomes

$$FOM \left(\theta_2 | \theta_1 \right) = \frac{g_2 \left(\theta_2 \right) \left| k_d \left(\theta_2 | \theta_1 \right) \right|}{\sum_i w_i^2} \,. \tag{153}$$

The absolute value notation is required because the monopulse slope may be either positive or negative depending on which side of θ_1 we are calculating.

We shall again assume a beam-shape from Sherman and Barton, namely

$$g_0(\theta) = \frac{\cos(\pi K_\theta \theta)}{1 - (2K_\theta \theta)^2} , \tag{154}$$

where

$$K_\theta = 1.189 / \theta_{ref} . \tag{155}$$

Figure 17 plots contours of the maximum FOM for various combinations of θ_1 and θ_2. Figure 18 plots the corresponding squint angle that yields the maximum FOM. From these plots we make the following observations.

- The 'best' squint angle depends very much on the specific null angles θ_1 and θ_2 for which we are optimizing.

- The farther apart the two null angles are, the greater is the optimum squint angle.

- For null angles that are near each other, there still is a dependence on just where in the beam the two angles are, although there is a maximum at a squint angle in the neighborhood of the nominal beamwidth.

The question remains "How do we choose 'which' optimal squint angle to choose?" To gain some insight to an answer, we choose to plot the monopulse slope over intervals of θ_1 and θ_2 for two sample squint angles.

Figure 19 illustrates contours of local monopulse slope at θ_2 for squint angles of ± 0.55 constituent beamwidths. Note that no singularities are apparent over the entire domains of θ_1 and θ_2. The local monopulse slope does not blow up.

Figure 20 illustrates contours of local monopulse slope at θ_2 for squint angles of ± 1.05 constituent beamwidths. Note the behavior in the region of $\theta_2 = \theta_1 \approx \pm 0.43$. We explore this somewhat in Figure 21, where we identify arcs of extremely high values of monopulse slope. These represent angle combinations where the matrix in Eq. (130) becomes ill-conditioned. The bad behavior comes about from an interaction of mainlobes with sidelobes of the squinted beams. This arc of angle combinations needs to be avoided.

In order to keep the local monopulse slope well-behaved, we may adjust some combination of the following.

1. We may reduce the squint angles to lesser magnitudes to push the arcs farther away from boresight.

2. We may restrict the allowable interval of angles for θ_1 and θ_2 to domains inside of the arc of ill-conditioning.
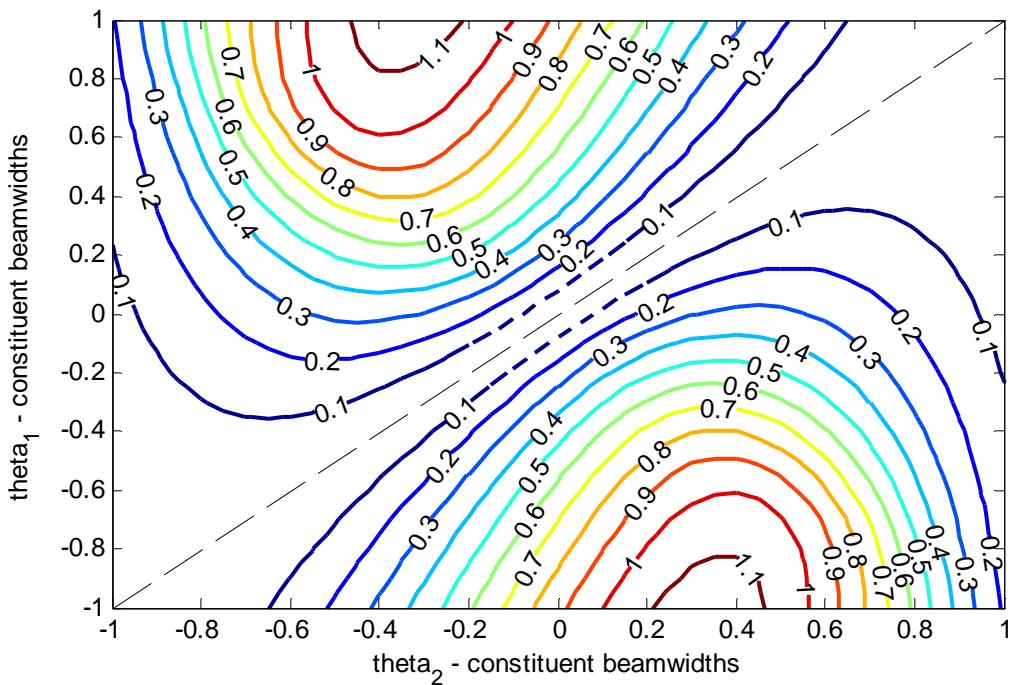
Figure 17. Maximum FOM over all squint angles.
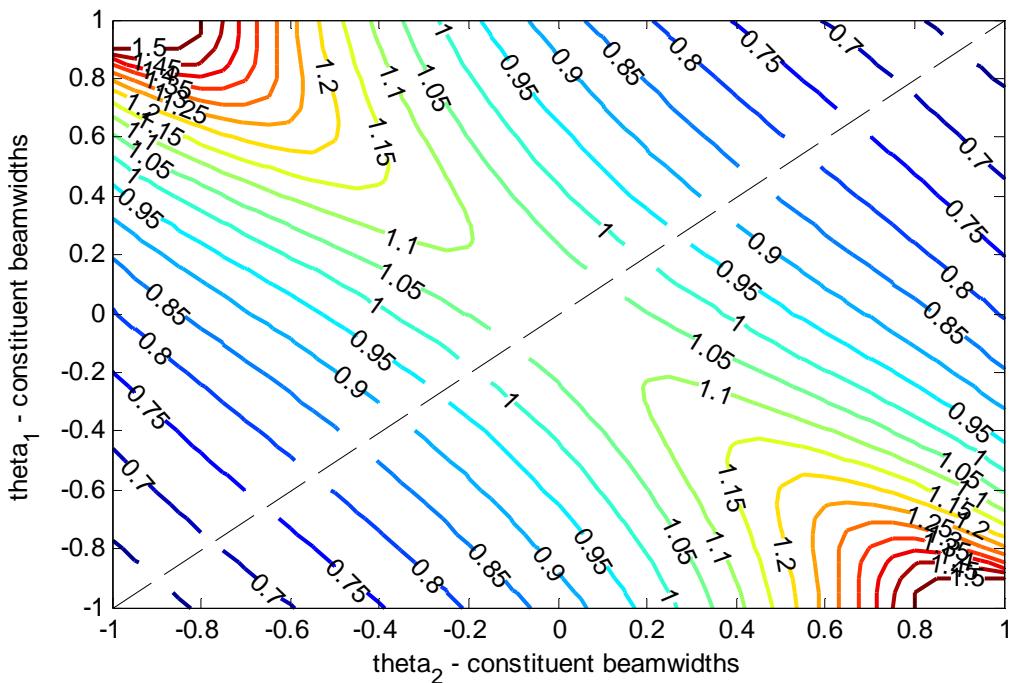
**Figure 18. Optimal squint angle in constituent beamwidths to maximize FOM.**
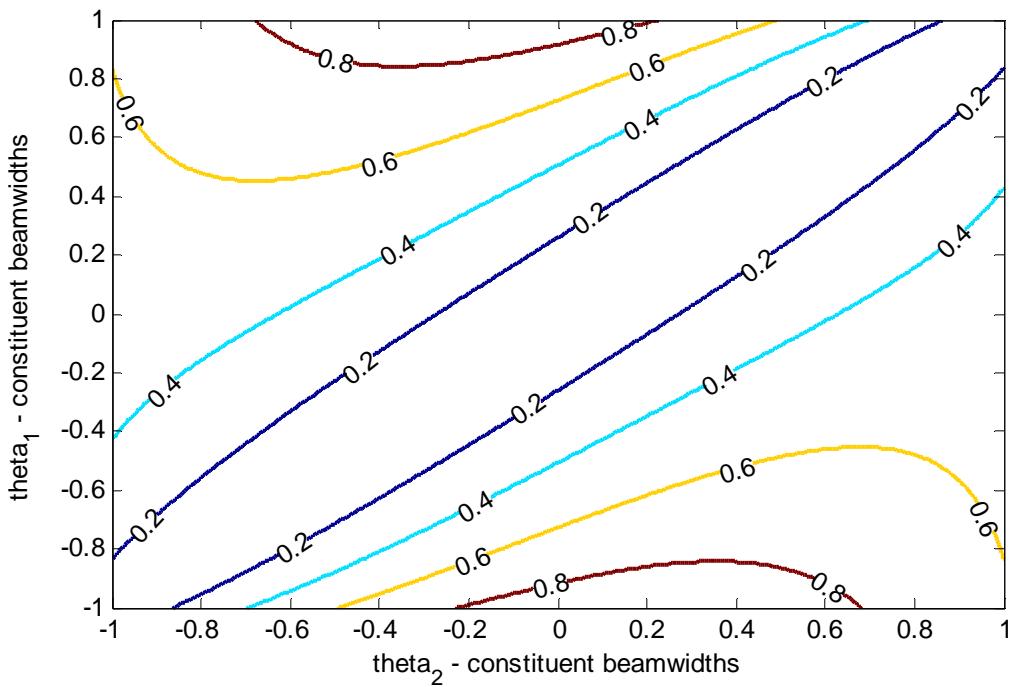
**Figure 19. Monopulse slope for squint angles of ±0.55 constituent beamwidths.**
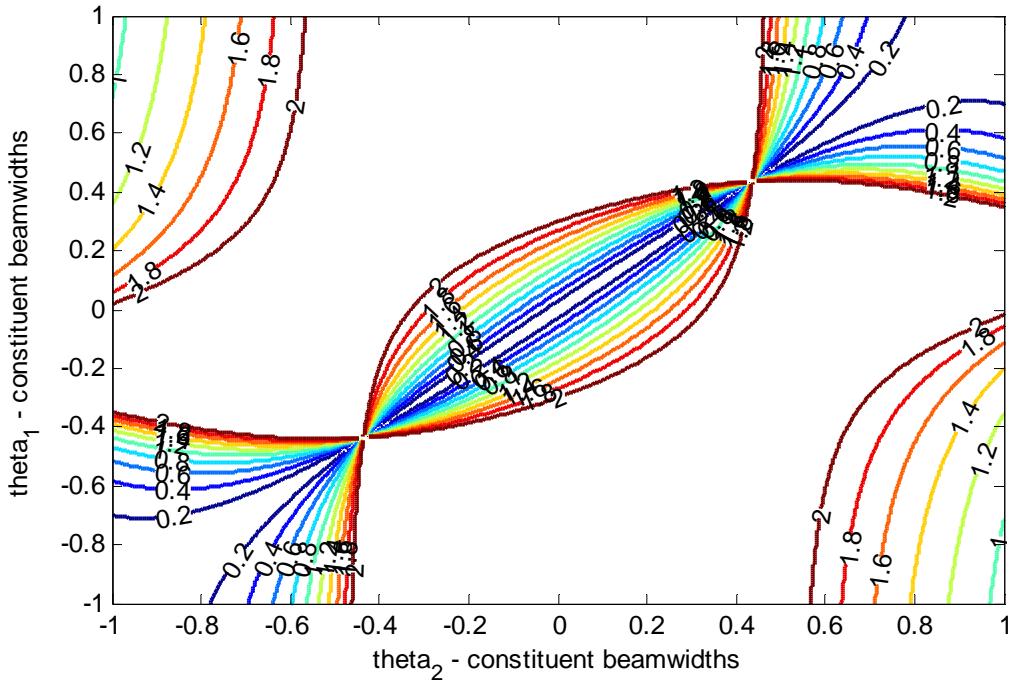
Figure 20. Monopulse slope for squint angles of ±1.05 constituent beamwidths.
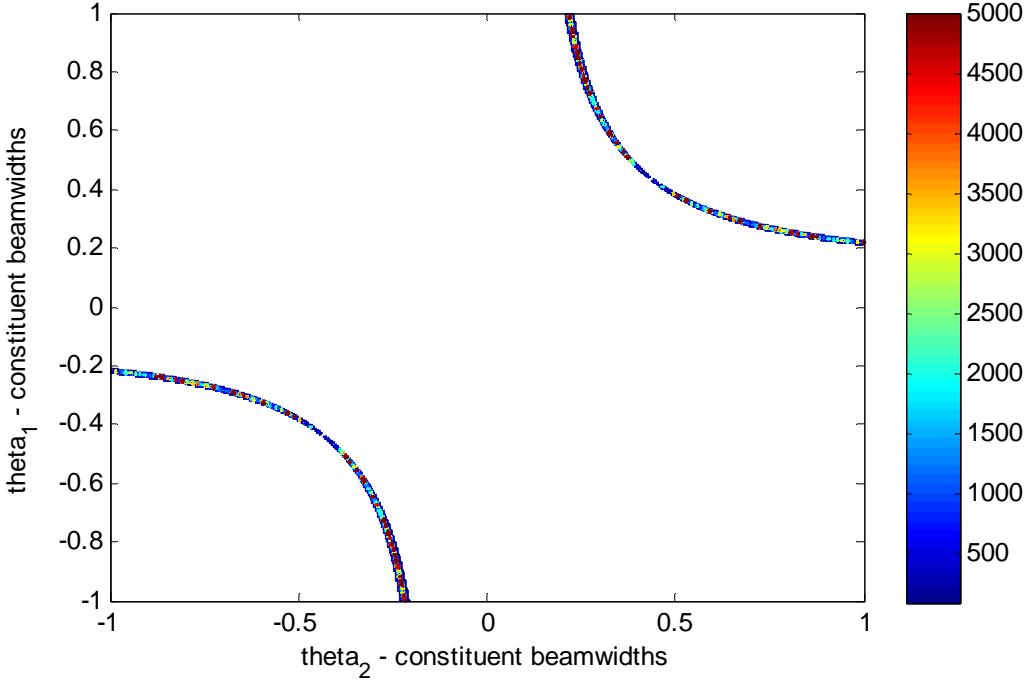
Figure 21. Monopulse slope for squint angles of ±1.05 constituent beamwidths, limited to values greater than 100.

Consequently, we desire the largest squint angles possible subject to the constraint that we avoid ill-conditioning for intervals of interest for angles for θ_1 and θ_2. This of course means that we need to define intervals of interest for angles for θ_1 and θ_2.

We point out that the solution for a closed-loop tracking radar is probably something quite different than for, say, an airborne GMTI radar. We are herein more interested in the latter.

We propose to define the intervals of interest θ_1 and θ_2 to be $\pm\theta_g$, that is, an interval of ±1 constituent beamwidth on either side of boresight for each null angle. This represents the approximately -15 dBc beamwidth of a constituent beam for the shape given in Eq. (154). With these criteria, we identify the optimum squint angle as approximately

$$\theta_{sq,optimum} \approx 0.657\,\theta_g. \tag{156}$$

Figure 22 plots contours of local monopulse slope at θ_2 for squint angles of ±0.657 constituent beamwidths, and Figure 23 plots contours of FOM for the same conditions. Recall that the FOM is proportional to the inverse of DOA angle estimation RMS noise.

Figure 24 plots the corresponding one-way receive antenna patterns.
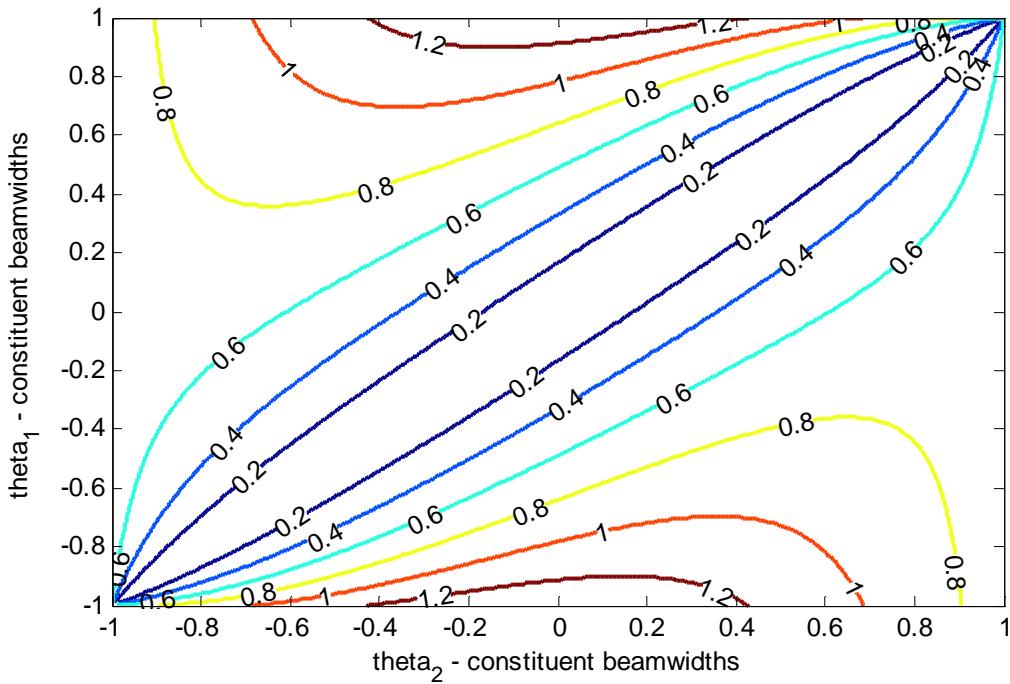
**Figure 22. Monopulse slope for squint angles of ±0.657 constituent beamwidths.**
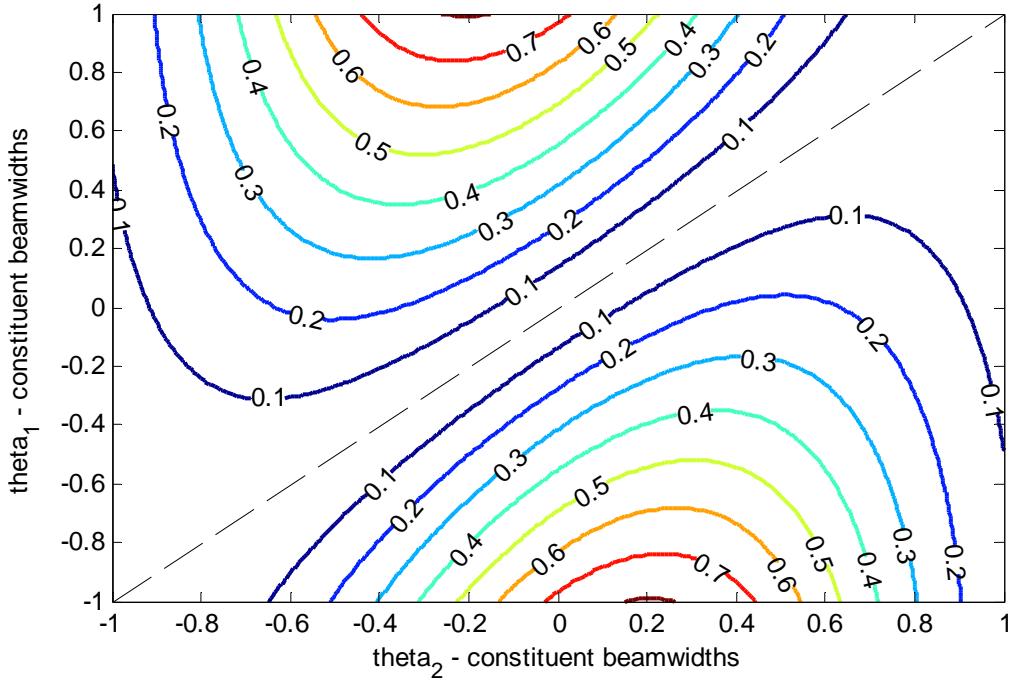
**Figure 23. FOM for squint angles of ±0.657 constituent beamwidths.**
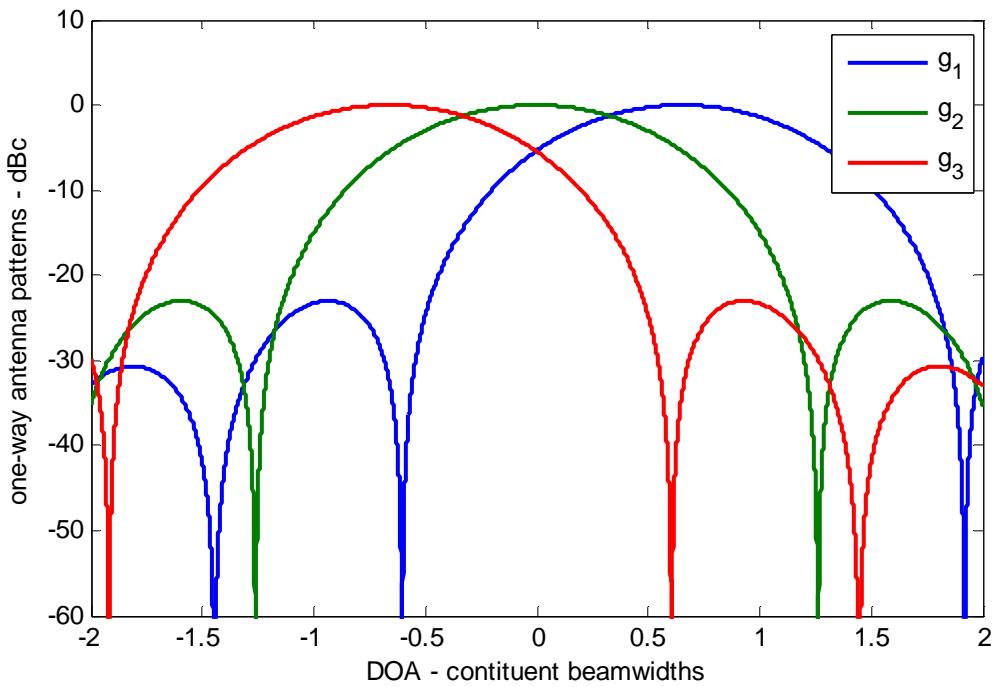
**Figure 24. One-way antenna patterns with squint angles of ±0.657 constituent beamwidths.**
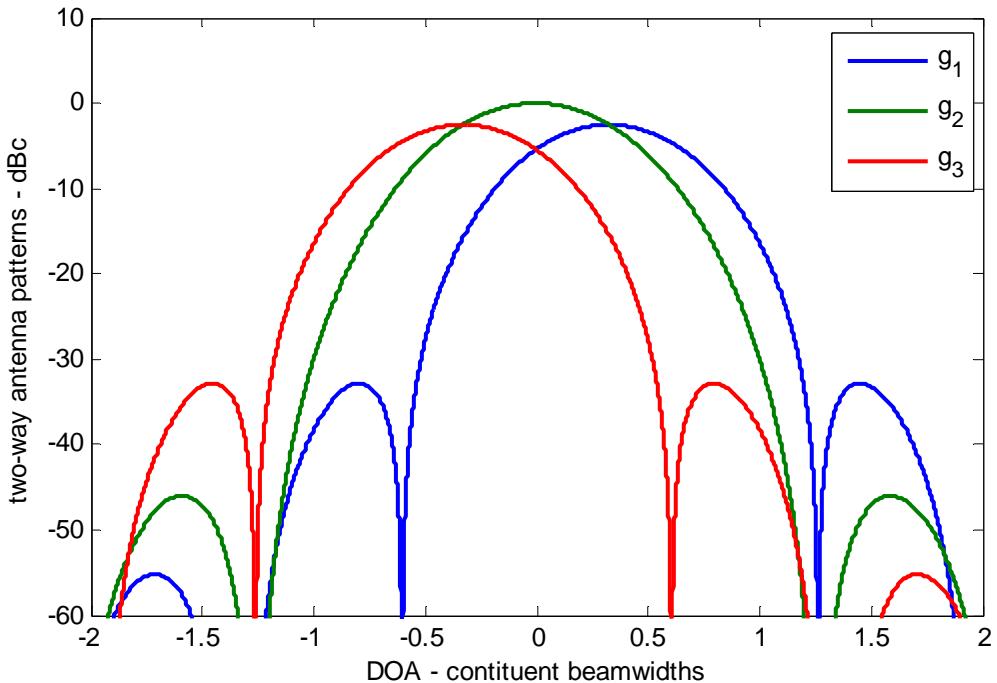
Figure 25. Two-way antenna patterns with squint angles of ±0.657 constituent beamwidths. Assumes transmit from the center beam only.

3.4 Comments and Notes

We make several comments in no particular order.

- We reiterate that the foregoing analysis is based on maximizing our ability to determine the DOA angle for a second target signal with the constraint of having a null placed in a precise first DOA direction.

- We contrast Figure 16 with Figure 2(b) and observe that they are essentially the same, except that the middle feed is also employed to receive in the three-beam case.

- We note that the angle combinations where the matrix in Eq. (130) becomes ill-conditioned depends on the scaling vector $\mathbf{v}_s$. A different scaling factor yields a different determinant, and hence different behavior of where Eq. (130) becomes ill-conditioned. However, the FOM divides this by the RMS weight values such that in the limit the FOM is independent of the particular scaling vector employed. The scaling vector then has no influence on FOM, and in fact no influence on DOA angle noise power σ_θ^2. Furthermore, it has no influence on the *normalized* monopulse slope $k_m\left(\theta_2 \middle| \theta_1\right)$.

- Since the choice of scaling vector doesn't influence DOA angle noise power σ_θ^2 anyway, this raises the question of whether the optimum squint angle limit in Eq. (156) is a valid concern. We argue that since the matrix inversion in Eq. (132) is a necessary step to find $\mathbf{w}_{opt}$, a necessary intermediate step, and we desire to keep $\mathbf{w}_{opt}$ bounded and well-behaved, then we do in fact wish to avoid Eq. (130) becoming ill-conditioned, requiring a corresponding limit on squint angle.

- The scaling vector does influence the particular optimum weight vector $\mathbf{w}_{opt}$ and any potential constraints that would keep the weight vector well-behaved. This raises the possibility of increasing the specific optimum squint angle limit in Eq. (156) by choosing a different scaling vector. Anecdotal evidence suggests that while a larger squint angle can indeed sometimes be allowed, this does not translate to an improved FOM for all angle combinations, and can in fact reduce FOM in large swaths of useful angle combinations. There is no free lunch. We argue that the scaling vector in Eq. (129) offers an attractive symmetry in the monopulse slope plots, and overall more desirable performance with less monopulse slope variation with target signal DOA angle difference.

The bottom line is that our choices do have consequences that we need to understand as we make them.

3.5 Target Angles

Consider two samples from a uniform distribution with zero mean. The Probability Density Function (PDF) is given by

$$f_u(x) = \text{rect}(x) = \begin{cases} 1 & |x| \le 1/2 \\ 0 & else \end{cases}. \tag{157}$$

The question is "What is the PDF (and its statistics) for the difference between two samples from this distribution?"

We can straight-forwardly calculate the PDF of the difference between two samples from the uniform distribution as

$$f_d(x) = \Lambda(x) = \begin{cases} 1+x & -1 \le x \le 0 \\ 1-x & 0 \le x \le 1 \\ 0 & else \end{cases}. \tag{158}$$

The absolute value of the difference has a PDF that is described by

$$f_{|d|}(x) = \begin{cases} 2 - 2x & 0 \le x \le 1 \\ 0 & else \end{cases}. \tag{159}$$

This has a mean value calculated as

$$\mu_{|d|} = 1 - \frac{\sqrt{2}}{2} \approx 0.3. \tag{160}$$

This suggests that the probability of being less than 0.3 is about 50% for the difference between two samples with PDFs given by Eq. (157). More generally, for uniform PDFs of some other common width, the mean difference is about 30% of the width of the uniform distribution.

With respect to two target signals uniformly distributed between ±1 constituent beamwidths from boresight, this means the mean difference between target signal DOA angles is approximately 0.6 constituent beamwidths.

4 Processing Three Amplitude Channels for DOA

Here we wish to make three measurements on each of three beams, and from them calculate the unknown direction of a target signal in the presence of another target signal in a known direction. We begin by defining several vectors as follows, namely

$$\mathbf{m}_1 = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} \end{bmatrix}^T = \text{real-valued signal vector for first signal,}$$

$$\mathbf{m}_2 = \begin{bmatrix} m_{2,1} & m_{2,2} & m_{2,3} \end{bmatrix}^T = \text{real-valued signal vector for second signal,}$$

$$\mathbf{M} = \mathbf{m}_1 + \mathbf{m}_2 = \text{combined signal vector,}$$

$$\mathbf{n} = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix}^T = \text{real-valued noise vector,}$$

$$\mathbf{x} = \mathbf{M} + \mathbf{n} = \text{measurement vector, and}$$

$$\mathbf{v}_s = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T = \text{scaling constraint vector.} \tag{161}$$

Furthermore, we define the signal vectors in terms of antenna gain parameters as

$$\mathbf{m}_1 = A_1 \begin{bmatrix} g_1(\theta_1) & g_2(\theta_1) & g_3(\theta_1) \end{bmatrix}^T, \text{ and}$$

$$\mathbf{m}_2 = A_2 \begin{bmatrix} g_1(\theta_2) & g_2(\theta_2) & g_3(\theta_2) \end{bmatrix}^T, \tag{162}$$

where

$$\theta_1 = \text{DOA of known target signal with amplitude } A_1, \text{ and}$$

$$\theta_2 = \text{DOA of unknown target signal with amplitude } A_2. \tag{163}$$

Essentially, vector $\mathbf{m}_2$ is unknown both in amplitude and DOA angle. We shall furthermore assume that both angles are located within the center beam, i.e. both $g_2(\theta_1)$ and $g_2(\theta_2)$ offer a significant response. Consequently, we may construct the matrix equation

$$\begin{bmatrix} \mathbf{v}_s^T \\ \mathbf{m}_1^T \\ \mathbf{x}^T \end{bmatrix} \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ \varepsilon \end{bmatrix}. \tag{164}$$

We make several observations.

- For the second row of the Matrix, we may assume an arbitrary $A_1 \neq 0$.

- Because of noise, we may assume that the matrix is of full rank. Consequently the matrix is invertible.

- Since the matrix is invertible, a solution exists for $\varepsilon = 0$.

Consequently, we may calculate the optimum weight vector as

$$\mathbf{w} = \begin{bmatrix} \mathbf{v}_s^T \\ \mathbf{m}_1^T \\ \mathbf{x}^T \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{165}$$

An implicit assumption here is that the beams are sufficiently close together that there is a reasonable response in more than the center beam, that is

$$\mathbf{m}_1^T \neq \beta \, \mathbf{v}_s^T \quad \text{for any scalar } \beta . \tag{166}$$

This means we need a significant response in more than one beam, essentially in all of them, for all target directions of interest. Otherwise the matrix is not invertible.

Once we have a weight vector solution, we may plot its response to various target signal angles by calculating the filter response

$$\xi = \mathbf{m}_{test}^T \, \mathbf{w} , \tag{167}$$

where

$$\mathbf{m}_{test} = \begin{bmatrix} g_1(\theta_{test}) & g_2(\theta_{test}) & g_3(\theta_{test}) \end{bmatrix} ,$$
$$\theta_{test} = \text{the test angle over an interval of interest.} \tag{168}$$

From this filter response, we should readily identify the known null at θ_1, since this is a constraint of the weight vector solution. In addition, with good SNR, a second null may manifest in the vicinity of θ_2.

We now offer an example to illustrate this.

Example

We shall assume a beam-shape consistent with earlier examples, namely

$$g_0(\theta) = \frac{\cos(\pi K_\theta \theta)}{1 - (2K_\theta \theta)^2}, \text{ and}$$

$$K_\theta = 1.189. \tag{169}$$

Note that the beam-shape has unity gain and a unity -3 dB beamwidth. The beams are squinted with angle

$$\theta_{sq} = 0.657 \text{ beamwidths.} \tag{170}$$

The unit "beamwidths" refers to the -3 dB beamwidth of $g_0(\theta)$, which for this example is unity. Now consider the conditions

$$\theta_1 = 0.1 \text{ beamwidths,}$$
$$\theta_2 = -0.2 \text{ beamwidths,}$$
$$\sigma_n^2 = E\left\{|n_1|^2\right\} = E\left\{|n_2|^2\right\} = E\left\{|n_3|^2\right\} = \frac{A_2^2}{2000}. \tag{171}$$

Note that the noise is -30 dBc with respect to the signal at the DOA of θ_2.

The three beams are illustrated in Figure 26. An example filter response for these conditions is illustrated in Figure 27. Two more examples are given in each of Figure 28 and Figure 29.

Of course, better SNR will cause less wandering of the second estimated null position.
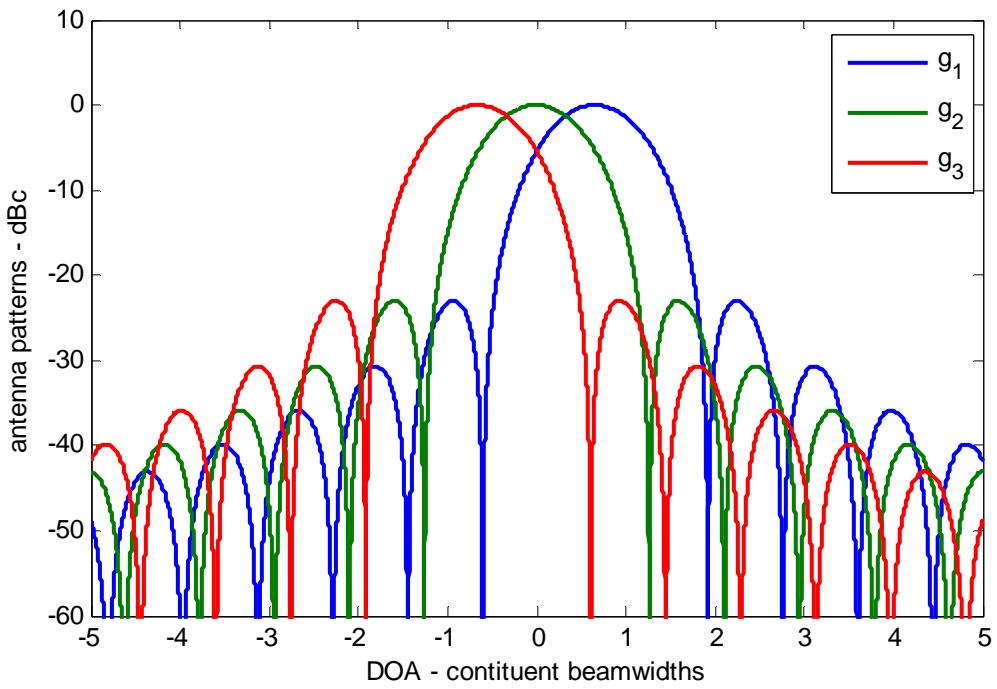
**Figure 26. Example beam patterns for three beams.**
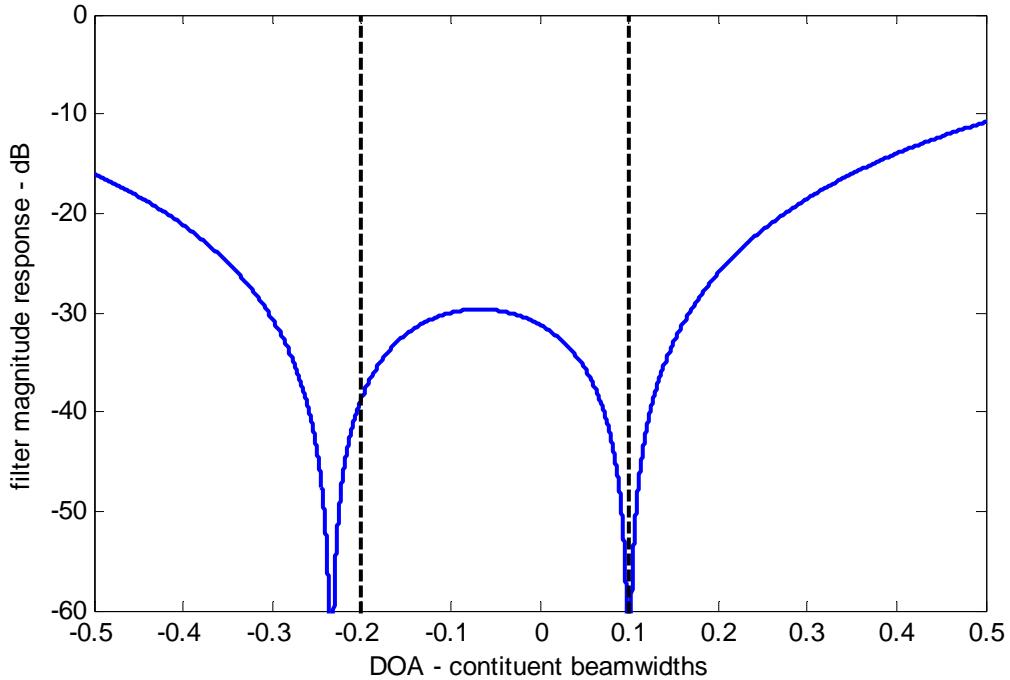
Figure 27. Example filter response. Note the specified null precisely at 0.1 beamwidths, and the calculated null at about –0.23 beamwidths, displaced by noise from the true DOA at –0.2 beamwidths.
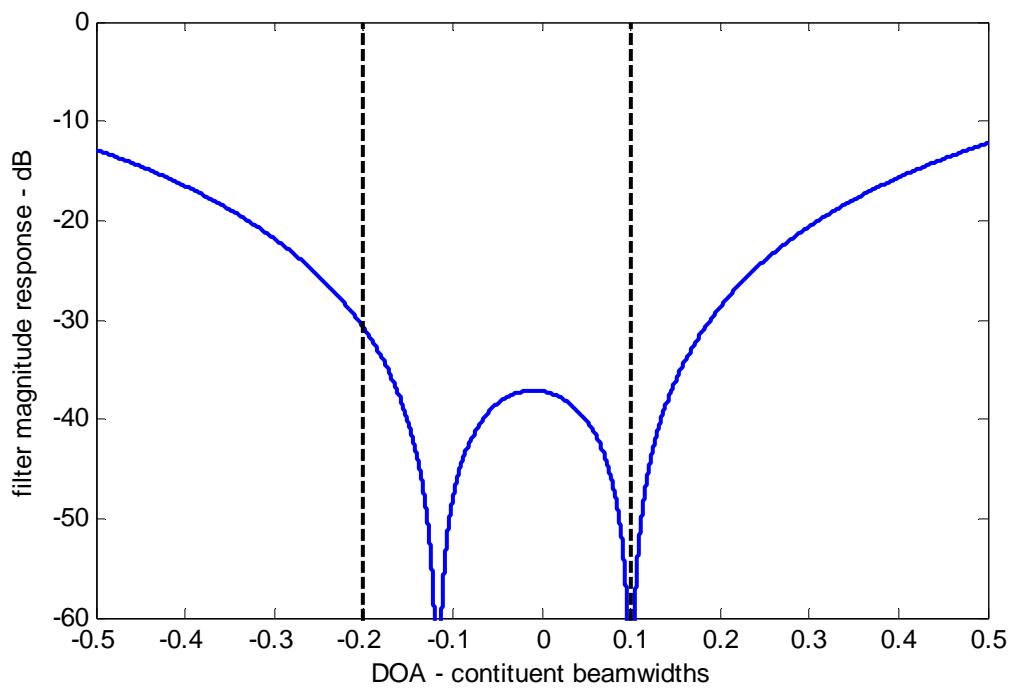
Figure 28. Example filter response due to different noise vector.
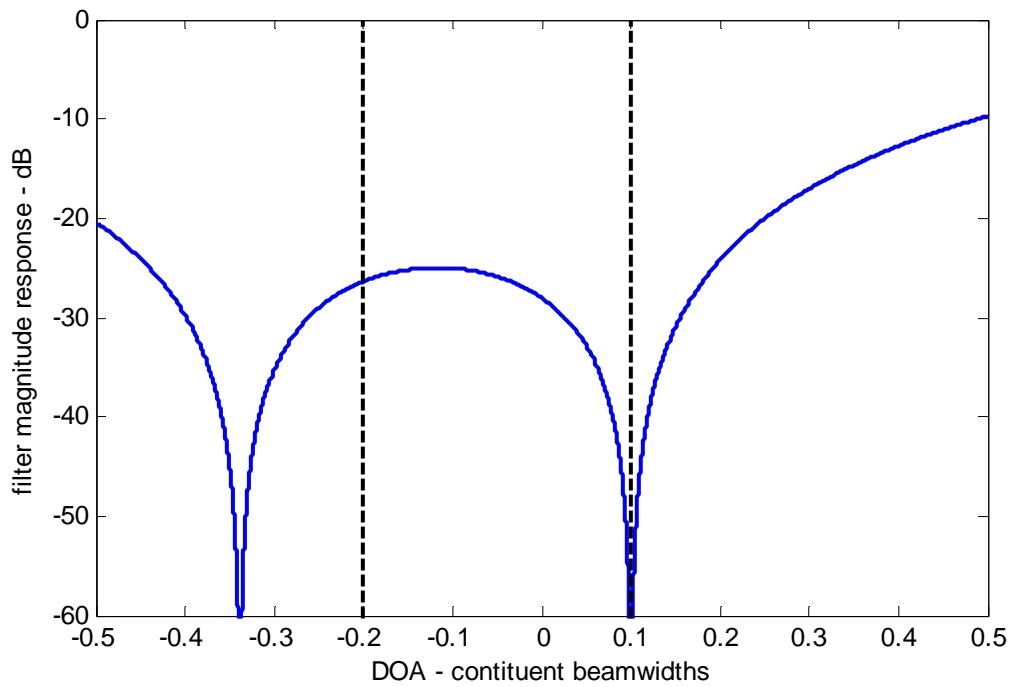
Figure 29. Example filter response due to yet a different noise vector.

"I'll try anything once, twice if I like it, three times to make sure."
-- Mae West

5 Conclusions

We summarize herein the following key points.

- Three beams can null two signals from different DOA angles.

- Three beams allow determining the DOA angle of a target signal in the presence of a second target signal.

- Maximizing the accuracy and precision of DOA calculations requires maximizing SNR as well as monopulse slope, and perhaps trading between the two.

- For three-beam monopulse, the transmit signal should use the center beam, and only the center beam, and not include any sum of other beams. Even in dual-beam monopulse, transmitting on the sum beam is merely a consolation if no center reference beam exists.

- Maximizing the accuracy and precision of DOA calculations requires significant overlap of the outside beam patterns.

- The optimum beam squint angles can be readily calculated from constituent beam patterns and the DOA interval of interest.

"Facts are stubborn things."
-- Ronald Reagan

References

[1] Samual M. Sherman, David K. Barton, Monopulse Principles and Techniques – second edition, ISBN-13 978-1-60807-174-6, Artech House, 2011.

[2] Armin W. Doerry, Douglas L. Bickel, "GMTI Direction of Arrival Measurements from Multiple Phase Centers," Sandia Report SAND2015-2310, Unlimited Release, March 2015.

[3] Armin W. Doerry, Douglas L. Bickel, "Limits to Clutter Cancellation in Multi-Aperture GMTI Data," Sandia Report SAND2015-2311, Unlimited Release, March 2015.

[4] Billy C. Brock, Steven E. Allen, "Correcting Cross-polarization Monopulse Response of Reflector Antennas," Sandia Report SAND2015-1316, Unlimited Release, December 2014.

[5] D. L. Bickel, "Precomparator and Postcomparator Errors in Monopulse," Sandia Report SAND20131287, Unlimited Release, February 2013.

[6] Armin W. Doerry, "Radar Channel Balancing with Commutation," Sandia Report SAND2014-1071, Unlimited Release, February 2014.

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