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Next-Generation Algorithms for Assessing Infrastructure Vulnerability and Optimizing System Resilience

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Abstract

This report summarizes the work performed under the project project “Next-Generation Algorithms for Assessing Infrastructure Vulnerability and Optimizing System Resilience.” The goal of the project was to improve mathematical programming-based optimization technology for infrastructure protection. In general, the owner of a network wishes to design a network a network that can perform well when certain transportation channels are inhibited (e.g. destroyed) by an adversary. These are typically bi-level problems where the owner designs a system, an adversary optimally attacks it, and then the owner can recover by optimally using the remaining network.

This project funded three years of Deon Burchett’s graduate research. Deon’s graduate advisor, Professor Jean-Philippe Richard, and his Sandia advisors, Richard Chen and Cynthia Phillips, supported Deon on other funds or volunteer time. This report is, therefore. essentially a replication of the Ph.D. dissertation it funded [12] in a format required for project documentation.

The thesis had some general polyhedral research. This is the study of the structure of the feasible region of mathematical programs, such as integer programs. For example, an integer program optimizes a linear objective function subject to linear constraints, and (nonlinear) integrality constraints on the variables. The feasible region without the integrality constraints is a convex polygon. Careful study of additional valid constraints can significantly improve computational performance.

Here is the abstract from the dissertation:

We perform a polyhedral study of a multi-commodity generalization of variable upper bound flow models. In particular, we establish some relations between facets of single- and multi-commodity models. We then introduce a new family of inequalities, which generalizes traditional flow cover inequalities to the multi-commodity context. We present encouraging numerical results.

We also consider the directed edge-failure resilient network design problem (DRNDP). This problem entails the design of a directed multi-commodity flow network that is capable of fulfilling a specified percentage of demands in the event that any Γ arcs are destroyed, where Γ is a constant parameter. We present a formulation of DRNDP and solve it in a branch-column-cut framework. We present computational results.

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Chapter 1

INTRODUCTION

Network design encompasses a large class of discrete optimization problems. A network design problem seeks to determine a network topology that minimizes a cost function (e.g., total link construction cost) while also satisfying certain requirements, chief among them being the ability to send a sufficient amount of flow between selected pairs of nodes. A network design problem is called *capacitated* if there are intrinsic limits on the amount of flow individual links can transmit. In the case of *fixed-charge capacitated* problems, when a link is constructed it incurs a fixed cost and is assigned a predetermined amount of capacity, both of which may vary by link. *Multi-commodity* problems are those that model the flow of two or more distinct commodities through a network. In this dissertation, we consider *multi-commodity fixed-charge capacitated network design* (MFCND), which is known to be NP-Hard [21].

MFCND is naturally defined in terms of an underlying graph which, depending on the application, can be chosen to be undirected or directed. We focus primarily on the latter, and we let $G = (N,A)$ denote a directed graph with node set N and arc set $A \subseteq N \times N$. For each node $i \in N$, let $\delta^+(i)$ and $\delta^-(i)$ be the sets of arcs entering i and leaving i, respectively. Let $K := \{1,\dots,|K|\}$ be the set of commodities. For each $i \in N$ and $k \in K$, let $b_i^k \in \mathbb{R}$ be the *supply* of commodity k at node i. We refer to $i \in N$ as a *supply node*, *demand node*, or *transhipment node* in relation to $k \in K$ if $b_i^k > 0$, $b_i^k < 0$, or $b_i^k = 0$, respectively. We assume that each commodity $k \in K$ has a unique supply node $s^k \in N$, which we call its *source*. We permit arcs to carry fractional amounts of flow. In addition, we assume that flows are *splittable*, that is, it is not required that all the flow of a given commodity be routed along a single path. Let $c_{ij} \in \mathbb{R}_+$ and $u_{ij} \in \mathbb{R}_+$ be the fixed construction cost and the capacity associated with arc (i, j), respectively. Also, we associate a cost of $h_{ij}^k \in \mathbb{R}_+$ with each unit of flow of commodity k on arc (i, j). The following is a standard MIP formulation of MFCND under these assumptions (see, *e.g.*, [24]):

$$\begin{aligned}
\text{MFCND} := \min_{x,y} \quad & \sum_{(i,j)\in A} c_{ij}x_{ij} + \sum_{(i,j)\in A}\sum_{k\in K} h_{ij}^k y_{ij}^k && && (1.1a)\\
\text{s.t.} \quad & \sum_{(i,j)\in\delta^+(i)} y_{ij}^k - \sum_{(j,i)\in\delta^-(i)} y_{ji}^k = b_i^k && \forall i \in N, \forall k \in K, && (1.1b)\\
& \sum_{k\in K} y_{ij}^k \le u_{ij}x_{ij} && \forall (i,j) \in A && (1.1c)\\
& y_{ij}^k \ge 0 && \forall (i,j) \in A, \forall k \in K && (1.1d)\\
& x \le 1 && \forall (i,j) \in A && (1.1e)\\
& x \ge 0 && \forall (i,j) \in A && (1.1f)\\
& x \in \mathbb{Z}^A. && && (1.1g)
\end{aligned}$$

In formulation (1.1), the *design variable* x_{ij} corresponds to the decision of whether or not to construct arc (i, j), while the *flow variable* y_{ij}^k represents the flow of commodity k on (i, j). The objective (1.1a) minimizes the total cost incurred by constructing links and routing flows through the resulting network. Constraints (1.1b) enforce the flow balance of each commodity $k \in K$ at each node $i \in N$. The *variable upper bound constraints* (1.1c) ensure that, for each arc $(i, j) \in A$, the total flow of all commodities across the arc does not exceed its allotted capacity. Constraints (1.1d) require y to be nonnegative. Finally, constraints (1.1e) and (1.1f) are upper and lower bound constraints on x_{ij}, respectively, while (1.1g) enforces the integrality of x.

As defined above, MFCND is concerned solely with constructing a network that supports a feasible multi-commodity flow. However, the *resilience* or *survivability* of a network, *i.e.*, its performance in the face of disruptions, is often a question of interest. There are multiple ways of characterizing network survivability. We focus on a criterion which we refer to as Γ-*edge failure resilience*, which requires that a network be able to meet some specified proportion of demand in the event that any subset of Γ edges fails, where $\Gamma \in \mathbb{Z}_+$ is a fixed parameter.

The task of constructing a multi-commodity network occurs in various domains. For instance, numerous aspects of transportation planning can be expressed in terms of network design. These vary in scope from strategic capital investments to day-to-day operational scheduling [30]. The telecommunications industry has also provided a fertile ground for the application of network design models. Typically, these models seek an optimal allocation of *facilities*, *i.e.*, data transmission equipment, to links in a fiber-optic network [3].

Although the most prevalent uses of network design could be characterized as "civilian," applications also arise in the realm of national security. The US government has designated 16 sectors of *critical infrastructure*, comprised of those sectors deemed integral to the proper functioning of American society [34]. Many of these classes of infrastructure are naturally viewed as capacitated networks. While certain systems can be modeled in terms of single-commodity flows (*e.g.*, water distribution systems), others require multi-commodity flows (*e.g.*, communications networks). In light of the overriding importance of critical infrastructure, it is very much in the nation's interest to ensure that these systems continue to operate in the face of catastrophic events such as natural disasters, sabotage, or acts of terrorism. This provides a strong incentive for the study of network design formulations which incorporate survivability requirements.

Our work is aimed at advancing the body of knowledge pertaining to multi-commodity network design in two areas. First, we seek to further understand the polyhedral structure of these problems. This is realized through a study of the multi-commodity variable upper bound flow model (MVF), which we address in Chapter 2. Whereas it is standard practice to derive cuts for multi-commodity problems by way of single-commodity relaxations obtained through aggregation (see *e.g.*, [1, 4]), our analysis reveals features that are obscured by such aggregation procedures. Specifically, obtain a large family of valid inequalities, based on structures we call *commodity hierarchies*, that cannot be obtained through aggregation and can be applied to general instances of MFCND.

The second focus of our investigation is in the area of network resilience. We consider an edge-failure resilient MFCND problem, which we discuss in Chapter 3. We propose a solution algorithm and perform computational experiments on randomly generated instances. We view this work as an important contribution towards making resilient network design a practical tool for decision makers. We give concluding remarks and discuss future research in Chapter 4.

Chapter 2

MULT-COMMODITY VARIABLE UPPER BOUND FLOW MODELS

2.1 Background

Variable upper bound flow models are well-studied in the mixed-integer programming literature. The practical value of these models stems from the fact that they often occur as substructures of MILPs. In particular, they can be viewed as single-node instances of the fixed-charge capacitated network design problem, which has applications in telecommunications and transportation [33].

For the case where a single commodity is considered, Padberg, van Roy, and Wolsey [38] study the polyhedral structure of variable upper bound flow models and identify the family of flow cover inequalities. Wolsey [52] shows that these inequalities can be derived using properties of submodular functions. Gu, Nemhauser, and Savelsbergh [26] describe a procedure for lifting flow cover inequalities. Various authors have studied generalizations of the classic variable upper bound flow model, which in turn, yield extensions of flow cover inequalities. For instance, Klabjan and Nemhauser [28] study variable upper bound flow models with general integer variable upper bounds, while Shebalov and Klabjan [44] consider models in which the variable upper bound constraints include constant terms. Atamtürk, Nemhauser and Savelsbergh [5] examine the case with additive variable upper bound constraints.

Various families of valid inequalities have been identified for variants of the multi-commodity capacitated network design problem (MCND). Bienstock et al. [9] apply *partition inequalities* and *flow-cutset inequalities* in a branch-and-cut framework to solve the minimum cost capacity installation problem. Günlük [27] uses mixing to obtain valid inequalities for the capacity expansion problem. Atamtürk [4] studies single- and multi-commodity cutset polyhedra arising from directed network design problems, while Raack et al. [40] consider cutset polyhedra for directed, undirected, and bi-directed arcs. These families of inequalities apply to network design problems with general integer design variables and therefore are valid for the 0-1 case. The model we study considers binary design variables specifically. We are therefore able to obtain families of facet-defining inequalities for the 0-1 *fixed-charge* problem described in [21].

More precisely, we perform a polyhedral study of a multi-commodity generalization of variable upper bound flow models. In Section 2, we present the model formulation, define notation used throughout the paper, and derive basic polyhedral results. In Section 3, we explore the use of commodity aggregation as a way to obtain strong valid inequalities for the model. In Section 4, we define hierarchical flow cover inequalities, which generalize flow cover inequalities. We provide a set of sufficient conditions under which these valid inequalities are facet-defining for the model. In Section 5, we lift hierarchical flow cover inequalities to obtain valid inequalities for a model with in- and outflows, and give conditions under which these lifted inequalities are facet-defining. We present computational results in Section 6.

2.2 Single Node Model without Inflow

Consider first a network node with outgoing arcs $A := \{1,\ldots,|A|\}$ and a set of distinct commodities $K := \{1,\ldots,|K|\}$. For $k \in K$, the node has an exogenous supply in the amount of b^k, where $b^k \in \mathbb{R}_+$. Further, each arc $j \in A$ has capacity $u_j \in \mathbb{R}_+$. Commodities can be routed on any of the arcs leaving the node, as long as they are not overused, and the consolidated flow on each arc respects its capacity. For $j \in A$ and $k \in K$, we let the binary variable x_j correspond to the decision of whether or not to open arc j, and we let the continuous variable y_j^k represent the flow of commodity k on arc j. We will show in Section 5 that strong inequalities for this set directly yield strong inequalities for the model with additional incoming arcs.

The aforementioned model can be expressed mathematically as follows:

$$\sum_{j \in A} y_j^k \le b^k, \qquad \forall k \in K, \tag{2.1a}$$

$$\sum_{k \in K} y_j^k \le u_j x_j, \qquad \forall j \in A, \tag{2.1b}$$

$$x_j \le 1, \qquad \forall j \in A, \tag{2.1c}$$

$$x_j \ge 0, \qquad \forall j \in A, \tag{2.1d}$$

$$y_j^k \ge 0, \qquad \forall j \in A, \quad k \in K. \tag{2.1e}$$

Constraints (2.1a) guarantee that the flow of each commodity k does not exceed its supply. Constraints (2.1b) ensure that the capacity of each arc j is respected. Constraints (2.1c) and (2.1d) are upper and lower bounds constraints on the x_j variables, respectively. Constraints (2.1e) enforce the nonnegativity of the flow variables y_j^k.

We define the *multi-commodity variable upper bound flow model* (MVF) as the set

$$P := \left\{(x,y) \in \mathbb{Z}^A \times \mathbb{R}^{A \times K} \mid (2.1\text{a}) - (2.1\text{e})\right\}.$$

Throughout this paper, we make the following assumptions:

A1. $u_j > 0$, for each $j \in A$.

A2. $b^k > 0$, for each $k \in K$.

Assumption **A1** is without loss of generality, since if $u_j = 0$ for some $j \in A$, then constraints (2.1b) require that $y_j^k = 0$ for each $k \in K$, effectively eliminating arc j from consideration. Similarly, there is no loss of generality in imposing **A2**, because if $b^k = 0$ for some $k \in K$, then (2.1a) implies that $y_j^k = 0$ for each $j \in A$.

To streamline the ensuing derivations, we next introduce some notation. Given $a \in \mathbb{R}$, we let $a^+ := \max\{a, 0\}$. We represent the vector with all zero components by $\mathbf{0}$, and we let $\boldsymbol{e_j} \in \mathbb{R}^A$ denote the elementary vector with a 1 in the j^{th} component, and zeros elsewhere. Similarly, for $j \in A$ and $k \in K$, we define $\boldsymbol{e_j^k} \in \mathbb{R}^{A \times K}$ as the vector y with

$$y_\ell^q = \begin{cases} 1 & \text{if } \ell = j, q = k \\ 0 & \text{otherwise.} \end{cases}$$

For $L \subseteq A$, we define the characteristic vector of L in $\mathbb{R}^A$ as $\mathbb{1}_L = \sum_{j \in L} \boldsymbol{e_j}$. Given a vector $v \in \mathbb{R}^A$, we let $v(L) := \sum_{j \in L} v_j$ when $L \neq \emptyset$, and $v(L) = 0$ otherwise. Likewise, for a function $f : A \mapsto \mathbb{R}$, we let $f(L) := \sum_{j \in L} f(j)$ if $L \neq \emptyset$, and $f(L) = 0$ otherwise. Finally, we omit braces when writing the set subtraction of a single element, *e.g.*, we write $A \setminus j$ instead of $A \setminus \{j\}$.

Example 1. *Consider a three-arc, three-commodity flow model with capacities* $m = (8, 13, 15)$ *and supplies* $b = (7, 9, 11)$*; a problem that is graphically represented in Figure 2.1. Let* $\widehat{A}$, $\widehat{K}$, *and* $\widehat{P}$ *denote the arcs, commodities, and feasible region of this model, respectively.*

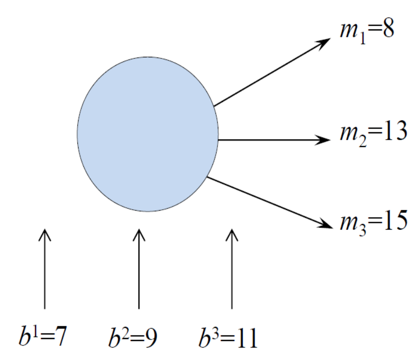

Figure 2.1. Graphical respresentation of MVF for Example 1

We used PORTA [18] to obtain the 398 inequalities of the linear description of $\text{conv}(\widehat{P})$. (See Appendix B.) The following facet-defining inequalities are part of this description:

$$y_1^1 \geq 0, \tag{2.2a}$$
$$x_1 \leq 1, \tag{2.2b}$$
$$y_1^1 + y_1^2 + y_1^3 \leq 8x_1, \tag{2.2c}$$
$$y_1^2 + y_2^2 + y_3^2 \leq 9, \tag{2.2d}$$
$$y_1^1 + y_1^2 + y_2^1 + y_2^2 \leq 16 - 3(1 - x_1) - 8(1 - x_2), \tag{2.2e}$$

$$y_1^1+y_1^2+y_1^3+y_2^3+y_3^2+y_3^3 \leq 27-7(1-x_1)-4(1-x_2)-8(1-x_3), \tag{2.2f}$$
$$y_1^1+y_1^2+y_2^2+y_2^3+y_3^2 \leq 27-7(1-x_1)-11(1-x_2)-6(1-x_3), \tag{2.2g}$$
$$y_1^1+y_1^2+y_1^3+y_2^2+y_3^1+y_3^2+y_3^3 \leq 27-3(1-x_1)-4(1-x_2)-10(1-x_3). \tag{2.2h}$$

□

We periodically refer back to Example 1 to illustrate the concepts presented in this paper.

We first present basic properties of $\text{conv}(P)$. Throughout this discussion, we let ε denote a small but positive number.

Proposition 1. *The set* $\text{conv}(P)$ *is a full-dimensional polytope.*

Proof. The fact that $\text{conv}(P)$ is a polytope is clear since P can be expressed as the union of a finite number of polytopes. Now, define $z_0 := (\mathbf{0},\mathbf{0})$, $z_j := (\boldsymbol{e_j},\mathbf{0})$ for $j \in A$, and $z_j^k := (\boldsymbol{e_j},\varepsilon\boldsymbol{e_j^k})$ for $(j,k) \in A \times K$. These $|K||A|+|A|+1$ affinely independent points of $\text{conv}(P)$ prove the result. □

The following variable upper bound constraint, which Crainic et al. [21] refer to as a *strong forcing constraint*, will be useful in upcoming proofs.

Proposition 2. *For any* $j \in A$ *and* $k \in K$*, the inequality*

$$y_j^k \leq b^k x_j \tag{2.3}$$

is valid for $\text{conv}(P)$*. Furthermore,* (2.3) *is facet-defining for* $\text{conv}(P)$ *if and only if (i)* $|K| = 1$ *and* $u_j \geq b^k$ *or (ii)* $u_j > b^k$*.*

Proof. First, we show the validity of (2.3). If $x_j = 0$, then (2.3) reduces to $y_j^k \leq 0$, which follows from (2.1b). Otherwise if $x_j = 1$, then (2.3) reduces to $y_j^k \leq b^k$, which is implied by (2.1a) and (2.1e). Therefore, (2.3) is valid for $\text{conv}(P)$.

Second, we prove the reverse implication. If either $|K| = 1$ and $u_j < b^k$ or $|K| > 1$ and $u_j \leq b^k$, then (2.3) is dominated by (2.1b) over the positive orthant, implying that (2.3) is not facet-defining for $\text{conv}(P)$.

Finally, we prove the forward implication. Construct the points $z_0 := (\mathbf{0},\mathbf{0})$, $\bar{z}_{j'} := (e_{j'},\mathbf{0})$ for $j' \in A \setminus j$, $\hat{z}_{j'}^{k'} := \bar{z}_{j'} + (\mathbf{0},\varepsilon\boldsymbol{e}_{j'}^{k'})$ for $(j',k') \in (A \setminus j) \times K$, $\dot{z}_0 := (\boldsymbol{e_j}, b^k\boldsymbol{e_j^k})$. Assume first that $|K| = 1$. In this case, $\dim(\text{conv}(P)) = |K||A|+|A| = 2|A|$. If $u_j \geq b^k$, the above $2|A|$ affinely independent points belong to P, and satisfy (2.3) at equality. This implies that (2.3) is facet-defining for $\text{conv}(P)$. Suppose second that $|K| > 1$ and $u_j > b^k$. In this case, we define the additional points $\breve{z}^{k'} = \dot{z}_0 + (0,\varepsilon\boldsymbol{e}_{\boldsymbol{j}}^{k'})$ for $k' \in K \setminus k$. We now have $|K||A|+|A|$ affinely independent points which belong to P and satisfy (2.3) at equality. Therefore, (2.3) is facet-defining for $\text{conv}(P)$. □

We call inequalities (2.1a)-(2.1e) *trivial inequalities*. In Example 1, observe that (2.2a)-(2.2d) are trivial inequalities for $\text{conv}(\widehat{P})$. We next derive necessary and sufficient conditions for the trivial inequalities (2.1a)-(2.1e) to be facet-defining for $\text{conv}(P)$. Since, by definition, these inequalities are valid for $\text{conv}(P)$, we only need to determine the dimension of the faces of $\text{conv}(P)$ they induce.

Proposition 3. *For $k \in K$,* (2.1a) *defines a facet of* $\operatorname{conv}(P)$ *if and only if $u(A \setminus j) \geq b^k$, for each $j \in A$.*

Proof. Suppose there exists $j \in A$ such that $u(A \setminus j) < b^k$. Then any solution $(x,y) \in P$ satisfying (2.1a) at equality must have $x_j = 1$. It follows that the dimension of the face induced by (2.1a) is at least two less than $\dim(\operatorname{conv}(P))$. We conclude that (2.1a) is not facet-defining for $\operatorname{conv}(P)$.

Now, assume that, for each $j \in A$, $u(A \setminus j) \geq b^k$. This condition implies that $|A| \geq 2$ and $b^k < u(A)$ because of Assumptions **A1** and **A2**. Define $z := \left(\mathbb{1}_A, b^k \cdot \sum_{\ell \in A} {}^{u_\ell}/_{u(A)} \boldsymbol{e}_{\boldsymbol{\ell}}^{\boldsymbol{k}}\right)$, $\tilde{z}_j := \left(\mathbb{1}_{A \setminus j}, b^k \cdot \sum_{\ell \in A \setminus j} {}^{u_\ell}/_{u(A \setminus j)} \boldsymbol{e}_{\boldsymbol{\ell}}^{\boldsymbol{k}}\right)$ for $j \in A$, $\bar{z}_j := \tilde{z}_j + (\boldsymbol{e_j}, \boldsymbol{0})$ for $j \in A$, $\hat{z}_j^{k'} := z + (\boldsymbol{0}, \varepsilon \boldsymbol{e}_{\boldsymbol{j}}^{\boldsymbol{k'}})$ for $(j,k') \in A \times (K \setminus k)$, and $\breve{z}_{j,j'} := z + (\boldsymbol{0}, \varepsilon \boldsymbol{e}_{\boldsymbol{j}}^{\boldsymbol{k}} - \varepsilon \boldsymbol{e}_{\boldsymbol{j'}}^{\boldsymbol{k}})$ for $j, j' \in A$ with $j \neq j'$. These points belong to P and satisfy (2.1a) at equality. Let

$$\sum_{j \in A} \alpha_j x_j + \sum_{j \in A} \sum_{k \in K} \beta_j^k y_j^k \leq \gamma, \tag{2.4}$$

where $(\alpha, \beta, \gamma) \in \mathbb{R}^A \times \mathbb{R}^{A \times K} \times \mathbb{R}$, be any inequality that is satisfied at equality by all of these points. We show that (2.4) is a scalar multiple of (2.1a), thereby proving that (2.1a) is facet-defining for $\operatorname{conv}(P)$. For $j \in A$, the points $\tilde{z}_j$ and $\bar{z}_j$ imply that $\alpha_j = 0$. For $j \in A$ and $k' \in K \setminus k$, points z and $\hat{z}_j^{k'}$ imply that $\beta_j^{k'} = 0$. Further, for $j, j' \in A$ with $j \neq j'$, points z and $\breve{z}_{j,j'}$ imply that $\beta_j^k = \beta_{j'}^k$. We conclude that $\beta_j^k = \bar{\beta}$ for each $j \in A$ for some $\bar{\beta} \in \mathbb{R}$. It follows that (2.4) can be written as

$$\bar{\beta} \cdot \sum_{j \in A} y_j^k \leq \gamma. \tag{2.5}$$

Since z satisfies (2.5) at equality, we obtain that $\gamma = \bar{\beta} b^k$, proving the result. □

Proposition 4. *For $j \in A$,* (2.1b) *is facet-defining for* $\operatorname{conv}(P)$ *if and only if (i) $u_j < b(K)$ or (ii) $u_j = b(K)$ and $|K| = 1$.*

Proof. Assume that $u_j < b(K)$. Consider the points $z_0 := (\boldsymbol{0}, \boldsymbol{0})$, $\bar{z}_\ell := (e_\ell, \boldsymbol{0})$ for $\ell \in A \setminus j$, $\hat{z}_\ell^k := \bar{z}_\ell + (\boldsymbol{0}, \varepsilon \boldsymbol{e}_{\boldsymbol{\ell}}^{\boldsymbol{k}})$ for $(\ell, k) \in (A \setminus j) \times K$, $\dot{z}_0 := (\boldsymbol{e_j}, u_j \cdot \sum_{k \in K} ({}^{b^k}/_{b(K)}) \boldsymbol{e}_{\boldsymbol{j}}^{\boldsymbol{k}})$, and $\breve{z}^k = \dot{z}_0 + (0, \varepsilon \boldsymbol{e}_{\boldsymbol{j}}^{\boldsymbol{k}} - \varepsilon \boldsymbol{e}_{\boldsymbol{j}}^{\boldsymbol{k+1}})$ for $k = 1, \ldots, |K| - 1$. These $|K||A| + |A|$ affinely independent points belong to P and satisfy (2.1b) at equality. Therefore, (2.1b) is facet-defining for $\operatorname{conv}(P)$.

Assume that $u_j = b(K)$. Observe that inequality

$$\sum_{k \in K} y_j^k \leq b(K) x_j, \tag{2.6}$$

can be obtained by summing inequalities (2.3) for each $k \in K$, and is therefore valid for $\operatorname{conv}(P)$. Observe also that when $u_j = b(K)$, (2.1b) is exactly (2.6). If $|K| = 1$, then (2.6), and hence (2.1b), is equivalent to (2.3). It then follows from Proposition 2 that (2.1b) is facet-defining for $\operatorname{conv}(P)$. Otherwise, if $|K| > 1$, then (2.1b) is obtained as a conic combination of multiple distinct inequalities of the form (2.3), and is therefore not facet-defining for $\operatorname{conv}(P)$, as P is full-dimensional.

Finally, assume that $u_j > b(K)$. In this case, it is clear that (2.6) dominates (2.1b), since they share the same left-hand side and the right-hand side of (2.6) has a smaller coefficient for x_j. It follows that (2.1b) is not facet-defining for $\text{conv}(P)$. □

For the remaining trivial inequalities, we identify in each case $\dim(\text{conv}(P)) = |K||A| + |A|$ affinely independent points that are tight for their respective inequality.

Proposition 5. *For $j \in A$, (2.1c) defines a facet of* $\text{conv}(P)$.

Proof. Construct $\bar{z} := (\boldsymbol{e_j}, \mathbf{0})$, $z^k := \bar{z} + (\mathbf{0}, \varepsilon \boldsymbol{e_j^k})$ for $k \in K$, $\tilde{z}_\ell := \bar{z} + (e_\ell, \mathbf{0})$ for $\ell \in A \setminus j$, and $\tilde{z}_\ell^k := \tilde{z}_\ell + (\mathbf{0}, \varepsilon e_\ell^k)$ for $(\ell, k) \in (A \setminus j) \times K$. □

For $j \in A$, the nonnegativity constraint (2.1d) is dominated by (2.1b) and (2.1e). It is therefore not facet-defining for $\text{conv}(P)$.

Proposition 6. *For $j \in A$ and $k \in K$, (2.1e) defines a facet of* $\text{conv}(P)$.

Proof. Construct the points $z_0 := (\mathbf{0}, \mathbf{0})$, $\bar{z}_\ell := (e_\ell, \mathbf{0})$ for $\ell \in A$, and $z_\ell^{k'} := \bar{z}_\ell + (\mathbf{0}, \varepsilon e_\ell^{k'})$ for $(\ell, k') \in (A \times K) \setminus (j, k)$. □

2.3 Commodity Aggregation

When studying multi-commodity flow problems, it is useful to define relaxations where a subset of commodities is *aggregated*; that is, treated as a single commodity whose supply is equal to the total supply of its constituent commodities. For instance, Achterberg and Raack [1] derive single-commodity relaxations of multi-commodity flow formulations by aggregating the flow balance constraints associated with some commodity subset $Q \subseteq K$. We illustrate such a procedure in the following example.

Example 2. *Consider the model of Example 1. By aggregating commodities* 1 *and* 2 *into a single commodity we call* $\{1,2\}$ *with supply equal to* $b^1 + b^2$ *and by keeping commodity* 3 *unchanged, we obtain the new two-commodity model depicted in Figure 2.2. Let $\dot{P}$ be the feasible region of this aggregated model.*

The following inequality can be verified to be facet-defining for $\text{conv}(\dot{P})$*:*

$$y_1^{\{1,2\}} + y_2^{\{1,2\}} \leq 16 - 3(1 - x_1) - 8(1 - x_2). \tag{2.7}$$

Observe that (2.7) *can be used to obtain a valid inequality for* $\text{conv}(\widehat{P})$ *by disaggregating the flow variables $y_1^{\{1,2\}}$ and $y_2^{\{1,2\}}$, i.e., by replacing $y_j^{\{1,2\}}$ with $y_j^1 + y_j^2$ for $j = 1, 2$. The resulting inequality is* (2.2e)*, which is facet-defining for* $\text{conv}(\widehat{P})$. □

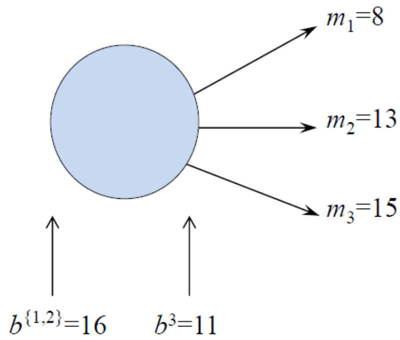

Figure 2.2. Graphical respresentation of MVF for Example 2

We next study this aggregation procedure in more detail. Let $\mathscr{Q} := \{Q_1, \dots, Q_p\}$, where $1 \leq p \leq |K|$, be a collection of nonempty, pairwise disjoint subsets of K. We do not require that $\bigcup_{i=1}^{p} Q_i = K$, and we define $Q_0 := K \setminus \left(\bigcup_{i=1}^{p} Q_i\right)$ to represent the set of commodities removed from the aggregated model. Now, define

$$\sum_{j \in A} y_j^Q \leq b(Q), \qquad \forall Q \in \mathscr{Q}, \tag{2.8a}$$
$$\sum_{Q \in \mathscr{Q}} y_j^Q \leq u_j x_j, \qquad \forall j \in A, \tag{2.8b}$$
$$x_j \leq 1, \qquad \forall j \in A, \tag{2.8c}$$
$$x_j \geq 0, \qquad \forall j \in A, \tag{2.8d}$$
$$y_j^Q \geq 0, \qquad \forall j \in A, \quad Q \in \mathscr{Q}. \tag{2.8e}$$

The *commodity aggregation* of P with respect to $\mathscr{Q}$ is the set

$$P(\mathscr{Q}) := \left\{ (x, y) \in \mathbb{Z}^A \times \mathbb{R}_+^{A \times \mathscr{Q}} \mid (2.8\text{a}) - (2.8\text{e}) \right\}.$$

Note that $P(\mathscr{Q})$ is an instance of MVF in which we associate each commodity subset $Q \in \mathscr{Q}$ with a single commodity having a supply of $b(Q)$. Naturally, $P = P\left(\{\{1\}, \{2\}, \dots, \{|K|\}\}\right)$.

Consider the following valid inequality

$$\sum_{j \in A} \alpha_j x_j + \sum_{j \in A} \sum_{Q \in \mathscr{Q}} \beta_j^Q y_j^Q \leq \gamma, \tag{2.9}$$

defined in the variable space of $P(\mathscr{Q})$. This inequality can be disaggregated with respect to each $Q \in \mathscr{Q}$, yielding

$$\sum_{j \in A} \alpha_j x_j + \sum_{j \in A} \sum_{Q \in \mathscr{Q}} \left(\beta_j^Q \sum_{k \in Q} y_j^k \right) \leq \gamma, \tag{2.10}$$

in the variable space of P.

Proposition 7. *If* (2.9) *is valid for* $\mathrm{conv}(P(\mathscr{Q}))$*, then* (2.10) *is valid for* $\mathrm{conv}(P)$.

Proof. Suppose, for a contradiction, that (2.10) is not valid for $\mathrm{conv}(P)$. Then there exists a point $(\bar{x},\bar{y}) \in \mathrm{conv}(P)$ that violates (2.10), *i.e.*,

$$\sum_{j\in A} \alpha_j \bar{x}_j + \sum_{j\in A}\sum_{Q\in\mathscr{Q}} \Big(\beta_j^Q \sum_{k\in Q} \bar{y}_j^k\Big) > \gamma.$$

Since $\mathrm{conv}(P)$ is a polytope, $(\bar{x},\bar{y})$ can be chosen to be one of its extreme points, and hence a point of P. Now, define $\hat{x} := \bar{x}$ and $\hat{y}_j^Q := \sum_{k\in Q} \bar{y}_j^k$ for $j \in A$ and for each $Q \in \mathscr{Q}$. It is simple to verify that $(\hat{x},\hat{y}) \in P(\mathscr{Q})$. However, observe that

$$\sum_{j\in A} \alpha_j \hat{x}_j + \sum_{j\in A}\sum_{Q\in\mathscr{Q}} \beta_j^Q \hat{y}_j^Q = \sum_{j\in A} \alpha_j \bar{x}_j + \sum_{j\in A}\sum_{Q\in\mathscr{Q}} \Big(\beta_j^Q \sum_{k\in Q} \bar{y}_j^k\Big) > \gamma,$$

which contradicts the fact that (2.9) is valid for $\mathrm{conv}(P(\mathscr{Q}))$. □

We now give sufficient conditions for a facet-defining inequality of $\mathrm{conv}(P(\mathscr{Q}))$ to disaggregate into a facet-defining inequality of $\mathrm{conv}(P)$. We define

$$\begin{aligned} F &:= \{(x,y) \in P \mid \text{(2.10) is satisfied at equality}\},\\ F(\mathscr{Q}) &:= \{(x,y) \in P(\mathscr{Q}) \mid \text{(2.9) is satisfied at equality}\}, \end{aligned}$$

the sets of mixed-integer solutions in the faces defined by (2.10) in the the original variable space and by (2.9) in the aggregated variable space, respectively. In addition, for $Q \in \mathscr{Q}$, we let

$$\bar{S}^Q := \left\{ j \in A \;\middle|\; \exists \bar{\sigma}_j^Q = (\bar{x},\bar{y}) \in F(\mathscr{Q}) \text{ with } \bar{y}_j^Q > 0 \text{ and } \sum_{\ell\in A} \bar{y}_\ell^Q < b(Q) \right\}, \tag{2.11a}$$

$$\check{S}^Q := \Big\{ (j,j') \in A\times A \;\Big|\; j \neq j', \beta_j^Q = \beta_{j'}^Q \text{ and } \exists \check{\sigma}_{j,j'}^Q = (\check{x},\check{y}) \in F(\mathscr{Q}) \text{ with }$$

$$\check{x}_j = \check{x}_{j'} = 1,\quad \sum_{Q\in\mathscr{Q}} \check{y}_j^Q < u_j, \text{ and } \check{y}_{j'}^Q > 0 \Big\}. \tag{2.11b}$$

We introduce the following two properties.

Property 1. Either (i) $Q_0 = \emptyset$, or (ii) for each $j \in A$, there exists a point $\dot{\sigma}_j = (\dot{x},\dot{y}) \in F(\mathscr{Q})$ such that $\dot{x}_j = 1$ and $\sum_{Q\in\mathscr{Q}} \dot{y}_j^Q < u_j$.

Property 2. For each $Q \in \mathscr{Q}$ such that $|Q| > 1$ and each $j' \in A$, either $j' \in \bar{S}^Q$ or there exists $j \in \bar{S}^Q$ such that $(j,j') \in \check{S}^Q$.

Theorem 1. *Assume that* (2.9) *is facet-defining for* $\mathrm{conv}(P(\mathscr{Q}))$ *and satisfies Properties 1 and 2. Then* (2.10) *is facet-defining for* $\mathrm{conv}(P)$.

Proof. We know that (2.10) is valid for conv(P), since (2.9) is valid for conv($P(\mathscr{Q})$); see Proposition 7.

We define a transformation $\Psi : \mathbb{R}^{A\times A\times \mathscr{Q}} \mapsto \mathbb{R}^{A\times A\times K}$ as follows:

$$\Psi[(x,y)] := \Big[x,\ \sum_{j\in A}\Big(\sum_{Q\in\mathscr{Q}} y_j^Q \cdot \sum_{k\in Q}\frac{b^k}{b(Q)}\boldsymbol{e_j^k}\Big)\Big].$$

It is easily shown that Ψ maps feasible points of $P(\mathscr{Q})$ to feasible points of P and, in addition, if a point $p \in P(\mathscr{Q})$ satisfies (2.9) at equality, then $\Psi(p)$ satisfies (2.10) at equality.

Let $d := \dim \mathrm{conv}(P(\mathscr{Q}))$. It is easily shown that since P satisfies **A1** and **A2**, $P(\mathscr{Q})$ does as well. It follows from Proposition 1 that conv($P(\mathscr{Q})$) is full-dimensional, *i.e.*, $d = |\mathscr{Q}||A| + |A|$. Since (2.9) defines a facet of conv($P(\mathscr{Q})$), we can select d affinely independent points $\{({}^i\hat{x}, {}^i\hat{y})\}_{i=1}^d \subseteq F(\mathscr{Q})$. It follows that the points

$$({}^i\breve{x}, {}^i\breve{y}) := \Psi[({}^i\hat{x}, {}^i\hat{y}_j)], \quad \text{for } i = 1,\ldots,d, \tag{2.12}$$

are feasible to P and satisfy (2.9) at equality. Next, using the vectors $\dot{\sigma}_j$, $\bar{\sigma}_j^Q$, and $\breve{\sigma}_{j,j'}^Q$ that are provided by Properties 1 and 2, we create the remaining points needed to prove that (2.10) is facet-defining. Specifically, let ε be a positive but sufficiently small real number. For each arc $j \in A$ and commodity $k \in Q_0$, if any, let

$$\dot{\tau}_j^k := \Psi(\dot{\sigma}_j) + \Big(\mathbf{0}, \varepsilon\boldsymbol{e_j^k}\Big) \tag{2.13}$$

be the point that modifies $\Psi(\dot{\sigma}_j)$ by sending a small amount of commodity k on arc j. The fact that $\Psi(\dot{\sigma}_j) \in F$ implies that $\dot{\tau}_j^k \in F$, as well. Next, suppose there exists $Q \in \mathscr{Q}$ such that $|Q| > 1$, and consider an arc $j \in A$. If $j \in \bar{S}^Q$, then for each distinct pair of commodities $k, k' \in Q$ we define the point

$$\bar{\tau}_{j,k,k'}^Q := \Psi(\bar{\sigma}_j^Q) + \big(\mathbf{0}, \varepsilon\boldsymbol{e_j^k} - \varepsilon\boldsymbol{e_j^{k'}}\big), \tag{2.14}$$

which transforms $\Psi(\bar{\sigma}_j^Q)$ by increasing the flow of k on j while decreasing the flow of k on j by the same amount. We see from the definition of $\bar{S}^Q$ that $\bar{\tau}_{j,k,k'}^Q \in F$. Otherwise, if $j \notin \bar{S}^Q$, then Property 2 implies that we can select an arc $j' \neq j$ such that $(j, j') \in \breve{S}^Q$ and for each $k \in Q$ define the point

$$\breve{\tau}_{j,j',k}^Q := \Psi(\breve{\sigma}_{j,j'}^Q) + \big(\mathbf{0}, \varepsilon\boldsymbol{e_j^k} - \varepsilon\boldsymbol{e_{j'}^k}\big), \tag{2.15}$$

which alters $\Psi(\breve{\sigma}_{j,j'}^Q)$ by increasing the flow of k on j and decreasing the flow of k' on j' by the same amount. It follows from the definition of $\breve{S}^Q$ that $\breve{\tau}_{j,j',k}^Q \in F$.

Let

$$\sum_{j\in A}\mu_j x_j + \sum_{j\in A}\sum_{k\in K}\nu_j^k y_j^k = \xi, \tag{2.16}$$

for some $(\mu, \nu, \xi) \in \mathbb{R}^A \times \mathbb{R}^{A\times K} \times \mathbb{R}$, be any representation of the face of $\text{conv}(P)$ defined by (2.10). We show that (μ, ν, ξ) is a scalar multiple of $(\alpha, \hat{\beta}, \gamma)$, where $\hat{\beta} := \sum_{j\in A} \sum_{Q\in\mathscr{Q}} \beta_j^Q \sum_{k\in Q} \boldsymbol{e_j^k}$, thereby proving that it is facet-defining for $\text{conv}(P)$.

We see from points $\dot{\tau}_j^k$ and $\Psi(\dot{\sigma}_j)$ that $\nu_j^k = 0$ for $j \in A$ and $k \in Q_0$. Now, consider $Q \in \mathscr{Q}$. If $Q = \{k\}$, for some $k \in K$, then $\nu_j^Q := \nu_j^k$ for $j \in A$. Otherwise, we have $|Q| > 1$. For $j \in \bar{S}^Q$, the points $\Psi(\bar{\sigma}_j^Q)$ and $\bar{\tau}_{j,k,k'}^Q$ imply that $\nu_j^k = \nu_j^{k'}$ for distinct $k, k' \in Q$. Next, for $j \in A \setminus \bar{S}^Q$, Property 2 allows us to choose an arc $j' \in \bar{S}^Q$ such that $(j', j) \in \breve{S}^Q$. In this case, the points $\Psi(\breve{\sigma}_{j',j}^Q)$ and $\breve{\tau}_{j',j,k}^Q$ imply that $\nu_j^k = \nu_{j'}^k$ for $k \in Q$. Since ν_j^k are identical for all $k \in Q$, it follows that $\nu_{j'}^k$ are equal for all $k \in Q$. Therefore, for each $j \in A$ we can set $\nu_j^Q := \nu_j^k$ for any $k \in Q$. Thus, (2.16) reduces to

$$\sum_{j\in A} \mu_j x_j + \sum_{j\in A} \nu_j^Q \Big(\sum_{Q\in\mathscr{Q}} \sum_{k\in Q} y_j^k \Big) = \xi. \tag{2.17}$$

Now, we substitute the points $({}^i\breve{x}, {}^i\breve{y})$ into (2.17) to obtain

$$\sum_{j\in A} \mu_j {}^i\breve{x}_j + \sum_{j\in A} \sum_{Q\in\mathscr{Q}} \Big(\nu_j^Q \sum_{k\in Q} {}^i\breve{y}_j^k \Big) = \xi, \quad \text{for } i = 1, \dots, d. \tag{2.18}$$

For each $i \in \{1, \dots, d\}$, ${}^i\hat{x} = {}^i\breve{x}$ and ${}^i\hat{y}_j^Q = \sum_{k\in Q} {}^i\breve{y}_j^k$, for $j \in A$ and $Q \in \mathscr{Q}$. Therefore, (2.18) becomes

$$\sum_{j\in A} \mu_j {}^i\hat{x}_j + \sum_{j\in A} \sum_{Q\in\mathscr{Q}} \nu_j^Q {}^i\hat{y}_j^Q = \xi, \quad \text{for } i = 1, \dots, d.$$

Since (2.9) was assumed to be facet-defining for $\text{conv}(P(\mathscr{Q}))$, $(\alpha, \hat{\beta}, \gamma)$ is the unique solution of this system up to scalar multiplication. We therefore must have $(\mu, \nu, \xi) = \theta(\alpha, \hat{\beta}, \gamma)$ for some scalar θ, yielding the desired result. $\square$

Next, we apply the result of Theorem 1 to obtain facet-defining inequalities for MVF through aggregation. In particular, we obtain these inequalities by converting the model into a single commodity model and constructing a valid inequality for the single commodity aggregation.

We use the notation P^1 to refer to instances of P with a single commodity. The set P^1 is a traditional variable upper bound flow model, for which we omit commodity superscripts. A subset $C \subseteq A$ is called a *flow cover* if $\lambda := u(C) - b > 0$. Given a flow cover C, the inequality

$$\sum_{j\in C} y_j \le b - \sum_{j\in C} (u_j - \lambda)^+ (1 - x_j) \tag{2.19}$$

is commonly referred to as a *flow cover inequality* [38]. A discussion of variable upper bound flow models and flow cover inequalities can be found in [33].

Proposition 8 ([38])**.** *Inequality* (2.19) *is valid for* $\text{conv}(P^1)$. *Furthermore, if* $\max_{j\in C} u_j > \lambda$, *then* (2.19) *is facet-defining for* $\text{conv}(P^1)$. $\square$

Now, let $Q \subseteq K$ be nonempty and let $P_Q := P(\{Q\})$. Note that P_Q is a single commodity model with a supply of $b(Q)$. Suppose there exists $C \subseteq A$ such that $u(C) > b(Q)$. We say that C is an *aggregated flow cover* (AFC) for P with respect to Q. We define the *aggregated flow cover inequality*

$$\sum_{j\in C}\sum_{k\in Q} y_j^k \leq b(Q) - \sum_{j\in C}(u_j - \lambda_Q)^+(1-x_j), \tag{2.20}$$

where $\lambda_Q := u(C) - b(Q)$. Referring back to Example 1, it can be verified that (2.2e) is an AFC inequality for $\widehat{P}$. Also, we note that (2.3) is an AFC inequality for P in the case where $|Q| = 1$.

Proposition 9. *Inequality* (2.20) *is valid for* $\operatorname{conv}(P)$*. Furthermore, if* $\max_{j\in C} u_j > \lambda_Q$*, then* (2.20) *is facet-defining for* $\operatorname{conv}(P)$ *if and only if (i)* $|C| = 1$ *and* $|Q| = 1$ *or (ii)* $|C| > 1$.

Proof. The set C is a flow cover for P_Q. By Proposition 8, its associated flow cover inequality (2.19) is valid for $\operatorname{conv}(P_Q)$. Disaggregating this flow cover inequality into the variable space of P yields (2.20), which is valid for $\operatorname{conv}(P)$ by Proposition 7.

Suppose that $|C| = 1$. If $|Q| = 1$, then (2.20) reduces to an inequality of the form (2.3), and it follows from Proposition 2 that (2.20) is facet-defining for $\operatorname{conv}(P)$. If $|Q| > 1$, then (2.20) can be obtained as the nonnegative combination of multiple distinct inequalities of the form (2.3). It is therefore not facet-defining for $\operatorname{conv}(P)$, since P is full-dimensional.

Now, assume that $|C| > 1$. By assumption, there exists an arc $\ell_1 \in C$ such that $u_{\ell_1} > \lambda_Q$. Therefore, Proposition 8 implies that (2.19) is facet-defining for $\operatorname{conv}(P_Q)$. To prove that (2.20) is facet-defining for $\operatorname{conv}(P)$, we construct the points required by Theorem 1. Select $\ell_2 \in C \setminus \ell_1$ arbitrarily, and define

$$\begin{aligned}
&z := \Big(\mathbb{1}_C, b(Q)\cdot \sum_{j\in C}\frac{u_j}{u(C)}\boldsymbol{e_j}\Big),\\
&\bar{z}_{\ell_1} := \Big(\mathbb{1}_{C\setminus \ell_1}, \sum_{j'\in C\setminus \ell_1} u_{j'}\boldsymbol{e}_{j'}\Big)\\
&\dot{\sigma}_j := \begin{cases} z & \text{if } j\in C\\ \bar{z}_{\ell_1} + (\boldsymbol{e_j}, \mathbf{0}) & \text{otherwise}\end{cases} && \forall j\in A,\\
&\bar{\sigma}_j := \begin{cases} \bar{z}_{\ell_1} & \text{if } j\in C\\ \dot{\sigma}_j + \Big(\mathbf{0}, \tfrac{1}{2}\min\big\{u_j, b(Q) - u(C\setminus \ell_1)\big\}\Big) & \text{otherwise}\end{cases} && \forall j\in A\setminus \ell_1,\\
&\breve{\sigma}_{\ell_1,\ell_2} := z.
\end{aligned}$$

Note that $\bar{z}_{\ell_1}$ is feasible, since $\lambda_Q = u(C) - b(Q) < m_{\ell_1}$. It can be verified that these points belong to $P(\mathscr{Q})$ and satisfy (2.19) at equality. Moreover, it is straightforward to show that the points $\dot{\sigma}_j$, $\bar{\sigma}_j$, and $\breve{\sigma}_{\ell_1,\ell_2}$ meet the requirements described in Properties 1 and 2, respectively. $\square$

2.4 Hierarchical Flow Cover Inequalities

In this section, we introduce hierarchical flow cover inequalities, a family of valid inequalities for MVF that is not obtained through commodity aggregation. These inequalities arise from structures we call *hierarchical flow covers*, which are a generalization of flow covers.

2.4.1 Ordered Sets

Here, we give an overview of order-theoretic concepts that are central to our presentation. We refer the reader to Trotter [51] for an in-depth treatment of the subject. Throughout this section, we use S to denote a generic set.

A *quasi-order* on S is a binary relation $\preceq$ on S that is reflexive, *i.e.*, $s \preceq s\ \forall s \in S$, and transitive, *i.e.*, $s \preceq t$ and $t \preceq u \implies s \preceq u$, $\forall s,t,u \in S$. The pair $\langle S, \preceq \rangle$ is a *quasi-ordered set*. Consider two (not necessarily distinct) elements $s,t \in S$ such that $s \preceq t$ and $t \preceq s$. We call s and t *equivalent* under $\preceq$, and denote this property by $s \sim t$. If $\preceq$ is antisymmetric, *i.e.*, $s \sim t \implies s = t\ \forall s,t \in S$, then it is called a *partial order*, and $\langle S, \preceq \rangle$ is called a *partially ordered set*, or *poset*. The *equivalence class* of an element $s \in S$ is $[s] := \{s' \in S \mid s' \sim s\}$. Let $S/\sim$ be the collection of equivalence classes of S under the relation $\sim$. It is well-known that $S/\sim$ forms a partition of S. Further, it is easy to show that the quasi-order $\preceq$ on S acts as a partial order on $S/\sim$. So, a quasi-order can be viewed as a partial order on the blocks of a partition of S.

Two elements $s,t \in S$ are called *comparable*, a property we denote by $s \perp t$, if $s \preceq t$ or $t \preceq s$. A quasi-ordered set $\langle S, \preceq \rangle$ is *connected* if for any pair of elements $s,t \in S$, there exists a finite sequence of elements $s = s_0, s_1, \ldots, s_p = t$ such that $s_i \perp s_{i+1}$ for $i = 0, 1, \ldots, p-1$. We define the *down-set* of s as $\downarrow s := \{s' \in S \mid s' \preceq s\}$, and the *up-set* of s as $\uparrow s := \{s' \in S \mid s' \succeq s\}$. Similarly, we define the *strict down-set* $\Downarrow s := \downarrow s \setminus [s]$ and the *strict up-set* $\Uparrow s := \uparrow s \setminus [s]$. We say that *t covers s*, written $s \sqsubset t$, if $s \not\sim t$ and for $u \in S$, $s \preceq u \preceq t$ implies that $s \sim u$ or $u \sim t$. Intuitively, if $s \sqsubset t$, then t is the next element above s in the quasi-order.

The following proposition describes a well-known class of posets.

Proposition 10. *The power set of S ordered by inclusion, $\langle 2^S, \subseteq \rangle$, is a poset.*

When we say that 2^S is ordered by inclusion, we mean that, for $A, B \subseteq S$, $A \preceq B$ if and only if $A \subseteq B$. A poset $\langle S, \preceq \rangle$ can be depicted using a *Hasse diagram*, a graph with vertex set S and edges $\{(s,s') : s \sqsubset s'\}$ in which "larger" elements are positioned higher along the vertical axis. Figure 2.3 shows the Hasse diagram of $\langle 2^S, \subseteq \rangle$, when $S = \{1,2,3\}$.

2.4.2 Hierarchical Flow Covers

A *commodity map* is a function $\mathscr{C} : A \mapsto 2^K$. By definition, a commodity map $\mathscr{C}$ assigns some subset of commodities to each arc. For $L \subseteq A$, we let $\mathscr{C}(L) := \bigcup_{j \in L} \mathscr{C}(j)$. We do not require that

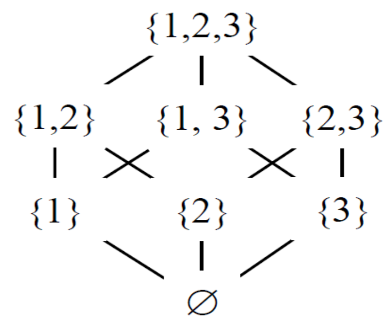

Figure 2.3. Hasse Diagram for $\langle 2^{\{1,2,3\}}, \subseteq \rangle$.

$\mathscr{C}(A) = K$. We define the *support* of $\mathscr{C}$ to be the set $\operatorname{supp}(\mathscr{C}) := \{j \in A \mid \mathscr{C}(j) \neq \emptyset\}$. Intuitively, $\operatorname{supp}(\mathscr{C})$ is the set of all arcs that are assigned at least one commodity. A commodity map induces a quasi-order $\preceq$ on $\operatorname{supp}(\mathscr{C})$ with respect to set inclusion. We relate two given arcs $j, j' \in \operatorname{supp}(\mathscr{C})$ as follows:

$$j \prec j' \iff \mathscr{C}(j) \subsetneq \mathscr{C}(j'), \tag{2.21a}$$
$$j \preceq j' \iff \mathscr{C}(j) \subseteq \mathscr{C}(j'), \tag{2.21b}$$
$$j \sim j' \iff \mathscr{C}(j) = \mathscr{C}(j'). \tag{2.21c}$$

The relations $j \succ j'$ and $j \succeq j'$ are defined analogously to (2.21a) and (2.21b) by reversing the direction of the set inclusions. Referring back to concepts introduced in Section 2.4.1, $[j]$ should be interpreted as the set of arcs that are assigned the same set of commodities as arc j.

Let $H := \operatorname{supp}(\mathscr{C})$. We define the associated *arc-commodity hierarchy* as the quasi-ordered set $\langle H, \preceq \rangle$, where $\preceq$ is defined as in (2.21a)-(2.21c).

Example 3. *For Example 1, we may construct the following commodity maps:*

$$\mathscr{C}_1 : 1 \mapsto \{1,2,3\},\ 2 \mapsto \{3\},\ 3 \mapsto \{2,3\}.$$
$$\mathscr{C}_2 : 1 \mapsto \{1,2\},\ 2 \mapsto \{2,3\},\ 3 \mapsto \{2\}.$$
$$\mathscr{C}_3 : 1 \mapsto \{1,2,3\},\ 2 \mapsto \{2\},\ 3 \mapsto \{1,2,3\}.$$

Let $\preceq_i$ be the quasi-order induced on $\widehat{A}$ by $\mathscr{C}_i$, for $i = 1,2,3$. Then $\langle \widehat{A}, \preceq_1 \rangle$, $\langle \widehat{A}, \preceq_2 \rangle$, and $\langle \widehat{A}, \preceq_3 \rangle$ are commodity hierarchies, depicted in Figure 2.4. We see that $2 \prec_1 3 \prec_1 1$, $3 \prec_2 1$, $3 \prec_2 2$, and $2 \prec_3 1 \sim_3 3$. □

In the remainder of this paper, we refer to the equivalence classes under $\preceq$ as *tiers*. In other words, a tier is a set of arcs that are assigned the same commodities. For each arc $j \in H$, we define the set of *marginal commodities* $\Delta(j) := \mathscr{C}(j) \setminus \mathscr{C}(\Downarrow j)$. The set $\Delta(j)$ contains those commodities assigned to j that are not assigned to any arcs strictly less than j in the arc-commodity hierarchy. We let $\Delta(L) := \bigcup_{j \in L} \Delta(j)$, for $L \subseteq A$. A commodity $k \in \mathscr{C}(H)$ is marginal for at least one arc in H, but can be marginal for several arcs. Also, for each $k \in \mathscr{C}(H)$, we define the *marginal arcs of* k as $\Delta^{-1}(k) := \{j \in H : k \in \Delta(j)\}$.

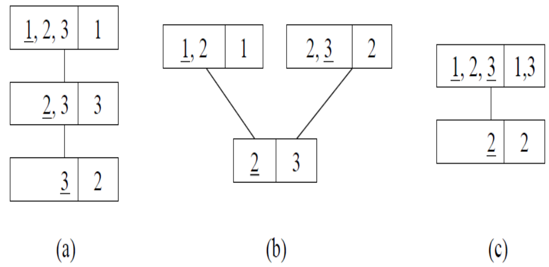

Figure 2.4. Arc-commodity hierarchies of Example 3. Each block corresponds to a tier, with commodities displayed on the left and arcs on the right. Marginal commodities are underlined.

Definition 1. An arc-commodity hierarchy $\langle H, \preceq \rangle$ is a *hierarchical flow cover* (HFC) if, for $j \in H$, $\Delta(j) \neq \emptyset$ and $u([j]) > b(\Delta(j))$.

An HFC is an arc-commodity hierarchy in which the total arc capacity in each tier is strictly greater than the supply of marginal commodities in the tier. Summing the relations $u([j]) > b(\Delta(j))$ over all tiers yields the following result.

Corollary 1. *If $\langle H, \preceq \rangle$ is a HFC, then $\mathscr{C}(H) \neq \emptyset$ and $u(H) > b(\mathscr{C}(H))$.* □

Example 4. *For Example 3, the arc-commodity hierarchy $\langle \widehat{A}, \preceq_1 \rangle$ is a HFC, as*

$$\begin{aligned} m_1 &= 8 > b(\Delta(1)) = b^1 = 7, \\ m_2 &= 13 > b(\Delta(2)) = b^3 = 11, \\ m_3 &= 15 > b(\Delta(3)) = b^2 = 9. \end{aligned}$$

It can be verified similarly that $\langle \widehat{A}, \preceq_2 \rangle$ is a HFC. For $\langle \widehat{A}, \preceq_3 \rangle$, we see that

$$\begin{aligned} u(\{1,3\}) &= 23 > b(\Delta(\{1,3\})) = b^1 + b^3 = 18, \\ m_2 &= 13 > b(\Delta(2)) = b^2 = 9, \end{aligned}$$

demonstrating that $\langle \widehat{A}, \preceq_3 \rangle$ is also a HFC. □

Given a HFC $\langle H, \preceq \rangle$, the *upward excess* of $j \in H$ is $\lambda_j := u(\uparrow j) - b(\Delta(\uparrow j))$. The value λ_j measures the gap between the capacity of the up-set of j and the supply of the marginal commodities associated with the up-set of j. Since equivalent elements in $\langle H, \preceq \rangle$ have identical up-sets, it follows that arcs in the same tier have the same upward excess. Given a HFC $\langle H, \preceq \rangle$, we define the *hierarchical flow cover inequality* as

$$\sum_{j \in H} \sum_{k \in \mathscr{C}(j)} y_j^k \leq b(\mathscr{C}(H)) - \sum_{j \in H} (u_j - \lambda_j)^+ (1 - x_j). \tag{2.22}$$

Example 5. *We next derive the HFC inequalities associated with the HFCs identified in Example 4. For $\langle \widehat{A}, \preceq_1 \rangle$, we compute*

$$\begin{aligned}
\lambda_1 &= u(\uparrow 1) - b(\Delta(1)) = m_1 - b^1 = 1, \\
\lambda_2 &= u(\uparrow 2) - b(\Delta(2)) = u(\{1,2,3\}) - b(\{1,2,3\}) = 9, \\
\lambda_3 &= u(\uparrow 3) - b(\Delta(3)) = u(\{1,3\}) - b(\{1,2\}) = 7.
\end{aligned}$$

The associated HFC inequality reduces to (2.2f)*, as $(m_1 - \lambda_1)^+ = 7$, $(m_2 - \lambda_2)^+ = 4$, and $(m_3 - \lambda_3)^+ = 8$. Similarly, it can be shown that the HFC inequality associated with $\langle \widehat{A}, \preceq_1 \rangle$ reduces to* (2.2g)*. For $\langle \widehat{A}, \preceq_3 \rangle$, we have*

$$\begin{aligned}
\lambda_1 &= u(\uparrow 1) - b(\Delta(\uparrow 1)) = u(\{1,3\}) - b(\{1,3\}) = 5, \\
\lambda_2 &= u(\uparrow 2) - b(\Delta(\uparrow 2)) = u(\{1,2,3\}) - b(\{1,2,3\}) = 9, \\
\lambda_3 &= \lambda_1 = 5.
\end{aligned}$$

The associated HFC inequality reduces to (2.2h)*, as $(m_1 - \lambda_1)^+ = 3$, $(m_2 - \lambda_2)^+ = 4$, and $(m_3 - \lambda_3)^+ = 10$.* □

In the remainder of this section, we present conditions under which the HFC inequality (2.22) is strong for conv(P). As a first step, we establish that (2.22) is valid for conv(P) under the following condition.

C1 For $j, j' \in H$, if there exists $k \in \Delta(j) \cap \Delta(j')$, then $j \sim j'$.

Condition **C1** requires that each commodity $k \in \mathscr{C}(H)$ be marginal in exactly one tier. We assume that **C1** holds throughout the remainder of the section.

Wolsey [52] uses submodularity to derive a general class of valid inequalities for fixed charge problems, which includes traditional flow cover inequalities. We apply the same technique to show that (2.22) is valid. A set function $f : 2^S \mapsto \mathbb{R}$ is *submodular* on S if

$$f(A \cap B) + f(A \cup B) \leq f(A) + f(B), \quad \forall A, B \subseteq S.$$

For vectors $u, v \in \mathbb{R}^A$, define $u \wedge v := \big(\min\{u_1, v_1\}, \ldots, \min\{u_n, v_n\}\big)$ and $u \vee v := \big(\max\{u_1, v_1\}, \ldots, \max\{u_n, v_n\}\big)$. The notion of submodularity can also be defined for functions defined on $\mathbb{R}^A$. Let $f : D \mapsto \mathbb{R}$, with $D \subseteq \mathbb{R}^A$. The function f is *submodular* on D if, for each $u, v \in D$ such that $u \wedge v, u \vee v \in D$, we have

$$f(u \wedge v) + f(u \vee v) \leq f(u) + f(v).$$

We refer to Topkis [50] for an extensive treatment of submodular functions. The following result is taken from Wolsey [52].

Proposition 11. *We have that*

(ii) If $g(u)$ is submodular and nonincreasing on $\mathbb{R}^m$, then $f(u,x) = g(u)x$ is submodular on $\mathbb{R}^m \times \mathbb{R}_+$,

(iii) If $\mathscr{L}(u,x)$ is submodular on $\mathbb{R}^m_+ \times \mathbb{R}^n_+$, then $w(x) = \min_{u \geq \mathbf{0}} \mathscr{L}(u,x)$ is submodular on $\mathbb{R}^n_+$. $\square$

Let $\mathscr{I}\{(j,k) \in A \times K \mid j \in H \text{ and } k \in \mathscr{C}(j)\}$. For $x \in [0,1]^A$, we define the following flow problem

$$w(x) := \max_{y \geq \mathbf{0}} \left\{ \sum_{j \in H} \sum_{k \in \mathscr{C}(j)} y^k_j \;\middle|\; (2.1\text{a}), (2.1\text{b}),\;\; y^k_j = 0 \;\; \forall (j,k) \in (A \times K) \setminus \mathscr{I} \right\},$$

which seeks to find the maximum flow in the network, given that certain arcs are (partially) closed.

Theorem 2. *The function $w(x)$ is submodular on $[0,1]^A$.*

Proof. We dualize constraints (2.1a) in the definition of $w(x)$, with corresponding dual variables $u^k \geq 0$, to obtain

$$\begin{aligned} \mathscr{L}(u,x) := \max_{y \geq \mathbf{0}} & \left\{ \sum_{j \in H} \sum_{k \in \mathscr{C}(j)} (1-u^k) y^k_j \;\middle|\; (2.1\text{b}),\;\; y^k_j = 0 \;\; \forall (j,k) \in (A \times K) \setminus \mathscr{I} \right\} \\ & + \sum_{k \in K} b^k u^k. \end{aligned}$$

It follows from strong duality in linear programming that $w(x) = \min_{u \geq \mathbf{0}} \mathscr{L}(u,x)$. For $j \in H$, select $\bar{k}(j) \in \operatorname{argmax}_{k' \in \mathscr{C}(j)} \{(1-u^{k'})^+\}$. An optimal flow $\hat{y}$ corresponding to the problem defining $\mathscr{L}(u,x)$ can be obtained as

$$\hat{y}^k_j = \begin{cases} u_j x_j & \text{if } j \in H, k = \bar{k}(j), \text{ and } u^k < 1, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\mathscr{L}(u,x) = \sum_{j \in H} \max_{k \in \mathscr{C}(j)} \{(1-u^k)^+\} u_j x_j + \sum_{k \in K} b^k u^k.$$

For $j \in H$, let $g_j(u) := \max_{k \in \mathscr{C}(j)} \{(1-u^k)^+\} u_j$, or equivalently, $g_j(u) = \left(1 - \min_{k \in \mathscr{C}(j)} \{u^k\}\right)^+ u_j$. We next show that $g_j(\cdot)$ is submodular over $\mathbb{R}^K_+$. Let $u_1, u_2 \in \mathbb{R}^K_+$. Assume without loss of generality that $u^1_1 = \min_{i=1,2} \{\min_{k \in \mathscr{C}(j)} \{u^k_i\}\}$. Then $g_j(u_1) = (1-u^1_1)^+ u_j$. Now, we have

$$\begin{aligned} g_j(u_1 \wedge u_2) &= \left(1 - \min_{k \in \mathscr{C}(j)} \left\{ \min\{u^k_1, u^k_2\} \right\}\right)^+ u_j \\ &= (1-u^1_1)^+ u_j \\ &= g_j(u_1), \end{aligned}$$

and

$$\begin{aligned} g_j(u_1 \vee u_2) &= \left(1 - \min_{k \in \mathscr{C}(j)} \left\{ \max\{u_1^k, u_2^k\} \right\}\right)^+ u_j \\ &\le \left(1 - \min_{k \in \mathscr{C}(j)} \{u_2^k\}\right)^+ u_j \\ &= g_j(u_2), \end{aligned}$$

where the inequality follows because $(\cdot)^+$ is an increasing function and, for $k \in \mathscr{C}(j)$, $\max\{u_1^k, u_2^k\} \ge u_2^k$, and therefore $\min_{k \in \mathscr{C}(j)} \left\{ \max\{u_1^k, u_2^k\} \right\} \ge \min_{k \in \mathscr{C}(j)} \{u_2^k\}$. So, $g_j(u_1 \wedge u_2) + g_j(u_1 \vee u_2) \le g_j(u_1) + g_j(u_2)$, implying that $g_j(\cdot)$ is submodular.

Since $\mathscr{L}(u,x) = \sum_{j \in H} g_j(u) x_j + \sum_{k \in K} b^k u^k$, and a sum of submodular functions is submodular, it follows from Proposition 11 that $\mathscr{L}(u,x)$ is submodular, as $g_j(u)$ is nonincreasing in u. Since $w(x) = \min_{u \ge \mathbf{0}} \mathscr{L}(u,x)$, Proposition 11 (ii) shows that $w(x)$ is submodular. □

For $L \subseteq A$, we define the set function

$$\nu(L) := w(\mathbb{1}_L). \tag{2.23}$$

Theorem 2 directly implies the following corollary.

Corollary 2. *$\nu(\cdot)$ is submodular on A.* □

The following proposition is again taken from Wolsey [52].

Proposition 12. *For some $x \in \{0,1\}^A$, let $T := \{j \in A \mid x_j = 1\}$. Then the following inequality holds:*

$$\nu(T) \le \nu(H) - \sum_{j \in H} \rho_j(H \setminus j)(1 - x_j) + \sum_{j \in A \setminus H} \rho_j(\emptyset) x_j, \tag{2.24}$$

where $\rho_j(A) := \nu(A \cup \{j\}) - \nu(A)$. □

We show that (2.22) is in fact (2.24) after computing the values of $\nu(T)$, $\nu(H)$, $\rho_j(H \setminus j)$ and $\rho_j(\emptyset)$ for a HFC. In order to compute these values, we solve (2.23). We do so by finding matching pairs of upper and lower bounds. Next, we introduce notation to define the points we use to establish the lower bounds.

First, for $Q \subseteq \mathscr{C}(H)$, we define

$$\eta(Q) := \begin{cases} \sum_{k \in Q} \sum_{j \in \Delta^{-1}(k)} \dfrac{u_j}{u(\Delta^{-1}(k))} \cdot b^k \boldsymbol{e_j^k} & \text{if } Q \neq \emptyset, \\ \mathbf{0} & \text{otherwise.} \end{cases} \tag{2.25}$$

The vector $\eta(Q) \in \mathbb{R}_+^{A \times K}$ corresponds to a flow in which the entire supply of each commodity in Q is distributed proportionally, with respect to capacity, across its marginal arcs. This construction is well-defined, as $\Delta^{-1}(k) \neq \emptyset$ for each $k \in \mathscr{C}(H)$. Fix $\ell \in H$, and let $\overline{\Delta}(\ell) := \mathscr{C}(H) \setminus \Delta(\ell)$, the set of commodities associated with H that are not marginal to arc ℓ. For $j \in H \setminus \ell$, we define

$$r_\ell(j) := u_j - \sum_{k \in K} [\eta(\overline{\Delta}(\ell))]^k_j. \tag{2.26}$$

This expression computes the capacity remaining on arc j when $y = \eta(\overline{\Delta}(\ell))$. Note that $\eta(\overline{\Delta}(\ell))$ corresponds to a flow in which the entire amount of commodities that are not marginal to ℓ are distributed among their marginal arcs, proportionally with respect to the arc capacities. Next, we define

$$\bar{\theta}_\ell := \begin{cases} \sum_{k \in \Delta(\ell)} \sum_{j \in \uparrow \ell \setminus \ell} \dfrac{r_\ell(j)}{r_\ell(\uparrow \ell \setminus \ell)} \cdot b^k \boldsymbol{e}^{\boldsymbol{k}}_{\boldsymbol{j}} & \text{if } \uparrow \ell \setminus \ell \neq \emptyset \text{ and } u_\ell \leq \lambda_\ell, \\ \sum_{k \in \Delta(\ell)} \sum_{j \in \uparrow \ell \setminus \ell} \dfrac{b^k}{b(\Delta(\ell))} \cdot r_\ell(j) \boldsymbol{e}^{\boldsymbol{k}}_{\boldsymbol{j}} & \text{if } \uparrow \ell \setminus \ell \neq \emptyset \text{ and } u_\ell > \lambda_\ell, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

This vector assigns the flow of marginal commodities of arc ℓ to other arcs in its tier and those in higher tiers, proportionally with respect to the arcs' residual capacities. Note that $\bar{\theta}_\ell = \mathbf{0}$ if $\uparrow \ell \setminus \ell = \emptyset$, since in this case, there is no arc $j \in H \setminus \ell$ with $j \succeq \ell$ to which the marginal commodities of ℓ can be assigned. Using the above notation, we construct the points

$$\zeta := (\mathbb{1}_H, \eta(\mathscr{C}(H))), \tag{2.27a}$$

$$\bar{\zeta}_\ell := (\mathbb{1}_{H \setminus \ell}, \eta(\overline{\Delta}(\ell)) + \bar{\theta}_\ell), \quad \forall \ell \in H. \tag{2.27b}$$

Finally, for $j \in H \setminus \ell$, we define

$$\mu_\ell(j) := \frac{\sum_{k \in K} (\bar{\theta}_\ell)^k_j}{r_\ell(j)},$$

a quantity that measures the proportion of the residual capacity $r_\ell(j)$ that is used when $y = \eta(\overline{\Delta}(\ell)) + \bar{\theta}_\ell$. We next provide alternate ways of computing $r_\ell(j)$ and $\mu_\ell(j)$ that allow us to show that points ζ and $\bar{\zeta}_\ell$ belong to P.

Lemma 1. *Let $\ell \in H$. We have*

$$r_\ell(j) = \left[1 - \frac{b(\Delta(j) \cap \overline{\Delta}(\ell))}{u([j])}\right] u_j, \tag{2.28}$$

and

$$\mu_\ell(j) = \begin{cases} \dfrac{b(\Delta(\ell) \cap \mathscr{C}(j))}{r_\ell(\uparrow \ell \setminus \ell)} & \text{if } \uparrow \ell \setminus \ell \neq \emptyset \text{ and } u_\ell \leq \lambda_\ell, \\ \dfrac{b(\Delta(\ell) \cap \mathscr{C}(j))}{b(\Delta(\ell))} & \text{if } \uparrow \ell \setminus \ell \neq \emptyset \text{ and } u_\ell > \lambda_\ell, \\ 0 & \text{otherwise,} \end{cases} \tag{2.29}$$

for $j \in H \setminus \ell$.

Proof. First, we compute $r_\ell(j)$. Observe that

$$\begin{aligned}\sum_{k\in K}[\eta(\overline{\Delta}(\ell))]_j^k &= \sum_{k\in\overline{\Delta}(\ell)}[\eta(\overline{\Delta}(\ell))]_j^k = \sum_{k\in\Delta(j)\cap\overline{\Delta}(\ell)}[\eta(\overline{\Delta}(\ell))]_j^k \\ &= \sum_{k\in\Delta(j)\cap\overline{\Delta}(\ell)}\frac{u_j}{u(\Delta^{-1}(k))}\cdot b^k = \sum_{k\in\Delta(j)\cap\overline{\Delta}(\ell)}\frac{u_j}{u([j])}\cdot b^k = \frac{b(\Delta(j)\cap\overline{\Delta}(\ell))}{u([j])}\cdot u_j,\end{aligned}$$

where the first equality holds because $[\eta(\overline{\Delta}(\ell))]_j^k = 0$ for each $k\in K\setminus\overline{\Delta}(\ell)$, the second equality follows from (2.25) and the fact that $j\in\Delta^{-1}(k)$ if and only if $k\in\Delta(j)$ for each $k\in\mathscr{C}(H)$, the third equality is obtained from the definition of $\eta(\cdot)$, and the fourth equality holds since **C1** implies that $\Delta^{-1}(k)=[j]$ for $j\in H$ and $k\in\Delta(j)$. This proves (2.28).

Next, we compute $\mu_\ell(j)$. If $\uparrow\ell\setminus\ell=\emptyset$, the definition of $\bar{\theta}_\ell$ clearly shows that $\mu_\ell(j)=0$. If $\uparrow\ell\setminus\ell\neq\emptyset$, then

$$\sum_{k\in K}(\bar{\theta}_\ell)_j^k = \sum_{k\in\Delta(\ell)}(\bar{\theta}_\ell)_j^k = \sum_{k\in\Delta(\ell)\cap\mathscr{C}(j)}(\bar{\theta}_\ell)_j^k, \tag{2.30}$$

where the first equality holds because $(\bar{\theta}_\ell)_j^k=0$ for $k\in K\setminus\Delta(\ell)$, and the second equality is verified as, for $j\in H$ and $j\neq\ell$, **C1** implies that $j\in\uparrow\ell\setminus\ell$ if and only if $k\in\mathscr{C}(j)$ for each $k\in\Delta(\ell)$. Using (2.30) and the definition of $\bar{\theta}_\ell$, we obtain

$$\sum_{k\in K}(\bar{\theta}_\ell)_j^k = \begin{cases}\dfrac{b(\Delta(\ell)\cap\mathscr{C}(j))}{r_\ell(\uparrow\ell\setminus\ell)}\cdot r_\ell(j) & \text{if } \uparrow\ell\setminus\ell\neq\emptyset \text{ and } u_\ell\le\lambda_\ell,\\ \dfrac{b(\Delta(\ell)\cap\mathscr{C}(j))}{b(\Delta(\ell))}\cdot r_\ell(j) & \text{if } \uparrow\ell\setminus\ell\neq\emptyset \text{ and } u_\ell>\lambda_\ell,\\ 0 & \text{otherwise,}\end{cases}$$

which implies (2.29). $\square$

Lemma 2. *Let $\ell\in H$. Then, for $j\in H\setminus\ell$, the following relations hold true:*

(i) $0<r_\ell(j)\le u_j$,

(ii) $0\le\mu_\ell(j)\le 1$.

Proof. Fix $j\in H\setminus\ell$. Observe that $0\le b(\Delta(j)\cap\overline{\Delta}(\ell))\le\ b(\Delta(j))<\ u([j])$, where the strict inequality follows from the fact that $\langle H,\preceq\rangle$ is a HFC. Therefore, $0\le\frac{b(\Delta(j)\cap\overline{\Delta}(\ell))}{u([j])}<1$. Lemma 1 then implies (i).

Once again, let $j\in H\setminus\ell$. It is clear from its definition that $\mu_\ell(j)\ge 0$. Assume first $u_\ell>\lambda_\ell$. Then Lemma 1 directly shows that

$$\mu_\ell(j)=\frac{b(\Delta(\ell)\cap\mathscr{C}(j))}{b(\Delta(\ell))}\le 1.$$

Next, suppose $u_\ell \le \lambda_\ell$. We conclude from **C1** that

$$\Delta(\ell) \cap \mathscr{C}(j) = \begin{cases} \Delta(\ell) & \text{if } j \in \uparrow \ell, \\ \emptyset & \text{otherwise.} \end{cases} \tag{2.31}$$

So, if $j \in H \setminus \uparrow \ell$, then from (2.31), we have $\mu_\ell(j) = 0$. Suppose $j \in \uparrow \ell \setminus \ell$. It follows from (2.26) that

$$r_\ell(\uparrow \ell \setminus \ell) = u(\uparrow \ell \setminus \ell) - b(\Delta(\Uparrow \ell)). \tag{2.32}$$

From the definitions of HFC and λ_ℓ, we have $u_\ell \le \lambda_\ell = u(\uparrow \ell) - b(\Delta(\uparrow \ell))$. This inequality directly yields

$$b(\Delta(\ell)) \le u(\uparrow \ell \setminus \ell) - b(\Delta(\Uparrow \ell)), \tag{2.33}$$

since $u(\uparrow \ell \setminus \ell) = u(\uparrow \ell) - m_\ell$ and $b(\Delta(\Uparrow \ell)) = b(\Delta(\uparrow \ell)) - b(\Delta(\ell))$. It follows that

$$\mu_\ell(j) = \frac{b(\Delta(\ell))}{r_\ell(\uparrow \ell \setminus \ell)} = \frac{b(\Delta(\ell))}{u(\uparrow \ell \setminus \ell) - b(\Delta(\Uparrow \ell))} \le 1,$$

where the first equality is established in Lemma 1 as $\Delta(\ell) \cap \mathscr{C}(j) = \Delta(\ell)$, the second equality follows from (2.32), and the inequality follows from (2.33). □

Proposition 13. *The points ζ and $\bar{\zeta}_\ell$, where $\ell \in H$, belong to P.*

Proof. Let $(\bar{x}, \bar{y})$ be the components of $\bar{\zeta}_\ell$. By definition, $(\bar{x}, \bar{y})$ satisfies the binary restriction on x and the nonnegativity requirement on y.

First, we show that $(\bar{x}, \bar{y})$ satisfies (2.1a). Fix $k \in K$. Note that since $\bar{y}^k_j = 0$ for $j \notin H \setminus \ell$, $\sum_{j \in A} \bar{y}^k_j = \sum_{j \in H \setminus \ell} \bar{y}^k_j$. Assume first that $k \in \overline{\Delta}(\ell)$. We have from (2.25) that

$$\sum_{j \in H \setminus \ell} \bar{y}^k_j = \sum_{j \in \Delta^{-1}(k)} \frac{u_j}{u(\Delta^{-1}(k))} \cdot b^k = b^k,$$

where the first equality follows from the fact that $(\bar{\theta}_\ell)^k_j = 0$ for all $j \in A$ and $k \in \overline{\Delta}(\ell)$. Assume next that $k \in \Delta(\ell)$. If $u_\ell \le \lambda_\ell$, then

$$\sum_{j \in H \setminus \ell} \bar{y}^k_j = \sum_{j \in \uparrow \ell \setminus \ell} \frac{r_\ell(j)}{r_\ell(\uparrow \ell \setminus \ell)} \cdot b^k = b^k,$$

where the first equality holds since $(\eta(\overline{\Delta}(\ell)))^k_j = 0$ for all $j \in A$ and $k \in \Delta(\ell)$. Now, assume $u_\ell > \lambda_\ell = u(\uparrow \ell) - b(\Delta(\uparrow \ell))$. Similar to (2.33), this expression can be written as

$$b(\Delta(\ell)) > u(\uparrow \ell \setminus \ell) - b(\Delta(\Uparrow \ell)), \tag{2.34}$$

It follows that

$$\sum_{j \in H \setminus \ell} \bar{y}^k_j = \sum_{j \in \uparrow \ell \setminus \ell} \frac{b^k}{b(\Delta(\ell))} \cdot r_\ell(j) = \frac{b^k}{b(\Delta(\ell))} \sum_{j \in \uparrow \ell \setminus \ell} r_\ell(j)$$

$$= \frac{r_\ell(\uparrow \ell \setminus \ell)}{b(\Delta(\ell))} \cdot b^k = \frac{u(\uparrow \ell \setminus \ell) - b(\Delta(\Uparrow \ell))}{b(\Delta(\ell))} \cdot b^k < b^k, \tag{2.35}$$

where the fourth equality follows from (2.32) and the inequality follows from (2.34).

Next, we show that $(\bar{x}, \bar{y})$ satisfies (2.1b). Fix $j \in A$. If $j \notin H \setminus \ell$, then $\bar{x}_j = 0$ and $\sum_{k \in K} \bar{y}_j^k = 0 = u_j \bar{x}_j$. Now, suppose $j \in H \setminus \ell$, and hence, $\bar{x}_j = 1$. We have

$$\begin{aligned} \sum_{k \in K} \bar{y}_j^k &= \sum_{k \in K} [\eta(\overline{\Delta}(\ell))]_j^k + \sum_{k \in K} (\bar{\theta}_\ell)_j^k = \big(u_j - r_\ell(j)\big) + \mu_\ell(j) \cdot r_\ell(j) \\ &= \left[\left(1 - \frac{r_\ell(j)}{u_j}\right) + \mu_\ell(j) \cdot \frac{r_\ell(j)}{u_j}\right] \cdot u_j \le u_j \bar{x}_j, \end{aligned} \tag{2.36}$$

where the first equality holds because of (2.27b), the second equality follows from the definitions of $r_\ell(\cdot)$ and $\mu_\ell(\cdot)$, respectively, and the last inequality holds since $0 \le \mu_\ell(j) \le 1$ and $0 < r_\ell(j) \le u_j$ by Lemma 2. Therefore, $\bar{\zeta}_\ell$ belongs to P. The proof for ζ is similar. □

Theorem 3. *The HFC inequality* (2.22) *is valid for* $\text{conv}(P)$.

Proof. Clearly, for any feasible point $(x, y) \in P$, we must have

$$\sum_{j \in H} \sum_{k \in \mathscr{C}(j)} y_j^k \le \nu(T),$$

where $T := \{j \in A \mid x_j = 1\}$. It follows from the flow balance constraints (2.1a) that $\nu(H) \le b(\mathscr{C}(H))$. Solution ζ demonstrates that $\nu(H) = b(\mathscr{C}(H))$.

Now, fixing $\ell \in H$, we determine the value of $\rho_\ell(H \setminus \ell)$. Suppose first that $u_\ell \le \lambda_\ell$. Observe that $\nu(H \setminus \ell) \le b(\mathscr{C}(H \setminus \ell)) \le b(\mathscr{C}(H))$ and $\bar{\zeta}_\ell$ is a feasible solution of value $b(\mathscr{C}(H))$ to the problem defining $\nu(H \setminus \ell)$ in (2.23). Therefore $\nu(H \setminus \ell) = \nu(H)$, and we conclude that $\rho_\ell(H \setminus \ell) = 0$.

Assume next that $u_\ell > \lambda_\ell$. Note that any feasible flow y satisfies

$$\begin{aligned} \sum_{j \in H \setminus \ell} \sum_{k \in \mathscr{C}(j)} y_j^k &= \sum_{j \in H \setminus \uparrow \ell} \sum_{k \in \mathscr{C}(j)} y_j^k + \sum_{j \in \uparrow \ell \setminus \ell} \sum_{k \in \mathscr{C}(j)} y_j^k \\ &\le b(\mathscr{C}(H \setminus \uparrow \ell)) + u(\uparrow \ell \setminus \ell). \end{aligned} \tag{2.37}$$

It follows that $\nu(H \setminus \ell) \le b(\mathscr{C}(H \setminus \uparrow \ell)) + u(\uparrow \ell \setminus \ell)$. Solution $\bar{\zeta}_\ell$ shows that equality holds in the previous relation, and therefore

$$\begin{aligned} \nu(H \setminus \ell) &= b(\mathscr{C}(H \setminus \uparrow \ell)) + u(\uparrow \ell \setminus \ell) \\ &= b(\mathscr{C}(H) \setminus \Delta(\uparrow \ell)) + u(\uparrow \ell \setminus \ell), \end{aligned}$$

where the second equality follows from **C1**. This allows us to compute

$$\begin{aligned} \rho_\ell(H \setminus \ell) &= b(\mathscr{C}(H)) - [b(\mathscr{C}(H) \setminus \Delta(\uparrow \ell)) + u(\uparrow \ell \setminus \ell)] \\ &= b(\Delta(\uparrow \ell)) - u(\uparrow \ell \setminus \ell) \end{aligned}$$

$$= u_\ell - \lambda_\ell.$$

We conclude that $\rho_j(H \setminus j) = (u_j - \lambda_j)^+$ for $j \in H$.

Finally, it is clear from the definition of (2.23) that, for $j \in A \setminus H$, $\nu(\{j\}) = \nu(\emptyset) = 0$, since the objective of (2.23) does not contain these values. We conclude that $\rho_j(\emptyset) = 0$, for $j \in A \setminus H$. Applying Proposition 12 yields the desired result. $\square$

We next show that HFC inequalities are, under mild assumptions, facet-defining for conv(P). To this end, we first establish that the points ζ and $\bar{\zeta}_\ell$ defined above lie on the face of conv(P) induced by (2.22). We then argue in Lemma 3 that certain perturbations of these points also belong to this face. These results are the building blocks for the facet proof given in Theorem 4.

Proposition 14. *The points ζ and $\bar{\zeta}_\ell$, for $\ell \in H$, satisfy* (2.22) *at equality.*

Proof. For a given $x \in \{0,1\}^A$, define $h(x) = b(\mathscr{C}(H)) - \sum_{j \in H}(u_j - \lambda_j)^+(1 - x_j)$, *i.e.*, $h(x)$ is the right-hand side of (2.22).

Let $(\hat{x}, \hat{y})$ be the components of ζ. Since $\hat{x} = \mathbb{1}_H$, $h(\hat{x}) = b(\mathscr{C}(H))$. We also see that $\sum_{j \in H} \sum_{k \in \mathscr{C}(j)} \hat{y}_j^k = b(\mathscr{C}(H)) = h(\hat{x})$.

Now, let $(\bar{x}, \bar{y})$ be the components of $\bar{\zeta}_\ell$, for some $\ell \in H$. There are two cases.
Case 1 ($u_\ell \leq \lambda_\ell$). In this case, $h(\bar{x}) = b(\mathscr{C}(H))$, since $(u_\ell - \lambda_\ell)^+ = 0$. Observe that

$$\begin{aligned}
&\sum_{j \in H} \sum_{k \in \mathscr{C}(j)} \bar{y}_j^k \\
&= \sum_{k \in \overline{\Delta}(\ell)} \sum_{j \in \Delta^{-1}(k)} \frac{u_j}{u(\Delta^{-1}(k))} \cdot b^k + \sum_{k \in \Delta(\ell)} \sum_{j \in \uparrow \ell \setminus \ell} \frac{r_\ell(j)}{r_\ell(\uparrow \ell \setminus \ell)} \cdot b^k \\
&= b(\overline{\Delta}(\ell)) + b(\Delta(\ell)) \\
&= b(\mathscr{C}(H)) \\
&= h(\bar{x}).
\end{aligned}$$

Case 2 ($u_\ell > \lambda_\ell$). Then $h(\bar{x}) = b(\mathscr{C}(H)) - (u_\ell - \lambda_\ell)$, and

$$\begin{aligned}
&\sum_{j \in H} \sum_{k \in \mathscr{C}(j)} \bar{y}_j^k \\
&= \sum_{k \in \overline{\Delta}(\ell)} \sum_{j \in \Delta^{-1}(k)} \frac{u_j}{u(\Delta^{-1}(k))} \cdot b^k + \sum_{k \in \Delta(\ell)} \sum_{j \in \uparrow \ell \setminus \ell} \frac{b^k}{b(\Delta(\ell))} \cdot r_\ell(j) \\
&= b(\overline{\Delta}(\ell)) + r_\ell(\uparrow \ell \setminus \ell) \\
&= [b(\mathscr{C}(H)) - b(\Delta(\ell))] + [u(\uparrow \ell \setminus \ell) - b(\Delta(\Uparrow \ell))] \\
&= b(\mathscr{C}(H)) + u(\uparrow \ell \setminus \ell) - b(\Delta(\uparrow \ell)) \\
&= b(\mathscr{C}(H)) - (u_\ell - \lambda_\ell) \\
&= h(\bar{x}),
\end{aligned}$$

where the third equality follows from the definition of $\overline{\Delta}(\ell)$ and (2.32), and the fifth equality holds by the definition of λ_ℓ. $\square$

We next introduce an additional condition on $\langle H, \preceq\rangle$, which we assume holds throughout the remainder of the section.

C2 For $j \in H$, $\max_{\ell\in[j]} u_\ell > \lambda_j$,

Condition **C2** requires that the maximum arc capacity in each tier exceed the upward excess of that tier.

For each commodity $k \in \mathscr{C}(H)$, we choose $\Delta^{-1}_{\max}(k) \in \operatorname{argmax}_{j\in\Delta^{-1}(k)} u_j$, *i.e.*, a marginal arc of k with maximum capacity. Using this notation, we construct the points

$$
\begin{aligned}
&\pi_j := \zeta + (\boldsymbol{e_j}, \mathbf{0}), && \forall j \in A\setminus H; && (2.38a)\\
&\bar{\pi}^k_j := \begin{cases} \bar{\zeta}_{\Delta^{-1}_{\max}(k)} + (\mathbf{0}, \varepsilon \boldsymbol{e^k_j}) & \text{if } k \in \mathscr{C}(H)\\ \zeta + (\mathbf{0}, \varepsilon \boldsymbol{e^k_j}) & \text{otherwise,}\end{cases} && \forall j \in H,\ k \in K\setminus\mathscr{C}(j); && (2.38b)\\
&\dot{\pi}^k_j := \bar{\zeta}_{\Delta^{-1}_{\max}(k)} + (\boldsymbol{e_j}, \mathbf{0}), && \forall j \in A\setminus H,\ k \in K; && (2.38c)\\
&\ddot{\pi}^k_j := \dot{\pi}^k_j + (\mathbf{0}, \varepsilon \boldsymbol{e^k_j}), && \forall j \in A\setminus H,\ k \in K; && (2.38d)\\
&\tilde{\pi}^{k,k'}_j := \bar{\zeta}_{\Delta^{-1}_{\max}(k)} + (\mathbf{0}, \varepsilon \boldsymbol{e^k_j} - \varepsilon \boldsymbol{e^{k'}_j}), && \forall j \in H,\ k \in \mathscr{C}(j),\ k' \in \Delta(j): &&\\
& && j \neq \Delta^{-1}_{\max}(k) \text{ and } k \neq k'; && (2.38e)\\
&\hat{\pi}^k_{j,j'} := \zeta + (\mathbf{0}, -\varepsilon \boldsymbol{e^k_j} + \varepsilon \boldsymbol{e^k_{j'}}), && \forall j,\ j' \in H,\ k \in \Delta(j)\cap\mathscr{C}(j'): &&\\
& && j \neq j'. && (2.38f)
\end{aligned}
$$

Note that each point in (2.38) is constructed as a perturbation of some point defined in (2.27). We do not claim that these points are affinely independent. However, we claim that these perturbations are points of P that belong to the face of conv(P) defined by (2.22) under the additional condition

C3 $\langle H, \preceq\rangle$ is connected,

which we assume holds in the remainder of this section.

Lemma 3. *The points defined in* (2.38) *belong to P and satisfy* (2.22) *at equality.*

Proof. (a) First, given arc $j \in A\setminus H$, the point π_j modifies ζ by setting $x_j = 1$. The corresponding solution is clearly feasible, and since the coefficient of x_j in (2.22) is zero, this solution satisfies (2.22) at equality.

(b) Now, let $j \in H$ and $k \in K \setminus \mathscr{C}(j)$. The point $\bar{\pi}_j^k$ takes a point $(\bar{x}, \bar{y}) \in \{\zeta, \bar{\zeta}_{\Delta_{\max}^{-1}(k)}\}$ and increases the flow of commodity k on arc j by ε. Since $k \notin \mathscr{C}(j)$, the coefficient of y_j^k in (2.22) is zero, and so $\bar{\pi}_j^k$ satisfies (2.22) at equality. In order to be feasible, we must have (i) $\bar{x}_j = 1$, (ii) $\sum_{q \in K} \bar{y}_j^q < u_j$, and (iii) $\sum_{j' \in A} \bar{y}_{j'}^k < b^k$.

Suppose $k \in \mathscr{C}(H)$, in which case $(\bar{x}, \bar{y}) = \bar{\zeta}_{\Delta_{\max}^{-1}(k)}$. We see that $\bar{x} = \mathbb{1}_{H \setminus \Delta_{\max}^{-1}(k)}$. Since $k \notin \mathscr{C}(j)$, we conclude that $j \neq \Delta_{\max}^{-1}(k)$, which implies (i). From **C2**, we have $u_{\Delta_{\max}^{-1}(k)} > \lambda_{\Delta_{\max}^{-1}(k)}$. From (2.29), setting $\ell = \Delta_{\max}^{-1}(k)$, we obtain

$$\mu_{\Delta_{\max}^{-1}(k)}(j) = \begin{cases} \frac{b(\Delta(\Delta_{\max}^{-1}(k)) \cap \mathscr{C}(j))}{b(\Delta(\Delta_{\max}^{-1}(k)))} & \text{if } \uparrow (\Delta_{\max}^{-1}(k)) \setminus (\Delta_{\max}^{-1}(k)) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{2.39}$$

Expression (2.39) shows that $\mu_{\Delta_{\max}^{-1}(k)}(j) = 0$ if $\uparrow (\Delta_{\max}^{-1}(k)) \setminus (\Delta_{\max}^{-1}(k)) = \emptyset$. Suppose that $\uparrow (\Delta_{\max}^{-1}(k)) \setminus (\Delta_{\max}^{-1}(k)) \neq \emptyset$. Since $k \notin \mathscr{C}(j)$, **C1** implies that $\Delta(\Delta_{\max}^{-1}(k)) \cap \mathscr{C}(j) = \emptyset$. In this case, (2.39) also implies that $\mu_{\Delta_{\max}^{-1}(k)}(j) = 0$. Now, using (2.36), again setting $\ell = \Delta_{\max}^{-1}(k)$, yields

$$\begin{aligned} \sum_{q \in K} \bar{y}_j^q &= \left[\left(1 - \frac{r_{\Delta_{\max}^{-1}(k)}(j)}{u_j}\right) + \underbrace{\mu_{\Delta_{\max}^{-1}(k)}(j)}_{=0} \cdot \frac{r_{\Delta_{\max}^{-1}(k)}(j)}{u_j} \right] \cdot u_j \\ &= \left(1 - \frac{r_{\Delta_{\max}^{-1}(k)}(j)}{u_j}\right) \cdot u_j < u_j, \end{aligned}$$

where the inequality follows from Lemma 2. Thus, (ii) is satisfied. Using the fact that $u_{\Delta_{\max}^{-1}(k)} > \lambda_{\Delta_{\max}^{-1}(k)}$ and setting $\ell = \Delta_{\max}^{-1}(k)$ in (2.35) gives (iii).

Next, assume $k \notin \mathscr{C}(H)$, and hence, $(\bar{x}, \bar{y}) = \zeta$. The fact that $\bar{x} = \mathbb{1}_H$ and $j \in H$ implies (i). From the construction of ζ and the definition of HFC, we see that

$$\sum_{q \in K} \bar{y}_j^q = \sum_{q \in \Delta(j)} \frac{b^q}{u([j])} \cdot u_j < u_j,$$

which gives (ii). Also, since $k \notin \mathscr{C}(H)$, (iii) holds, as $\sum_{j' \in A} y_{j'}^k = 0 < b^k$.

(c-d) Let $j \in A \setminus H$ and $k \in K$. The point $\dot{\pi}_j^k$ is obtained from $\bar{\zeta}_{\Delta_{\max}^{-1}(k)}$ by setting $x_j = 1$. It follows that $\dot{\pi}_j^k$ is feasible. Since $j \notin H$, $\dot{\pi}_j^k$ also satisfies (2.22) at equality. The point $\ddot{\pi}_j^k$ is derived from $\dot{\pi}_j^k$ by increasing the flow of commodity k on arc j by ε. Letting $(\bar{x}, \bar{y})$ be the components of $\dot{\pi}_j^k$, we see that $\ddot{\pi}_j^k$ is feasible if $(\bar{x}, \bar{y})$ meets requirements (i), (ii), and (iii) defined above. By construction, $\bar{x}_j = 1$ and $\sum_{q \in K} \bar{y}_j^q = 0 < u_j$, which shows (i) and (ii). As in the previous case, using (2.35) with $\ell = \Delta_{\max}^{-1}(k)$ implies (iii). Since the coefficient of y_j^k in (2.22) is zero, it follows that $\ddot{\pi}_j^k$ also satisfies (2.22) at equality.

(e) Let $j \in H$, $k \in \mathscr{C}(j)$, and $k' \in \Delta(j)$ be such that $j \neq \Delta^{-1}_{\max}(k)$ and $k \neq k'$. The point $\tilde{\pi}^{k,k'}_{j}$ modifies $\bar{\zeta}_{\Delta^{-1}_{\max}(k)}$ by increasing the flow of commodity k on arc j by ε and simultaneously decreasing the flow of commodity k' on arc j by ε. This perturbation is neutral with respect to the total flow over H, which implies that it satisfies (2.22) at equality. Letting $(\bar{x},\bar{y})$ be the components of $\bar{\zeta}_{\Delta^{-1}_{\max}(k)}$, we see that this perturbation is feasible if (iv) $\bar{x}_j = 1$, (v) $\sum_{j' \in A} \bar{y}^{k}_{j'} < b^k$, and (vi) $\bar{y}^{k'}_{j} > 0$.

Since $\bar{x} = \mathbb{1}_{H \setminus \Delta^{-1}_{\max}(k)}$ and $j \neq \Delta^{-1}_{\max}(k)$, (iv) holds. As in the previous cases, (v) follows from (2.35). We see that (vi) holds, since $k' \in \Delta(j)$ and the point $\bar{\zeta}_{\Delta^{-1}_{\max}(k)}$ assigns each $j' \in H \setminus \Delta^{-1}_{\max}(k)$ a positive amount of flow for each of its marginal commodities. (f) Let $j, j' \in H$, with $j \neq j'$, and let $k \in \Delta(j) \cap \mathscr{C}(j')$. The point $\hat{\pi}^{k}_{j,j'}$ modifies ζ by decreasing the flow of commodity k on arc j by ε while increasing the flow of commodity k on arc j'. Since this perturbation is commodity-neutral with respect to the variables that appear in (2.22), the corresponding point still satisfies (2.22) at equality. Let $(\bar{x},\bar{y})$ be the components of ζ. To verify that these perturbations are feasible, we must show that (vii) $\bar{x}_{j'} = 1$, (viii) $\sum_{q \in K} \bar{y}^{q}_{j'} < u_{j'}$, and (ix) $\bar{y}^{k}_{j} > 0$. Since $\bar{x} = \mathbb{1}_H$ and $j' \in H$, (vii) holds. Next, from the definition of ζ, we have

$$\sum_{q \in K} \bar{y}^{q}_{j'} = \sum_{q \in \Delta(j')} \frac{u_{j'}}{u(\Delta^{-1}(q))} \cdot b^q = \frac{b(\Delta(j'))}{u([j'])} \cdot u_{j'} < u_{j'},$$

where the strict inequality follows from the definition of HFC. This shows (viii). Finally, the definition of ζ and the fact that $k \in \Delta(j)$ shows (ix), as

$$y^{k}_{j} = \frac{u_j}{u(\Delta^{-1}(k))} \cdot b^k > 0. \qquad \square$$

We next use Lemma 3 to show that when Conditions **C1**, **C2**, and **C3** are met, (2.22) is facet-defining for $\mathrm{conv}(P)$.

Theorem 4. *The HFC inequality* (2.22) *is facet-defining for* $\mathrm{conv}(P)$ *if and only if (i)* $|H| = 1$ *and* $|\mathscr{C}(H)| = 1$*; or (ii)* $|H| > 1$.

Proof. We consider two cases.

Case 1 ($|H| = 1$). Then $H = \{j\}$ for some $j \in A$, and (2.22) reduces to

$$\sum_{k \in \mathscr{C}(j)} y^{k}_{j} \leq b(\mathscr{C}(j)) - (u_j - \lambda_j)^{+}(1 - x_j). \tag{2.40}$$

Condition **C2** requires that $u_j - \lambda_j > 0$. Consequently, we can rewrite (2.40) as

$$\sum_{k \in \mathscr{C}(j)} y^{k}_{j} \leq b(\mathscr{C}(j)) x_j, \tag{2.41}$$

since $\lambda_j = u_j - b(\mathscr{C}(j))$. We now consider two subcases.

Case 1.1 ($|\mathscr{C}(j)| = 1$). In this case, $\mathscr{C}(j) = \{k_0\}$ for some $k_0 \in K$, and (2.41) reduces to

$$y_j^{k_0} \leq b^{k_0} x_j. \tag{2.42}$$

Note that (2.42) is of the form (2.3), and **C2** implies that $u_j > b^{k_0}$. It follows from Proposition 2 that (2.22) is facet-defining for conv(P).

Case 1.2 ($|\mathscr{C}(j)| > 1$). We see that (2.41) can be obtained as a conic combination of distinct inequalities of the form (2.42) for $k \in \mathscr{C}(j)$. It is therefore not facet-defining for conv(P), as conv(P) is full-dimensional.

Case 2 ($|H| > 1$). Consider any face-defining inequality

$$\sum_{j \in A} \alpha_j x_j + \sum_{j \in A} \sum_{k \in K} \beta_j^k y_j^k \leq \gamma, \tag{2.43}$$

that is satisfied at equality by points (2.27) and (2.38), where $(\alpha, \beta, \gamma) \in \mathbb{R}^A \times \mathbb{R}^{A \times K} \times \mathbb{R}$. We next show that (2.43) is a scalar multiple of (2.22).

For $j \in A \setminus H$, the points ζ and π_j imply that $\alpha_j = 0$. Also, for $j \in H$ and $k \in K \setminus \mathscr{C}(j)$, $\bar{\pi}_j^k$ and the point from which it is derived, either ζ or $\bar{\zeta}_{\Delta_{\max}^{-1}(k)}$, imply that $\beta_j^k = 0$. Similarly, for $j \in A \setminus H$ and $k \in K$, the points $\dot{\pi}_j^k$ and $\ddot{\pi}_j^k$ imply that $\beta_j^k = 0$. As a result, (2.43) reduces to

$$\sum_{j \in H} \alpha_j x_j + \sum_{j \in H} \sum_{k \in \mathscr{C}(j)} \beta_j^k y_j^k \leq \gamma,$$

or equivalently,

$$\sum_{j \in H} \sum_{k \in \mathscr{C}(j)} \beta_j^k y_j^k \leq \gamma - \alpha(H) + \sum_{j \in H} \alpha_j (1 - x_j). \tag{2.44}$$

For each $j \in H$, we next show that the coefficients β_j^k in (2.44) are equal for all $k \in \mathscr{C}(j)$. To prove this claim, we consider two cases.

Case 2.1 ($\Delta(j) \subsetneq \mathscr{C}(j)$). Since $0 < |\Delta(j)| < |\mathscr{C}(j)|$, we have $\mathscr{C}(j) \setminus \Delta(j) \neq \emptyset$ and Condition **C2** also requires that $\Delta(j) \neq \emptyset$. Consider any $k_0' \in \Delta(j)$ and any $k_0 \in \mathscr{C}(j) \setminus \Delta(j)$. Clearly, $j \neq \Delta_{\max}^{-1}(k_0)$. We conclude from points $\bar{\zeta}_{\Delta_{\max}^{-1}(k_0)}$ and $\tilde{\pi}_j^{k_0, k_0'}$ that $\beta_j^{k_0'} = \beta_j^{k_0}$. It follows that all coefficients β_j^k, for $k \in \mathscr{C}(j)$, are equal.

Case 2.2 ($\Delta(j) = \mathscr{C}(j)$). In this case, it follows from the definition of $\Delta(j)$ that $\mathscr{C}(\Downarrow j) = \emptyset$, and therefore, $\Downarrow j = \emptyset$. Suppose first that $\Uparrow j \neq \emptyset$. Then we can select an arc $j' \in \Uparrow j$. Since, by the definition of HFC, $\Delta(j') \neq \emptyset$ and $\mathscr{C}(j) \cup \Delta(j') \subseteq \mathscr{C}(j')$, we have that $\Delta(j') \subsetneq \mathscr{C}(j')$. It follows from the argument of Case 2.1 that the coefficients $\beta_{j'}^k$, for $k \in \mathscr{C}(j')$, are all equal. Denote this common value by $\bar{\beta}_{j'}$. For each $k \in \mathscr{C}(j)$, the points ζ and $\hat{\pi}_{j,j'}^k$ imply that $\beta_j^k = \bar{\beta}_{j'}$. Therefore, the coefficients β_j^k, for $k \in \mathscr{C}(j)$, are all equal to each other.

Now, assume $\Uparrow j = \emptyset$. Recall that, by **C3**, $\langle H, \preceq \rangle$ is connected. This implies that $\langle H, \preceq \rangle$ consists of a single tier, and therefore, $\mathscr{C}(j') = \mathscr{C}(H)$ for each $j' \in H$. If $|\mathscr{C}(H)| = 1$, then $|\mathscr{C}(j)| = 1$

and the set $\{\beta_j^k \mid k \in \mathscr{C}(j)\}$ contains a single element, trivially establishing the result. Suppose $|\mathscr{C}(H)| > 1$. Since $|H| > 1$, $H \setminus \Delta_{\max}^{-1}(k) \neq \emptyset$ and we see that $\bar{\zeta}_{\Delta_{\max}^{-1}(k)}$ and $\tilde{\pi}_{j'}^{k,k'}$ imply that $\beta_{j'}^k = \beta_{j'}^{k'}$ for $j' \in H \setminus \Delta_{\max}^{-1}(k)$ and for distinct $k, k' \in \mathscr{C}(j') = \Delta(j')$. Also, for $k \in \mathscr{C}(j)$ and $j' \in H \setminus \Delta_{\max}^{-1}(k)$, ζ and $\hat{\pi}_{\Delta_{\max}^{-1}(k),j'}^k$ imply that $\beta_{\Delta_{\max}^{-1}(k)}^k = \beta_{j'}^k$. It follows that the coefficients β_j^k, for $k \in \mathscr{C}(j)$, are all equal.

At this point, we have shown that for any $j \in H$, the coefficients β_j^k, for $k \in \mathscr{C}(j)$, equal a single value $\bar{\beta}_j$. Now, select any two distinct arcs $\ell, \ell' \in H$. Since $\langle H, \preceq \rangle$ is connected, there is a sequence of arcs $\{\ell_i\}_{i=1}^p \subseteq H$, where $1 \leq p \leq |A|$, such that

$$\ell = \ell_1 \perp \ell_2 \perp \cdots \perp \ell_p = \ell'.$$

Consider any two consecutive arcs ℓ_i and ℓ_{i+1} in the sequence. Since $\ell_i \perp \ell_{i+1}$, it must hold that $\mathscr{C}(\ell_i) \subseteq \mathscr{C}(\ell_{i+1})$ or $\mathscr{C}(\ell_i) \supsetneq \mathscr{C}(\ell_{i+1})$. Suppose $\mathscr{C}(\ell_i) \subseteq \mathscr{C}(\ell_{i+1})$, and therefore, $\Delta(\ell_i) \subseteq \mathscr{C}(\ell_{i+1})$. Taking a commodity $k \in \Delta(\ell_i)$, the points ζ and $\hat{\pi}_{\ell_i,\ell_{i+1}}^k$ imply that $\beta_{\ell_i}^k = \beta_{\ell_{i+1}}^k$. This, in turn, implies that $\bar{\beta}_{\ell_i} = \bar{\beta}_{\ell_{i+1}}$. Alternatively, if $\mathscr{C}(\ell_i) \supsetneq \mathscr{C}(\ell_{i+1})$, we take $k \in \Delta(\ell_{i+1})$ and use the points ζ and $\hat{\pi}_{\ell_{i+1},\ell_i}^k$ to obtain the same result. We conclude that

$$\bar{\beta}_\ell = \bar{\beta}_{\ell_1} = \cdots = \bar{\beta}_{\ell_p} = \bar{\beta}_{\ell'}.$$

Therefore, the coefficients β_j^k in (2.44) are equal. If we denote their common value by $\bar{\beta}$, (2.44) can then be written as

$$\bar{\beta} \sum_{j \in H} \sum_{k \in \mathscr{C}(j)} y_j^k \leq \gamma - \alpha(H) + \sum_{j \in H} \alpha_j (1 - x_j). \tag{2.45}$$

Substituting ζ into (2.45), we obtain

$$\bar{\beta} b(\mathscr{C}(H)) = \gamma - \alpha(H). \tag{2.46}$$

Next, fix an arc $\ell \in H$ and let $(\bar{x}, \bar{y})$ be the components of $\bar{\zeta}_\ell$. Substituting $\bar{\zeta}_\ell$ into (2.45) therefore yields

$$\bar{\beta} b(\mathscr{C}(H)) + \alpha_\ell = \sum_{j \in H} \sum_{k \in \mathscr{C}(j)} \bar{y}_j^k \tag{2.47}$$

It was shown in Theorem 3 that

$$\sum_{j \in H} \sum_{k \in \mathscr{C}(j)} \bar{y}_j^k = \begin{cases} b(\mathscr{C}(H)) - (u_\ell - \lambda_\ell) & \text{if } u_\ell > \lambda_\ell, \\ b(\mathscr{C}(H)) & \text{otherwise.} \end{cases} \tag{2.48}$$

Combining (2.47) and (2.48), we therefore obtain $\alpha_j = -\bar{\beta}(u_j - \lambda_j)^+$ for $j \in H$. Substituting (2.46) and the values just obtained for α_j into (2.45) yields

$$\bar{\beta} \sum_{j \in H} \sum_{k \in \mathscr{C}(j)} y_j^k = \bar{\beta} b(\mathscr{C}(H)) - \bar{\beta} \cdot \sum_{j \in H} (u_j - \lambda_j)^+ (1 - x_j),$$

which establishes the desired result. $\square$

To the best of our knowledge, HFC inequalities have not been described in the literature. However, they generalize traditional flow cover inequalities, as well as inequalities obtained from flow covers through aggregation. We formalize this observation next.

Proposition 15. *Let $C \subseteq A$ be an AFC for $Q \subseteq K$. Then $\langle C, \preceq \rangle$ is a HFC, where $\preceq$ is obtained from the map $\mathscr{C}(j) = Q$ for $j \in C$ and $\mathscr{C}(j) = \emptyset$ for $j \in A \setminus C$.*

Proof. Since C is an AFC for Q, we must have $u(C) > b(Q)$. Let $j \in C$. It follows from the definition of $\mathscr{C}$ that $\langle C, \preceq \rangle$ contains a single tier, and therefore, $[j] = C$ and $\Delta(j) = Q$. We write that $u([j]) = u(C) > b(Q) = b(\Delta(j))$, showing that $\langle C, \preceq \rangle$ is a HFC. $\square$

Since AFCs can be viewed as HFCs with a single tier, we obtain the following corollary.

Corollary 3. *Aggregated flow cover inequalities are HFC inequalities.* $\square$

We note that Proposition 9, which establishes that AFC inequalities are facet-defining under mild assumptions, can be obtained by applying the results we derived in this section for HFC inequalities. In fact, **C1** is satisfied by arc-commodity hierarchies with a single tier and **C2** reduces to the condition of Proposition 9. Further, **C3** is satisfied by hierarchies with a single tier. It follows that the conditions under which Theorem 4 is established naturally reduce to those used in Proposition 9 in the case that $\langle H, \preceq \rangle$ consists of a single tier.

2.5 Lifting HFC Inequalities

In the preceding sections, we derived valid inequalities for MVF under the assumption that all arcs are directed outward from a network node. We next use lifting (see, *e.g.*, [26, 42]) to extend HFC inequalities for situations where incoming arcs are also present. Let $A = A^+ \cup A^-$, where A^+ and A^- are the sets of outgoing and incoming arcs, respectively. In this case, we replace (2.1a) with the flow balance constraint

$$\sum_{j \in A^+} y_j^k - \sum_{j \in A^-} y_j^k \le b^k \quad \forall k \in K, \tag{2.49}$$

and define the set

$$\widetilde{P} := \left\{ (x,y) \in \mathbb{Z}^A \times \mathbb{R}^{A \times K} \mid (2.1\text{b}) - (2.1\text{e}), (2.49) \right\}.$$

We consider the flow model $\widetilde{P}$, which is shown in Figure 2.5. We assume, as before, that $m_j > 0$ for $j \in A$ and $b^k > 0$ for $k \in K$. It can be shown using a similar argument to the one presented in Proposition 1 that $\widetilde{P}$ is full-dimensional. For $L \subseteq A^-$, we define $\widetilde{P}_L$ to be the projection of

$$\left\{ (x,y) \in \widetilde{P} \,\middle|\, x_j = 0 \text{ and } y_j^k = 0 \ \forall j \in L, \forall k \in K \right\}$$

onto the set of non-fixed variables. Note that $\widetilde{P}_\emptyset = \widetilde{P}$, while $\widetilde{P}_{A^-}$ is of the form of P, a MVF model without inflow.

Assume that

$$\sum_{j\in A^+} \alpha_j x_j + \sum_{j\in H}\sum_{k\in Q_j} y_j^k \leq \delta, \tag{2.50}$$

where $H \subseteq A^+$, $Q_j \subseteq K$ for $j \in H$, and $\bigcup_{j\in H} Q_j \neq \emptyset$, is a valid inequality for $\mathrm{conv}(\widetilde{P}_{A^-})$ that is satisfied at equality by at least one solution of $\widetilde{P}_{A^-}$. An example of such an inequality is (2.22), which under Condition **C1** is satisfied at equality by the point ζ. We wish to lift (2.50) into a valid

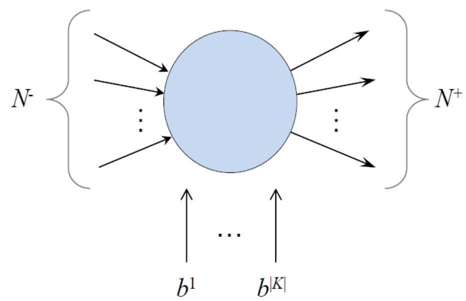

Figure 2.5. Graphical respresentation of $\widetilde{P}$

inequality of $\mathrm{conv}(\widetilde{P})$ of the form

$$\sum_{j\in A^+} \alpha_j x_j + \sum_{\ell\in A^-} \alpha_\ell x_\ell + \sum_{j\in H}\sum_{k\in Q_j} y_j^k + \sum_{\ell\in A^-}\sum_{k\in K} \beta_\ell^k y_\ell^k \leq \delta, \tag{2.51}$$

for suitable coefficients α_ℓ and β_ℓ^k, where $\ell \in A^-$ and $k \in K$. To this end, we define

(i) the function $f(y) := \sum_{j\in H}\sum_{k\in Q_j} y_j^k$ for $y \in \mathbb{R}^{A^+\times K}$,

(ii) the function $g(x) := \sum_{j\in A^+} \alpha_j x_j$ for $x \in \mathbb{R}^{A^+}$,

(iii) the set

$$S(u,\bar{x}) := \left\{ y \in \mathbb{R}^{A^+\times K} \;\middle|\; \begin{array}{ll} \sum_{j\in A^+} y_j^k \leq b^k + u^k & \forall k \in K, \\ \sum_{k\in K} y_j^k \leq m_j \bar{x}_j & \forall j \in A^+, \\ y_j^k \geq 0 & \forall j \in A^+, \forall k \in K \end{array} \right\},$$

where $u \in \mathbb{R}^K$ and $\bar{x} \in \{0,1\}^{A^+}$,

(iv) the lifting function

$$\phi(u) := \max\left\{ f(y) + g(x) - \delta \;\middle|\; x \in \{0,1\}^{A^+}, y \in S(u,x) \right\},$$

where $u \in \mathbb{R}^K$.

Our assumption that (2.50) is valid for $\mathrm{conv}(\widetilde{P}_{A^-})$ implies that $\phi(0) \leq 0$. Further, the assumption that this inequality is satisfied at equality by one solution shows that $\phi(0) = 0$. In order for (2.51) to be valid for $\mathrm{conv}(\widetilde{P})$, the inequality

$$\sum_{\ell \in A^-} \alpha_\ell x_\ell + \sum_{\ell \in A^-} \sum_{k \in K} \beta_\ell^k y_\ell^k + \phi\left(\sum_{\ell \in A^-} y_\ell \right) \leq 0, \tag{2.52}$$

where $y_\ell = (y_\ell^1, y_\ell^2, \ldots, y_\ell^{|K|})$ for $\ell \in A^-$, must hold for all (x_ℓ, y_ℓ) such that (i) $\sum_{k \in K} y_\ell^k \leq m_\ell x_\ell$, (ii) $y_\ell^k \geq 0$ for $k \in K$, and (iii) $x_\ell \in \{0,1\}$. Observe that the condition $y_\ell^k \geq 0$ for $k \in K$ implies that it is sufficient to study the function $\phi(u)$ on $\mathbb{R}_+^K$.

Lemma 4. *For $u \in \mathbb{R}_+^K$, $\phi(u) \leq u^T \mathbb{1}_Q$ where $Q := \bigcup_{j \in A} Q_j$, and $\overline{Q} := K \setminus Q$.*

Proof. First, we establish that, for $\bar{x} \in \{0,1\}^{A^+}$ and $u \in \mathbb{R}_+^K$,

$$z_u^* \leq z_\ell^* + u^T \mathbb{1}_Q, \tag{2.53}$$

where $z_u^* := \max\{f(y) \mid y \in S(u,\bar{x})\}$ and $z_\ell^* := \max\{f(y) \mid y \in S(0,\bar{x})\}$. Clearly, the problems defining z_u^* and z_ℓ^* are feasible and have optimal solutions. Let $\bar{y}$ be an optimal solution to the problem defining z_u^*. We may assume that $\bar{y}_j^k = 0$ for all (j,k) such that (i) $j \in A^+ \setminus H$ or (ii) $j \in H$ and $k \in K \setminus Q_j$, since these variables do not appear in the objective. Define $\gamma^k := (\sum_{j \in A^+} \bar{y}_j^k - b^k)^+$ for $k \in K$. Clearly, since $y \in S(u,\bar{x})$, we have $\gamma^k \leq u^k$ for $k \in K$. Also, it is clear from the definition that $\gamma^k \geq 0$. Now, define the solution $\tilde{y}$, where for $j \in H$ and $k \in K$,

$$\tilde{y}_j^k := \begin{cases} \bar{y}_j^k \left(1 - \frac{\gamma^k}{b^k + \gamma^k}\right) & \text{if } k \in Q_j, \\ 0 & \text{if } k \in K \setminus Q_j. \end{cases}$$

We claim that $\tilde{y} \in S(0,\bar{x})$. It is clear that $0 \leq \frac{\gamma^k}{b^k + \gamma^k} \leq 1$, since $b^k \geq 0$. It follows that $0 \leq \tilde{y}_j^k \leq \bar{y}_j^k$ for $j \in H$. Now, we compute, for $j \in H$,

$$\sum_{k \in K} \tilde{y}_j^k = \sum_{k \in Q_j} \bar{y}_j^k \left(1 - \frac{\gamma^k}{b^k + \gamma^k}\right) \leq \sum_{k \in Q_j} \bar{y}_j^k = \sum_{k \in K} \bar{y}_j^k \leq m_j \bar{x}_j,$$

and for $k \in Q$,

$$\sum_{j \in A^+} \tilde{y}_j^k = \sum_{j \in H} \bar{y}_j^k \left(1 - \frac{\gamma^k}{b^k + \gamma^k}\right) = \left(1 - \frac{\gamma^k}{b^k + \gamma^k}\right) \sum_{j \in H} \bar{y}_j^k$$

$$\leq \left(1-\frac{\gamma^k}{b^k+\gamma^k}\right)(b^k+\gamma^k) = b^k,$$

while, for $k \in \overline{Q}$, $\sum_{j\in A^+} \tilde{y}_j^k = 0$. Solution $\tilde{y}$ being feasible to $S(0,\bar{x})$, we next verify that

$$\begin{aligned}
z_\ell^* &\geq f(\tilde{y}) = \sum_{j\in H}\sum_{k\in Q_j} \tilde{y}_j^k = \sum_{j\in H}\sum_{k\in Q} \tilde{y}_j^k = \sum_{j\in H}\sum_{k\in Q} \bar{y}_j^k \left(1-\frac{\gamma^k}{b^k+\gamma^k}\right) \\
&= \sum_{j\in H}\sum_{k\in Q} \bar{y}_j^k - \sum_{j\in H}\sum_{k\in Q} \bar{y}_j^k \frac{\gamma^k}{b^k+\gamma^k} = \sum_{j\in H}\sum_{k\in Q_j} \bar{y}_j^k - \sum_{j\in H}\sum_{k\in Q} \bar{y}_j^k \frac{\gamma^k}{b^k+\gamma^k} \\
&= f(\bar{y}) - \sum_{k\in Q} \frac{\gamma^k}{b^k+\gamma^k}\left(\sum_{j\in A^+} \bar{y}_j^k\right) = z_u^* - \sum_{k\in Q} \frac{\gamma^k}{b^k+\gamma^k}\left(\sum_{j\in A^+} \bar{y}_j^k\right) \\
&\geq z_u^* - \sum_{k\in Q} \gamma^k = z_u^* - u^T \mathbb{1}_Q,
\end{aligned}$$

showing (2.53). We are now ready to prove the result. We write that

$$\begin{aligned}
\phi(u) &= \max_{\bar{x}\in\{0,1\}^{A^+}} \left[\max_{y\in S(u,\bar{x})} \left\{f(y)\right\} + g(\bar{x}) - \delta\right] \\
&\leq \max_{\bar{x}\in\{0,1\}^{A^+}} \left[\max_{y\in S(0,\bar{x})} \left\{f(y)\right\} + u^T\mathbb{1}_Q + g(\bar{x}) - \delta\right] \\
&= u^T\mathbb{1}_Q + \max_{\bar{x}\in\{0,1\}^{A^+}} \left[\max_{y\in S(0,\bar{x})} \left\{f(y)\right\} + g(\bar{x}) - \delta\right] \\
&= u^T\mathbb{1}_Q + \phi(0) \\
&\leq u^T\mathbb{1}_Q,
\end{aligned}$$

where the first inequality follows from (2.53) and the last inequality holds since $\phi(0) \leq 0$. □

Lemma 5. *The coefficients $\alpha_\ell = 0$ for $\ell \in A^-$, $\beta_\ell^k = -1$ for $\ell \in A^-$ and $k \in Q$, and $\beta_\ell^k = 0$ for $\ell \in A^-$ and $k \in \overline{Q}$ are valid lifting coefficients in* (2.51).

Proof. We need to verify (2.52) for all $(x_\ell, y_\ell)_{\ell\in A^-}$ such that $\sum_{k\in K} y_\ell^k \leq m_\ell x_\ell$ and $y_\ell^k \geq 0$ for $k \in K$. For the given values of α_ℓ and β_ℓ^k, we write that

$$\sum_{\ell\in A^-} \alpha_\ell x_\ell + \sum_{\ell\in A^-}\sum_{k\in K} \beta_\ell^k y_\ell^k = -\sum_{\ell\in A^-}\sum_{k\in Q} y_\ell^k.$$

Since Lemma 4 shows that $\phi\left(\sum_{\ell\in A^-} y_\ell\right) \leq \sum_{\ell\in A^-}\sum_{k\in Q} y_\ell^k$, as $y_\ell^k \geq 0$ for $\ell \in A^-$ and $k \in K$, we obtain that

$$\sum_{\ell\in A^-} \alpha_\ell x_\ell + \sum_{\ell\in A^-}\sum_{k\in K} \beta_\ell^k y_\ell^k + \phi\left(\sum_{\ell\in A^-} y_\ell\right) \leq 0,$$

as desired. □

Given a HFC $\langle H, \preceq \rangle$, we define the *lifted hierarchical flow cover inequality* (LHFCI) to be

$$\sum_{j \in H} \sum_{k \in \mathscr{C}(j)} y_j^k - \sum_{j \in A^-} \sum_{k \in \mathscr{C}(H)} y_j^k \leq b(\mathscr{C}(H)) - \sum_{j \in H} (u_j - \lambda_j)^+ (1 - x_j). \tag{2.54}$$

Lemma 5 directly implies the following result, since HFC inequalities are of the form (2.50).

Theorem 5. *Inequality* (2.54) *is valid for* $\operatorname{conv}(\widetilde{P})$. □

Although Lemma 4 only derives an approximation of the lifting function of (2.50), we next show that this approximation is sufficient to perform exact lifting (and therefore to obtain facet-defining inequalities) in the case of HFCs. We first prove a more general result in Proposition 16. To this end, we introduce

Property 3. There exists $\varepsilon > 0$ such that $\phi(u) = u^T \mathbb{1}_Q$ for all $u \in [0, \varepsilon]^K$.

Proposition 16. *If* (2.50) *is facet-defining for* $\operatorname{conv}(\widetilde{P}_{A^-})$ *and satisfies Property 3, then* (2.51)*, with* $\alpha_\ell = 0$ *for* $\ell \in A^-$*,* $\beta_\ell^k = -1$ *for* $\ell \in A^-$ *and* $k \in Q$*, and* $\beta_\ell^k = 0$ *for* $\ell \in A^-$ *and* $k \in \overline{Q}$*, is facet-defining for* $\operatorname{conv}(\widetilde{P})$.

Proof. Let $d := \dim \operatorname{conv}(\widetilde{P}_{A^-})$, and let $\{(x^t, y^t)\}_{t=1}^d$ be a set of affinely independent points of $\widetilde{P}_{A^-}$ showing that (2.50) is facet-defining for $\operatorname{conv}(\widetilde{P}_{A^-})$. We construct the points $(\bar{x}^t, \bar{x}^t_{A^-}, \bar{y}^t, \bar{y}^t_{A^-})$, where $\bar{x}^t_{A^-} = \mathbf{0}$ and $\bar{y}^t_{A^-} = \mathbf{0}$ for $t = 1, \ldots, d$. These points belong to $\widetilde{P}$ and satisfy (2.51) at equality. Next, we find $|A^-|(|K|+1)$ additional points that are affinely independent from these and from each other. Clearly, we can choose $(\bar{x}^1, e_j, \bar{y}^1, \mathbf{0})$ for $j \in A^-$ to form $|A^-|$ of these points. For the other $|A^-||K|$ points, we proceed as follows. For each $q \in \{1, \ldots, |K|\}$, let $(\tilde{x}^q, \tilde{y}^q)$ be any solution of

$$\mathscr{L}^q := \left\{ (x,y) \;\middle|\; x \in \{0,1\}^{|A^+|},\ y \in S(\varepsilon e_q, x),\ \phi\left(\varepsilon e_q\right) = \varepsilon e_q^T \mathbb{1}_Q \right\}. \tag{2.55}$$

In other words, $\mathscr{L}^q$ is the set of optimal solutions defining the value of $\phi\left(\varepsilon e_q\right)$. This set is nonempty because of Property 3. We then construct the solutions $(\tilde{x}^q, e_j, \tilde{y}^q, \varepsilon e_q^j)$ for $q \in \{1, \ldots, |K|\}$ and $j \in A^-$. These points belong to $\widetilde{P}$ and satisfy (2.51) at equality and are affinely independent from all previously constructed points. It follows that the dimension of the face of $\operatorname{conv}(\widetilde{P})$ defined by (2.51) is $|A^-|(|K|+1)$ more than that of (2.50), showing that (2.51) is facet-defining for $\operatorname{conv}(\widetilde{P})$. □

We conclude this section by showing that (2.22) satisfies Property 3, thereby showing that (2.54) is facet-defining for $\operatorname{conv}(\widetilde{P})$ under Conditions **C1**, **C2**, and **C3**.

Lemma 6. *There exists* $\varepsilon > 0$ *for which* (2.22) *satisfies Property 3.*

Proof. Lemma 4 shows that $\phi(u) \leq u^T \mathbb{1}_Q$. We define $(\bar{x}, \bar{y}) := (\mathbb{1}_H, \eta(Q))$ and $\gamma := \min_{j \in A} \{m_j - \sum_{k \in K} \bar{y}_j^k\}$. Observe that $(\bar{x}, \bar{y}) \in S(0, \mathbb{1}_H)$ and $\gamma > 0$. We will show that there exists a feasible solution to the problem defining $\phi(u)$ that achieves the value $u^T \mathbb{1}_Q$ for all $u \in [0, \varepsilon]$, where $\varepsilon = \gamma/|K|$. Consider the solution $(\hat{x}, \hat{y}) = (\mathbb{1}_H, \eta^*(Q))$, where

$$\eta^*(Q) := \sum_{k \in Q} \sum_{j \in \Delta^{-1}(k)} \frac{u_j}{u(\Delta^{-1}(k))} \cdot (b^k + u^k) \boldsymbol{e}_{\boldsymbol{j}}^{\boldsymbol{k}}.$$

This solution has value $u^T \mathbb{1}_Q$ and belongs to $S(u, \mathbb{1}_H)$, since $(\bar{x}, \bar{y}) = (\mathbb{1}_H, \eta(Q)) \in S(0, \mathbb{1}_H)$ and $\sum_{k \in K} \bar{y}_j^k + \gamma \leq m_j$ for $j \in H$, and $\sum_{k \in K} \hat{y}_j^k \leq \sum_{k \in Q} \bar{y}_j^k + \varepsilon |K|$. □

Example 6. *Consider a five-arc, three-commodity model with* $A^+ = \{1, 2, 3\}$, $A^- = \{4, 5\}$, $m = (8, 13, 15, 9, 12)$ *and supplies* $b = (7, 9, 11)$. *Note that this model can be obtained from the model of Example 1 by adding two incoming arcs* $\{4, 5\}$ *with capacities* $m_4 = 9$ *and* $m_5 = 12$*, respectively. We find that the three hierarchical flow covers of Example 4 produce the following LHFCIs:*

$$\begin{aligned}
&y_1^1 + y_1^2 + y_1^3 + y_2^3 + y_3^2 + y_3^3 - y_4^1 - y_4^2 - y_4^3 - y_5^1 - y_5^2 - y_5^3 \\
&\leq 27 - 7(1 - x_1) - 4(1 - x_2) - 8(1 - x_3), \qquad (2.56a) \\
&y_1^1 + y_1^2 + y_2^2 + y_2^3 + y_3^2 - y_4^1 - y_4^2 - y_4^3 - y_5^1 - y_5^2 - y_5^3 \\
&\leq 27 - 7(1 - x_1) - 11(1 - x_2) - 6(1 - x_3), \qquad (2.56b) \\
&y_1^1 + y_1^2 + y_1^3 + y_2^2 + y_3^1 + y_3^2 + y_3^3 - y_4^1 - y_4^2 - y_4^3 - y_5^1 - y_5^2 - y_5^3 \\
&\leq 27 - 3(1 - x_1) - 4(1 - x_2) - 10(1 - x_3). \qquad (2.56c)
\end{aligned}$$

Inequalities (2.56a)-(2.56c)*, which are obtained by lifting* (2.2f)-(2.2h) *respectively, are facet-defining for the convex hull of this model.* □

Theorem 6. *The lifted HFC inequality* (2.54) *is facet-defining for* $\operatorname{conv}(\widetilde{P})$ *under Conditions* ***C1****,* ***C2****, and* ***C3****.*

2.6 Computation

In this section, we show that LHFCIs have the potential to be very useful in the solution of network design problems with multiple commodities. These preliminary results would have to be confirmed through a full-fledged computational study, which we reserve for future work.

2.6.1 Instances

We perform our computational experiments on a test bed of randomly generated instances of the *multi-commodity fixed charge capacitated network design problem* (MFCND). Each instance consists of a directed graph $G = (V, A)$, a commodity set K, arc construction costs $c \in \mathbb{R}_+^A$, variable arc flow costs $h \in \mathbb{R}_+^A$, arc capacities $u \in \mathbb{R}_+^A$, and node supplies/demands $b \in \mathbb{R}^{V \times K}$. We classify instances by the number of nodes $|V|$, the order of magnitude of $|A|$, and the order of magnitude of $|K|$. For a given number of nodes $|V|$, we select $|A|$ uniformly over (i) $\left[\lceil |V|^{1.3} \rceil, \lceil |V|^{1.5} \rceil\right]$ to obtain instances with a medium number of arcs, or (ii) $\left[\lceil|V|^{1.6}\rceil,\lceil|V|^{1.8}\rceil\right]$ to obtain instances with a large number of arcs. Similarly, we select $|K|$ uniformly over (i) $\left[\lceil .5|V|\rceil,\lceil .8|V|\rceil\right]$ to obtain instances with a medium number of commodities, or (ii) $\left[.9|V|,\lceil 1.2|V|\rceil\right]$ to obtain instances with a large number of commodities.

Once the the dimensions $|V|$, $|A|$, and $|K|$ of an instance are set, we generate arc and commodity data. We iteratively add random pairs $(i,j) \in V \times V$ to A until we have reached the predetermined number of arcs. During this process, we forbid the creation of duplicate arcs and loops. For each arc $(i,j) \in A$, we generate values $u_{ij} \in \{10,\ldots,30\}$, $c_{ij} \in \{100,\ldots,200\}$, and $h_{ij} \in \{30,\ldots,50\}$ uniformly at random. For each commodity $k \in K$, we uniformly generate a value $\bar{b}^k \in \{5,\ldots,10\}$. We then (i) uniformly select parameters $(s,d) \in \{1,2,3\}^2$; (ii) randomly add nodes $i \in V$ to (initially empty) sets of source nodes $S(k)$ and demand nodes $D(k)$ so that $|S(k)| = s$, $|D(k)| = d$, and $S(k) \cap D(k) = \emptyset$; and (iii) randomly assign values $b_i^k \in \{0,\ldots,\bar{b}^k\}$ to the nodes such that $\sum_{i\in S(k)} b_i^k = \bar{b}^k$, $b_i^k > 0$ for $i \in S(k)$, $\sum_{i\in D(k)} b_i^k = -\bar{b}^k$, $b_i^k < 0$ for $i \in D(k)$, and $b_i^k = 0$ for $i \in V \setminus (S(k) \cup D(k))$. It is easy to verify that an MFCND instance is feasible if and only if the linear program obtained by opening all arcs, *i.e.*, fixing $x_{ij} = 1$ for each $(i,j) \in A$, is feasible. After generating each instance, we therefore verify its feasibility by solving the aforementioned linear program. If an instance is infeasible, we discard it and generate a new instance with the same node-arc-commodity setting, repeating this process until a feasible instance is obtained. We create instances for $|V| = 15$, with 10 instances for each of the arc-commodity settings medium-medium, medium-large, and large-medium. The test instances can be found online at [13].

2.6.2 Separation

Separating flow cover inequalities is known to be NP-Hard [33]. It follows that separating HFC inequalities is NP-Hard, as well, since the set of HFC inequalities is exactly the set of flow cover inequalities when a single commodity appears in the model. We therefore propose a greedy heuristic for identifying a most violated lifted HFC inequality. Let $(\tilde{x},\tilde{y})$ be a fractional solution to (2.1), and let $\langle H,\preceq\rangle$ be a HFC of (2.1), induced by a commodity map $\mathscr{C}(\cdot)$. The procedure starts by heuristically solving the following knapsack problem to identify an AFC for $Q \subseteq K$

$$\max\left\{\sum_{j\in A}(\tilde{x}_j - 1)\omega_j \;\middle|\; \sum_{j\in A} u_j\omega_j > b(Q), \quad \omega_j \in \{0,1\} \quad \forall j \in A\right\}. \tag{2.57}$$

Next, commodities are iteratively removed from $\mathscr{C}(j)$, for $j \in H$, whenever doing so increases the violation of (2.54) by $(\tilde{x},\tilde{y})$. We compute the violation of inequality (2.54) by $(\tilde{x},\tilde{y})$ as

$$\tilde{v}(H,\mathscr{C}) := \sum_{j\in H}\sum_{k\in\mathscr{C}(j)} \tilde{y}_j^k - \sum_{j\in A^-}\sum_{k\in\mathscr{C}(H)} \tilde{y}_j^k - [b(\mathscr{C}(H)) - \sum_{j\in H}(u_j-\lambda_j)^+(1-\tilde{x}_j)].$$

The separation procedure is described formally in Algorithm 1.

Algorithm 1 Separation heuristic for LHFCIs

Input: A MVF, a commodity subset $Q \subseteq K$ with $b^k > 0$ for $k \in Q$, and a fractional point $(\tilde{x}, \tilde{y})$.
Output: A lifted HFC inequality of the form (2.54).

Solve (2.57) greedily to obtain $H \subseteq A^+$.
Set $\mathscr{C}(j) \leftarrow Q, \forall j \in H$.
for all $k \in Q$ **do**
    repeat
        $changed \leftarrow False$.
        for all $j \in H$ such that $k \in \mathscr{C}(j)$ **do**
            Set $H' \leftarrow H$ and $\mathscr{C}' \leftarrow \mathscr{C}$.
            Set $\mathscr{C}'(j) \leftarrow \mathscr{C}'(j) \setminus k$.
            //Remove j if it has no associated commodities
            if $\mathscr{C}'(j) = \emptyset$ **then**
                $H' \leftarrow H' \setminus j$.
            end if
            Use $\mathscr{C}'(\cdot)$ to compute $\Delta'(j)$ and $\lambda'_j, \forall j \in H'$.
            //Let $\langle H', \preceq \rangle$ denote the hierarchy induced on H' by $\mathscr{C}'$.
            if $\langle H', \preceq \rangle$ satisfies the definition of HFC and Condition **C1 then**
                if $\tilde{v}(H', \mathscr{C}') > \tilde{v}(H, \mathscr{C})$ **then**
                    Set $H \leftarrow H'$ and $\mathscr{C} \leftarrow \mathscr{C}'$.
                    $changed \leftarrow True$.
                end if
            end if
        end for
    until $\neg$*changed*
end for

2.6.3 Computational Setup

We implement our separation algorithm in C++ using CPLEX 12.3 via ILOG Concert Technology. We disable CPLEX's built-in cuts, impose a run time limit of 3 hours, set the MIP optimality gap parameter to 0.1%, and leave the other parameters at their default values. Our experiments are performed on a PC with a 2.50GHz Intel Core Processor and 6 GB RAM. When a fractional solution is found at any node in the branch-and-bound tree, we execute the separation algorithm from each node $i \in V$ that has a positive supply for at least one commodity $k \in K$, using the initial commodity subset $Q := \{k \in K \mid b_i^k > 0\}$. A cut is added to the formulation if its violation by the fractional solution is at least 1×10^{-3}, and we allow at most 50 cuts to be added in total. For comparison purposes, we also solve each instance using (i) branch-and-bound (*i.e.*, no cuts are generated), and (ii) lifted AFC inequalities (LAFCIs). In the latter case, we obtain an AFC by solving (2.57). We then add its corresponding lifted HFC inequality (2.54) to the formulation.

2.6.4 Results

The results of our computational experiments are summarized in Tables 2.1-2.4 (see Appendix A for detailed results). Table 2.1 shows the performance of pure branch-and-bound in solving the test instances. Each row corresponds to a particular arc-commodity category (with 10 instances), which is indicated in the "Arcs/Coms." column. The "B&B Nodes" column shows the average number of nodes in the branch-and-bound tree. We also include the average, minimum, and maximum time, in seconds, elapsed for each category. We note that the branch-and-bound procedure failed to solve two of the large arc-medium commodity instances due to insufficient memory, and therefore the statistics in this category are computed only for the eight instances that were successfully solved. Table 2.2 and Table 2.3 show the results obtained using LAFCI and LHFCI algorithms, respectively. Recall that the arc-commodity hierarchy associated with a LAFCI must consist of a single tier, whereas the hierarchy of an LHFCI may include one or many tiers. Therefore, some of the cuts produced by Algorithm 1 can also be obtained through commodity aggregation. For this reason, in Tables 2.2 and 2.3, the column marked "STC" displays the average number of single-tier cuts generated in each category, and in Table 2.3, the "MTC" column shows the average number of multiple-tier cuts.

Table 2.1. Branch-and-Bound

Arcs/Coms.	B&B Nodes	Time (sec.)		
		Avg	Min	Max
M/M	114,513.50	14.78	0.11	68.50
M/L	258,896.50	45.78	0.17	268.05
L/M	25,092,499.50	5,685.06	133.06	10,807.20

We note that most of the medium arc-medium commodity (M/M) and medium arc-large commodity (M/L) instances solve in less than a minute, regardless of the choice of algorithm, whereas

Table 2.2. LAFCI Algorithm

Arcs/Coms.	B&B Nodes	STC	Time (sec.)		
			Avg	Min	Max
M/M	29,034.50	20.50	7.26	0.09	47.38
M/L	82,303.00	22.70	28.77	0.29	174.86
L/M	11,003,869.40	49.60	3,330.54	69.60	10,807.00

Table 2.3. LHFCI Algorithm

Arcs/Coms.	B&B Nodes	STC	MTC	Time (sec.)		
				Avg	Min	Max
M/M	19,167.50	27.90	0.50	4.38	0.07	24.18
M/L	43,309.70	33.60	2.60	11.94	0.18	61.87
L/M	8,145,980.00	47.50	2.50	2,427.92	27.42	10,807.30

the large arc-medium commodity (L/M) instances typically require much more time. As one might expect, both the LAFCI- and LHFCI-based branch-and-cut algorithms tend to solve faster than branch-and-bound across all three instance categories. More intriguingly, we observe that, on average, the LHFCI algorithm has shorter run times and requires the evaluation of fewer branch-and-bound nodes than the LAFCI algorithm. This difference is especially pronounced for the L/M instances. The LHFCI algorithm produces multi-tier cuts for only nine of the thirty instances: four L/M, three M/L, and two M/M instances, respectively. Table 2.4 shows the average performance of both branch-and-cut algorithms for this subset of instances. In each category, the average LHFCI algorithm run time is roughly half that of the LAFCI algorithm. We also find that the average percentage of cuts that are multi-tiered is highest in the M/L category (21.0%) and lowest in the M/M category (9.1%). This is unsurprising, since an instance with a larger set of commodities has more arc-commodity hierarchies, and therefore, more potential multi-tiered LHFCIs. This is also encouraging, as our numerical results show that, when multi-tier cuts can be identified, they tend to have a significant impact on the solution times of these problems.

Table 2.4. Instances in which multi-tiered LHFCIs were identified.

	AFC			HFC			
Arcs/Coms	B&B Nodes	STC	Time	B&B Nodes	STC	MTC	Time
M/M	1,294.50	10.50	0.38	882.50	27.50	2.50	0.19
M/L	128,979.33	31.00	30.59	54,285.67	41.33	8.67	13.57
L/M	7,062,398.00	50.00	2,049.39	3,473,272.50	43.75	6.25	1,027.77

Chapter 3

DIRECTED EDGE-FAILURE RESILIENT NETWORK DESIGN

3.1 Background

Transportation plays an indispensable role in the United States' economy. In 2013, approximately $48 billion worth of freight was shipped throughout the country [2]. As such, the national transportation network, comprised of assets including highways, waterways, and railways, is widely considered to be critical infrastructure whose continued operation is vital to the nation's well-being [35]. The condition of links in the network is a cause for great concern, since individual link failures can degrade the performance of the network as a whole. While these failures are often the result of natural wear and tear, they may also occur as a consequence of sudden catastrophic events. For instance, in the wake of the 1994 Northridge earthquake, the Los Angeles metropolitan area suffered considerable economic setbacks, with $1.5 billion in losses attributable to transport disruptions alone [25]. In 2005, Hurricanes Katrina and Rita devastated the Gulf Coast and placed enormous stress on the region's transportation system [29]. In addition to natural disasters, deliberate acts of destruction, undertaken by terrorists or saboteurs, are another source of danger. In light of these risks, *resilience*, the ability of a system to recover after a shock, is an essential attribute of critical infrastructure [37]. In this chapter, which is based on joint work with R.L.-Y. Chen and C.A. Phillips, we consider a problem concerning the design of transportation networks with resilience requirements.

There are various works in the literature that apply network design to the construction of transportation infrastructure. For instance, Meng and Yang [31] study a road network design problem that attempts to determine optimal expansions of road capacity while accounting for the equilibrium behavior of users. The authors provide bilevel linear programming formulations and use simulated annealing to solve the Sioux Falls benchmark instance described in [49]. Solanki et al. [47] use a decomposition-based heuristic to design a large-scale highway network. They do not model road congestion, however, arguing that its effects are negligible in the rural highway setting they consider.

Whereas the aforementioned papers consider network design in the absence of failures, there are various approaches to guaranteeing some measure of resilience (or *survivability*) in a network. One criterion is *network connectivity*, commonly defined as the number of edge-disjoint paths existing between pairs of nodes in the network. For example, Balakrishan et al. [7] present valid inequalities, a cutting plane algorithm, and a heuristic procedure for solving a network design problem with connectivity requirements. Alternatively, many formulations explicitly take into account the failure of elements in the network. Stoer and Dahl [22, 48] apply cutting plane algorithms to designing networks which support feasible flows in the presence of a single node or edge failure. Bienstock and Muratore [10] derive valid inequalities by studying the polyhedra associated with various survivable network design formulations. Rajan and Atamtürk study [6, 41] network design problems in which spare capacity is reserved along directed cycles in order to reroute flows in the event of arc failures.

Multi-level optimization provides another framework for assessing network resilience. *Network interdiction* is a two-player game in which an *attacker* acts on a network, typically by destroying arcs, in order to minimize the maximum utility the *defender* gains by sending flow through the network (see, *e.g.*, [46, 54]). This game is naturally formulated as a bilevel optimization problem. Various authors incorporate network interdiction as a subproblem in survivable network design. For instance, Brown et al. [11] solve bi- and tri-level models related to the defense of critical infrastructure. Scaparra and Church [43] solve a bilevel problem in which resources are assigned to protect facilities in a distribution network. Cappanera and Scaparra [14] devise an algorithm for protecting arcs in a network so as to minimize the shortest path between a supply node and a demand node after some subset of unprotected arcs have been interdicted. Garg and Smith [23] present algorithms for designing and operating networks in the presence of an interdictor capable of destroying arcs. Chen and Phillips [15] formulate an undirected resilient network design problem and evaluate both a cutting plane algorithm and a column-and-cut algorithm for its solution.

We study a variant of multi-commodity fixed-charge capacitated network design (MFCND), which we call the *directed edge-failure resilient network design problem* (DRNDP). This problem seeks to determine a directed network with minimum arc construction costs that is capable of transporting a pre-specified proportion of a set of commodities between their respective origins and destinations in the event that any fixed size subset of arcs is destroyed. The problem allows for fractional commodity flows, and we assume that arc failures are total - *i.e.*, when an arc is destroyed, it is rendered incapable of carrying any flow. Ideally, an optimal network should be able to accommodate the entire set of commodity flows even after some arcs are destroyed. However, we anticipate that real world planners, faced with limited resources, will accept some decrease in network performance when failures occur. Therefore, we incorporate *demand shedding* into our model, which allows for solutions in which a limited amount of commodity demands is left unsatisfied. Although we consider this problem specifically in the context of transportation infrastructure, with arcs representing links to be built (e.g., stretches of highway or railroad track), due to its generality, DRNDP can also be applied to other applications (e.g., supply chain planning).

The remainder of the chapter is organized as follows. In Section 3.2, we describe the model in more detail and provide a mixed-integer linear programming (MILP) formulation. In Section 3.3, we describe an algorithm for solving DRNDP. In Section 3.4, we present computational results.

3.2 Problem Description

We consider a directed graph $G := (N,A)$, where N is the set of nodes and A is the set of arcs. Given a node $i \in N$, let $\delta^+(i)$ and $\delta^-(i)$ be the sets of arcs leaving and entering i, respectively. We associate with each arc $(i,j) \in A$ a fixed construction cost c_{ij} and a capacity u_{ij}. Let K be the set of commodities, where, for each commodity $k \in K$, there is a unique origin node $s^k \in N$, a unique destination node $t^k \in N$, and a demand b^k. We assume that, in any failure scenario, at most Γ arcs can be destroyed, where $\Gamma \in \{0,1,\ldots,|A|\}$ is a fixed parameter that we refer to as the *failure budget*. We represent each scenario with a binary vector $\tilde{d}$, where $\tilde{d}_{ij}$ equals one if arc (i,j) is destroyed and zero otherwise. The set of all possible failure scenarios can be described as $D := \{\tilde{d} \in \{0,1\}^{|A|} \mid \sum_{(i,j)\in A} \tilde{d}_{ij} \leq \Gamma\}$. We use $\tilde{d}^l$ to denote the l-th scenario, where $l \in L := \{1,\ldots,|D|\}$. We define the *shed threshold* to be a fixed parameter Z, with $0 \leq Z < \sum_{k\in K} b^k$, that corresponds to the maximum aggregate demand shedding allowed across all commodities.

Next, we define x_{ij} to be a binary variable determining whether arc (i,j) is open or closed, and use the continuous variable y^k_{ij} to represent the flow of commodity k on arc (i,j). In addition, we let the continuous variable z^k_l represent the amount of demand shed for commodity k in the l-th failure scenario. Now, we formulate DRNDP as:

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{3.1a}$$

$$\text{s.t.} \sum_{(i,j)\in\delta^+(i)} y^{kl}_{ij} - \sum_{(j,i)\in\delta^-(i)} y^{kl}_{ji} = \begin{cases} b^k - z^k_l & i = s^k \\ -b^k + z^k_l & i = t^k \\ 0 & \text{otherwise} \end{cases} \qquad \forall i \in N, k \in K, l \in L \tag{3.1b}$$

$$\sum_{k\in K} y^{kl}_{ij} \leq u_{ij}(x_{ij} - \tilde{d}^l_{ij})^+ \qquad \forall (i,j) \in A, l \in L \tag{3.1c}$$

$$\sum_{k\in K} z^k_l \leq Z \qquad \forall l \in L, \tag{3.1d}$$

$$z^k_l \leq b^k \qquad \forall k \in K, \forall l \in L, \tag{3.1e}$$

$$z^k_l \geq 0 \qquad \forall k \in K, \forall l \in L, \tag{3.1f}$$

$$y^{kl}_{ij} \geq 0 \qquad \forall (i,j) \in A, k \in K, \forall l \in L \tag{3.1g}$$

$$x_{ij} \in \{0,1\} \qquad \forall (i,j) \in A. \tag{3.1h}$$

The objective (3.1a) minimizes the total arc construction cost. Constraints (3.1b) enforce the flow balance of each commodity at each node in each scenario. Next, we have variable upper bound constraints (3.1c) corresponding to each arc-scenario combination. These constraints allow the aggregate flow on an arc $(i,j) \in A$ in scenario l to be at most its capacity u_{ij} if the arc is constructed ($x_{ij} = 1$) and not destroyed in the scenario ($\tilde{d}^l_{ij} = 0$), and otherwise force this flow to zero. Constraints (3.1d) ensure that the total demand shed in any scenario does not exceed the shed threshold Z, while constraints (3.1e) guarantee that the amount of commodity k shed in scenario l is at most equal to its demand b^k. Constraints (3.1f) and (3.1g) are nonnegativity restrictions on the shed variables z and flow variables y, respectively. Finally, constraints (3.1h) are binary restrictions on the design variables x.

It is known that MFCND is NP-Hard [17]. Since MFCND is the case of DRNDP obtained when $\Gamma = 0$ and $Z = 0$, it follows that DRNDP is also NP-Hard. Note that the number of variables and constraints in (3.1) is proportional to $\binom{|A|}{\Gamma}$, the number of (maximal) failure scenarios. As a consequence, the formulation becomes quite large even for networks of modest size. This motivates the use of a decomposition approach.

3.3 Branch-Price-and-Cut Algorithm

In this section, we present a branch-price-and-cut algorithm for solving DRNDP. Throughout, we let $\tilde{x} \in \{0,1\}^{|A|}$ denote a candidate network design. Further, we let $A_{\tilde{x}} := \{(i,j) \in A : \tilde{x}_{ij} = 1\}$, the set of arcs constructed in the solution $\tilde{x}$, and $G_{\tilde{x}} := (N, A_{\tilde{x}})$, the subgraph of G induced by $A_{\tilde{x}}$. The algorithm is structured as follows:

Algorithm BPC

1. Initialize *master problem* (MP).

2. Branch until finding an integer solution $\tilde{x}$ to MP.

3. Pass $\tilde{x}$ to a *separation problem* to identify a failure scenario $\tilde{d}$ such that no feasible flow exists.

4. If such $\tilde{d}$ found, generate valid inequalities and columns to add to MP and return to step 2. Otherwise, exit.

We note that column generation within BPC occurs specifically in the context of generating compact cutset inequalities, which we describe in Section 3.3.4.

The algorithm begins by constructing a master problem, which is strengthened over successive iterations of the algorithm. For $i \in N$, let $b_i^- := \sum_{k \in K | i = t^k} b^k$ and $b_i^+ := \sum_{k \in K | i = s^k} b^k$ be quantities corresponding to the aggregate demand directed into and out of node i, respectively. We use this notation in the following proposition.

Proposition 17. *Let $\tilde{x} \in \{0,1\}^A$ be a feasible network design and let $i \in N$. We must have that (i) if $b_i^- > Z$, there are least $\Gamma + 1$ incoming arcs incident to i in $G_{\tilde{x}}$, and (ii) if $b_i^+ > Z$, there are at least $\Gamma + 1$ outgoing arcs incident to i in $G_{\tilde{x}}$.*

Proof. First, assume that $b_i^- > Z$. Suppose, for a contradiction, that there are at most Γ arcs entering i in the network $G_{\tilde{x}}$. Then there exists a scenario $\tilde{d} \in D$ in which each of the arcs entering i are destroyed. Under this scenario, at least b_i^- units of demand must be shed, and therefore $\tilde{x}$ is infeasible since (3.1d) is violated. This implies (i). An analogous argument shows (ii). $\square$

We formulate the initial master problem as:

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{3.2a}$$

$$\text{s.t.} \sum_{(i,j)\in\delta^-(i)} x_{ij} \geq \Gamma + 1 \qquad \forall i \in N : b_i^- > Z \tag{3.2b}$$

$$\sum_{(j,i)\in\delta^+(i)} x_{ji} \geq \Gamma + 1 \qquad \forall i \in N : b_i^+ > Z \tag{3.2c}$$

$$x_{ij} \in \{0,1\} \qquad \forall (i,j) \in A. \tag{3.2d}$$

Note that (3.2a) and (3.2d) are the objective and binary restrictions on x contained in formulation (3.1), while constraints (3.2b) and (3.2c) enforce the in- and out-degree requirements specified in Proposition 17.

We utilize valid inequalities in order to strengthen the master problem. These inequalities belong to various families, which we describe next.

3.3.1 Benders Cuts

The first family of inequalities we address are *Benders (feasibility) cuts*, which were originally introduced in the context of Benders decomposition [8]. These cuts are frequently utilized in the solution of large MILPs, and, in particular, they have been applied extensively to network design problems [19].

In order to generate a violated Benders cut for the master problem, we must first identify a scenario under which $\tilde{x}$ is infeasible. To this end, we formulate the following network interdiction problem

$$\max_{d\in D} \min_{y,z} \sum_{k\in K} z^k \tag{3.3a}$$

$$\text{s.t.} \sum_{(i,j)\in\delta^+(i)} y_{ij}^k - \sum_{(j,i)\in\delta^-(i)} y_{ji}^k = \begin{cases} b^k - z^k & i = s^k \\ -b^k + z^k & i = t^k \\ 0 & \text{otherwise} \end{cases} \qquad \forall i \in N, k \in K, \tag{3.3b}$$

$$\sum_{k\in K} y_{ij}^k \leq u_{ij}\tilde{x}_{ij}(1-d_{ij}) \qquad \forall (i,j) \in A, \tag{3.3c}$$

$$z^k \leq b^k \qquad \forall k \in K, \tag{3.3d}$$

$$z^k \geq 0 \qquad \forall k \in K, \tag{3.3e}$$

$$y_{ij}^k \geq 0 \qquad \forall (i,j) \in A, k \in K \tag{3.3f}$$

$$d_{ij} \in \{0,1\} \qquad \forall (i,j) \in A. \tag{3.3g}$$

Note that (3.3) is a bilevel formulation in which the outer problem seeks a destruction scenario $\tilde{d}$ that maximizes the objective of the inner problem. The inner problem, in turn, seeks a flow in $G_{\tilde{x}}$

under scenario d that minimizes the aggregate demand shedding. In order to solve this problem, we apply standard techniques to convert it to a single level MILP. First, if we fix the failure scenario vector d to a constant $\tilde{d}$, the inner minimization problem becomes a linear program. Its dual is the maximization problem

$$\max \sum_{k\in K} b^k(\pi^k_{s^k}-\pi^k_{t^k})+\sum_{(i,j)\in A_{\tilde{x}}} u_{ij}\tilde{x}_{ij}(1-\tilde{d}_{ij})\alpha_{ij} \tag{3.4a}$$

$$\text{s.t.}\ \pi^k_i-\pi^k_j+\alpha_{ij}\le 0 \qquad \forall (i,j)\in A_{\tilde{x}}, k\in K \tag{3.4b}$$

$$\pi^k_{s^k}-\pi^k_{t^k}\le 1 \qquad \forall k\in K \tag{3.4c}$$

$$\alpha_{ij}\le 0 \qquad \forall (i,j)\in A_{\tilde{x}}. \tag{3.4d}$$

We substitute (3.4) for the inner problem of (3.3) and unfix d to obtain the equivalent single level problem

$$\max \sum_{k\in K} b^k(\pi^k_{s^k}-\pi^k_{t^k})+\sum_{(i,j)\in A_{\tilde{x}}} u_{ij}\tilde{x}_{ij}(1-d_{ij})\alpha_{ij} \tag{3.5a}$$

$$\text{s.t.}\ \sum_{(i,j)\in A_{\tilde{x}}} d_{ij}\le\Gamma \tag{3.5b}$$

$$\pi^k_i-\pi^k_j+\alpha_{ij}\le 0 \qquad \forall (i,j)\in A_{\tilde{x}}, k\in K \tag{3.5c}$$

$$\pi^k_{s^k}-\pi^k_{t^k}\le 1 \qquad \forall k\in K \tag{3.5d}$$

$$\alpha_{ij}\le 0 \qquad \forall (i,j)\in A_{\tilde{x}} \tag{3.5e}$$

$$d_{ij}\in\{0,1\} \qquad \forall (i,j)\in A_{\tilde{x}}. \tag{3.5f}$$

Observe that (3.5) is a mixed-integer program with linear constraints. However, there are bilinear terms $-u_{ij}d_{ij}\alpha_{ij}\tilde{x}_{ij}$ in the objective (3.5a). We introduce auxiliary variables v_{ij} and associated linearization constraints (3.6e)-(3.6g) to replace these bilinear terms, producing an MILP we call the *flow separation problem*

$$\max \sum_{k\in K} b^k(\pi^k_{s^k}-\pi^k_{t^k})+\sum_{(i,j)\in A_{\tilde{x}}} u_{ij}v_{ij} \tag{3.6a}$$

$$\text{s.t.}\ \sum_{(i,j)\in A_{\tilde{x}}} d_{ij}\le\Gamma \tag{3.6b}$$

$$\pi^k_i-\pi^k_j+\alpha_{ij}\le 0 \qquad \forall (i,j)\in A_{\tilde{x}}, k\in K \tag{3.6c}$$

$$\pi^k_{s^k}-\pi^k_{t^k}\le 1 \qquad \forall k\in K \tag{3.6d}$$

$$v_{ij}\ge\alpha_{ij}-d_{ij} \qquad \forall (i,j)\in A_{\tilde{x}} \tag{3.6e}$$

$$v_{ij}\ge d_{ij}-\tilde{x}_{ij} \qquad \forall (i,j)\in A_{\tilde{x}} \tag{3.6f}$$

$$v_{ij}\le\alpha_{ij}+d_{ij} \qquad \forall (i,j)\in A_{\tilde{x}} \tag{3.6g}$$

$$\alpha_{ij}\le 0 \qquad \forall (i,j)\in A_{\tilde{x}} \tag{3.6h}$$

$$v_{ij}\le 0 \qquad \forall (i,j)\in A_{\tilde{x}} \tag{3.6i}$$

$$d_{ij}\in\{0,1\} \qquad \forall (i,j)\in A_{\tilde{x}}. \tag{3.6j}$$

Let $\tilde{d}\in D$ be a fixed arc destruction scenario. We define the corresponding *feasibility problem* as follows

$$\min \sum_{k\in K} z^k \tag{3.7a}$$

$$\text{s.t.} \sum_{(i,j)\in\delta^+(i)} y_{ij}^k - \sum_{(j,i)\in\delta^-(i)} y_{ji}^k = \begin{cases} b^k - z^k & i = s^k \\ -b^k + z^k & i = t^k \\ 0 & \text{otherwise} \end{cases} \qquad \forall i \in N, k \in K, \tag{3.7b}$$

$$\sum_{k\in K} y_{ij}^k \le u_{ij}\tilde{x}_{ij}(1-\tilde{d}_{ij}) \qquad \forall (i,j)\in A, \tag{3.7c}$$

$$z^k \le b^k \qquad \forall k \in K, \tag{3.7d}$$

$$z^k \ge 0 \qquad \forall k \in K, \tag{3.7e}$$

$$y_{ij}^k \ge 0 \qquad \forall (i,j)\in A, k \in K. \tag{3.7f}$$

The objective (3.7a) minimizes the total demand shed. If the optimal objective value exceeds Z, then the design $\tilde{x}$ violates (3.1d), and is therefore infeasible.

Now, we associate dual variables π_i^k, α_{ij}, and γ^k with constraints (3.7b)-(3.7d), respectively, to obtain the *dual feasibility problem*:

$$\max \sum_{k\in K} b^k(\pi_{s^k}^k - \pi_{t^k}^k + \gamma^k) + \sum_{(i,j)\in A_{\tilde{x}}} u_{ij}(\tilde{x}_{ij} - \tilde{d}_{ij})\alpha_{ij} \tag{3.8a}$$

$$\text{s.t. } \pi_i^k - \pi_j^k + \alpha_{ij} \le 0 \qquad \forall (i,j)\in A_{\tilde{x}}, k\in K \tag{3.8b}$$

$$\pi_{s^k}^k - \pi_{t^k}^k + \gamma^k \le 1 \qquad \forall k \in K \tag{3.8c}$$

$$\alpha_{ij} \le 0 \qquad \forall (i,j)\in A_{\tilde{x}} \tag{3.8d}$$

$$\gamma^k \le 0 \qquad \forall k \in K. \tag{3.8e}$$

Given an optimal solution $(\tilde{\alpha}, \tilde{\pi}, \tilde{\gamma})$ of (3.8), there is an associated Benders feasibility cut of the form

$$\sum_{k\in K} b^k(\tilde{\pi}_{s^k}^k - \tilde{\pi}_{t^k}^k + \tilde{\gamma}^k) + \sum_{(i,j)\in A_{\tilde{x}}} u_{ij}(1-\tilde{d}_{ij})^+\tilde{\alpha}_{ij}x_{ij} \le Z. \tag{3.9}$$

We note that the family of Benders feasibility cuts (3.9) is necessary and sufficient to describe DRNDP. That is, $\tilde{x}$ is feasible if and only if it is not violated by any Benders cut. However, computational experience suggests that using these cuts alone leads to slow convergence for large instances. Therefore we incorporate additional families of cuts into our algorithm.

3.3.2 Multi-commodity Disjoint Paths Cuts

Two paths in a network are *disjoint* if any arc contained in one path is not contained in the other, and vice versa. Figure 3.1 shows a pair of disjoint paths in a 10 node network. Given a commodity $k \in K$, we will make use of the following condition

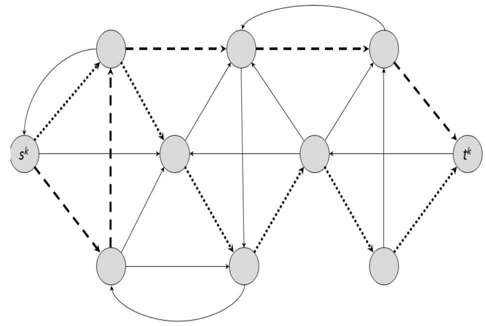

Figure 3.1. Two disjoint paths between a commodity's origin and destination nodes.

Condition 1. There exist at least $\Gamma+1$ disjoint paths from s^k to t^k in $G_{\tilde{x}}$.

Note that if Condition 1 does not hold for some $k \in K$, it is possible to disconnect t^k from s^k by destroying a single arc in each of the disjoint $s^k - t^k$ paths, thereby ensuring no flow of commodity k reaches t^k (see Menger's Theorem [32]). Therefore, in the case that $Z = 0$, Condition 1 must be satisfied by each commodity. However, if $Z > 0$ and $b^k \leq Z$, for some $k \in K$, there may exist feasible network designs in which all of commodity k is shed in some failure scenarios.

Suppose there is a subset $Q \subseteq K$ such that $\sum_{k \in Q} b^k > Z$. We make the following observation.

Observation 1. *If there is a failure scenario in which each $k \in Q$ is completely shed, then $\tilde{x}$ is infeasible.*

This motivates our use of *multi-commodity disjoint paths cuts*. Let $H^1_{\tilde{x}}$ be the graph obtained from $G_{\tilde{x}}$ by assigning each arc $(i,j) \in A_{\tilde{x}}$ a capacity of 1. In a unit capacity network, the number of disjoint paths between a source node and a destination node is equal to the total units of flow that can be sent from the former to the latter. This implies the following proposition.

Proposition 18. *A commodity $k \in K$ satisfies Condition 1 if and only if it is possible to send $\Gamma+1$ units of flow from s^k to t^k in $H^1_{\tilde{x}}$.* □

Consider a multi-commodity flow problem (with demand shedding) defined on $H^1_{\tilde{x}}$, in which each commodity $k \in Q$ has a demand of $\Gamma+1$ units and the commodities $k \in K \setminus Q$ are excluded. We call this the *multi-commodity disjoint paths separation problem* and formulate it as

$$\min \sum_{k \in Q} \zeta^k \tag{3.10a}$$

$$\text{s.t.} \sum_{(i,j)\in\delta^+(i)} p^k_{ij} - \sum_{(j,i)\in\delta^-(i)} p^k_{ji} = \begin{cases} \Gamma+1-\zeta^k & i = s^k \\ -(\Gamma+1)+\zeta^k & i = t^k \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in N, \quad k \in Q \tag{3.10b}$$

$$\sum_{k \in K} p^k_{ij} \leq \tilde{x}_{ij} \qquad \forall (i,j) \in A, \tag{3.10c}$$

$$p_{ij}^k \geq 0 \qquad \forall (i,j) \in A, \quad k \in Q, \tag{3.10d}$$

$$\zeta^k \geq 0 \qquad \forall k \in Q, \tag{3.10e}$$

where the variables p_{ij}^k and ζ^k represent commodity flows and demand shedding, respectively.

Proposition 19. *If $k \in Q$ satisfies Condition 1, then the optimal objective value of* (3.10) *is at most* $(|Q|-1)(\Gamma+1)$.

Proof. It follows from Proposition 18 that there exists a solution $(\tilde{p}, \tilde{\zeta})$ in which all $\Gamma+1$ units of demand for commodity k are sent from s^k to t^k. We assume without loss of generality that $\tilde{\zeta}^k = 0$. Next, we note that in the worst case, the flow given by $\tilde{p}$ fails to send any of the demand associated with commodities $k' \in Q \setminus k$ to their respective destinations. In this case, the constraints (3.10b) and (3.10e) require that $\tilde{\zeta}^{k'} \geq \Gamma+1$ for $k' \in Q \setminus k$. We assume without loss of generality that this relation is satisfied at equality for $k' \in Q \setminus k$, which gives the result. □

The dual of (3.10) is:

$$\max \sum_{k \in Q} (\Gamma+1)(\gamma_{s^k}^k - \gamma_{t^k}^k) + \sum_{(i,j) \in A_{\tilde{x}}} \tilde{x}_{ij}\eta_{ij} \tag{3.11a}$$

$$\text{s.t. } \gamma_i^k - \gamma_j^k + \eta_{ij} \leq 0 \qquad \forall (i,j) \in A, \quad k \in K \tag{3.11b}$$

$$\gamma_{s^k}^k - \gamma_{t^k}^k \leq 1 \qquad \forall k \in Q \tag{3.11c}$$

$$\eta_{ij} \leq 0 \qquad \forall (i,j) \in A. \tag{3.11d}$$

Let $(\tilde{\gamma}, \tilde{\eta})$ be an optimal solution of (3.11). Then we define the following *multi-commodity disjoint paths cut*

$$\sum_{k \in Q} (\Gamma+1)(\tilde{\gamma}_{s^k}^k - \tilde{\gamma}_{t^k}^k) + \sum_{(i,j) \in A} \tilde{\eta}_{ij} x_{ij} \leq (|Q|-1)(\Gamma+1). \tag{3.12}$$

Observation (1) implies that for $\tilde{x}$ to be feasible, at least one commodity in Q must satisfy Condition 1. Proposition 19 shows that (3.12) is necessary for this to occur. In general, (3.12) is not sufficient to guarantee all commodities in Q satisfy Condition 1. However, if $|Q| = 1$, then (3.12) reduces to the k-paths inequality described in [15], which forces Condition 1 to be satisfied for the single commodity in Q.

In order to generate a multi-commodity disjoint paths cut, we first need to identify a subset of commodities whose aggregate demand exceeds the shed threshold. Such a subset corresponds to a cover of the knapsack set

$$\left\{ w \in \{0,1\}^{|K|} \,\middle|\, \sum_{k \in K} b^k w^k \leq Z \right\}.$$

It follows that identifying a subset Q is equivalent to *cover separation*, which is NP-Hard [53]. Therefore, we find minimal covers $Q \subseteq K$ greedily by sorting the demands $\{b^k\}_{k \in K}$ in non-increasing order and grouping successive commodities in the list such that the aggregate demand of each group is greater than Z.

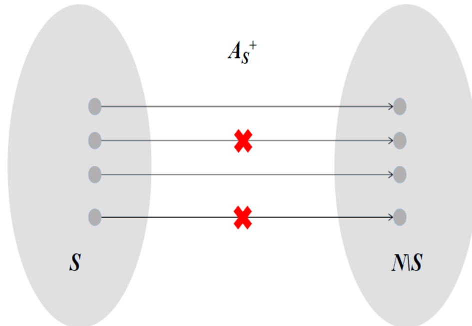

Figure 3.2. A directed cut $(S, N \setminus S)$ and its forward arcs in $G_{\tilde{x}}$ under a particular failure scenario.

3.3.3 Cutset Inequalities

Cutset inequalities are a well known family of strong valid inequalities for capacitated network design problems [16, 40]. These inequalities are obtained from relaxations in which the nodes of a network are partitioned into a nonempty subset $S \subsetneq N$ and its complement $N \setminus S$. For such a subset S, we define the *forward arcs* as $A_S^+ := \{(i,j) \in A \mid i \in S \text{ and } j \in N \setminus S\}$. Similarly, we define the *forward commodities* as $K_S^+ := \{k \in K \mid s^k \in S \text{ and } t^k \in N \setminus S)\}$. In directed MFCND, a cutset inequality takes the form

$$\sum_{(i,j)\in A_S^+} u_{ij}x_{ij} - \sum_{k\in K_S^+} b^k \geq 0. \tag{3.13}$$

We note that cutset inequalities are necessary but not sufficient in MFCND [20]. The validity of (3.13) follows from the fact that, in a feasible network design, the total capacity of the arcs crossing from S to $N \setminus S$ must be large enough to carry those commodities whose origins and destinations lie in S and $N \setminus S$, respectively. Inequality (3.13) can be easily modified to obtain a cutset inequality specific to DRNDP. First, we consider a particular failure scenario $\tilde{d} \in D$. Since arcs that fail in this scenario have zero flow, we exclude them from the summation that calculates the total capacity of forward arcs crossing the cut. Second, we adjust the right-hand side to reflect the fact that up to Z units of demand may be shed from the commodities crossing the cut. This gives the cutset inequality

$$\sum_{(i,j)\in A_S^+|\tilde{d}_{ij}=0} u_{ij}x_{ij} - \sum_{k\in K_S^+} b^k \geq -Z. \tag{3.14}$$

From here on, we use cutset inequality to mean (3.14), and refer to (3.13) as the "standard cutset inequality."

In order to determine if (3.14) is violated for a failure scenario $\tilde{d} \in D$ and a set S, we construct a weighted graph $H_{\tilde{x}}^l = (N, A'_{\tilde{x}})$. We obtain $H_{\tilde{x}}^l$ by including those arcs $(i,j) \in A_{\tilde{x}}$ such that $\tilde{d}_{ij} = 0$, letting the weight of each be its respective capacity u_{ij}, and, for each $k \in K$, adding an artificial arc (s^k, t^k) with weight $-b^k$. Figure 3.3 shows an example of this construction. If the total weight of arcs in A_S^+ in the graph $H_{\tilde{x}}^l$ is less than $-Z$, then (3.14) is violated.

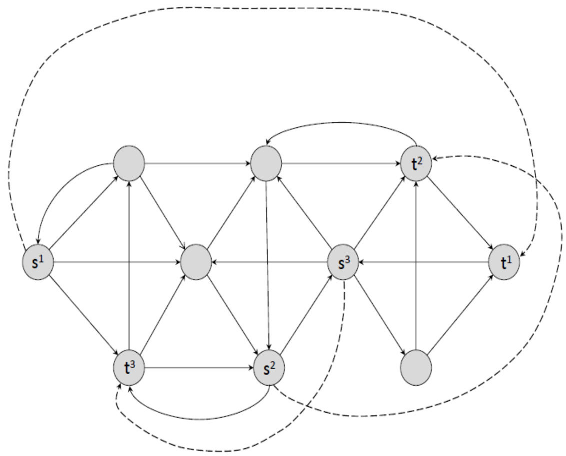

Figure 3.3. The graph $H^l_{\tilde{x}}$ with artificial origin-destination arcs represented by dotted lines.

This observation forms the basis of our cutset inequality separation problem, which we call the *multi-commodity network inhibition problem* (MNIP). MNIP seeks a minimum cut in $H^l_{\tilde{x}}$. For each node $i \in N$, we define a variable ρ_i which equals zero if $i \in S$, and one otherwise. For each arc $(i,j) \in A$, we define a binary variable w_{ij} such that $w_{ij} = 1$ if $(i,j) \in A^+_S$, and zero otherwise. The variable d_{ij} determines whether or not (i,j) is destroyed. Finally, for each $k \in K$ we define a binary variable w^k, which equals one if $k \in K^+_S$, and zero otherwise. We formulate MNIP as the following 0-1 integer linear program

$$\begin{aligned}
\min \quad & \sum_{(i,j)\in A_{\tilde{x}}} u_{ij} w_{ij} - \sum_{k\in K} b^k w^k && && (3.15a)\\
\text{s.t.} \quad & \sum_{(i,j)\in A_{\tilde{x}}} d_{ij} \le \Gamma && && (3.15b)\\
& w_{ij} \ge \rho_j - \rho_i - d_{ij} && \forall (i,j) \in A && (3.15c)\\
& w^k \le \rho_{t^k} && \forall k \in K && (3.15d)\\
& w^k \le 1 - \rho_{s^k} && \forall k \in K && (3.15e)\\
& \rho_i \in \{0,1\} && \forall i \in N && (3.15f)\\
& w^k \in \{0,1\} && \forall k \in K && (3.15g)\\
& w_{ij},\ d_{ij} \in \{0,1\} && \forall (i,j) \in A. && (3.15h)
\end{aligned}$$

The objective (3.15a) minimizes the weight of arcs crossing the cut. Note that only those arcs constructed in the candidate solution $\tilde{x}$ are considered in the objective. Constraint (3.15b) requires that the number of arcs destroyed be at most Γ. Constraint (3.15c) ensures that an arc $(i,j) \in A_{\tilde{x}}$ is counted as belonging to A^+_S ($w_{ij} = 1$) if $i \in S$ ($\rho_i = 0$), $j \in N \setminus S$ ($\rho_j = 1$), and (i,j) is not destroyed ($d_{ij} = 0$). Constraints (3.15d) and (3.15e) ensure that, for $k \in K$, we can have $k \in K^+_S$ ($w^k = 1$) only if $s^k \in S$ ($\rho_{s^k} = 0$) and $t^k \in N \setminus S$ ($\rho_{t^k} = 1$). Since the coefficient of w^k in (3.15a) is negative, an optimal solution will set $w^k = 1$ whenever this condition is met.

Let $(\tilde{\rho}, \tilde{w}, \tilde{d})$ be an optimal solution of MNIP and ν^* the optimal objective value. Then, if

$\nu^* < -Z$, setting $S := \{i \in N \mid \tilde{\rho}_i = 0\}$ gives a violated cutset inequality (3.14).

3.3.4 Compact Cutset Inequalities

In MFCND, a standard cutset inequality ensures there is sufficient capacity to carry commodities across a cut. However, in the case of DRNDP, a cutset inequality only applies to a single failure scenario, and therefore several inequalities may be required to achieve the same result. The following example illustrates this point.

Example 7. *Consider an instance of* DRNDP in which $\Gamma = 1$. Suppose G contains a set of nodes S with n outgoing arcs, where $n > \Gamma$, all of which have the same capacity u. Further, suppose that $(n-2)u < \sum_{k \in K_S^+} b^k - Z \leq (n-1)u$. We see that n cutset inequalities, one corresponding to the failure of each forward arc, will guarantee there is enough capacity crossing the cut. Also, any strict subset of these n inequalities will leave some scenario unaccounted for. □

As Example 7 shows, during the execution of BPC, several cutset inequalities associated with the same directed cut, but different failure scenarios. In order to avoid revisiting the same cutset, we use the *compact cutset inequalities* described in [15]. These inequalities are derived as follows. First, note that solving the following problem will identify a failure scenario that minimizes the capacity of forward arcs in the cutset:

$$\max \quad \sum_{(i,j) \in A_S^+} u_{ij} \tilde{x}_{ij} d_{ij} \tag{3.16a}$$

$$\text{s.t.} \quad \sum_{ij \in A_S^+} d_{ij} \leq \Gamma, \tag{3.16b}$$

$$d_{ij} \leq 1 \qquad \forall (i,j) \in A_S^+ \tag{3.16c}$$

$$d_{ij} \geq 0 \qquad \forall (i,j) \in A_S^+ \tag{3.16d}$$

$$d_{ij} \in \mathbb{Z} \quad \forall (i,j) \in A_S^+. \tag{3.16e}$$

Since each arc has unit weight in (3.16b), we obtain an optimal solution to (3.16) by setting $d_{ij} = 1$ for arcs in A_S^+, in order of non-increasing capacity, until we have chosen Γ arcs, and then setting $d_{ij} = 0$ for the remaining arcs. Further, since (3.16) can be viewed as a 0-1 knapsack problem, this greedy solution is also optimal to its linear relaxation [33]. Therefore, relaxing (3.16e) does not change the optimal objective value of (3.16). We associate dual variables λ^s and μ_{ij}^s with constraints (3.16b) and (3.16c), respectively. (We include the s superscripts because these variables are defined specifically for the set S.) The dual of the linear relaxation of (3.16) is then

$$\min \quad \Gamma \lambda^s + \sum_{ij \in A_S^+} \mu_{ij}^s \tag{3.17a}$$

$$\text{s.t.} \quad \lambda^s + \mu_{ij}^s \geq u_{ij} \tilde{x}_{ij} \qquad \forall (i,j) \in A_S^+, \tag{3.17b}$$

$$\mu_{ij}^s \geq 0 \qquad \forall (i,j) \in A_S^+ \tag{3.17c}$$

$$\lambda^s \geq 0. \tag{3.17d}$$

Now, we define the compact cutset inequalities associated with S as

$$\sum_{(i,j)\in A_S^+} u_{ij}x_{ij} - \Gamma\lambda^s - \sum_{(i,j)\in A_S^+} \mu_{ij}^s \geq \sum_{k\in K_S^+} b^k - Z \tag{3.18a}$$

$$\lambda^s + \mu_{ij}^s \geq u_{ij}x_{ij} \qquad \forall (i,j) \in A_S^+ \tag{3.18b}$$

$$\mu_{ij}^s \geq 0 \qquad \forall (i,j) \in A_S^+ \tag{3.18c}$$

$$\lambda^s \geq 0. \tag{3.18d}$$

The variables and constraints in (3.18) guarantee that there is sufficient capacity crossing the directed cut $(S, N\setminus S)$ under all failure scenarios of size Γ. Note, however, that the variables λ^s and μ_{ij}^s are associated with individual directed cuts in G, of which there are exponentially many. This makes it impractical to include all of these variables *a priori*, and therefore, we introduce them into master problem as needed.

3.4 Computational Experiments

We consider three variants of the algorithm: (i) BC1, in which only Benders feasibility cuts are applied, (ii) BC2, which utilizes all cuts except compact cutset inequalities, and (iii) BCP, which uses all cuts. BC1 and BC2 are branch-and-cut algorithms. In BC2, cuts are prioritized in order of multi-commodity disjoint paths, cutset inequalities, and Benders cuts. That is, first the algorithm seeks violated multi-commodity disjoint paths cuts, and if none are found, cutset inequalities, and so forth. BCP uses the same prioritization, but with compact cutset inequalities following multi-commodity disjoint paths. When any of the algorithm terminates with a solution $\tilde{x}$, we verify its feasibility by solving the dual feasiblity problem (3.8), using $\tilde{x}$ as input. Recall that the objective (3.8a) computes the maximum aggregate demand shed under all potential failure scenarios $\tilde{d} \in D$. Therefore, it is sufficient to check that the optimal objective value is less than Z.

We implement the algorithms in C++ using CPLEX 12.3 via ILOG Concert Technology. We impose a run time limit of 2 hours and set the MIP optimality gap parameter to 0.1%. Our experiments are performed on a PC with a 2.50GHz Intel Core Processor and 6 GB RAM.

We note that CPLEX does not permit the creation of new decision variables once the branch-and-cut process has begun. Therefore, in order to generate compact cutset inequalities in BCP, we create enough λ^s and μ_{ij}^s variables to accommodate a fixed number of sets of compact cutset inequalities. In our computational experiments, we set that number to $|K|$ for each problem instance. When MNIP is solved for the i-th time, identifying a set S, if $i \leq |K|$, BCP uses λ^i and $\{\mu_1^i, \ldots, \mu_{|A|}^i\}$ to generate compact cutset inequalities associated with S. Once the limit has been exceeded, BCP generates cutset inequalities instead.

We evaluate the algorithms on a set of random test instances generated using a code from Smith et al. [45]. They fall into four groups, consisting of five instances each. Table 3.1 shows the number of nodes, commodities, and minimum and maximum number of arcs in each group.

Table 3.1. Instance characteristics

$\|N\|$	$\|K\|$	$\|A\|$	
		Min.	Max.
10	10	48	54
10	20	54	61
15	10	101	106
15	20	103	117

For brevity, we refer to a particular subgroup of instances by the ordered pair $(|N|,|K|)$. We apply the three algorithms to each instance for $\Gamma \in \{1,2,3\}$ and Z set to 0, 5% of total demand, and 10% of total demand, respectively.

First, we note that in several cases, one or more of the algorithms terminate prematurely, either due to reaching the 2 hour time limit or exceeding available memory. We record these in Table 3.2. Each row corresponds to a particular instance group/parameter combination, as indicated by the columns marked $|N|$, $|K|$, Γ, and Z. We omit rows corresponding to those settings for which all three algorithms solved the instances to optimality. For BC1, BC2, and BCP, there are columns showing the number of instances in which the respective algorithm runs out of memory ("OM"). When one of the algorithms reaches the time limit, we compute the optimality gap for its solution. This requires knowledge of the optimal objective value, which we determine by either referring to the solution obtained by an algorithm that succeeds in solving the instance in less than 2 hours, or, if this does not occur, running BC2 with no time limit to determine the optimal solution. For BC1 and BC2, the columns labeled "> 2 hrs." and "Avg. Gap" respectively show the number of instances in which the algorithm reaches the time limit and the average optimality gap for those instances. All unsuccessful BCP runs are due to insufficient memory, so we do not include these last two columns.

Table 3.3 shows average performance statistics for BC1. As in the previous table, rows are associated with specific instance group/parameter combinations. The columns labeled "B&B," "Benders," and "Time" display the average number of nodes in the branch-and-bound tree, the average number of Benders cuts, and the average wall clock time required by the algorithm. The results for BC2 are summarized in Table 3.4, which, in addition to the columns contained in Table 3.3, also has columns showing the average number of multi-commodity disjoint paths cuts ("MC-DP"), and cutset inequalities ("CS") generated by the algorithm. Finally, Table 3.5 shows the results for BCP, using the same columns as the previous table, as well as a column corresponding to the average number of compact cutset inequalities ("C-CS"). The entries in each table are calculated only from those instances which solve to optimality within the 2 hour time limit. We note that all three algorithms fail due to insufficient memory for the (15,20) instances with 10% demand shedding, for each value of Γ, and therefore we omit those rows from the tables.

Note that BC2 solves the most instances successfully, with BPC coming second and BC1 last. In fact, BC2 and BPC consistently require less time and fewer branch-and-bound nodes than BC1, and the difference becomes more dramatic as the size of the instances increases. This demonstrates the computational advantage of supplementing Benders cuts with additional families of valid inequalities. We also see that, in most cases, larger values of Z are associated with longer solution

Table 3.2. Instances in which the algorithms terminate prematurely.

				BC1			BC2			BCP
$\|N\|$	$\|K\|$	Γ	Z	OM	>2 hrs.	Avg. Gap	OM	>2 hrs.	Avg. Gap	OM
10	20	1	0.1	3	1	1.53%	0	0		1
		2	0.05	0	0		0	0		0
			0.1	5	0		0	0		1
		3	0.1	1	0		0	0		1
15	10	1	0.05	3	0		0	0		0
			0.1	4	0		0	0		0
		2	0.05	2	0		0	0		0
			0.1	4	1	0.63%	0	0		0
		3	0.05	1	0		0	0		0
			0.1	2	1	1.36%	0	0		0
	20	1	0	1	2	2.07%	0	0		0
			0.05	4	0		0	3	1.39%	2
			0.1	5	0		5	0		5
		2	0	0	2	2.33%	0	0		0
			0.05	4	0		0	1		3
			0.1	5	0		5	0		5
		3	0	0	1	1.43%	0	0		0
			0.05	3	0		0	1	0.52%	2
			0.1	5	0		5	0		5
Total:				52	8		15	5		25

times. This is not surprising, since allowing more demand shedding leads to a larger set of feasible network designs.

Now, we turn our attention specifically to BC2 and BPC. Table 3.6 compares the average performance of these algorithms for those instances that are solved by both within the time limit. Next to the columns showing the average number of branch-and-bound nodes and solution time for each algorithm, we include columns ("% BC2" and "% BPC", respectively) showing these as a percentage of the number of nodes (respectively, solution time) used by the algorithm. We see that, in most cases, BPC takes longer to solve these instances. Also, as Table 3.2 shows, the algorithm runs out of memory more frequently than BC2. We attribute both these behaviors to the additional overhead required by the λ^i and $\{\mu_1^i, \ldots, \mu_{|A|}^i\}$ variables associated with compact cutset inequalities. However, we see that there are problem classes for which BPC tends to explore fewer branch-and-bound nodes than BC2. In particular, this occurs for many of the 20 commodity instances. Recall that BPC adds at most $|K|$ sets of compact cutset inequalities. It seems that allowing a larger number of compact cutset inequalities can significantly reduce the size of the branch-and-bound tree as compared to using only (noncompact) cutset inequalities.

Considering its relative speed and lower incidence of memory issues, BC2 seems to be the most effective of our algorithms. BPC, despite its notable reduction of the size of the branch-and-bound tree in several instances, is hindered by its need to create a fixed number of compact cutset variables at the outset. However, we suspect that an implementation of BPC in which compact cutset variables are generated in a truly dynamic fashion could overcome these limitations and potentially outperform BC2.

Table 3.3. BC1 Results

$\|N\|$	$\|K\|$	Γ	Z	B&B	Benders	Time
10	10	1	0	5047.60	105.60	1.44
			0.05	9940.80	151.80	2.36
			0.1	174071.40	668.00	53.34
		2	0	2321.20	46.60	1.10
			0.05	7859.20	95.20	2.62
			0.1	148593.60	705.60	45.68
		3	0	101.60	15.00	1.39
			0.05	128.20	20.80	2.29
			0.1	28535.00	357.60	14.53
	20	1	0	1947.60	49.40	1.23
			0.05	176144.20	368.00	37.77
			0.1	2265674.00	3218.00	2323.82
		2	0	941.80	26.80	2.83
			0.05	104015.20	250.00	23.47
		3	0	173.00	17.60	13.17
			0.05	3757.80	108.20	18.88
			0.1	743022.33	1715.00	377.08
15	10	1	0	2361710.80	549.00	1224.78
			0.05	371414.33	1282.67	250.90
			0.1	106801.00	870.00	47.49
		2	0	783021.00	352.20	251.89
			0.05	223873.33	799.33	83.94
		3	0	226992.20	183.40	58.53
			0.05	870719.00	813.00	482.60
			0.1	2137369.00	1840.50	1156.09
	20	1	0	5982139.00	486.50	1962.16
			0.05	783442.00	682.00	218.52
		2	0	2099730.67	280.33	385.69
			0.05	60449.00	197.00	24.02
		3	0	1032940.00	266.00	467.06
			0.05	18445.00	107.00	44.45

Table 3.4. BC2 Results

$\|N\|$	$\|K\|$	Γ	Z	B&B	MC-DP	CS	Benders	Time
10	10	1	0	14.20	21.00	6.20	0.40	0.15
			0.05	6.80	18.20	4.60	0.20	0.11
			0.1	2259.40	43.40	35.80	3.20	0.77
		2	0	1.20	14.00	3.20	0.00	0.14
			0.05	1.20	16.20	4.40	0.00	0.14
			0.1	1895.60	42.20	49.20	6.20	1.15
		3	0	0.00	6.60	1.20	0.00	0.79
			0.05	0.00	9.20	2.40	0.00	0.39
			0.1	1123.00	21.40	48.00	2.00	1.13
	20	1	0	55.00	13.20	8.40	0.40	0.63
			0.05	5898.00	35.00	41.80	8.00	3.01
			0.1	1514802.20	102.60	448.60	81.40	415.62
		2	0	32.20	9.00	6.80	0.00	0.91
			0.05	2342.80	21.20	37.60	5.20	2.66
			0.1	2198664.00	105.00	647.50	72.75	769.18
		3	0	7.40	4.20	4.60	0.00	4.44
			0.05	232.00	15.20	33.00	3.00	3.41
			0.1	634859.50	56.25	617.75	70.25	177.82
15	10	1	0	422.20	67.40	29.60	1.60	1.14
			0.05	292.00	63.60	24.00	2.20	0.65
			0.1	7480.40	115.40	91.00	16.60	3.83
		2	0	1187.20	44.40	27.60	0.40	1.89
			0.05	795.00	40.00	16.20	0.20	0.69
			0.1	5207.60	82.20	60.20	14.60	3.19
		3	0	1411.00	36.20	25.80	0.00	5.08
			0.05	571.00	46.40	29.60	0.60	2.35
			0.1	3738.00	94.60	97.40	7.00	4.31
	20	1	0	359250.40	134.00	182.60	21.60	131.60
			0.05	137497.00	304.00	390.00	138.50	157.50
		2	0	132163.60	73.00	105.00	0.60	40.61
			0.05	3863079.50	218.50	433.00	40.00	1431.51
		3	0	105336.60	63.20	123.20	0.00	98.07
			0.05	1850235.00	181.00	456.00	32.75	808.70

Table 3.5. BPC Results

$\lvert N\rvert$	$\lvert K\rvert$	Γ	Z	B&B	MC-DP	CS	C-CS	Benders	Time
10	10	1	0	23.60	20.40	0.40	62.80	0.00	0.13
			0.05	18.40	19.00	0.40	62.60	0.20	0.15
			0.1	2000.20	41.60	21.00	116.00	5.20	0.94
		2	0	13.20	15.60	0.00	48.20	0.00	0.24
			0.05	8.40	17.60	0.00	52.60	0.00	0.19
			0.1	2202.60	37.80	27.80	100.80	5.40	1.30
		3	0	1.60	6.20	0.00	14.40	0.00	0.77
			0.05	12.80	8.20	0.00	31.60	0.00	0.46
			0.1	1485.60	17.40	17.40	66.60	2.60	1.05
	20	1	0	314.40	14.20	0.00	108.60	1.40	0.94
			0.05	2362.60	31.20	11.40	184.60	4.80	2.14
			0.1	579926.50	83.75	317.50	287.25	72.50	324.63
		2	0	29.80	7.00	0.00	20.80	0.00	0.61
			0.05	7556.40	16.40	0.00	150.80	6.80	4.21
			0.1	835152.00	60.33	380.00	281.00	106.67	516.70
		3	0	20.40	3.80	0.00	15.80	0.00	5.13
			0.05	294.20	11.80	0.00	53.60	3.80	3.59
			0.1	384744.75	28.00	217.50	264.25	57.50	193.98
15	10	1	0	448.80	59.40	15.00	222.40	3.80	1.30
			0.05	313.60	66.20	13.40	226.20	2.40	0.74
			0.1	4942.60	121.40	95.20	245.40	26.20	5.42
		2	0	966.20	39.80	13.40	141.80	0.20	1.65
			0.05	794.40	38.40	8.00	189.40	0.20	0.87
			0.1	4191.20	87.20	59.00	241.20	11.80	3.87
		3	0	450.20	25.60	4.00	109.20	0.00	3.56
			0.05	375.80	40.20	11.40	136.60	0.20	1.74
			0.1	4051.40	85.00	73.40	234.20	8.80	5.49
	20	1	0	162394.40	123.60	104.00	419.60	16.80	162.23
			0.05	185161.50	314.50	376.50	452.50	109.00	379.67
		2	0	63756.00	59.40	11.40	284.20	1.80	53.56
			0.05	424406.00	212.50	235.50	258.00	32.50	695.00
		3	0	109786.80	37.60	2.80	236.60	0.00	153.26
			0.05	245548.50	60.00	156.00	230.50	46.00	272.20

Table 3.6. Relative performance of BC2 and BPC

				BC2				BPC			
$\|N\|$	$\|K\|$	Γ	Z	B&B	% BPC	Time	% BPC	B&B	% BC2	Time	% BC2
10	10	1	0	14.20	60%	0.15	112%	23.60	166.20%	0.13	89%
			0.05	6.80	37%	0.11	73%	18.40	270.59%	0.15	137%
			0.1	2259.40	113%	0.77	82%	2000.20	88.53%	0.94	122%
		2	0	1.20	9%	0.14	58%	13.20	1100.00%	0.24	172%
			0.05	1.20	14%	0.14	76%	8.40	700.00%	0.19	131%
			0.1	1895.60	86%	1.15	88%	2202.60	116.20%	1.30	113%
		3	0	0.00	0%	0.79	102%	1.60	n/a	0.77	98%
			0.05	0.00	0%	0.39	84%	12.80	n/a	0.46	118%
			0.1	1123.00	76%	1.13	108%	1485.60	132.29%	1.05	93%
	20	1	0	55.00	17%	0.63	66%	314.40	571.64%	0.94	150%
			0.05	5898.00	250%	3.01	141%	2362.60	40.06%	2.14	71%
			0.1	811132.00	140%	209.58	65%	579926.50	71.50%	324.63	155%
		2	0	32.20	108%	0.91	149%	29.80	92.55%	0.61	67%
			0.05	2342.80	31%	2.66	63%	7556.40	322.54%	4.21	158%
			0.1	1072014.33	128%	280.01	54%	835152.00	77.90%	516.70	185%
		3	0	7.40	36%	4.44	86%	20.40	275.68%	5.13	116%
			0.05	232.00	79%	3.41	95%	294.20	126.81%	3.59	105%
			0.1	634859.50	165%	177.82	92%	384744.75	60.60%	193.98	109%
15	10	1	0	422.20	94%	1.14	88%	448.80	106.30%	1.30	114%
			0.05	292.00	93%	0.65	88%	313.60	107.40%	0.74	113%
			0.1	7480.40	151%	3.83	71%	4942.60	66.07%	5.42	142%
		2	0	1187.20	123%	1.89	115%	966.20	81.38%	1.65	87%
			0.05	795.00	100%	0.69	79%	794.40	99.92%	0.87	127%
			0.1	5207.60	124%	3.19	82%	4191.20	80.48%	3.87	121%
		3	0	1411.00	313%	5.08	143%	450.20	31.91%	3.56	70%
			0.05	571.00	152%	2.35	135%	375.80	65.81%	1.74	74%
			0.1	3738.00	92%	4.31	79%	4051.40	108.38%	5.49	127%
	20	1	0	359250.40	221%	131.60	81%	162394.40	45.20%	162.23	123%
			0.05	137497.00	74%	157.50	41%	185161.50	134.67%	379.67	241%
		2	0	132163.60	207%	40.61	76%	63756.00	48.24%	53.56	132%
			0.05	434721.00	102%	273.81	39%	424406.00	97.63%	695.00	254%
		3	0	31369.50	90%	82.81	92%	34910.50	111.29%	90.41	109%
			0.05	2202191.00	194%	672.52	71.65%	1137715.50	51.66%	938.57	139.56%

Chapter 4

CONCLUSION

With this dissertation, we have sought to better understand the structure of network design problems and to design new methodologies for their solution. In Chapter 2, we studied the polyhedral structure of MVF, a multi-commodity extension of variable upper bound flow models. We described basic features of the MVF polytope and considered relaxations obtained through commodity aggregation. We presented conditions under which facets of these relaxed models disaggregate into facets of the original model. We identified hierarchical flow cover inequalities. We showed that these inequalities, which are tied to ordered structures we call arc-commodity hierarchies, generalize flow cover inequalities to a multi-commodity setting. Our computational tests suggest that HFC inequalities provide an advantage over flow cover inequalities obtained through aggregation alone.

In Chapter 3, we discussed DRNDP, a model motivated by the need to incorporate resilience in the design of critical infrastructure. We devised a branch-and-cut algorithm that utilizes multiple families of cuts to solve the problem. We evaluated the performance of this algorithm on a collection of randomly generated instances. We believe the DRNDP model, in conjunction with our solution algorithm, can provide a powerful framework for assessing and improving the resilience of large scale infrastructure systems.

This work can serve as the foundation for multiple avenues of future research. One possible direction is the continued study of valid inequalities for network design that are based on arc-commodity hierarchies. This would include the identification of families of valid inequalities beyond those described in Chapter 2, as well as additional research into lifting techniques and separation procedures for these inequalities. As for our work in resilient network design, we are interested to see how DRNDP performs on practical data. To this end, we have constructed a large instance based on the highway network in Florida, using the the Oak Ridge National Laboratory CTA Transportation Networks [39] and the US Commodity Flow survey [36] as sources. We will attempt to solve this instance using a mainframe computer located at Sandia National Laboratories. In addition, we think there is potential for further development of our DRNDP solution algorithm. This might include implementing the algorithm in a framework that allows both columns and cuts to be generated dynamically, which could significantly reduce the memory overhead associated with the additional variables used by compact cutset inequalities. It would also be worthwhile to explore ways of solving the multi-commodity network inhibition and flow separation problems more efficiently.

APPENDIX A: MCFND COMPUTATIONAL RESULTS

Instance	$\|V\|$	$\|A\|$	$\|K\|$	Algorithm	B&B Nodes	STC	MTC	Time
LM1	15	124	12	BB	Out of Memory			
				AFC	35,683,497	50		10807.00
				HFC	35,497,417	50	0	10807.30
LM2	15	82	11	BB	850,198			133.06
				AFC	161,457	46		69.60
				HFC	119,079	50	0	27.42
LM3	15	104	12	BB	47,491,586			10806.80
				AFC	14,792,470	50		4496.19
				HFC	5,740,471	43	7	1773.77
LM4	15	105	12	BB	Out of Memory			
				AFC	33,919,085	50		10806.70
				HFC	22,741,099	50	0	6908.67
LM5	15	54	18	BB	7,012,430			1241.83
				AFC	325,177	50		77.59
				HFC	155,056	49	1	38.87
LM6	15	122	10	BB	45,806,604			10807.20
				AFC	11,207,194	50		3220.22
				HFC	8,486,065	50	0	2254.52
LM7	15	99	11	BB	5,114,697			961.39
				AFC	383,886	50		104.00
				HFC	388,306	50	0	94.67
LM8	15	94	10	BB	14,505,178			2599.66
				AFC	433,983	50		100.29
				HFC	334,744	50	0	75.55
LM9	15	112	11	BB	38,103,545			8123.30
				AFC	6,945,917	50		1919.61
				HFC	5,943,382	41	9	1602.18
LM10	15	118	12	BB	41,855,758			10807.20
				AFC	6,186,028	50		1704.18
				HFC	2,054,181	42	8	696.25
ML1	15	45	16	BB	1,582			0.36
				AFC	1,051	7		0.47
				HFC	1,026	8	0	0.52
ML2	15	47	18	BB	3,272			0.84
				AFC	2,131	5		1.00
				HFC	1,269	16	0	0.66
ML3	15	52	14	BB	170,180			26.12
				AFC	56,306	50		10.84
				HFC	30,373	50	0	11.00
ML4	15	52	14	BB	5,125			0.96
				AFC	1,582	20		0.74
				HFC	1,288	37	13	0.74

Instance	$\|V\|$	$\|A\|$	$\|K\|$	Algorithm	B&B Nodes	STC	MTC	Time
ML5	15	54	18	BB	824,623			149.99
				AFC	352,532	28		174.86
				HFC	224,058	50	0	61.87
ML6	15	47	17	BB	688			0.17
				AFC	587	11		0.29
				HFC	316	12	0	0.18
ML7	15	47	15	BB	13,184			2.38
				AFC	6,953	16		2.77
				HFC	4,831	50	0	1.39
ML8	15	49	17	BB	21,458			4.69
				AFC	18,601	23		8.98
				HFC	8,744	41	9	2.81
ML9	15	52	18	BB	1,521,978			268.05
				AFC	366,755	50		82.05
				HFC	152,825	46	4	37.17
ML10	15	44	14	BB	26,875			4.26
				AFC	16,532	17		5.72
				HFC	8,367	26	0	3.10
MM1	15	59	10	BB	28,989			3.85
				AFC	3,405	20		1.12
				HFC	1,366	27	0	0.52
MM2	15	41	11	BB	690			0.11
				AFC	228	8		0.09
				HFC	172	9	1	0.07
MM3	15	54	12	BB	45,344			5.85
				AFC	8,226	25		2.85
				HFC	8,073	23	0	2.70
MM4	15	50	12	BB	4,779			0.76
				AFC	3,942	18		1.30
				HFC	3,914	19	0	1.35
MM5	15	47	8	BB	11,233			1.37
				AFC	3,935	17		1.06
				HFC	3,252	23	0	0.87
MM6	15	57	10	BB	509,651			65.01
				AFC	100,031	50		15.96
				HFC	34,778	46	0	11.53
MM7	15	54	8	BB	1,538			0.24
				AFC	447	14		0.15
				HFC	426	18	0	0.14
MM8	15	55	10	BB	527,196			68.50
				AFC	160,619	25		47.38
				HFC	130,372	50	0	24.18
MM9	15	41	11	BB	5,309			0.72
				AFC	2,361	13		0.67
				HFC	1,593	46	4	0.31
MM10	15	51	9	BB	10,406			1.42
				AFC	7,151	15		1.98
				HFC	7,729	18	0	2.11

APPENDIX B: LINEAR DESCRIPTION OF MODEL 1

$$-y_1^1 \leq 0 \tag{1.1}$$
$$-y_1^2 \leq 0 \tag{1.2}$$
$$-y_1^3 \leq 0 \tag{1.3}$$
$$-y_2^1 \leq 0 \tag{1.4}$$
$$-y_2^2 \leq 0 \tag{1.5}$$
$$-y_2^3 \leq 0 \tag{1.6}$$
$$-y_3^1 \leq 0 \tag{1.7}$$
$$-y_3^3 \leq 0 \tag{1.8}$$
$$-x_3 + 1/9y_3^2 \leq 0 \tag{1.9}$$
$$-y_3^2 \leq 0 \tag{1.10}$$
$$-5x_3 + \frac{1}{3}y_3^1 + \frac{1}{3}y_3^2 + \frac{1}{3}y_3^3 \leq 0 \tag{1.11}$$
$$-7x_3 + 1y_3^1 \leq 0 \tag{1.12}$$
$$-7x_2 + 1y_2^1 \leq 0 \tag{1.13}$$
$$-x_3 + 1/11y_3^3 \leq 0 \tag{1.14}$$
$$-13x_2 + 1y_2^1 + 1y_2^2 + 1y_2^3 \leq 0 \tag{1.15}$$
$$-9x_2 + 1y_2^2 \leq 0 \tag{1.16}$$
$$-x_2 + 1/11y_2^3 \leq 0 \tag{1.17}$$
$$-8x_1 + 1y_1^1 + 1y_1^2 + 1y_1^3 \leq 0 \tag{1.18}$$
$$-x_1 + 1/7y_1^1 \leq 0 \tag{1.19}$$
$$+1/7y_1^1 + 1/7y_2^1 + 1/7y_3^1 \leq 1 \tag{1.20}$$
$$+1x_3 \leq 1 \tag{1.21}$$
$$+1x_1 \leq 1 \tag{1.22}$$
$$+1x_2 \leq 1 \tag{1.23}$$
$$-12x_2 - 14x_3 + 1y_2^1 + 1y_2^2 + 1y_2^3 + 1y_3^1 + 1y_3^2 + 1y_3^3 \leq 1 \tag{1.24}$$
$$+1/11y_1^3 + 1/11y_2^3 + 1/11y_3^3 \leq 1 \tag{1.25}$$
$$-5/6x_2 - 3/2x_3 + \frac{1}{6}y_2^3 + \frac{1}{6}y_3^2 + \frac{1}{6}y_3^3 \leq 1 \tag{1.26}$$
$$-7/6x_2 - 5/6x_3 + \frac{1}{6}y_2^1 + \frac{1}{6}y_2^3 + \frac{1}{6}y_3^3 \leq 1 \tag{1.27}$$
$$-3/8x_2 - 3/8x_3 + \frac{1}{8}y_1^3 + \frac{1}{8}y_2^3 + \frac{1}{8}y_3^3 \leq 1 \tag{1.28}$$
$$-9/4x_2 - 7/4x_3 + \frac{1}{4}y_2^2 + \frac{1}{4}y_2^3 + \frac{1}{4}y_3^3 \leq 1 \tag{1.29}$$
$$+1/9y_1^2 + 1/9y_2^2 + 1/9y_3^2 \leq 1 \tag{1.30}$$
$$-5/8x_2 - 7/8x_3 + \frac{1}{8}y_2^2 + \frac{1}{8}y_2^3 + \frac{1}{8}y_3^2 + \frac{1}{8}y_3^3 \leq 1 \tag{1.31}$$
$$-5/4x_2 - 11/4x_3 + \frac{1}{4}y_2^2 + \frac{1}{4}y_3^2 + \frac{1}{4}y_3^3 \leq 1 \tag{1.32}$$
$$-3/8x_2 - 7/8x_3 + \frac{1}{8}y_2^3 + \frac{1}{8}y_3^1 + \frac{1}{8}y_3^3 \leq 1 \tag{1.33}$$
$$-3/8x_2 + \frac{1}{8}y_1^3 + \frac{1}{8}y_2^3 \leq 1 \tag{1.34}$$
$$-3/8x_3 + \frac{1}{8}y_1^3 + \frac{1}{8}y_3^3 \leq 1 \tag{1.35}$$
$$-7/6x_2 - 1/2x_3 + \frac{1}{6}y_2^1 + \frac{1}{6}y_2^2 + \frac{1}{6}y_3^2 \leq 1 \tag{1.36}$$
$$-1/8x_2 - 1/8x_3 + \frac{1}{8}y_1^2 + \frac{1}{8}y_2^2 + \frac{1}{8}y_3^2 \leq 1 \tag{1.37}$$
$$-1/4x_1 - 1/4x_2 - 1/4x_3 + \frac{1}{12}y_1^1 + \frac{1}{12}y_1^3 + \frac{1}{12}y_2^1 + \frac{1}{12}y_2^3 + \frac{1}{12}y_3^1 + \frac{1}{12}y_3^3 \leq 1 \tag{1.38}$$
$$-3/5x_1 - 2x_3 + \frac{1}{5}y_1^1 + \frac{1}{5}y_1^3 + \frac{1}{5}y_3^1 + \frac{1}{5}y_3^3 \leq 1 \tag{1.39}$$

$$-15/46x_1-15/46x_2-29/46x_3+3/46y_1^1+5/46y_1^3+3/46y_2^1+5/46y_2^3+5/46y_3^1+5/46y_3^3\leq 1 \tag{1.40}$$

$$-3/8x_1-3/8x_2-7/8x_3+3/56y_1^1+\frac{1}{8}y_1^3+\frac{1}{8}y_2^3+\frac{1}{8}y_3^1+\frac{1}{8}y_3^3\leq 1 \tag{1.41}$$

$$-5/6x_1-5/6x_2-3/2x_3+5/48y_1^2+\frac{1}{6}y_1^3+\frac{1}{6}y_2^3+\frac{1}{6}y_3^2+\frac{1}{6}y_3^3\leq 1 \tag{1.42}$$

$$-1/2x_1-1/2x_2-1/2x_3+\frac{1}{10}y_1^2+\frac{1}{10}y_1^3+\frac{1}{10}y_2^2+\frac{1}{10}y_2^3+\frac{1}{10}y_3^2+\frac{1}{10}y_3^3\leq 1 \tag{1.43}$$

$$-5/7x_1-5/7x_2-8/7x_3+1/7y_1^2+1/7y_1^3+1/7y_2^2+1/7y_3^2+1/7y_3^3\leq 1 \tag{1.44}$$

$$-5/6x_1-7/6x_2-5/6x_3+5/42y_1^1+\frac{1}{6}y_1^3+\frac{1}{6}y_2^1+\frac{1}{6}y_2^3+\frac{1}{6}y_3^3\leq 1 \tag{1.45}$$

$$-5/8x_1-5/8x_2-5/8x_3+\frac{1}{8}y_1^1+\frac{1}{8}y_1^3+\frac{1}{8}y_2^1+\frac{1}{8}y_2^3+\frac{1}{16}y_3^1+\frac{1}{8}y_3^3\leq 1 \tag{1.46}$$

$$-1/2x_1-3/10x_2-1/2x_3+\frac{1}{10}y_1^1+\frac{1}{10}y_1^3+\frac{1}{10}y_2^1+\frac{1}{10}y_2^3+\frac{1}{10}y_3^1+\frac{1}{10}y_3^3\leq 1 \tag{1.47}$$

$$-1/2x_1-1/2x_2-1/2x_3+9/70y_1^1+\frac{1}{10}y_1^3+9/70y_2^1+\frac{1}{10}y_2^3+\frac{1}{10}y_3^1+\frac{1}{10}y_3^3\leq 1 \tag{1.48}$$

$$-3/5x_1-8/5x_2+\frac{1}{5}y_1^1+\frac{1}{5}y_1^2+\frac{1}{5}y_2^1+\frac{1}{5}y_2^2\leq 1 \tag{1.49}$$

$$-1/3x_1-1/3x_2-2/3x_3+1/9y_1^1+1/9y_1^3+1/9y_2^1+1/9y_3^1+1/9y_3^3\leq 1 \tag{1.50}$$

$$-15/38x_1-15/38x_2-37/38x_3+5/38y_1^1+3/38y_1^3+5/38y_2^1+3/38y_2^3+5/38y_3^1+5/38y_3^3\leq 1 \tag{1.51}$$

$$-1/2x_1-2x_2-3/2x_3+\frac{1}{6}y_1^1+\frac{1}{6}y_1^3+\frac{1}{6}y_2^1+\frac{1}{6}y_2^2+\frac{1}{6}y_2^3+\frac{1}{6}y_3^1+\frac{1}{6}y_3^3\leq 1 \tag{1.52}$$

$$-3/10x_2-1/2x_3+\frac{1}{10}y_2^1+\frac{1}{10}y_2^3+\frac{1}{10}y_3^1+\frac{1}{10}y_3^3\leq 1 \tag{1.53}$$

$$-3/4x_2-11/4x_3+\frac{1}{4}y_2^1+\frac{1}{4}y_3^1+\frac{1}{4}y_3^3\leq 1 \tag{1.54}$$

$$-3/4x_1-11/4x_3+\frac{1}{4}y_1^1+\frac{1}{4}y_3^1+\frac{1}{4}y_3^3\leq 1 \tag{1.55}$$

$$-3/4x_1-3/4x_2-11/4x_3+\frac{1}{4}y_1^1+\frac{1}{4}y_2^1+\frac{1}{4}y_3^1+\frac{1}{4}y_3^3\leq 1 \tag{1.56}$$

$$-3/4x_1-33/62x_2-11/4x_3+\frac{1}{4}y_1^1+11/62y_2^1+3/62y_2^3+\frac{1}{4}y_3^1+\frac{1}{4}y_3^3\leq 1 \tag{1.57}$$

$$-3/4x_1-12x_2-14x_3+\frac{1}{4}y_1^1+1y_2^1+1y_2^2+3/4y_2^3+1y_3^1+3/4y_3^2+1y_3^3\leq 1 \tag{1.58}$$

$$-3/4x_1-39/7x_2-53/7x_3+\frac{1}{4}y_1^1+4/7y_2^1+3/7y_2^2+3/7y_2^3+4/7y_3^1+9/28y_3^2+4/7y_3^3\leq 1 \tag{1.59}$$

$$-1/12x_2-1/4x_3+\frac{1}{12}y_2^1+\frac{1}{12}y_2^2+\frac{1}{12}y_3^1+\frac{1}{12}y_3^2\leq 1 \tag{1.60}$$

$$-9/4x_2-3/4x_3+\frac{1}{4}y_2^1+\frac{1}{4}y_2^2+\frac{1}{4}y_3^1\leq 1 \tag{1.61}$$

$$-3/4x_1-9/4x_2-3/2x_3+\frac{1}{4}y_1^1+\frac{1}{4}y_2^1+\frac{1}{4}y_2^2+\frac{1}{6}y_3^1+\frac{1}{6}y_3^2\leq 1 \tag{1.62}$$

$$-3/4x_1-9/4x_2-3/4x_3+\frac{1}{4}y_1^1+\frac{1}{4}y_2^1+\frac{1}{4}y_2^2+\frac{1}{4}y_3^1\leq 1 \tag{1.63}$$

$$-3/4x_1-9/4x_2+\frac{1}{4}y_1^1+\frac{1}{4}y_2^1+\frac{1}{4}y_2^2\leq 1 \tag{1.64}$$

$$-3/4x_1-9/4x_2-9/20x_3+\frac{1}{4}y_1^1+\frac{1}{4}y_2^1+\frac{1}{4}y_2^2+3/20y_3^1+1/20y_3^2\leq 1 \tag{1.65}$$

$$-3/4x_1-9/4x_2-45/16x_3+\frac{1}{4}y_1^1+\frac{1}{4}y_2^1+\frac{1}{4}y_2^2+\frac{1}{4}y_3^1+3/16y_3^2+3/16y_3^3\leq 1 \tag{1.66}$$

$$-3/4x_1-45/7x_2-45/7x_3+\frac{1}{4}y_1^1+4/7y_2^1+4/7y_2^2+9/28y_2^3+4/7y_3^1+3/7y_3^2+3/7y_3^3\leq 1 \tag{1.67}$$

$$-3/4x_1-9/4x_2-441/236x_3+\frac{1}{4}y_1^1+\frac{1}{4}y_2^1+\frac{1}{4}y_2^2+\frac{1}{4}y_3^1+33/236y_3^2+6/59y_3^3\leq 1 \tag{1.68}$$

$$-1/4x_1-1/12x_2-1/4x_3+\frac{1}{12}y_1^1+\frac{1}{12}y_1^2+\frac{1}{12}y_2^1+\frac{1}{12}y_2^2+\frac{1}{12}y_3^1+\frac{1}{12}y_3^2\leq 1 \tag{1.69}$$

$$-3/10x_1-3/10x_2-3/10x_3+\frac{1}{10}y_1^1+\frac{1}{10}y_1^2+\frac{1}{10}y_2^1+\frac{1}{10}y_2^2+1/15y_3^1+\frac{1}{10}y_3^2\leq 1 \tag{1.70}$$

$$-1/14x_1-1/14x_2-1/14x_3+1/14y_1^1+1/14y_1^2+1/14y_2^1+1/14y_2^2+1/14y_3^1+1/14y_3^2\leq 1 \tag{1.71}$$

$$-1/4x_1-1/4x_2-1/4x_3+3/28y_1^1+\frac{1}{12}y_1^2+3/28y_2^1+\frac{1}{12}y_2^2+\frac{1}{12}y_3^1+\frac{1}{12}y_3^2\leq 1 \tag{1.72}$$

$$-1/2x_1-7/6x_2-7/3x_3+\frac{1}{6}y_1^1+\frac{1}{6}y_1^2+\frac{1}{6}y_2^1+\frac{1}{6}y_2^2+\frac{1}{6}y_3^1+\frac{1}{6}y_3^2+\frac{1}{6}y_3^3\leq 1 \tag{1.73}$$

$$-24/35x_1-3/4x_2-11/4x_3+8/35y_1^1+3/35y_1^3+\frac{1}{4}y_2^1+\frac{1}{4}y_3^1+\frac{1}{4}y_3^3\leq 1 \tag{1.74}$$

$$-21/31x_1-21/31x_2-74/31x_3+7/31y_1^1+3/31y_1^3+7/31y_2^1+7/31y_3^1+7/31y_3^3\leq 1 \tag{1.75}$$

$$-15/22x_1-15/22x_2-53/22x_3+5/22y_1^1+1/11y_1^3+5/22y_2^1+1/66y_2^3+5/22y_3^1+5/22y_3^3\leq 1 \tag{1.76}$$

$$-24/35x_1-12/7x_2-26/7x_3+8/35y_1^1+3/35y_1^3+11/35y_2^1+3/35y_2^2+3/35y_2^3+11/35y_3^1+3/35y_3^2+11/35y_3^3\leq 1 \tag{1.77}$$

$$-24/35x_1-9/4x_2-3/4x_3+8/35y_1^1+3/35y_1^2+\frac{1}{4}y_2^1+\frac{1}{4}y_2^2+\frac{1}{4}y_3^1\leq 1 \tag{1.78}$$

$$-24/35x_1-108/35x_2-66/35x_3+8/35y_1^1+3/35y_1^2+11/35y_2^1+11/35y_2^2+3/35y_2^3+11/35y_3^1+3/35y_3^2+3/35y_3^3\leq 1 \tag{1.79}$$

$$-24/35x_1-45/7x_2-45/7x_3+8/35y_1^1+3/35y_1^2+4/7y_2^1+4/7y_2^2+12/35y_2^3+4/7y_3^1+3/7y_3^2+3/7y_3^3\leq 1 \tag{1.80}$$

$$-24/35x_1-39/7x_2-53/7x_3+8/35y_1^1+3/35y_1^3+4/7y_2^1+3/7y_2^2+3/7y_2^3+4/7y_3^1+12/35y_3^2+4/7y_3^3\leq 1 \tag{1.81}$$

$$-5/3x_1-5/6x_2-4x_3+5/24y_1^2+\frac{1}{3}y_1^3+\frac{1}{6}y_2^3+\frac{1}{3}y_3^2+\frac{1}{3}y_3^3\leq 1 \tag{1.82}$$

$$-5/3x_1-4x_3+\frac{1}{3}y_1^2+\frac{1}{3}y_1^3+\frac{1}{3}y_3^2+\frac{1}{3}y_3^3\leq 1 \tag{1.83}$$

$$-5/4x_1-3x_2-11/4x_3+\frac{1}{4}y_1^2+\frac{1}{4}y_1^3+\frac{1}{4}y_2^1+\frac{1}{4}y_2^2+\frac{1}{4}y_2^3+\frac{1}{4}y_3^2+\frac{1}{4}y_3^3\leq 1 \tag{1.84}$$

$$-x_1-3/8x_2-51/20x_3+\frac{1}{8}y_1^2+\frac{1}{4}y_1^3+\frac{1}{8}y_2^3+\frac{1}{5}y_3^2+\frac{1}{4}y_3^3\leq 1 \tag{1.85}$$

$$-3/5x_1-2x_2-2x_3+\frac{1}{5}y_1^1+\frac{1}{5}y_1^3+\frac{1}{5}y_2^1+\frac{1}{5}y_2^3+\frac{1}{5}y_3^1+\frac{1}{5}y_3^3\leq 1 \tag{1.86}$$

$$-3/5x_1-36/25x_2-2x_3+3/25y_1^1+\frac{1}{5}y_1^3+3/25y_2^1+\frac{1}{5}y_2^3+\frac{1}{5}y_3^1+\frac{1}{5}y_3^3\leq 1 \tag{1.87}$$

$$-5/3x_1-10/3x_2+\frac{1}{3}y_1^1+\frac{1}{3}y_1^3+\frac{1}{3}y_2^1+\frac{1}{3}y_2^3\leq 1 \tag{1.88}$$

$$-5/4x_1-9/4x_2-7/2x_3+\frac{1}{4}y_1^1+\frac{1}{4}y_1^3+\frac{1}{4}y_2^1+\frac{1}{4}y_2^3+\frac{1}{4}y_3^1+\frac{1}{4}y_3^2+\frac{1}{4}y_3^3\leq 1 \tag{1.89}$$

$$-5/6x_1-7/6x_2-1/2x_3+\frac{1}{6}y_1^1+\frac{1}{6}y_1^3+\frac{1}{6}y_2^1+\frac{1}{6}y_2^3+\frac{1}{10}y_3^1+\frac{1}{10}y_3^3\leq 1 \tag{1.90}$$

$$-5/3x_1-10/3x_2-2x_3+\frac{1}{3}y_1^1+\frac{1}{3}y_1^3+\frac{1}{3}y_2^1+\frac{1}{3}y_2^3+\frac{1}{5}y_3^1+\frac{1}{5}y_3^3\leq 1 \tag{1.91}$$

$$-25/32x_1-33/32x_2-43/32x_3+5/32y_1^1+5/32y_1^3+5/32y_2^1+5/32y_2^3+5/32y_3^1+\frac{1}{16}y_3^2+5/32y_3^3\leq 1 \tag{1.92}$$

$$-x_1-43/20x_2-3/8x_3+\frac{1}{8}y_1^1+\frac{1}{4}y_1^3+\frac{1}{5}y_2^1+\frac{1}{4}y_2^3+\frac{1}{8}y_3^3\leq 1 \tag{1.93}$$

$$-5/3x_1-31/8x_2-33/8x_3+5/24y_1^1+\frac{1}{3}y_1^3+3/8y_2^1+1/24y_2^2+3/8y_2^3+\frac{1}{4}y_3^1+\frac{1}{4}y_3^2+3/8y_3^3\leq 1 \tag{1.94}$$

$$-3/2x_1-243/80x_2-51/16x_3+3/16y_1^1+5/16y_1^3+3/10y_2^1+5/16y_2^3+3/16y_3^1+3/16y_3^2+5/16y_3^3\leq 1 \tag{1.95}$$

$$-15/23x_1-30/23x_2-52/23x_3+5/23y_1^1+3/23y_1^3+5/23y_2^1+3/23y_2^3+5/23y_3^1+5/23y_3^3\leq 1 \tag{1.96}$$

$$-5/3x_1-12x_2-14x_3+\frac{1}{3}y_1^3+1y_2^1+2/3y_2^2+1y_2^3+2/3y_3^1+1y_3^2+1y_3^3\leq 1 \tag{1.97}$$

$$-3/5x_1-36/25x_2-42/25x_3+\frac{1}{5}y_1^3+3/25y_2^1+\frac{1}{5}y_2^3+3/25y_3^2+\frac{1}{5}y_3^3\leq 1 \tag{1.98}$$

$$-5/3x_1-10/3x_2-11/3x_3+2/9y_1^1+\frac{1}{3}y_1^3+\frac{1}{3}y_2^1+\frac{1}{3}y_2^3+2/9y_3^1+2/9y_3^2+\frac{1}{3}y_3^3\leq 1 \tag{1.99}$$

$$-12/19x_1-39/19x_2-41/19x_3+4/19y_1^1+3/19y_1^3+4/19y_2^1+3/19y_2^2+3/19y_2^3+4/19y_3^1+4/19y_3^3\leq 1 \tag{1.100}$$

$$-5/3x_1-29/8x_2-37/8x_3+5/24y_1^2+\frac{1}{3}y_1^3+\frac{1}{4}y_2^1+\frac{1}{4}y_2^2+3/8y_2^3+1/24y_3^1+3/8y_3^2+3/8y_3^3\leq 1 \tag{1.101}$$

$$-3/2x_1-45/16x_2-291/80x_3+3/16y_1^2+5/16y_1^3+3/16y_2^1+3/16y_2^2+5/16y_2^3+3/10y_3^2+5/16y_3^3\leq 1 \tag{1.102}$$

$$-35/52x_1-35/52x_2-53/52x_3+5/52y_1^2+7/52y_1^3+5/52y_2^2+7/52y_2^3+7/52y_3^2+7/52y_3^3\leq 1 \tag{1.103}$$

$$-11/2x_2-7/2x_3+\frac{1}{2}y_2^2+\frac{1}{2}y_2^3+\frac{1}{2}y_3^2\leq 1 \tag{1.104}$$

$$-5/8x_1-57/56x_2-7/8x_3+5/56y_1^2+\frac{1}{8}y_1^3+\frac{1}{8}y_2^2+9/56y_2^3+\frac{1}{8}y_3^2+\frac{1}{8}y_3^3\leq 1 \tag{1.105}$$

$$-35/48x_1-35/48x_2-19/16x_3+7/48y_1^2+5/48y_1^3+7/48y_2^2+5/48y_2^3+7/48y_3^2+7/48y_3^3\leq 1 \tag{1.106}$$

$$-5/3x_1-46/15x_2-4x_3+5/24y_1^2+\frac{1}{3}y_1^3+\frac{1}{5}y_2^1+\frac{1}{5}y_2^2+\frac{1}{3}y_2^3+\frac{1}{3}y_3^2+\frac{1}{3}y_3^3\leq 1 \tag{1.107}$$

$$-5/3x_1-29/9x_2-4x_3+2/9y_1^2+\frac{1}{3}y_1^3+2/9y_2^1+2/9y_2^2+\frac{1}{3}y_2^3+\frac{1}{3}y_3^2+\frac{1}{3}y_3^3\leq 1 \tag{1.108}$$

$$-3/4x_1-111/76x_2-11/4x_3+\frac{1}{4}y_1^1+\frac{1}{4}y_2^1+3/38y_2^2+9/76y_2^3+\frac{1}{4}y_3^1+\frac{1}{4}y_3^3\leq 1 \tag{1.109}$$

$$-3/4x_1-39/16x_2-11/4x_3+\frac{1}{4}y_1^1+\frac{1}{4}y_2^1+3/16y_2^2+3/16y_2^3+\frac{1}{4}y_3^1+\frac{1}{4}y_3^3\leq 1 \tag{1.110}$$

$$-x_1-8/5x_2-x_3+\frac{1}{5}y_1^1+\frac{1}{5}y_1^3+\frac{1}{5}y_2^1+\frac{1}{5}y_2^3+\frac{1}{5}y_3^1\leq 1 \tag{1.111}$$

$$-25/34x_1-31/34x_2-25/34x_3+5/34y_1^1+5/34y_1^3+5/34y_2^1+5/34y_2^3+5/34y_3^1+3/34y_3^3\leq 1 \tag{1.112}$$

$$-1/2x_1-23/40x_2-1/2x_3+\frac{1}{10}y_1^1+\frac{1}{8}y_1^3+\frac{1}{10}y_2^1+\frac{1}{8}y_2^3+\frac{1}{10}y_3^1+\frac{1}{10}y_3^3\leq 1 \tag{1.113}$$

$$-1/8x_1-1/8x_2-7/8x_3+\frac{1}{8}y_1^1+1/24y_1^2+\frac{1}{8}y_2^1+1/24y_2^2+\frac{1}{8}y_3^1+\frac{1}{8}y_3^2\leq 1 \tag{1.114}$$

$$-11/2x_2-5/2x_3+\frac{1}{2}y_2^1+\frac{1}{2}y_2^3+\frac{1}{2}y_3^1\leq 1 \tag{1.115}$$

$$-3/10x_1-37/50x_2-1/2x_3+3/50y_1^1+\frac{1}{10}y_1^3+\frac{1}{10}y_2^1+7/50y_2^3+\frac{1}{10}y_3^1+\frac{1}{10}y_3^3\leq 1 \tag{1.116}$$

$$-3/10x_1-29/50x_2-1/2x_3+\frac{1}{10}y_1^1+3/50y_1^3+7/50y_2^1+\frac{1}{10}y_2^3+\frac{1}{10}y_3^1+\frac{1}{10}y_3^3\leq 1 \tag{1.117}$$

$$-5/2x_1-11/2x_2-11/4x_3+\frac{1}{2}y_1^1+\frac{1}{2}y_2^1+\frac{1}{2}y_2^3+\frac{1}{4}y_3^1+\frac{1}{4}y_3^3\leq 1 \tag{1.118}$$

$$-5/2x_1-11/2x_2+\frac{1}{2}y_1^1+\frac{1}{2}y_2^1+\frac{1}{2}y_2^3\leq 1 \tag{1.119}$$

$$-5/2x_1-11/2x_2-5/2x_3+\frac{1}{2}y_1^1+\frac{1}{2}y_2^1+\frac{1}{2}y_2^3+\frac{1}{2}y_3^1\leq 1 \tag{1.120}$$

$$-5/2x_1-11/2x_2-55/62x_3+\frac{1}{2}y_1^1+\frac{1}{2}y_2^1+\frac{1}{2}y_2^3+11/62y_3^1+5/62y_3^3\leq 1 \tag{1.121}$$

$$-3/4x_1-33/20x_2-11/4x_3+\frac{1}{4}y_1^1+\frac{1}{4}y_2^1+3/20y_2^3+\frac{1}{4}y_3^1+\frac{1}{4}y_3^3\leq 1 \tag{1.122}$$

$$-40/21x_1-11/2x_2-5/2x_3+8/21y_1^1+5/21y_1^3+\frac{1}{2}y_2^1+\frac{1}{2}y_2^3+\frac{1}{2}y_3^1\leq 1 \tag{1.123}$$

$$-35/19x_1-72/19x_2-35/19x_3+7/19y_1^1+5/19y_1^3+7/19y_2^1+7/19y_2^3+7/19y_3^1\leq 1 \tag{1.124}$$

$$-295/158x_1-609/158x_2-295/158x_3+59/158y_1^1+20/79y_1^3+59/158y_2^1+59/158y_2^3+59/158y_3^1+5/158y_3^3\le 1 \tag{1.125}$$
$$-40/21x_1-148/21x_2-110/21x_3+8/21y_1^1+5/21y_1^3+13/21y_2^1+5/21y_2^2+13/21y_2^3+13/21y_3^1+5/21y_3^2+5/21y_3^3\le 1 \tag{1.126}$$
$$-5/2x_1-12x_2-14x_3+\frac{1}{2}y_1^1+1y_2^1+\frac{1}{2}y_2^2+1y_2^3+1y_3^1+1y_3^2+\frac{5}{6}y_3^3\le 1 \tag{1.127}$$
$$-5/2x_1-71/7x_2-75/7x_3+\frac{1}{2}y_1^1+6/7y_2^1+5/14y_2^2+6/7y_2^3+6/7y_3^1+\frac{5}{7}y_3^2+\frac{5}{7}y_3^3\le 1 \tag{1.128}$$
$$-5/2x_1-11/2x_2-25/4x_3+\frac{1}{2}y_1^1+\frac{1}{2}y_2^1+\frac{1}{2}y_2^3+\frac{1}{2}y_3^1+5/12y_3^2+5/12y_3^3\le 1 \tag{1.129}$$
$$-5/2x_1-11/2x_2-655/118x_3+\frac{1}{2}y_1^1+\frac{1}{2}y_2^1+\frac{1}{2}y_2^3+\frac{1}{2}y_3^1+20/59y_3^2+45/118y_3^3\le 1 \tag{1.130}$$
$$-1/8x_2-7/8x_3+\frac{1}{8}y_2^2+\frac{1}{8}y_3^1+\frac{1}{8}y_3^2\le 1 \tag{1.131}$$
$$-1/7x_1-8/7x_2-8/7x_3+1/7y_1^1+1/7y_1^2+1/7y_2^1+1/7y_2^2+1/7y_3^1+1/7y_3^2\le 1 \tag{1.132}$$
$$-3/28x_1-3/28x_2-17/28x_3+1/28y_1^1+3/28y_1^2+1/28y_2^1+3/28y_2^2+3/28y_3^1+3/28y_3^2\le 1 \tag{1.133}$$
$$-1/7x_1-8/7x_3+1/7y_1^1+1/7y_1^2+1/7y_3^1+1/7y_3^2\le 1 \tag{1.134}$$
$$-1/8x_1-3/2x_2-7/8x_3+\frac{1}{8}y_1^1+\frac{1}{8}y_1^2+\frac{1}{8}y_2^1+\frac{1}{8}y_2^2+\frac{1}{8}y_2^3+\frac{1}{8}y_3^1+\frac{1}{8}y_3^2\le 1 \tag{1.135}$$
$$-1/8x_2+\frac{1}{8}y_1^2+\frac{1}{8}y_2^2\le 1 \tag{1.136}$$
$$-5/8x_1-53/56x_2-7/8x_3+\frac{1}{8}y_1^2+5/56y_1^3+9/56y_2^2+\frac{1}{8}y_2^3+\frac{1}{8}y_3^2+\frac{1}{8}y_3^3\le 1 \tag{1.137}$$
$$-5/4x_1-5/4x_2-11/4x_3+\frac{1}{4}y_1^2+5/32y_1^3+\frac{1}{4}y_2^2+\frac{1}{4}y_3^2+\frac{1}{4}y_3^3\le 1 \tag{1.138}$$
$$-1/3x_1-4/9x_2-1/3x_3+1/9y_1^1+1/9y_1^2+1/9y_2^1+1/9y_2^2+1/9y_3^1\le 1 \tag{1.139}$$
$$-21/31x_1-60/31x_2-21/31x_3+7/31y_1^1+3/31y_1^2+7/31y_2^1+7/31y_2^2+7/31y_3^1\le 1 \tag{1.140}$$
$$-1/12x_1-7/12x_2-1/4x_3+1/36y_1^1+\frac{1}{12}y_1^2+\frac{1}{12}y_2^1+5/36y_2^2+\frac{1}{12}y_3^1+\frac{1}{12}y_3^2\le 1 \tag{1.141}$$
$$-19/28x_1-163/84x_2-19/28x_3+19/84y_1^1+2/21y_1^2+19/84y_2^1+19/84y_2^2+19/84y_3^1+1/84y_3^2\le 1 \tag{1.142}$$
$$-7/20x_1-31/60x_2-1/4x_3+7/60y_1^1+7/60y_1^2+7/60y_2^1+7/60y_2^2+\frac{1}{12}y_3^1+\frac{1}{12}y_3^2\le 1 \tag{1.143}$$
$$-1/4x_1-13/48x_2-1/4x_3+\frac{1}{12}y_1^1+5/48y_1^2+\frac{1}{12}y_2^1+5/48y_2^2+\frac{1}{12}y_3^1+\frac{1}{12}y_3^2\le 1 \tag{1.144}$$
$$-21/68x_1-23/68x_2-21/68x_3+7/68y_1^1+7/68y_1^2+7/68y_2^1+7/68y_2^2+7/68y_3^1+5/68y_3^2\le 1 \tag{1.145}$$
$$-1/8x_1-1/8x_2-7/8x_3+1/56y_1^1+\frac{1}{8}y_1^2+\frac{1}{8}y_2^2+\frac{1}{8}y_3^1+\frac{1}{8}y_3^2\le 1 \tag{1.146}$$
$$-3/5x_1-8/5x_2-8/7x_3+\frac{1}{5}y_1^1+\frac{1}{5}y_1^2+\frac{1}{5}y_2^1+\frac{1}{5}y_2^2+1/7y_3^1+1/7y_3^2\le 1 \tag{1.147}$$
$$-1/7x_1-10/21x_2-8/7x_3+1/21y_1^1+1/7y_1^2+1/21y_2^1+1/7y_2^2+1/7y_3^1+1/7y_3^2\le 1 \tag{1.148}$$
$$-12/19x_1-33/19x_2-45/19x_3+4/19y_1^1+3/19y_1^2+4/19y_2^1+4/19y_2^2+4/19y_3^1+3/19y_3^2+3/19y_3^3\le 1 \tag{1.149}$$
$$-21/62x_1-29/62x_2-43/62x_3+7/62y_1^1+7/62y_1^2+7/62y_2^1+7/62y_2^2+7/62y_3^1+7/62y_3^2+1/31y_3^3\le 1 \tag{1.150}$$
$$-1/6x_2-3/2x_3+\frac{1}{6}y_2^1+\frac{1}{6}y_3^1+\frac{1}{6}y_3^2\le 1 \tag{1.151}$$
$$-1/6x_1-1/6x_2-3/2x_3+\frac{1}{6}y_1^1+\frac{1}{6}y_2^1+\frac{1}{6}y_3^1+\frac{1}{6}y_3^2\le 1 \tag{1.152}$$
$$-1/6x_1-3/2x_3+\frac{1}{6}y_1^1+\frac{1}{6}y_3^1+\frac{1}{6}y_3^2\le 1 \tag{1.153}$$
$$-1/6x_1-3/20x_2-3/2x_3+\frac{1}{6}y_1^1+3/20y_2^1+1/60y_2^2+\frac{1}{6}y_3^1+\frac{1}{6}y_3^2\le 1 \tag{1.154}$$
$$-1/6x_1-13/12x_2-3/2x_3+\frac{1}{6}y_1^1+\frac{1}{6}y_2^1+\frac{1}{12}y_2^2+\frac{1}{12}y_2^3+\frac{1}{6}y_3^1+\frac{1}{6}y_3^2\le 1 \tag{1.155}$$
$$-1/6x_1-13/7x_2-23/7x_3+\frac{1}{6}y_1^1+2/7y_2^1+1/7y_2^2+1/7y_2^3+2/7y_3^1+2/7y_3^2+5/42y_3^3\le 1 \tag{1.156}$$
$$-1/6x_1-41/114x_2-3/2x_3+\frac{1}{6}y_1^1+\frac{1}{6}y_2^1+11/342y_2^2+1/57y_2^3+\frac{1}{6}y_3^1+\frac{1}{6}y_3^2\le 1 \tag{1.157}$$
$$-1/6x_1-1/2x_2-3/2x_3+\frac{1}{6}y_1^1+\frac{1}{6}y_2^1+1/18y_2^2+\frac{1}{6}y_3^1+\frac{1}{6}y_3^2\le 1 \tag{1.158}$$
$$-1/12x_1-17/36x_2-1/4x_3+\frac{1}{12}y_1^1+1/36y_1^2+5/36y_2^1+\frac{1}{12}y_2^2+\frac{1}{12}y_3^1+\frac{1}{12}y_3^2\le 1 \tag{1.159}$$
$$-1/6x_1-38/15x_2-4x_3+\frac{1}{6}y_1^1+\frac{1}{3}y_2^1+\frac{1}{6}y_2^2+\frac{1}{5}y_2^3+\frac{1}{3}y_3^1+\frac{1}{3}y_3^2+\frac{1}{6}y_3^3\le 1 \tag{1.160}$$
$$-3/19x_1-8/19x_2-26/19x_3+3/19y_1^1+1/19y_1^2+3/19y_2^1+1/19y_2^2+3/19y_3^1+3/19y_3^2\le 1 \tag{1.161}$$
$$-8/49x_1-1/6x_2-3/2x_3+8/49y_1^1+1/49y_1^2+\frac{1}{6}y_2^1+\frac{1}{6}y_3^1+\frac{1}{6}y_3^2\le 1 \tag{1.162}$$
$$-8/49x_1-20/49x_2-86/49x_3+8/49y_1^1+1/49y_1^2+9/49y_2^1+1/49y_2^2+1/49y_2^3+9/49y_3^1+9/49y_3^2+1/49y_3^3\le 1 \tag{1.163}$$
$$-8/49x_1-13/7x_2-23/7x_3+8/49y_1^1+1/49y_1^2+2/7y_2^1+1/7y_2^2+1/7y_2^3+2/7y_3^1+2/7y_3^2+6/49y_3^3\le 1 \tag{1.164}$$
$$-30/17x_1-61/17x_2-75/17x_3+6/17y_1^1+5/17y_1^3+6/17y_2^1+6/17y_2^3+6/17y_3^1+5/17y_3^2+5/17y_3^3\le 1 \tag{1.165}$$
$$-40/21x_1-71/7x_2-75/7x_3+8/21y_1^1+5/21y_1^3+6/7y_2^1+10/21y_2^2+6/7y_2^3+6/7y_3^1+\frac{5}{7}y_3^2+\frac{5}{7}y_3^3\le 1 \tag{1.166}$$
$$-5/3x_1-10/3x_2-5/6x_3+3/14y_1^1+\frac{1}{3}y_1^3+\frac{1}{3}y_2^1+\frac{1}{3}y_2^3+\frac{1}{6}y_3^3\le 1 \tag{1.167}$$
$$-5/3x_1-10/3x_2-7/10x_3+7/30y_1^1+\frac{1}{3}y_1^3+\frac{1}{3}y_2^1+\frac{1}{3}y_2^3+1/25y_3^1+7/50y_3^3\le 1 \tag{1.168}$$

$$-x_1-8/5x_2-1/2x_3+\frac{1}{5}y_1^1+\frac{1}{10}y_1^3+\frac{1}{5}y_2^1+\frac{1}{5}y_2^3+\frac{1}{10}y_3^1+\frac{1}{10}y_3^3\leq 1 \tag{1.169}$$

$$-5/3x_1-10/3x_2-8/3x_3+5/21y_1^1+\frac{1}{3}y_1^3+\frac{1}{3}y_2^1+\frac{1}{3}y_2^3+\frac{1}{3}y_3^3\leq 1 \tag{1.170}$$

$$-5/3x_1-10/3x_2-10/3x_3+\frac{1}{3}y_1^1+\frac{1}{3}y_1^3+\frac{1}{3}y_2^1+\frac{1}{3}y_2^3+\frac{1}{6}y_3^1+\frac{1}{3}y_3^3\leq 1 \tag{1.171}$$

$$-x_1-2x_2-2x_3+9/35y_1^1+\frac{1}{5}y_1^3+9/35y_2^1+\frac{1}{5}y_2^3+\frac{1}{5}y_3^1+\frac{1}{5}y_3^3\leq 1 \tag{1.172}$$

$$-2x_1-21/5x_2-2x_3+2/5y_1^1+\frac{1}{5}y_1^3+2/5y_2^1+2/5y_2^3+\frac{1}{5}y_3^1+\frac{1}{5}y_3^3\leq 1 \tag{1.173}$$

$$-55/24x_1-119/24x_2-5/6x_3+11/24y_1^1+\frac{1}{12}y_1^3+11/24y_2^1+11/24y_2^3+\frac{1}{6}y_3^1+\frac{1}{12}y_3^3\leq 1 \tag{1.174}$$

$$-65/36x_1-133/36x_2-5/6x_3+13/36y_1^1+5/18y_1^3+13/36y_2^1+13/36y_2^3+\frac{1}{6}y_3^1+\frac{1}{12}y_3^3\leq 1 \tag{1.175}$$

$$-5/3x_1-10/3x_2-14/3x_3+\frac{1}{3}y_1^1+\frac{1}{3}y_2^1+\frac{1}{3}y_2^3+\frac{1}{3}y_3^1+\frac{1}{3}y_3^2+\frac{1}{3}y_3^3\leq 1 \tag{1.176}$$

$$-8/7x_1-13/7x_2-23/7x_3+2/7y_1^1+1/7y_1^2+2/7y_2^1+1/7y_2^2+1/7y_2^3+2/7y_3^1+2/7y_3^2+1/7y_3^3\leq 1 \tag{1.177}$$

$$-35/18x_1-12x_2-14x_3+7/18y_1^1+2/9y_1^3+1y_2^1+11/18y_2^2+1y_2^3+1y_3^1+1y_3^2+\frac{5}{6}y_3^3\leq 1 \tag{1.178}$$

$$-7x_1-10x_3+1y_1^1+1y_1^3+1y_3^3\leq 1 \tag{1.179}$$

$$-7x_1-10/3x_2-10x_3+1y_1^1+1y_1^3+\frac{1}{3}y_2^1+\frac{1}{3}y_2^3+1y_3^3\leq 1 \tag{1.180}$$

$$-3/2x_1-3/5x_2-2x_3+23/70y_1^1+\frac{1}{5}y_1^3+\frac{1}{5}y_2^1+1/30y_2^3+\frac{1}{5}y_3^1+\frac{1}{5}y_3^3\leq 1 \tag{1.181}$$

$$-7/5x_1-3/5x_2-2x_3+11/35y_1^1+\frac{1}{5}y_1^3+\frac{1}{5}y_2^1+\frac{1}{5}y_3^1+\frac{1}{5}y_3^3\leq 1 \tag{1.182}$$

$$-19/20x_1-39/20x_2-2x_3+\frac{1}{4}y_1^1+\frac{1}{5}y_1^3+\frac{1}{5}y_2^1+3/20y_2^2+3/20y_2^3+\frac{1}{5}y_3^1+\frac{1}{5}y_3^3\leq 1 \tag{1.183}$$

$$-3/8x_1-5/8x_2-1/4x_3+\frac{1}{8}y_1^1+\frac{1}{12}y_1^2+\frac{1}{8}y_2^1+\frac{1}{8}y_2^2+\frac{1}{12}y_3^1+\frac{1}{12}y_3^2\leq 1 \tag{1.184}$$

$$-3/5x_1-8/5x_2-8/5x_3+\frac{1}{5}y_1^1+\frac{1}{5}y_1^2+\frac{1}{5}y_2^1+\frac{1}{5}y_2^2+2/15y_3^1+\frac{1}{5}y_3^2\leq 1 \tag{1.185}$$

$$-1/2x_1-7/6x_2-1/2x_3+1/14y_1^1+\frac{1}{6}y_1^2+\frac{1}{6}y_2^1+\frac{1}{6}y_2^2+\frac{1}{6}y_3^2\leq 1 \tag{1.186}$$

$$-9/14x_1-25/14x_2-8/7x_3+3/14y_1^1+1/7y_1^2+3/14y_2^1+3/14y_2^2+1/7y_3^1+1/7y_3^2\leq 1 \tag{1.187}$$

$$-183/280x_1-513/280x_2-3/7x_3+61/280y_1^1+9/70y_1^2+61/280y_2^1+61/280y_2^2+1/7y_3^1+3/56y_3^2\leq 1 \tag{1.188}$$

$$-159/224x_1-465/224x_2-3/7x_3+53/224y_1^1+3/56y_1^2+53/224y_2^1+53/224y_2^2+1/7y_3^1+3/56y_3^2\leq 1 \tag{1.189}$$

$$-3/7x_1-8/7x_2-8/7x_3+9/49y_1^1+1/7y_1^2+9/49y_2^1+1/7y_2^2+1/7y_3^1+1/7y_3^2\leq 1 \tag{1.190}$$

$$-3/5x_1-8/5x_2-14/5x_3+\frac{1}{5}y_1^1+\frac{1}{5}y_2^1+\frac{1}{5}y_2^2+\frac{1}{5}y_3^1+\frac{1}{5}y_3^2+\frac{1}{5}y_3^3\leq 1 \tag{1.191}$$

$$-9/25x_1-14/25x_2-4/5x_3+3/25y_1^1+2/25y_1^2+3/25y_2^1+3/25y_2^2+3/25y_3^1+3/25y_3^2+1/25y_3^3\leq 1 \tag{1.192}$$

$$-8x_1-13x_2-23x_3+8/7y_1^1+1y_1^2+8/7y_2^1+1y_2^2+1y_2^3+1y_3^1+2y_3^2+1y_3^3\leq 1 \tag{1.193}$$

$$-156/217x_1-459/217x_2-387/217x_3+52/217y_1^1+9/217y_1^2+52/217y_2^1+52/217y_2^2+52/217y_3^1+30/217y_3^2+3/31y_3^3\leq 1 \tag{1.194}$$

$$-60/91x_1-13/7x_2-137/91x_3+20/91y_1^1+11/91y_1^2+20/91y_2^1+20/91y_2^2+20/91y_3^1+11/91y_3^2+1/13y_3^3\leq 1 \tag{1.195}$$

$$-1/8x_3+\frac{1}{8}y_1^2+\frac{1}{8}y_3^2\leq 1 \tag{1.196}$$

$$-7x_1-8x_3+1y_1^1+1y_1^2+1y_3^2\leq 1 \tag{1.197}$$

$$-5/3x_1-45/58x_2-4x_3+37/174y_1^2+\frac{1}{3}y_1^3+1/29y_2^2+9/58y_2^3+\frac{1}{3}y_3^2+\frac{1}{3}y_3^3\leq 1 \tag{1.198}$$

$$-5/9x_1-5/9x_2-2/3x_3+1/9y_1^2+1/9y_1^3+1/9y_2^3+1/9y_3^2+1/9y_3^3\leq 1 \tag{1.199}$$

$$-1/13x_1-1/13x_2-2/13x_3+1/13y_1^1+1/13y_1^2+1/13y_2^1+1/13y_3^1+1/13y_3^2\leq 1 \tag{1.200}$$

$$-7/43x_1-7/43x_2-62/43x_3+7/43y_1^1+1/43y_1^2+7/43y_2^1+7/43y_3^1+7/43y_3^2\leq 1 \tag{1.201}$$

$$-x_1-1/7x_2-8/7x_3+13/49y_1^1+1/7y_1^2+1/7y_2^1+1/7y_3^1+1/7y_3^2\leq 1 \tag{1.202}$$

$$-43/264x_1-43/264x_2-127/88x_3+43/264y_1^1+1/44y_1^2+43/264y_2^1+1/264y_2^2+43/264y_3^1+43/264y_3^2\leq 1 \tag{1.203}$$

$$-3/5x_1-3/8x_2-2x_3+3/40y_1^1+\frac{1}{5}y_1^3+\frac{1}{8}y_2^3+\frac{1}{5}y_3^1+\frac{1}{5}y_3^3\leq 1 \tag{1.204}$$

$$-15/2x_1-5/3x_2-10x_3+15/14y_1^1+1y_1^3+5/21y_2^1+\frac{1}{6}y_2^3+1y_3^3\leq 1 \tag{1.205}$$

$$-3/5x_1-6/5x_2-2x_3+3/35y_1^1+\frac{1}{5}y_1^3+\frac{1}{5}y_2^3+\frac{1}{5}y_3^1+\frac{1}{5}y_3^3\leq 1 \tag{1.206}$$

$$-7x_1-10x_2-10x_3+1y_1^1+1y_1^3+1y_2^3+1y_3^3\leq 1 \tag{1.207}$$

$$-7/2x_1-7/6x_2-9/2x_3+\frac{1}{2}y_1^1+\frac{1}{2}y_1^3+\frac{1}{6}y_2^1+\frac{1}{6}y_2^3+\frac{1}{2}y_3^3\leq 1 \tag{1.208}$$

$$-165/232x_1-15/29x_2-593/232x_3+55/232y_1^1+3/58y_1^3+5/29y_2^1+3/58y_2^3+55/232y_3^1+55/232y_3^3\leq 1 \tag{1.209}$$

$$-39/58x_1-15/29x_2-137/58x_3+13/58y_1^1+3/29y_1^3+5/29y_2^1+3/58y_2^3+13/58y_3^1+13/58y_3^3\leq 1 \tag{1.210}$$

$$-135/58x_1-15/29x_2-90/29x_3+25/58y_1^1+9/29y_1^3+5/29y_2^1+3/58y_2^3+5/29y_3^1+9/29y_3^3\leq 1 \tag{1.211}$$

$$-13/14x_1-3/7x_2-8/7x_3+3/14y_1^1+1/7y_1^3+1/7y_2^1+1/14y_2^3+1/7y_3^1+1/7y_3^3\leq 1 \tag{1.212}$$

$$-419/290x_1-15/29x_2-2x_3+93/290y_1^1+\frac{1}{5}y_1^3+5/29y_2^1+3/58y_2^3+\frac{1}{5}y_3^1+\frac{1}{5}y_3^3\leq 1 \tag{1.213}$$

$$-7x_1 - 10x_2 + 1y_1^1 + 1y_1^3 + 1y_2^3 \leq 1 \tag{1.214}$$
$$-7x_1 - 10x_2 - 2x_3 + 1y_1^1 + 1y_1^3 + 1y_2^3 + \frac{1}{5}y_3^1 + \frac{1}{5}y_3^3 \leq 1 \tag{1.215}$$
$$-59/8x_1 - 10x_2 - 5/4x_3 + 59/56y_1^1 + 1y_1^3 + 1y_2^3 + 5/28y_3^1 + \frac{1}{8}y_3^3 \leq 1 \tag{1.216}$$
$$-19/10x_1 - 5/2x_2 - 1/2x_3 + 3/10y_1^1 + 3/10y_1^3 + \frac{1}{10}y_2^1 + 3/10y_2^3 + \frac{1}{10}y_3^1 + \frac{1}{10}y_3^3 \leq 1 \tag{1.217}$$
$$-59/12x_1 - 20/3x_2 - 5/6x_3 + 3/4y_1^1 + 2/3y_1^3 + \frac{1}{6}y_2^1 + 2/3y_2^3 + \frac{1}{6}y_3^1 + \frac{1}{12}y_3^3 \leq 1 \tag{1.218}$$
$$-35/8x_1 - 47/8x_2 - 7/8x_3 + 5/8y_1^1 + 5/8y_1^3 + 5/8y_2^3 + \frac{1}{8}y_3^1 + \frac{1}{8}y_3^3 \leq 1 \tag{1.219}$$
$$-7/3x_1 - 10/3x_2 - 5/3x_3 + 3/7y_1^1 + \frac{1}{3}y_1^3 + \frac{1}{3}y_2^1 + \frac{1}{3}y_2^3 + \frac{1}{3}y_3^1 \leq 1 \tag{1.220}$$
$$-59/24x_1 - 10/3x_2 - 5/3x_3 + 25/56y_1^1 + \frac{1}{3}y_1^3 + \frac{1}{3}y_2^1 + \frac{1}{3}y_2^3 + \frac{1}{3}y_3^1 + 1/24y_3^3 \leq 1 \tag{1.221}$$
$$-37/18x_1 - 10/3x_2 - 25/6x_3 + 7/18y_1^1 + \frac{1}{3}y_1^3 + \frac{1}{3}y_2^1 + \frac{1}{3}y_2^3 + \frac{1}{3}y_3^1 + 5/18y_3^2 + 5/18y_3^3 \leq 1 \tag{1.222}$$
$$-9/4x_1 - 10/3x_2 - 5/6x_3 + 5/12y_1^1 + \frac{1}{3}y_1^3 + \frac{1}{3}y_2^1 + \frac{1}{3}y_2^3 + \frac{1}{6}y_3^1 + \frac{1}{12}y_3^3 \leq 1 \tag{1.223}$$
$$-1/7x_1 - 1/8x_2 - 8/7x_3 + 1/56y_1^1 + 1/7y_1^2 + \frac{1}{8}y_2^2 + 1/7y_3^1 + 1/7y_3^2 \leq 1 \tag{1.224}$$
$$-43/42x_1 - 4/21x_2 - 8/7x_3 + 79/294y_1^1 + 1/7y_1^2 + 22/147y_2^1 + 1/42y_2^2 + 1/7y_3^1 + 1/7y_3^2 \leq 1 \tag{1.225}$$
$$-43/6x_1 - 4/3x_2 - 8x_3 + 43/42y_1^1 + 1y_1^2 + 4/21y_2^1 + \frac{1}{6}y_2^2 + 1y_3^2 \leq 1 \tag{1.226}$$
$$-1/7x_1 - 2/7x_2 - 8/7x_3 + 1/49y_1^1 + 1/7y_1^2 + 1/7y_2^2 + 1/7y_3^1 + 1/7y_3^2 \leq 1 \tag{1.227}$$
$$-7x_1 - 8x_2 - 8x_3 + 1y_1^1 + 1y_1^2 + 1y_2^2 + 1y_3^2 \leq 1 \tag{1.228}$$
$$-7x_1 - 8x_2 - 8/7x_3 + 1y_1^1 + 1y_1^2 + 1y_2^2 + 1/7y_3^1 + 1/7y_3^2 \leq 1 \tag{1.229}$$
$$-57/8x_1 - 8x_2 - x_3 + 57/56y_1^1 + 1y_1^2 + 1y_2^2 + 1/7y_3^1 + \frac{1}{8}y_3^2 \leq 1 \tag{1.230}$$
$$-171/56x_1 - 24/7x_2 - 3/7x_3 + 29/56y_1^1 + 3/7y_1^2 + 1/7y_2^1 + 3/7y_2^2 + 1/7y_3^1 + 3/56y_3^2 \leq 1 \tag{1.231}$$
$$-49/8x_1 - 55/8x_2 - 7/8x_3 + 7/8y_1^1 + 7/8y_1^2 + 7/8y_2^2 + \frac{1}{8}y_3^1 + \frac{1}{8}y_3^2 \leq 1 \tag{1.232}$$
$$-7x_1 - 8x_2 + 1y_1^1 + 1y_1^2 + 1y_2^2 \leq 1 \tag{1.233}$$
$$-17/12x_1 - 19/12x_2 - 1/4x_3 + \frac{1}{4}y_1^1 + \frac{1}{4}y_1^2 + \frac{1}{12}y_2^1 + \frac{1}{4}y_2^2 + \frac{1}{12}y_3^1 + \frac{1}{12}y_3^2 \leq 1 \tag{1.234}$$
$$-7/5x_1 - 8/5x_2 - 3/5x_3 + 11/35y_1^1 + \frac{1}{5}y_1^2 + \frac{1}{5}y_2^1 + \frac{1}{5}y_2^2 + \frac{1}{5}y_3^1 \leq 1 \tag{1.235}$$
$$-57/40x_1 - 8/5x_2 - 3/5x_3 + 89/280y_1^1 + \frac{1}{5}y_1^2 + \frac{1}{5}y_2^1 + \frac{1}{5}y_2^2 + \frac{1}{5}y_3^1 + 1/40y_3^2 \leq 1 \tag{1.236}$$
$$-19/20x_1 - 8/5x_2 - 9/4x_3 + \frac{1}{4}y_1^1 + \frac{1}{5}y_1^2 + \frac{1}{5}y_2^1 + \frac{1}{5}y_2^2 + \frac{1}{5}y_3^1 + 3/20y_3^2 + 3/20y_3^3 \leq 1 \tag{1.237}$$
$$-53/324x_1 - 4/27x_2 - 157/108x_3 + 53/324y_1^1 + 1/54y_1^2 + 4/27y_2^1 + 1/54y_2^2 + 53/324y_3^1 + 53/324y_3^2 \leq 1 \tag{1.238}$$
$$-35/6x_1 - 7/6x_2 - 13/2x_3 + \frac{5}{6}y_1^1 + \frac{5}{6}y_1^2 + \frac{1}{6}y_2^1 + \frac{1}{6}y_2^2 + \frac{5}{6}y_3^2 \leq 1 \tag{1.239}$$
$$-49/40x_1 - 8/5x_2 - 3/7x_3 + 81/280y_1^1 + \frac{1}{5}y_1^2 + \frac{1}{5}y_2^1 + \frac{1}{5}y_2^2 + 1/7y_3^1 + 3/56y_3^2 \leq 1 \tag{1.240}$$
$$-51/40x_1 - 8/5x_2 - 43/35x_3 + 83/280y_1^1 + \frac{1}{5}y_1^2 + \frac{1}{5}y_2^1 + \frac{1}{5}y_2^2 + \frac{1}{5}y_3^1 + 29/280y_3^2 + 2/35y_3^3 \leq 1 \tag{1.241}$$
$$-57/56x_1 - 1/7x_2 - 8/7x_3 + 15/56y_1^1 + 1/7y_1^2 + 1/7y_2^1 + 1/56y_2^2 + 1/7y_3^1 + 1/7y_3^2 \leq 1 \tag{1.242}$$
$$-43/54x_1 - 4/27x_2 - 11/9x_3 + 13/54y_1^1 + 1/9y_1^2 + 4/27y_2^1 + 1/54y_2^2 + 4/27y_3^1 + 4/27y_3^2 \leq 1 \tag{1.243}$$
$$-61/378x_1 - 4/27x_2 - 179/126x_3 + 61/378y_1^1 + 2/63y_1^2 + 4/27y_2^1 + 1/54y_2^2 + 61/378y_3^1 + 61/378y_3^2 \leq 1 \tag{1.244}$$
$$-2/7x_2 - 9/14x_3 + 1/14y_1^1 + 1/14y_1^3 + 1/14y_2^1 + 1/14y_2^3 + 1/14y_3^1 + 1/14y_3^2 + 1/14y_3^3 \leq 1 \tag{1.245}$$
$$-11/10x_2 - 3/5x_3 + \frac{1}{10}y_1^1 + \frac{1}{10}y_1^2 + \frac{1}{10}y_2^1 + \frac{1}{10}y_2^2 + \frac{1}{10}y_2^3 + \frac{1}{10}y_3^1 + \frac{1}{10}y_3^2 \leq 1 \tag{1.246}$$
$$-3/4x_2 - 1/2x_3 + \frac{1}{12}y_1^1 + \frac{1}{12}y_1^3 + \frac{1}{12}y_2^1 + \frac{1}{12}y_2^2 + \frac{1}{12}y_2^3 + \frac{1}{12}y_3^1 + \frac{1}{12}y_3^3 \leq 1 \tag{1.247}$$
$$-1/3x_2 - 11/12x_3 + \frac{1}{12}y_1^1 + \frac{1}{12}y_1^2 + \frac{1}{12}y_2^1 + \frac{1}{12}y_2^2 + \frac{1}{12}y_3^1 + \frac{1}{12}y_3^2 + \frac{1}{12}y_3^3 \leq 1 \tag{1.248}$$
$$-5/3x_1 - 10/3x_2 - 52/15x_3 + 22/105y_1^1 + \frac{1}{3}y_1^3 + \frac{1}{3}y_2^1 + \frac{1}{3}y_2^3 + \frac{1}{5}y_3^1 + \frac{1}{5}y_3^2 + \frac{1}{3}y_3^3 \leq 1 \tag{1.249}$$
$$-5/3x_1 - 10/3x_2 - 496/141x_3 + 10/47y_1^1 + \frac{1}{3}y_1^3 + \frac{1}{3}y_2^1 + \frac{1}{3}y_2^3 + 10/47y_3^1 + 29/141y_3^2 + \frac{1}{3}y_3^3 \leq 1 \tag{1.250}$$
$$-10/11x_1 - 15/11x_2 - 19/11x_3 + 2/11y_1^1 + 1/11y_1^3 + 2/11y_2^1 + 2/11y_2^3 + 2/11y_3^1 + 1/11y_3^2 + 2/11y_3^3 \leq 1 \tag{1.251}$$
$$-x_1 - 4/7x_2 - 9/7x_3 + 1/7y_1^1 + 1/7y_1^3 + 1/7y_2^3 + 1/7y_3^2 + 1/7y_3^3 \leq 1 \tag{1.252}$$
$$-310/169x_1 - 49/13x_2 - 625/169x_3 + 62/169y_1^1 + 45/169y_1^3 + 62/169y_2^1 + 62/169y_2^3 + 62/169y_3^1 + 35/169y_3^2 + 45/169y_3^3 \leq 1 \tag{1.253}$$
$$-260/109x_1 - 567/109x_2 - 575/109x_3 + 52/109y_1^1 + 5/109y_1^3 + 52/109y_2^1 + 52/109y_2^3 + 52/109y_3^1 + 35/109y_3^2 + 40/109y_3^3 \leq 1 \tag{1.254}$$
$$-409/174x_1 - 10/3x_2 - 280/87x_3 + 25/58y_1^1 + \frac{1}{3}y_1^3 + \frac{1}{3}y_2^1 + \frac{1}{3}y_2^3 + \frac{1}{3}y_3^1 + 5/29y_3^2 + 41/174y_3^3 \leq 1 \tag{1.255}$$
$$-2/13x_1 - x_2 - 17/13x_3 + 2/13y_1^1 + 1/13y_1^2 + 2/13y_2^1 + 1/13y_2^2 + 1/13y_2^3 + 2/13y_3^1 + 2/13y_3^2 \leq 1 \tag{1.256}$$
$$-9/14x_1 - 13/14x_2 - 8/7x_3 + 3/14y_1^1 + 1/7y_1^2 + 1/7y_2^1 + 1/14y_2^2 + 1/14y_2^3 + 1/7y_3^1 + 1/7y_3^2 \leq 1 \tag{1.257}$$

$$-3/5x_1-31/8x_2-31/8x_3+3/40y_1^1+\frac{1}{5}y_1^2+3/8y_2^1+3/8y_2^2+7/40y_2^3+\frac{1}{4}y_3^1+3/8y_3^2+\frac{1}{4}y_3^3\leq 1 \tag{1.258}$$

$$-1/5x_1-37/60x_2-1/8x_3+1/40y_1^1+3/20y_1^2+1/15y_2^1+3/20y_2^2+\frac{1}{8}y_3^2\leq 1 \tag{1.259}$$

$$-3/5x_1-8/5x_2-1/2x_3+17/210y_1^1+\frac{1}{5}y_1^2+\frac{1}{5}y_2^1+\frac{1}{5}y_2^2+\frac{1}{6}y_3^2\leq 1 \tag{1.260}$$

$$-3/5x_1-8/5x_2-7/18x_3+29/270y_1^1+\frac{1}{5}y_1^2+\frac{1}{5}y_2^1+\frac{1}{5}y_2^2+1/27y_3^1+7/54y_3^2\leq 1 \tag{1.261}$$

$$-3/5x_1-8/5x_2-4/5x_3+3/35y_1^1+\frac{1}{5}y_1^2+\frac{1}{5}y_2^1+\frac{1}{5}y_2^2+\frac{1}{5}y_3^2\leq 1 \tag{1.262}$$

$$-7x_1-8/5x_2-8x_3+1y_1^1+1y_1^2+\frac{1}{5}y_2^1+\frac{1}{5}y_2^2+1y_3^2\leq 1 \tag{1.263}$$

$$-1/4x_1-71/96x_2-19/32x_3+1/32y_1^1+5/32y_1^2+\frac{1}{12}y_2^1+5/32y_2^2+1/32y_3^1+5/32y_3^2+1/32y_3^3\leq 1 \tag{1.264}$$

$$-3/5x_1-8/5x_2-26/25x_3+2/25y_1^1+\frac{1}{5}y_1^2+\frac{1}{5}y_2^1+\frac{1}{5}y_2^2+1/25y_3^1+\frac{1}{5}y_3^2+1/25y_3^3\leq 1 \tag{1.265}$$

$$-3/5x_1-8/5x_2-368/265x_3+26/265y_1^1+\frac{1}{5}y_1^2+\frac{1}{5}y_2^1+\frac{1}{5}y_2^2+26/265y_3^1+\frac{1}{5}y_3^2+19/265y_3^3\leq 1 \tag{1.266}$$

$$-3/5x_1-8/5x_2-31/15x_3+2/15y_1^1+\frac{1}{5}y_1^2+\frac{1}{5}y_2^1+\frac{1}{5}y_2^2+2/15y_3^1+\frac{1}{5}y_3^2+2/15y_3^3\leq 1 \tag{1.267}$$

$$-8/57x_1-1/3x_2-21/19x_3+1/57y_1^1+8/57y_1^2+1/57y_2^1+8/57y_2^2+1/57y_2^3+8/57y_3^1+8/57y_3^2\leq 1 \tag{1.268}$$

$$-1/7x_1-12/7x_2-8/7x_3+1/7y_1^1+1/7y_2^1+1/7y_2^2+1/7y_2^3+1/7y_3^1+1/7y_3^2\leq 1 \tag{1.269}$$

$$-10/61x_1-21/61x_2-89/61x_3+10/61y_1^1+1/61y_1^2+10/61y_2^1+2/61y_2^2+1/61y_2^3+10/61y_3^1+10/61y_3^2\leq 1 \tag{1.270}$$

$$-58/359x_1-113/359x_2-511/359x_3+58/359y_1^1+11/359y_1^2+58/359y_2^1+11/359y_2^2+5/359y_2^3+58/359y_3^1+58/359y_3^2\leq 1 \tag{1.271}$$

$$-1/2x_2-3/7x_3+1/14y_1^2+1/14y_1^3+1/14y_2^1+1/14y_2^2+1/14y_2^3+1/14y_3^2+1/14y_3^3\leq 1 \tag{1.272}$$

$$-5/16x_2-15/16x_3+\frac{1}{16}y_1^1+\frac{1}{16}y_1^2+\frac{1}{16}y_2^1+\frac{1}{16}y_2^2+\frac{1}{16}y_2^3+\frac{1}{16}y_3^1+\frac{1}{8}y_3^2+\frac{1}{16}y_3^3\leq 1 \tag{1.273}$$

$$-4/17x_2-6/17x_3+1/17y_1^1+1/17y_1^2+1/17y_1^3+1/17y_2^1+1/17y_2^2+1/17y_2^3+1/17y_3^1+1/17y_3^2+1/17y_3^3\leq 1 \tag{1.274}$$

$$-1/13x_1-4/13x_2-9/13x_3+1/13y_1^1+1/13y_1^3+1/13y_2^3+1/13y_3^1+1/13y_3^2+1/13y_3^3\leq 1 \tag{1.275}$$

$$-24/7x_1-39/7x_2-75/7x_3+4/7y_1^1+3/7y_1^3+4/7y_2^1+3/7y_2^2+3/7y_2^3+4/7y_3^1+3/7y_3^2+6/7y_3^3\leq 1 \tag{1.276}$$

$$-8x_1-13x_2-25x_3+8/7y_1^1+1y_1^3+8/7y_2^1+1y_2^2+1y_2^3+1y_3^1+1y_3^2+2y_3^3\leq 1 \tag{1.277}$$

$$-8/5x_1-3x_2-5x_3+12/35y_1^1+\frac{1}{5}y_1^3+2/5y_2^1+\frac{1}{5}y_2^2+\frac{1}{5}y_2^3+2/5y_3^1+\frac{1}{5}y_3^2+2/5y_3^3\leq 1 \tag{1.278}$$

$$-8/5x_1-39/7x_2-53/7x_3+12/35y_1^1+\frac{1}{5}y_1^3+4/7y_2^1+3/7y_2^2+3/7y_2^3+4/7y_3^1+13/35y_3^2+4/7y_3^3\leq 1 \tag{1.279}$$

$$-7/5x_1-4/5x_2-11/5x_3+\frac{1}{5}y_1^1+\frac{1}{5}y_1^2+\frac{1}{5}y_2^2+\frac{1}{5}y_3^2+\frac{1}{5}y_3^3\leq 1 \tag{1.280}$$

$$-7/4x_1-9/4x_2-7/4x_3+7/32y_1^2+\frac{1}{4}y_1^3+\frac{1}{4}y_2^2+\frac{1}{4}y_2^3+\frac{1}{4}y_3^3\leq 1 \tag{1.281}$$

$$-7/8x_1-5/8x_2-7/8x_3+\frac{1}{8}y_1^2+\frac{1}{8}y_1^3+\frac{1}{8}y_2^2+\frac{1}{8}y_2^3+\frac{1}{8}y_3^2+\frac{1}{8}y_3^3\leq 1 \tag{1.282}$$

$$-7/4x_1-9/4x_2-7/4x_3+\frac{1}{4}y_1^2+\frac{1}{4}y_1^3+\frac{1}{4}y_2^2+\frac{1}{4}y_2^3+\frac{1}{4}y_3^2+\frac{1}{12}y_3^3\leq 1 \tag{1.283}$$

$$-7/8x_1-31/32x_2-7/8x_3+\frac{1}{8}y_1^2+5/32y_1^3+\frac{1}{8}y_2^2+5/32y_2^3+\frac{1}{8}y_3^2+\frac{1}{8}y_3^3\leq 1 \tag{1.284}$$

$$-24/43x_1-51/43x_2-77/43x_3+3/43y_1^1+8/43y_1^3+3/43y_2^1+3/43y_2^2+8/43y_2^3+8/43y_3^1+8/43y_3^3\leq 1 \tag{1.285}$$

$$-7/8x_1-29/32x_2-7/8x_3+5/32y_1^2+\frac{1}{8}y_1^3+5/32y_2^2+\frac{1}{8}y_2^3+\frac{1}{8}y_3^2+\frac{1}{8}y_3^3\leq 1 \tag{1.286}$$

$$-21/16x_1-23/16x_2-21/16x_3+3/16y_1^2+3/16y_1^3+3/16y_2^2+3/16y_2^3+\frac{1}{16}y_3^2+3/16y_3^3\leq 1 \tag{1.287}$$

$$-7/5x_1-8/5x_2-7/5x_3+\frac{1}{5}y_1^2+\frac{1}{5}y_1^3+\frac{1}{5}y_2^2+\frac{1}{5}y_2^3+\frac{1}{5}y_3^3\leq 1 \tag{1.288}$$

$$-7/3x_1-10/3x_2-7/3x_3+\frac{1}{3}y_1^2+\frac{1}{3}y_1^3+\frac{1}{3}y_2^2+\frac{1}{3}y_2^3+\frac{1}{3}y_3^2\leq 1 \tag{1.289}$$

$$-3/5x_1-12/5x_2-2x_3+\frac{1}{5}y_1^1+\frac{1}{5}y_2^1+\frac{1}{5}y_2^2+\frac{1}{5}y_2^3+\frac{1}{5}y_3^1+\frac{1}{5}y_3^3\leq 1 \tag{1.290}$$

$$-24/7x_1-9x_2-45/7x_3+4/7y_1^1+3/7y_1^2+4/7y_2^1+6/7y_2^2+3/7y_2^3+4/7y_3^1+3/7y_3^2+3/7y_3^3\leq 1 \tag{1.291}$$

$$-8x_1-21x_2-15x_3+8/7y_1^1+1y_1^2+1y_2^1+2y_2^2+1y_2^3+8/7y_3^1+1y_3^2+1y_3^3\leq 1 \tag{1.292}$$

$$-8/5x_1-21/5x_2-17/5x_3+12/35y_1^1+\frac{1}{5}y_1^2+2/5y_2^1+2/5y_2^2+\frac{1}{5}y_2^3+2/5y_3^1+\frac{1}{5}y_3^2+\frac{1}{5}y_3^3\leq 1 \tag{1.293}$$

$$-8/5x_1-45/7x_2-45/7x_3+12/35y_1^1+\frac{1}{5}y_1^2+4/7y_2^1+4/7y_2^2+13/35y_2^3+4/7y_3^1+3/7y_3^2+3/7y_3^3\leq 1 \tag{1.294}$$

$$-47/20x_1-12x_2-14x_3+9/20y_1^1+\frac{1}{5}y_1^2+1y_2^1+1y_2^2+4/5y_2^3+1y_3^1+3/4y_3^2+1y_3^3\leq 1 \tag{1.295}$$

$$-11/4x_1-21/4x_2-11/4x_3+\frac{1}{2}y_1^1+\frac{1}{4}y_1^2+\frac{1}{4}y_2^1+\frac{1}{2}y_2^2+\frac{1}{4}y_2^3+\frac{1}{4}y_3^1+\frac{1}{4}y_3^3\leq 1 \tag{1.296}$$

$$-35/4x_1-21x_2-14x_3+5/4y_1^1+1y_1^2+1y_2^1+2y_2^2+1y_2^3+1y_3^1+3/4y_3^2+1y_3^3\leq 1 \tag{1.297}$$

$$-35/4x_1-219/16x_2-77/16x_3+5/4y_1^1+1y_1^2+\frac{1}{4}y_2^1+23/16y_2^2+7/16y_2^3+\frac{1}{4}y_3^1+7/16y_3^3\leq 1 \tag{1.298}$$

$$-35/4x_1-45/4x_2-11/2x_3+5/4y_1^1+1y_1^2+5/4y_2^2+\frac{1}{4}y_2^3+\frac{1}{4}y_3^1+\frac{1}{2}y_3^3\leq 1 \tag{1.299}$$

$$-35/4x_1-12x_2-25x_3+5/4y_1^1+1y_1^3+1y_2^1+1y_2^2+3/4y_2^3+1y_3^1+1y_3^2+2y_3^3\leq 1 \tag{1.300}$$

$$-47/20x_1-12x_2-14x_3+9/20y_1^1+\frac{1}{5}y_1^3+1y_2^1+1y_2^2+3/4y_2^3+1y_3^1+4/5y_3^2+1y_3^3\le 1 \tag{1.301}$$
$$-11/4x_1-9/4x_2-25/4x_3+\frac{1}{2}y_1^1+\frac{1}{4}y_1^3+\frac{1}{4}y_2^1+\frac{1}{4}y_2^2+\frac{1}{4}y_3^1+\frac{1}{4}y_3^2+\frac{1}{2}y_3^3\le 1 \tag{1.302}$$
$$-35/4x_1-9/2x_2-55/4x_3+5/4y_1^1+1y_1^3+\frac{1}{4}y_2^1+\frac{1}{2}y_2^2+\frac{1}{4}y_3^2+5/4y_3^3\le 1 \tag{1.303}$$
$$-35/4x_1-63/16x_2-265/16x_3+5/4y_1^1+1y_1^3+\frac{1}{4}y_2^1+7/16y_2^2+\frac{1}{4}y_3^1+7/16y_3^2+23/16y_3^3\le 1 \tag{1.304}$$
$$-13/16x_2-7/16x_3+\frac{1}{16}y_1^1+\frac{1}{16}y_1^2+\frac{1}{16}y_2^1+\frac{1}{8}y_2^2+\frac{1}{16}y_2^3+\frac{1}{16}y_3^1+\frac{1}{16}y_3^2+\frac{1}{16}y_3^3\le 1 \tag{1.305}$$
$$-1/3x_1-x_2-2/3x_3+1/9y_1^1+1/9y_1^3+1/9y_2^1+1/9y_2^2+1/9y_2^3+1/9y_3^3\le 1 \tag{1.306}$$
$$-5/16x_2-17/16x_3+\frac{1}{16}y_1^1+\frac{1}{16}y_1^3+\frac{1}{16}y_2^1+\frac{1}{16}y_2^2+\frac{1}{16}y_2^3+\frac{1}{16}y_3^1+\frac{1}{16}y_3^2+\frac{1}{8}y_3^3\le 1 \tag{1.307}$$
$$-1/3x_1-4/9x_2-11/9x_3+1/9y_1^1+1/9y_1^2+1/9y_2^2+1/9y_3^1+1/9y_3^2+1/9y_3^3\le 1 \tag{1.308}$$
$$-5/3x_1-53/8x_2-67/8x_3+\frac{1}{3}y_1^2+5/24y_1^3+\frac{1}{2}y_2^1+5/8y_2^2+\frac{1}{2}y_2^3+7/24y_3^1+5/8y_3^2+5/8y_3^3\le 1 \tag{1.309}$$
$$-5/3x_1-5/4x_2-4x_3+\frac{1}{3}y_1^2+5/24y_1^3+\frac{1}{4}y_2^2+\frac{1}{3}y_3^2+\frac{1}{3}y_3^3\le 1 \tag{1.310}$$
$$-5/3x_1-22/9x_2-4x_3+\frac{1}{3}y_1^2+5/24y_1^3+1/9y_2^1+\frac{1}{3}y_2^2+1/9y_2^3+\frac{1}{3}y_3^2+\frac{1}{3}y_3^3\le 1 \tag{1.311}$$
$$-1/2x_1-15/16x_2-103/80x_3+3/16y_1^2+\frac{1}{16}y_1^3+\frac{1}{16}y_2^1+3/16y_2^2+\frac{1}{16}y_2^3+3/16y_3^2+\frac{1}{10}y_3^3\le 1 \tag{1.312}$$
$$-1/3x_1-1/8x_2-9/10x_3+\frac{1}{6}y_1^2+1/24y_1^3+\frac{1}{8}y_2^2+\frac{1}{6}y_3^2+1/15y_3^3\le 1 \tag{1.313}$$
$$-5/3x_1-11/12x_2-4x_3+\frac{1}{3}y_1^2+13/60y_1^3+11/60y_2^2+1/15y_2^3+\frac{1}{3}y_3^2+\frac{1}{3}y_3^3\le 1 \tag{1.314}$$
$$-5/3x_1-12x_2-14x_3+\frac{1}{3}y_1^2+2/15y_1^3+1y_2^1+1y_2^2+4/5y_2^3+2/3y_3^1+1y_3^2+1y_3^3\le 1 \tag{1.315}$$
$$-3/5x_1-34/5x_2-188/25x_3+\frac{1}{5}y_1^2+3/5y_2^1+3/5y_2^2+2/5y_2^3+2/5y_3^1+3/5y_3^2+13/25y_3^3\le 1 \tag{1.316}$$
$$-5/3x_1-53/15x_2-4x_3+\frac{1}{3}y_1^2+4/15y_1^3+4/15y_2^1+\frac{1}{3}y_2^2+4/15y_2^3+\frac{1}{3}y_3^2+\frac{1}{3}y_3^3\le 1 \tag{1.317}$$
$$-1/7x_1-10/21x_2-16/35x_3+1/7y_1^2+1/21y_2^1+1/7y_2^2+1/7y_3^2+1/35y_3^3\le 1 \tag{1.318}$$
$$-5/3x_1-22/7x_2-4x_3+5/21y_1^2+\frac{1}{3}y_1^3+5/21y_2^2+\frac{1}{3}y_2^3+\frac{1}{3}y_3^2+\frac{1}{3}y_3^3\le 1 \tag{1.319}$$
$$-5/3x_1-4x_2-4x_3+\frac{1}{3}y_1^2+\frac{1}{3}y_1^3+\frac{1}{3}y_2^2+\frac{1}{3}y_2^3+\frac{1}{3}y_3^2+\frac{1}{3}y_3^3\le 1 \tag{1.320}$$
$$-7x_1-12x_2-63/8x_3+7/8y_1^2+1y_1^3+1y_2^2+1y_2^3+\frac{1}{2}y_3^1+\frac{1}{2}y_3^2+5/8y_3^3\le 1 \tag{1.321}$$
$$-56/11x_1-93/11x_2-105/11x_3+7/11y_1^2+8/11y_1^3+8/11y_2^2+8/11y_2^3+7/11y_3^1+7/11y_3^2+8/11y_3^3\le 1 \tag{1.322}$$
$$-7x_1-12x_2-7/4x_3+7/8y_1^2+1y_1^3+1y_2^2+1y_2^3+\frac{1}{4}y_3^3\le 1 \tag{1.323}$$
$$-7x_1-12x_2-63/52x_3+47/52y_1^2+1y_1^3+1y_2^2+1y_2^3+1/13y_3^2+9/52y_3^3\le 1 \tag{1.324}$$
$$-3x_1-75/14x_2-3/8x_3+3/8y_1^2+\frac{1}{2}y_1^3+3/7y_2^2+\frac{1}{2}y_2^3+\frac{1}{8}y_3^3\le 1 \tag{1.325}$$
$$-21/8x_1-31/8x_2-7/8x_3+3/8y_1^2+3/8y_1^3+3/8y_2^2+3/8y_2^3+\frac{1}{8}y_3^2+\frac{1}{8}y_3^3\le 1 \tag{1.326}$$
$$-7x_1-12x_2-4x_3+1y_1^2+1y_1^3+1y_2^2+1y_2^3+\frac{1}{3}y_3^2+\frac{1}{3}y_3^3\le 1 \tag{1.327}$$
$$-21/10x_1-29/10x_2-7/2x_3+3/10y_1^2+3/10y_1^3+3/10y_2^2+3/10y_2^3+\frac{1}{5}y_3^1+3/10y_3^2+3/10y_3^3\le 1 \tag{1.328}$$
$$-3/5x_1-48/35x_2-2x_3+\frac{1}{5}y_1^3+3/35y_2^2+\frac{1}{5}y_2^3+\frac{1}{5}y_3^1+\frac{1}{5}y_3^3\le 1 \tag{1.329}$$
$$-7x_1-12x_2-14x_3+4/5y_1^2+1y_1^3+1y_2^2+1y_2^3+1y_3^1+4/5y_3^2+1y_3^3\le 1 \tag{1.330}$$
$$-7x_1-12x_2+1y_1^2+1y_1^3+1y_2^2+1y_2^3\le 1 \tag{1.331}$$
$$-7/2x_1-11/2x_2-7x_3+\frac{1}{2}y_1^2+\frac{1}{2}y_1^3+\frac{1}{2}y_2^2+\frac{1}{2}y_2^3+\frac{1}{2}y_3^1+\frac{1}{2}y_3^2+\frac{1}{2}y_3^3\le 1 \tag{1.332}$$
$$-1/4x_2-7/16x_3+\frac{1}{16}y_1^2+\frac{1}{16}y_1^3+\frac{1}{16}y_2^2+\frac{1}{16}y_2^3+\frac{1}{16}y_3^1+\frac{1}{16}y_3^2+\frac{1}{16}y_3^3\le 1 \tag{1.333}$$
$$-5/3x_1-62/21x_2-4x_3+\frac{1}{3}y_1^2+5/21y_1^3+\frac{1}{3}y_2^2+5/21y_2^3+\frac{1}{3}y_3^2+\frac{1}{3}y_3^3\le 1 \tag{1.334}$$
$$-7/5x_1-9/5x_2-6/5x_3+\frac{1}{5}y_1^1+\frac{1}{5}y_1^3+\frac{1}{5}y_2^2+\frac{1}{5}y_2^3+\frac{1}{5}y_3^3\le 1 \tag{1.335}$$
$$-60/89x_1-105/89x_2-211/89x_3+20/89y_1^1+9/89y_1^3+20/89y_2^1+5/89y_2^2+9/89y_2^3+20/89y_3^1+20/89y_3^3\le 1 \tag{1.336}$$
$$-150/209x_1-15/11x_2-541/209x_3+50/209y_1^1+9/209y_1^3+50/209y_2^1+15/209y_2^2+24/209y_2^3+50/209y_3^1+50/209y_3^3\le 1 \tag{1.337}$$
$$-131/90x_1-13/15x_2-2x_3+29/90y_1^1+\frac{1}{5}y_1^3+\frac{1}{5}y_2^1+4/135y_2^2+7/90y_2^3+\frac{1}{5}y_3^1+\frac{1}{5}y_3^3\le 1 \tag{1.338}$$
$$-3/10x_1-2/5x_2-x_3+\frac{1}{10}y_1^1+\frac{1}{10}y_1^2+\frac{1}{10}y_1^3+\frac{1}{10}y_2^2+\frac{1}{10}y_3^1+\frac{1}{10}y_3^2+\frac{1}{10}y_3^3\le 1 \tag{1.339}$$
$$-3/10x_1-4/5x_2-3/5x_3+\frac{1}{10}y_1^1+\frac{1}{10}y_1^2+\frac{1}{10}y_1^3+\frac{1}{10}y_2^1+\frac{1}{10}y_2^2+\frac{1}{10}y_2^3+\frac{1}{10}y_3^3\le 1 \tag{1.340}$$
$$-5/6x_1-5/3x_2-x_3+\frac{1}{6}y_1^1+\frac{1}{6}y_1^2+\frac{1}{6}y_1^3+\frac{1}{6}y_2^1+\frac{1}{6}y_2^2+\frac{1}{6}y_2^3+\frac{1}{6}y_3^2\le 1 \tag{1.341}$$
$$-5/12x_1-11/12x_2-1/2x_3+\frac{1}{12}y_1^1+\frac{1}{12}y_1^2+\frac{1}{12}y_1^3+\frac{1}{6}y_2^1+\frac{1}{12}y_2^2+\frac{1}{12}y_2^3+\frac{1}{12}y_3^1+\frac{1}{12}y_3^2\le 1 \tag{1.342}$$

$$-3/8x_1-5/8x_2-11/8x_3+\frac{1}{8}y_1^1+\frac{1}{8}y_2^2+\frac{1}{8}y_3^1+\frac{1}{8}y_3^2+\frac{1}{8}y_3^3\le 1 \tag{1.343}$$

$$-3/14x_1-2/7x_2-17/14x_3+1/14y_1^1+1/14y_1^2+1/14y_1^3+1/14y_2^2+1/14y_2^3+1/14y_3^1+1/14y_3^2+1/7y_3^3\le 1 \tag{1.344}$$

$$-3/8x_1-9/8x_2-7/8x_3+\frac{1}{8}y_1^1+\frac{1}{8}y_2^1+\frac{1}{8}y_2^2+\frac{1}{8}y_2^3+\frac{1}{8}y_3^3\le 1 \tag{1.345}$$

$$-3/14x_1-13/14x_2-3/7x_3+1/14y_1^1+1/14y_1^2+1/14y_1^3+1/14y_2^1+1/7y_2^2+1/14y_2^3+1/14y_3^2+1/14y_3^3\le 1 \tag{1.346}$$

$$-3/14x_1-2/7x_2-13/14x_3+1/14y_1^1+1/14y_1^2+1/14y_1^3+1/14y_2^1+1/14y_2^2+1/7y_3^1+1/14y_3^2+1/14y_3^3\le 1 \tag{1.347}$$

$$-5/6x_1-2/3x_2-2x_3+\frac{1}{6}y_1^1+\frac{1}{6}y_1^2+\frac{1}{6}y_1^3+\frac{1}{6}y_2^1+\frac{1}{6}y_3^1+\frac{1}{6}y_3^2+\frac{1}{6}y_3^3\le 1 \tag{1.348}$$

$$-5/12x_1-1/3x_2-5/4x_3+\frac{1}{12}y_1^1+\frac{1}{12}y_1^2+\frac{1}{12}y_1^3+\frac{1}{12}y_2^1+\frac{1}{12}y_2^2+\frac{1}{12}y_3^1+\frac{1}{6}y_3^2+\frac{1}{12}y_3^3\le 1 \tag{1.349}$$

$$-7/2x_1-11/2x_2-7/2x_3+\frac{1}{2}y_1^2+7/16y_1^3+\frac{1}{2}y_2^2+\frac{1}{2}y_2^3+\frac{1}{2}y_3^2\le 1 \tag{1.350}$$

$$-40/7x_1-115/7x_2-75/7x_3+6/7y_1^1+\frac{5}{7}y_1^3+6/7y_2^1+\frac{5}{7}y_2^2+10/7y_2^3+6/7y_3^1+\frac{5}{7}y_3^2+\frac{5}{7}y_3^3\le 1 \tag{1.351}$$

$$-49/6x_1-33/2x_2-6x_3+7/6y_1^1+1y_1^3+\frac{1}{2}y_2^2+3/2y_2^3+\frac{1}{6}y_3^1+2/3y_3^2\le 1 \tag{1.352}$$

$$-3/2x_1-23/6x_2-3/2x_3+\frac{1}{3}y_1^1+\frac{1}{6}y_1^3+\frac{1}{6}y_2^1+\frac{1}{6}y_2^2+\frac{1}{3}y_2^3+\frac{1}{6}y_3^1+\frac{1}{6}y_3^2\le 1 \tag{1.353}$$

$$-49/6x_1-211/12x_2-21/4x_3+7/6y_1^1+1y_1^3+\frac{1}{6}y_2^1+7/12y_2^2+19/12y_2^3+\frac{1}{6}y_3^1+7/12y_3^2\le 1 \tag{1.354}$$

$$-8x_1-23x_2-15x_3+8/7y_1^1+1y_1^3+1y_2^1+1y_2^2+2y_2^3+8/7y_3^1+1y_3^2+1y_3^3\le 1 \tag{1.355}$$

$$-49/6x_1-23x_2-14x_3+7/6y_1^1+1y_1^3+1y_2^1+1y_2^2+2y_2^3+1y_3^1+1y_3^2+\frac{5}{6}y_3^3\le 1 \tag{1.356}$$

$$-8/3x_1-23/3x_2-19/3x_3+10/21y_1^1+\frac{1}{3}y_1^3+2/3y_2^1+\frac{1}{3}y_2^2+2/3y_2^3+2/3y_3^1+\frac{1}{3}y_3^2+\frac{1}{3}y_3^3\le 1 \tag{1.357}$$

$$-8/3x_1-71/7x_2-75/7x_3+10/21y_1^1+\frac{1}{3}y_1^3+6/7y_2^1+11/21y_2^2+6/7y_2^3+6/7y_3^1+\frac{5}{7}y_3^2+\frac{5}{7}y_3^3\le 1 \tag{1.358}$$

$$-17/6x_1-12x_2-14x_3+\frac{1}{2}y_1^1+\frac{1}{3}y_1^3+1y_2^1+2/3y_2^2+1y_2^3+1y_3^1+1y_3^2+\frac{5}{6}y_3^3\le 1 \tag{1.359}$$

$$-51/14x_1-12x_2-14x_3+9/14y_1^1+1/7y_1^2+1y_2^1+\frac{1}{2}y_2^2+1y_2^3+1y_3^1+1y_3^2+6/7y_3^3\le 1 \tag{1.360}$$

$$-21/2x_1-22/3x_2-21/2x_3+3/2y_1^1+1y_1^2+\frac{1}{2}y_2^1+2/3y_2^3+7/6y_3^2+\frac{1}{6}y_3^3\le 1 \tag{1.361}$$

$$-13/2x_1-11/2x_2-23/2x_3+1y_1^1+\frac{1}{2}y_1^2+\frac{1}{2}y_2^1+\frac{1}{2}y_2^3+\frac{1}{2}y_3^1+1y_3^2+\frac{1}{2}y_3^3\le 1 \tag{1.362}$$

$$-21/2x_1-12x_2-23x_3+3/2y_1^1+1y_1^2+1y_2^1+\frac{1}{2}y_2^2+1y_2^3+1y_3^1+2y_3^2+1y_3^3\le 1 \tag{1.363}$$

$$-21/2x_1-77/12x_2-67/4x_3+3/2y_1^1+1y_1^2+\frac{1}{2}y_2^1+7/12y_2^3+\frac{1}{2}y_3^1+19/12y_3^2+7/12y_3^3\le 1 \tag{1.364}$$

$$-7/3x_1-11/3x_2-2x_3+\frac{1}{3}y_1^1+\frac{1}{3}y_1^2+\frac{1}{3}y_2^2+\frac{1}{3}y_2^3+\frac{1}{3}y_3^2\le 1 \tag{1.365}$$

$$-15/16x_2-7/16x_3+\frac{1}{16}y_1^1+\frac{1}{16}y_1^3+\frac{1}{16}y_2^1+\frac{1}{16}y_2^2+\frac{1}{8}y_2^3+\frac{1}{16}y_3^1+\frac{1}{16}y_3^2+\frac{1}{16}y_3^3\le 1 \tag{1.366}$$

$$-x_1-11/5x_2-6/5x_3+\frac{1}{5}y_1^1+\frac{1}{5}y_1^2+\frac{1}{5}y_2^1+\frac{1}{5}y_2^2+\frac{1}{5}y_2^3+\frac{1}{5}y_3^2\le 1 \tag{1.367}$$

$$-5/4x_1-11/4x_2-7/4x_3+\frac{1}{4}y_1^1+\frac{1}{4}y_2^1+\frac{1}{4}y_2^2+\frac{1}{4}y_2^3+\frac{1}{4}y_3^2\le 1 \tag{1.368}$$

$$-5/12x_1-5/4x_2-1/2x_3+\frac{1}{12}y_1^1+\frac{1}{12}y_1^2+\frac{1}{12}y_1^3+\frac{1}{12}y_2^1+\frac{1}{12}y_2^2+\frac{1}{6}y_2^3+\frac{1}{12}y_3^2+\frac{1}{12}y_3^3\le 1 \tag{1.369}$$

$$-5/3x_1-472/153x_2-4x_3+32/153y_1^2+\frac{1}{3}y_1^3+31/153y_2^1+32/153y_2^2+\frac{1}{3}y_2^3+\frac{1}{3}y_3^2+\frac{1}{3}y_3^3\le 1 \tag{1.370}$$

$$-7x_1-12x_2-14x_3+1y_1^2+6/7y_1^3+1y_2^2+1y_2^3+1y_3^1+1y_3^2+6/7y_3^3\le 1 \tag{1.371}$$

$$-7x_1-12x_2-7/2x_3+1y_1^2+7/8y_1^3+1y_2^2+1y_2^3+\frac{1}{2}y_3^2\le 1 \tag{1.372}$$

$$-7x_1-12x_2-91/8x_3+1y_1^2+7/8y_1^3+1y_2^2+1y_2^3+3/4y_3^1+7/8y_3^2+3/4y_3^3\le 1 \tag{1.373}$$

$$-56/9x_1-95/9x_2-35/3x_3+8/9y_1^2+7/9y_1^3+8/9y_2^2+8/9y_2^3+7/9y_3^1+8/9y_3^2+7/9y_3^3\le 1 \tag{1.374}$$

$$-1/7x_1-18/49x_2-8/7x_3+1/7y_1^2+1/7y_2^2+1/49y_2^3+1/7y_3^1+1/7y_3^2\le 1 \tag{1.375}$$

$$-x_1-51/28x_2-1/8x_3+\frac{1}{4}y_1^2+\frac{1}{8}y_1^3+\frac{1}{4}y_2^2+1/7y_2^3+\frac{1}{8}y_3^2\le 1 \tag{1.376}$$

$$-7x_1-12x_2-77/58x_3+1y_1^2+53/58y_1^3+1y_2^2+1y_2^3+11/58y_3^2+3/29y_3^3\le 1 \tag{1.377}$$

$$-5/3x_1-454/159x_2-4x_3+\frac{1}{3}y_1^2+34/159y_1^3+9/53y_2^1+\frac{1}{3}y_2^2+34/159y_2^3+\frac{1}{3}y_3^2+\frac{1}{3}y_3^3\le 1 \tag{1.378}$$

$$-1/14x_1-2/7x_2-4/7x_3+1/14y_1^1+1/14y_1^2+1/14y_1^3+1/14y_2^3+1/14y_3^1+1/14y_3^2+1/14y_3^3\le 1 \tag{1.379}$$

$$-5/12x_1-1/3x_2-17/12x_3+\frac{1}{12}y_1^1+\frac{1}{12}y_1^2+\frac{1}{12}y_1^3+\frac{1}{12}y_2^1+\frac{1}{12}y_2^3+\frac{1}{12}y_3^1+\frac{1}{12}y_3^2+\frac{1}{6}y_3^3\le 1 \tag{1.380}$$

$$-3/14x_1-11/14x_2-3/7x_3+1/14y_1^1+1/14y_1^2+1/14y_1^3+1/7y_2^1+1/14y_2^2+1/14y_2^3+1/14y_3^1+1/14y_3^3\le 1 \tag{1.381}$$

$$-1/12x_1-5/12x_2-3/4x_3+\frac{1}{12}y_1^1+\frac{1}{12}y_2^3+\frac{1}{12}y_3^1+\frac{1}{12}y_3^2+\frac{1}{12}y_3^3\le 1 \tag{1.382}$$

$$-1/16x_1-1/4x_2-15/16x_3+\frac{1}{16}y_1^1+\frac{1}{16}y_1^2+\frac{1}{16}y_1^3+\frac{1}{16}y_2^2+\frac{1}{16}y_2^3+\frac{1}{16}y_3^1+\frac{1}{8}y_3^2+\frac{1}{16}y_3^3\le 1 \tag{1.383}$$

$$-7/8x_1-1/2x_2-x_3+\frac{1}{8}y_1^1+\frac{1}{8}y_1^2+\frac{1}{8}y_1^3+\frac{1}{8}y_2^3+\frac{1}{8}y_3^2+\frac{1}{8}y_3^3\le 1 \tag{1.384}$$

$$-7/10x_1-2/5x_2-3/5x_3+\frac{1}{10}y_1^1+\frac{1}{10}y_1^2+\frac{1}{10}y_1^3+\frac{1}{10}y_2^2+\frac{1}{10}y_2^3+\frac{1}{10}y_3^2+\frac{1}{10}y_3^3\le 1 \tag{1.385}$$

$$-7/4x_1 - 5/2x_2 - 3/2x_3 + \frac{1}{4}y_1^1 + \frac{1}{4}y_1^2 + \frac{1}{4}y_1^3 + \frac{1}{4}y_2^2 + \frac{1}{4}y_2^3 + \frac{1}{4}y_3^2 \leq 1 \tag{1.386}$$

$$-7/6x_1 - 4/3x_2 - x_3 + \frac{1}{6}y_1^1 + \frac{1}{6}y_1^2 + \frac{1}{6}y_1^3 + \frac{1}{6}y_2^2 + \frac{1}{6}y_2^3 + \frac{1}{6}y_3^3 \leq 1 \tag{1.387}$$

$$-7/6x_1 - 2/3x_2 - 5/3x_3 + \frac{1}{6}y_1^1 + \frac{1}{6}y_1^2 + \frac{1}{6}y_1^3 + \frac{1}{6}y_2^2 + \frac{1}{6}y_3^2 + \frac{1}{6}y_3^3 \leq 1 \tag{1.388}$$

$$-7/2x_1 - 5x_2 - 4x_3 + \frac{1}{2}y_1^1 + \frac{1}{2}y_1^2 + \frac{1}{2}y_1^3 + \frac{1}{2}y_2^3 + \frac{1}{2}y_3^2 \leq 1 \tag{1.389}$$

$$-7/12x_1 - 5/6x_2 - 7/6x_3 + \frac{1}{6}y_1^1 + \frac{1}{6}y_1^2 + \frac{1}{12}y_1^3 + \frac{1}{12}y_2^1 + \frac{1}{12}y_2^2 + \frac{1}{12}y_2^3 + \frac{1}{12}y_3^1 + \frac{1}{6}y_3^2 \leq 1 \tag{1.390}$$

$$-7/16x_1 - 7/8x_2 - 1/2x_3 + \frac{1}{8}y_1^1 + \frac{1}{16}y_1^2 + \frac{1}{8}y_1^3 + \frac{1}{16}y_2^1 + \frac{1}{8}y_2^3 + \frac{1}{16}y_3^1 + \frac{1}{16}y_3^2 + \frac{1}{16}y_3^3 \leq 1 \tag{1.391}$$

$$-7/2x_1 - 4x_2 - 5x_3 + \frac{1}{2}y_1^1 + \frac{1}{2}y_1^2 + \frac{1}{2}y_1^3 + \frac{1}{2}y_2^2 + \frac{1}{2}y_3^3 \leq 1 \tag{1.392}$$

$$-1/2x_1 - 6/7x_2 - 5/7x_3 + 1/7y_1^1 + 1/7y_1^2 + 1/14y_1^3 + 1/14y_2^1 + 1/7y_2^2 + 1/14y_3^1 + 1/14y_3^2 + 1/14y_3^3 \leq 1 \tag{1.393}$$

$$-1/2x_1 - 4/7x_2 - 8/7x_3 + 1/7y_1^1 + 1/14y_1^2 + 1/7y_1^3 + 1/14y_2^1 + 1/14y_2^2 + 1/14y_2^3 + 1/14y_3^1 + 1/7y_3^3 \leq 1 \tag{1.394}$$

$$-1/16x_1 - 1/4x_2 - 13/16x_3 + \frac{1}{16}y_1^1 + \frac{1}{16}y_1^2 + \frac{1}{16}y_1^3 + \frac{1}{16}y_2^1 + \frac{1}{16}y_2^3 + \frac{1}{8}y_3^1 + \frac{1}{16}y_3^2 + \frac{1}{16}y_3^3 \leq 1 \tag{1.395}$$

$$-7/2x_1 - 6x_2 - 3x_3 + \frac{1}{2}y_1^1 + \frac{1}{2}y_1^2 + \frac{1}{2}y_1^3 + \frac{1}{2}y_2^1 + \frac{1}{2}y_2^2 + \frac{1}{2}y_2^3 + \frac{1}{2}y_3^1 \leq 1 \tag{1.396}$$

$$-7/10x_1 - 13/10x_2 - 3/5x_3 + \frac{1}{10}y_1^1 + \frac{1}{10}y_1^2 + \frac{1}{10}y_1^3 + \frac{1}{10}y_2^1 + \frac{1}{5}y_2^2 + \frac{1}{10}y_2^3 + \frac{1}{10}y_3^1 + \frac{1}{10}y_3^2 \leq 1 \tag{1.397}$$

$$-7/10x_1 - 3/2x_2 - 3/5x_3 + \frac{1}{10}y_1^1 + \frac{1}{10}y_1^2 + \frac{1}{10}y_1^3 + \frac{1}{10}y_2^1 + \frac{1}{10}y_2^2 + \frac{1}{5}y_2^3 + \frac{1}{10}y_3^1 + \frac{1}{10}y_3^3 \leq 1 \tag{1.398}$$

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