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CPLOAS_2 V2.10 Verification Report

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Abstract

A series of test cases designed to verify the correct implementation of several features of the CPLOAS_2 program are documented. CPLOAS_2 is used to calculate the probability of loss of assured safety (PLOAS) for a weak link (WL)/strong link (SL) system. CPLOAS_2 takes physical properties (e.g., temperature, pressure, etc.) of a WL/SL system and uses these properties and definitions of link failure properties in probabilistic calculations to determine PLOAS. The features being tested include (i) six aleatory distribution forms, (ii) five numerical procedures for the determination of PLOAS (i.e., one quadrature procedure, two simple random sampling procedures, and two importance sampling procedures), and (iii) time and environmental margin calculations. All tests were performed with CPLOAS_2 version 2.10.

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Contents

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NOMENCLATURE

α	aleatory uncertainty associated with p
β	aleatory uncertainty associated with q
CPLOAS_2	Refers to a software package for Calculation Probability Loss of Assure Safety
CDF	Cumulative distribution function
IMP	Importance sampling
LOAS	Loss of Assured Safety
MC	Monte Carlo sampling
PLOAS	Probability of Loss of Assured Safety
P1	Pattern number 1
p	Property value
q	Failure value
SL	Strong Link
WL	Weak Link

1. Introduction

The program CPLOAS_2 is used to calculate the probability of loss of assured safety (PLOAS) for a weak link (WL)/strong link (SL) system [1]. Specifically, CPLOAS_2 takes physical properties (e.g., temperature, pressure, etc.) of a WL/SL system as input and uses these properties and definitions of link failure properties in probabilistic calculations to determine PLOAS. As described in App. C, CPLOAS_2 implements five numerical procedures to calculate PLOAS: one quadrature procedure, two Monte Carlo (i.e., simple random) sampling procedures (MC1 and MC2), and two importance sampling procedures (IMP1 and IMP2).

It is anticipated that most calculations of PLOAS with CPLOAS_2 will be performed with the quadrature procedure.

The sampling procedures MC1, MC2, IMP1 and IMP2 are included in CPLOAS_2 primarily as a means to verify the implementation of the quadrature procedure for the determination of PLOAS. With respect to verification, MC1 and IMP1 use the same pre-calculated cumulative distribution functions (CDFs) for link failure time that are used in the quadrature calculation. In contrast, MC2 and IMP2 avoid use of these pre-calculated CDFs by (i) sampling the α's and β's (i.e., the aleatory uncertainty in property values and failure values, respectively) for individual links used in the definition of the indicated CDFs and (ii) then determining the resultant link failure times (see App. C for details). This results in MC2 and IMP2 being less computationally efficient than MC1 and IMP1. However, the real purpose of having MC2 and IMP2 is to have a verification check on MC1, IMP1 and the quadrature procedure that does not involve the use of the pre-calculated CDFs for link failure time.

In CPLOAS_2, PLOAS calculated by quadrature is presented as a point value at each time step. For the four sampling techniques, CPLOAS_2 has two output options. One option outputs the mean and standard deviation (i.e., standard error) of the mean for the sampled distribution at each time step, and the second option outputs a confidence interval (two-sided) for the mean at each time step (see App. C for details). The test cases in this document use 95% confidence intervals (centered intervals between 2.5% and 97.5% percentiles; see App. C for details). For the sampling techniques, a confidence interval is truncated if results are outside of the interval of [0,1]. Quadrature results are not truncated in CPLOAS_2. In WL/SL systems with extremely low PLOAS values, time and environmental margins are more informative results than PLOAS values. Time and environmental margin results are provided as sets of sampled values. All results in CPLOAS_2 are reported to four significant places. CPLOAS_2 output is further described in [2].

1.1. Purpose of report

This report documents a series of test cases designed to verify correct implementation of several features of the CPLOAS_2 program. All test cases were performed with CPLOAS_2 version 2.10 [2].

The test cases in this report are designed to verify the determination of a range of PLOAS values. Specifically, the test cases in this report are designed to include PLOAS values in each order of magnitude between 10^{-6} and 1.0. In the results descriptions, the terms high/moderate/low are used to describe PLOAS values. The term "high probability" is used to describe cases producing PLOAS values on the order of 10^0 to 10^{-2}. The term "moderate probability" is used to describe cases producing PLOAS values on the order of 10^{-2} to 10^{-4}. The term "low probability" is used to describe cases producing PLOAS values on the order of 10^{-4} to 10^{-6}.

The test cases in Sections 2 and 3 are designed to verify correct implementation of the six aleatory distribution forms: normal, lognormal, triangular, log triangular, uniform, and log uniform. The test cases in Section 2 are designed to produce results that can be compared to analytical solutions; verification is achieved by comparing the results of the five numerical techniques to the analytical solutions. The test cases in Section 3 do not have analytical solutions. For these results, verification is achieved by comparing PLOAS produced by quadrature to the results produced by the sampling-based techniques (within the appropriate application domains of the techniques).

The test cases in Section 4 are designed to verify the correct implementation of the time and environmental margin calculations in CPLOAS_2. In this set of problems, verification is achieved by comparing the mean time and environmental margins from CPLOAS_2 to expected value analyses performed in the Excel macro "Race Metrics V2" [3].

Section 5 contains a full scope test problem designed to match the size and scope of planned CPLOAS_2 uses. For these results, verification is achieved by verifying that the five numerical techniques provide similar results.

The report then ends with (i) a summary of programming errors and desired program enhancements identified in the testing of CPLOAS_2 and associated corrections/enhancements to CPLOAS_2 (Section 6) and (ii) a brief summary section (Section 7).

1.2. Range of applicability of numerical techniques

The five numerical techniques have different ranges of applicability and different expected accuracy within those ranges. Approximate ranges are provided in Table 1. For the sampling techniques, the applicability is conditional on the use of a sample of size 1 million (i.e., 10^6).

Table 1: Approximate PLOAS ranges that can be correctly calculated by the indicated sampling technique

PLOAS range	Quad	MC1	MC2	IMP1	IMP2
High (1 to 10^{-2})	Yes	Yes	Yes	No	No
Moderate (10^{-2} to 10^{-4})	Yes	Yes	Yes	Yes	No
Low (10^{-4} to 10^{-6})	Yes	Yes	Yes	Yes	Yes
Very Low (10-^{6} to 10^{-8})	Yes	No	No	Yes	Yes
Below (10^{-8})	Yes	No	No	?	?

For user cases with very low PLOAS or lower, quadrature may be the only technique that will calculate a PLOAS below 10^{-8}. The two importance sampling procedures may also be effective for such low PLOAS values but have not been extensively tested for PLOAS values of this size. For these cases, and cases in the "very low category", time and environmental margins calculations can provide more insight into system safety than PLOAS values.

The accuracy of sampling procedures is limited by the number of samples. Due to computational limits, it is anticipated that users will run at most one million (10^6) samples. Monte Carlo sampling accuracy becomes limited below *PLOAS* = 10/*nsamples*; so MC1 and MC2 are most suited to calculate PLOAS values greater than 10^{-5}. It is unlikely that MC1 or MC2 can be used to produce reliable PLOAS estimates below 10^{-6}.

Importance sampling must be defined to enhance the observed occurrence of rare events that are important to the value of the quantity being calculated. The two importance sampling procedures implemented in CPLOAS_2 are intended for problems that have low PLOAS values (e.g., less than 10^{-4}). In such situations, the WLs tend to fail much earlier in time that the SLs. Thus, to increase the number of observed situations in which SLs fail earlier than WLs, the two implemented importance sampling procedures are defined so that they overemphasize early SL failures and late WL failures[1].

However, when WLs and SLs have similar failure times or when the SLs are likely to fail earlier that the WLs, this importance sampling procedure inappropriately over skews the sampling of the failure times for the WLs and SLs. In turn, this leads to poor estimates of PLOAS. An inappropriate choice of an importance sampling distribution will actually produce less accurate results than would be obtained with simple random sampling. This effect results because an inappropriate choice of an importance sampling distribution will force an excessive number of observations into regions of the sample space under consideration that do not need to be over or under emphasized. In turn, this reduces the convergence rate of the resultant estimate of interest.

The preceding importance sampling procedures should work reasonably well when the PLOAS value being calculated is low and a large number of links that are unimportant to the determination of PLOAS are not present. When the PLOAS value is large or a large number of links are present, there can be poor convergence of the resulting approximations to PLOAS. Results may not be valid for IMP1 for probabilities above 10^{-2} or for IMP2 for probabilities above 10^{-4}; however, the indicated ranges of validity will also depend to some extent on the number of links involved and the properties of the individual links.

Figure 1 summarizes the results of PLOAS calculations obtained in this verification study for the four sampling-based techniques with samples of size 1,000,000. As indicated by the size of

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[1] Two importance sampling procedures are implemented in CPLOAS_2: IMP1 and IMP2. In IMP1, the CDF for failure time for each WL is sampled with a right triangular importance sampling distribution, and the CDF for failure time for each SL is sampled with a left triangular importance sampling distribution. In IMP2, right triangular importance sampling distributions are used for sampling from the CDFs for WL β's and SL α's and left triangular importance sampling distributions are used for sampling from the CDFs for WL α's and SL β's. For both IMP1 and IMP2, the indicated importance sampling distributions result in an overemphasis for large (i.e., late) WL failure times and small (i.e., early) SL failure times. See App. C for additional details.

the standard deviations for PLOAS obtained with IMP1 and IMP2, these procedures perform very poorly in the estimation of large PLOAS values.

While CPLOAS_2 will output PLOAS values for sampling results outside of the appropriate domain of a technique, it is not meaningful to use these results to verify a quadrature calculation. In this report, PLOAS values calculated outside what is felt to be the appropriate application domain of a sampling technique are highlighted in yellow to indicate that they should not be used to test the accuracy of the quadrature procedure.

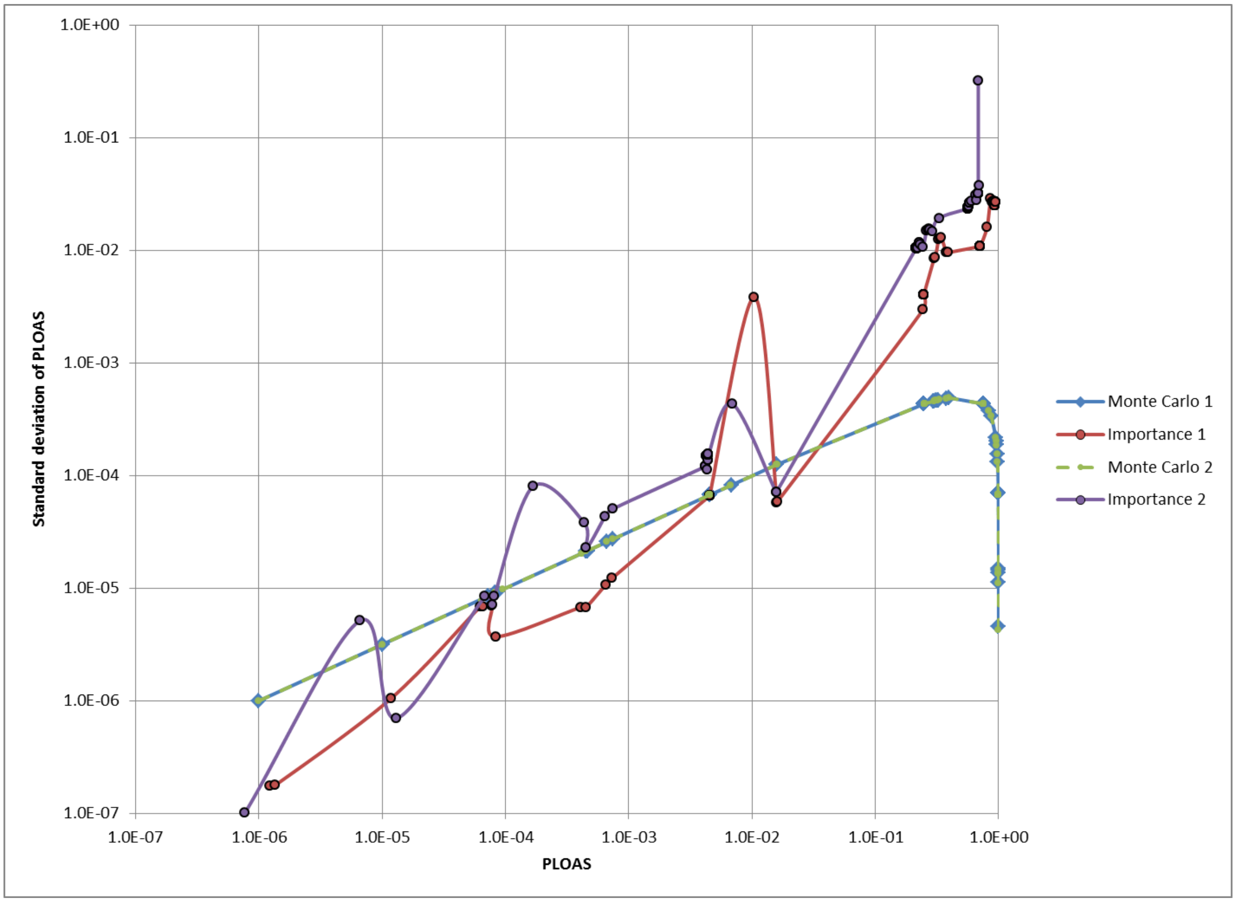

Figure 1: Summary of PLOAS calculations obtained in this verification study for the four sampling techniques with samples of size 1,000,000.

1.3. Test Acceptability criteria

1.3.1. Exact cases

For tests with known analytical (exact) PLOAS values, the PLOAS values predicted by the five CPLOAS_2 numerical techniques are compared to the analytical result. The performance of CPLOAS_2 is considered to be acceptable if the percent difference between the calculated PLOAS (either the quadrature value or the mean value from the sampling technique) and the analytical approach is less than 10% (in absolute value). The percent change is calculated as:

$$\frac{PLOAS_{CPLOAS2} - PLOAS_{analytic}}{PLOAS_{analytic}} \times 100\%.$$

Any PLOAS results obtained with sampling techniques used outside of their ranges of applicability are highlighted in yellow to indicate that they should not be used to test the accuracy of the quadrature procedure.

Any test cases that do not meet the acceptability criteria are discussed in the results section for those test cases.

1.3.2. All other cases

For tests without analytical results, the four sampling results are compared to the result of the quadrature calculation. This percent difference is calculated as indicated in Section 1.3.1. For probabilities greater than 10^{-4}, sampling-based results are expected to either (i) match the quadrature result within 10% or (ii) contain the quadrature result within their associated 95% confidence intervals. For probabilities below 10^{-4}, quadrature results should be within the confidence intervals provided by the appropriate sampling techniques.

Test cases that do not meet the acceptability criteria are discussed in the results section for those test cases.

Confidence intervals are used as an additional test where appropriate. The confidence interval calculations are documented in App. C. The performance for a case is considered to be acceptable if the quadrature result is within the confidence intervals calculated by those sampling techniques that are being used within the their domains of applicability. Specifically, (i) for medium probability cases, verification is achieved provided that the quadrature PLOAS falls within the MC1 and MC2 confidence intervals; (ii) for moderate probability cases,, verification is achieved provided that the quadrature PLOAS falls within the MC1 and MC2 confidence intervals; and (iii) for low probability cases, verification is achieved provided that the quadrature PLOAS falls within the MC1, MC2, IMP1 and IMP2 confidence intervals. These results are illustrated in App. B.

2. Verification of PLOAS calculations when all links have the same properties

2.1. Description of Problem Set

This verification problem set involves the assignment of the same properties to all links, with the result that PLOAS has a simple algebraic form for each WL/SL configuration that is a function of the numbers nWL and nSL of WLs and SLs.

As described in Section 6 of SAND 2012-8248 [1], when all of the links are assigned the same properties, it is possible to define an analytical solution for any combination of SLs and WLs for the four failure patterns modeled by CPLOAS_2:

- Pattern 1: Failure of **all SLs** before failure of **any WL**:
$$pF_1(\infty) = nWL!\,nSL!/(nWL + nSL)!$$
- Pattern 2: Failure of **any SL** before failure of **any WL**:
$$pF_2(\infty) = nSL/(nWL + nSL)$$
- Pattern 3: Failure of **all SLs** before failure of **all WLs**:
$$pF_3(\infty) = nWL/(nWL + nSL)$$
- Pattern 4: Failure of **any SL** before failure of **all WLs**:
$$pF_4(\infty) = 1 - [nWL!\,nSL!/(nWL + nSL)!]$$

In the example in Section 6 of the indicated SAND report, each link is defined with a uniform uncertainty on the property value ($\alpha \sim U[0.85, 1.15]$) and a triangular uncertainty on the failure value ($\beta \sim T[0.9,1.0,1.10]$). It is proposed to add more distributions combinations to the indicated example and confirm that the use of different distributions does not affect the estimate. Table 2 defines the five test cases designed to exercise each of the aleatory distributions.

Table 2: Distributions for aleatory variables α and β for test sets with suffixes a-e

Test set #	Distribution on alpha	Distribution on beta
…a	Normal(mean=1.0,stdev=0.1,trunc.=0.001)	Normal(mean=1.0,stdev=0.05,trunc.=0.001)
…b	Log Uniform(min=0.85,max=1.15)	Log Uniform(min=0.90,max=1.10)
…c	Lognormal(mean=0.0, stdev=0.1, trunc=0.001)	Lognormal(mean=0.0, stdev=0.05, trunc=0.001)
…d	Log Ttriangular(min=0.85 mode=1.0, max=1.15)	Log Triangular(min=0.9 mode=1.0, max=1.1)
…e	Uniform(min=0.85,max=1.15)	Triangular(min=0.9,mode=1.0,max=1.1)

In each test, there are 3 WLs and 9 SLs. Each link is assigned the same nominal properties and failure values from one of three sets of data. Table 3 contains the analytical solutions for the 3WL, 9SL systems under consideration. Each test case evaluates each of the four failure patterns implemented in CPLOAS_2 for each pair of variable distributions in Table 2.

Table 3: Analytical PLOAS values for a system containing 3 WLs and 9 SLs

Pattern #	Number of WLs and SLs	PLOAS (analytical)
1 (all SLs, any WL)	*nWL=3, nSL=9*	1/220
2 (any SL, any WL)	*nWL=3, nSL=9*	3/4
3 (all SLs, all WLs)	*nWL=3, nSL=9*	1/4
4 (any SL, all WLs)	*nWL=3, nSL=9*	219/220

2.2. Linear *p*, constant *q*

2.2.1. Linear *p*, constant *q*: Problem description

In this set of five test cases, $\bar{p}(t)$ is an increasing linear function, and $\bar{q}(t)$ is a constant function as indicated below

$$\bar{p}(t) = 3t + 100 \text{ and } \bar{q}(t) = 500.0.$$

and also in Figure 2. For input to CPLOAS_2, $\bar{p}(t)$ is discretized with time-steps of 0.1 min.

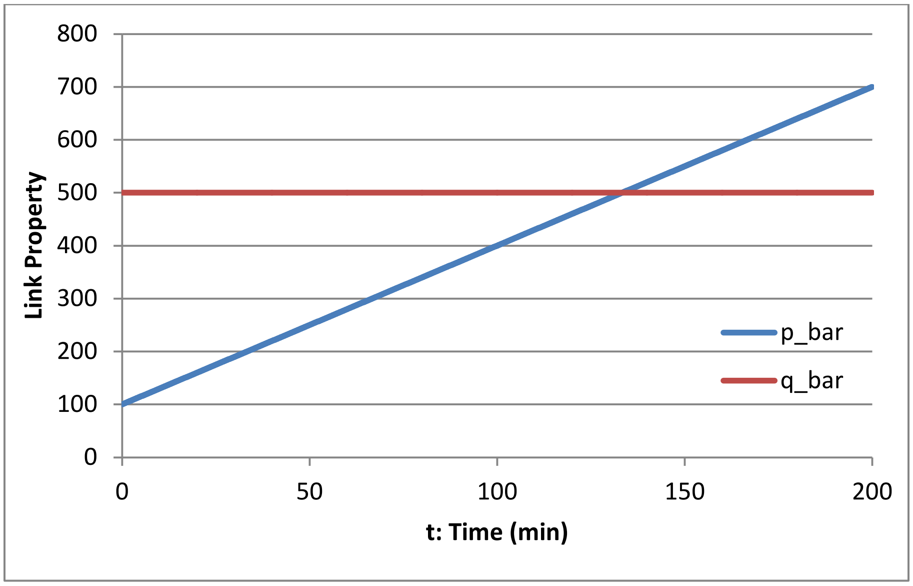

Figure 2: Nominal physical property $\bar{p}(t)$ and failure value $\bar{q}(t)$ for the linear *p*(*t*) and constant *q*(t) cases (units: temperature).

For the property set defined in Figure 2, five test cases were run to exercise each of the six uncertainty distribution forms handled by CPLOAS_2. The distributions for aleatory variables α and β associated with each of the five tests are defined in Table 2.

For each test case, CPLOAS_2 executed five different numerical procedures: a quadrature procedure and four variants of sampling-based procedures (i.e., MC1, MC2, IMP1, IMP2). The quadrature procedure was run with *nCDF*=20,000 CDF discretization steps and *nQUAD*=10,000 quadrature discretization steps. The sampling-based techniques were run with values of *nFT* = 1,000,000 and *nFTC* = 1,000,000 specified for the sampling control parameters *nFT* and *nFTC*. Values assigned to *nFT* and *nFTC* affect the calculations in the following ways: (i) For MC1, *nFT* failure times are sampled with simple random sampling for each link and then randomly combined to produce *nFTC* vectors of link failure times (see Sect. C.3 for detailed description), (ii) For MC2, *nFTC* vectors of link α and β values are sampled with simple random sampling (i.e., there is only one step in the sampling; see Sect. C.4 for detailed description), (iii) For IMP1, *nFT* failure times are sampled with importance sampling for each link and then randomly combined to produce *nFTC* vectors of link failure times (see Sect. C.5 for detailed description), and (iv) For IMP2, *nFTC* vectors of link α and β values are sampled with importance sampling (i.e., there is only one step in the sampling; see Sect. C.6 for detailed description). The random seed used to initiate each set of samplings was *nSEED*=32.

2.2.2. Linear *p*, constant *q*: Results

Results of the test cases are presented in Table 4. Owing to the high PLOAS values involved, the two importance sampling procedures are not appropriate for test cases P2, P3 and P4; and the results of their application are highlighted in yellow to distinguish these results from results obtained with the other more appropriate procedures.

The quadrature results deviate from the true values for PLOAS by less than 0.6% in all cases. MC1 predictions deviate from the true values for PLOAS by less than 0.2% in all cases. Both Monte Carlo sampling techniques deviate from the true values for PLOAS by less than 1.1% in all cases. The two MC techniques resulted in nearly identical performance for the high probability patterns (P2, P3, P4). MC1 performed slightly better than MC2 for the moderate probability pattern (P1). IMP1 showed small deviation (0.4%) for the moderate probability pattern (P1). IMP2 under-predicted the PLOAS for pattern P1 by less than 7.9% in all cases; however, IMP2 is near the limits of its applicability for pattern P1.

The performance of the quadrature, MC1, MC2, IMP1 and IMP2 techniques is acceptable for all test cases within their domains of applicability in this set of verification problems and performed similarly for all distribution forms of α and β. As indicated above, the two importance sampling procedures are inappropriate for P2, P3 and P4 and are so indicated in Table 4.

Table 4: Results from CPLOAS calculations for the system of 3 WLs, 9 SLs defined in Section 2.1 (Results highlighted in yellow obtained with procedures not appropriate for the problem under consideration)

	Analytical	Quadrature	MC1	MC2	IMP1	IMP2
Pattern 1						
Exact1a	1/220 (≈4.55E-03)	4.53E-03 (-0.4%)	4.55E-03 (0.1%)	4.56E-03 (0.3%)	4.56E-03 (0.4%)	4.42E-03 (-2.7%)
Exact1b	1/220 (≈4.55E-03)	4.53E-03 (-0.4%)	4.55E-03 (0.1%)	4.52E-03 (-0.5%)	4.56E-03 (0.4%)	4.19E-03 (-7.9%)
Exact1c	1/220 (≈4.55E-03)	4.53E-03 (-0.4%)	4.55E-03 (0.1%)	4.53E-03 (-0.3%)	4.56E-03 (0.4%)	4.41E-03 (-2.9%)
Exact1d	1/220 (≈4.55E-03)	4.52E-03 (-0.6%)	4.55E-03 (0.1%)	4.50E-03 (-1.1%)	4.56E-03 (0.4%)	4.39E-03 (-3.4%)
Exact1e	1/220 (≈4.55E-03)	4.53E-03 (-0.4%)	4.55E-03 (0.1%)	4.53E-03 (-0.3%)	4.56E-03 (0.4%)	4.27E-03 (-6.0%)
Pattern 2						
Exact1a	3/4 (=0.75)	0.750 (0.1%)	0.7495 (-0.1%)	0.7510 (0.1%)	0.7422 (-1.0%)	0.5786 (-22.9%)
Exact1b	3/4 (=0.75)	0.750 (0.1%)	0.7495 (-0.1%)	0.7509 (0.1%)	0.7422 (-1.0%)	0.5703 (-24.0%)
Exact1c	3/4 (=0.75)	0.750 (0.0%)	0.7495 (-0.1%)	0.7510 (0.1%)	0.7422 (-1.0%)	0.5621 (-25.1%)
Exact1d	3/4 (=0.75)	0.751 (0.1%)	0.7495 (-0.1%)	0.7511 (0.1%)	0.7422 (-1.0%)	0.5764 (-23.1%)
Exact1e	3/4 (=0.75)	0.751 (0.1%)	0.7495 (-0.1%)	0.7507 (0.1%)	0.7422 (-1.0%)	0.5660 (-24.5%)
Pattern 3						
Exact1a	1/4 (=0.25)	0.250 (-0.2%)	0.2505 (0.2%)	0.2502 (0.1%)	0.2706 (8.2%)	0.2149 (-14.0%)
Exact1b	1/4 (=0.25)	0.250 (-0.2%)	0.2505 (0.2%)	0.2503 (0.1%)	0.2706 (8.2%)	0.2280 (-8.8%)
Exact1c	1/4 (=0.25)	0.250 (-0.2%)	0.2505 (0.2%)	0.2503 (0.1%)	0.2706 (8.2%)	0.2164 (-13.4%)
Exact1d	1/4 (=0.25)	0.249 (-0.3%)	0.2505 (0.2%)	0.2503 (0.1%)	0.2706 (8.2%)	0.2192 (-12.3%)
Exact1e	1/4 (=0.25)	0.249 (-0.2%)	0.2505 (0.2%)	0.2504 (0.2%)	0.2706 (8.2%)	0.2149 (-14.0%)
Pattern 4						
Exact1a	219/220 (≈0.996)	0.996 (0.1%)	0.9954 (-0.0%)	0.9955 (0.0%)	1.0310 (3.6%)	0.6834 (-31.3%)
Exact1b	219/220 (≈0.996)	0.997 (0.1%)	0.9954 (-0.0%)	0.9955 (0.0%)	1.0310 (3.6%)	0.6834 (-31.3%)
Exact1c	219/220 (≈0.996)	0.996 (0.1%)	0.9954 (-0.0%)	0.9955 (0.0%)	1.0310 (3.6%)	0.6834 (-31.3%)
Exact1d	219/220 (≈0.996)	0.997 (0.1%)	0.9954 (-0.0%)	0.9955 (0.0%)	1.0310 (3.6%)	0.6835 (-31.3%)
Exact1e	219/220 (≈0.996)	0.997 (0.1%)	0.9954 (-0.0%)	0.9954 (-0.0%)	1.0310 (3.6%)	0.6834 (-31.3%)

2.3. Nonlinear *p* and *q*

2.3.1. Nonlinear *p* and *q*: Problem description

In this set of test cases, $\bar{p}(t)$ is defined as an increasing non-linear function, and $\overline{q}(t)$ is defined as a decreasing non-linear function. The specific forms of $\bar{p}(t)$ and $\overline{q}(t)$ are shown below

$$\bar{p}(t) = \frac{\bar{p}(\infty)\bar{p}(0)}{\bar{p}(0)+(\bar{p}(\infty)-\bar{p}(0))e^{-r_1 t}} \text{ and } \bar{q}(t) = \bar{q}(0)e^{-r_2 t},$$

$$\text{where } \bar{p}(\infty) = 700\ K, \bar{p}(0) = 100\ K, \bar{q}(0) = 500\ K, r_1 = 0.025, r_2 = 0.001$$

and also in Fig.3. For input to CPLOAS_2, $\bar{p}(t)$ and $\overline{q}(t)$ are discretized with time-steps of 0.1 min. The same 5 test cases used in Section 2.2 are also used to verify that CPLOAS_2 results hold for this more general non-linear case.

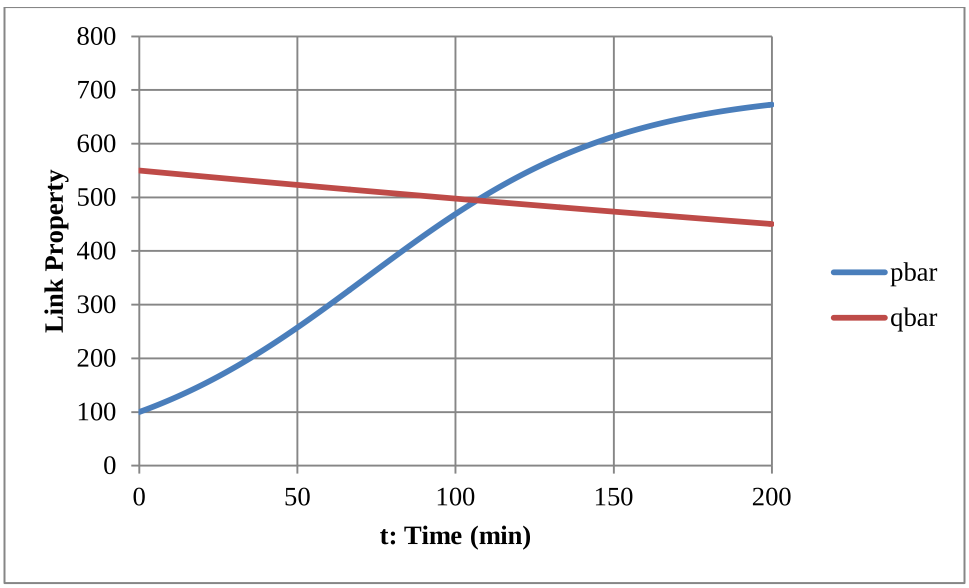

Figure 3: Nominal physical property $\bar{p}(t)$ and failure value $\overline{q}(t)$ for the non-linear *p*(*t*) and *q*(*t*) cases (units: temperature).

For the property set defined in Figure 3, five test cases were run to exercise each of the six uncertainty distribution forms handled by CPLOAS_2. The distributions for aleatory variables α and β associated with each of the five tests are shown in Table 2.

For each test case, CPLOAS_2 executed five different numerical procedures: a quadrature procedure and four variants of sampling-based procedures (i.e., MC1, MC2, IMP1, IMP2). The test cases were run with the same parameters as the linear cases in Section 2.2.1: *nCDF* = 20,000; *nQUAD* = 10,000; *nFT* = 1,000,000; *nFTC* = 1,000,000; *nSEED* = 32.

2.3.2. Nonlinear *p* and *q*: Results

Results of the test cases are presented in Table 5. Owing to the high PLOAS values involved, the two importance sampling procedures are not appropriate for test cases P2, P3 and P4; and the results of their application are highlighted in yellow to distinguish these results from results obtained with the other more appropriate procedures.

Each of the five techniques performed similarly for all distribution forms of α and β. The quadrature results deviate from the true values for PLOAS by less than 0.8% in all cases (and by less than 0.5% in 19 of the 20 pattern-case combinations). MC1 predictions deviate from the true PLOAS by less than 0.2% in all cases. Both Monte Carlo sampling techniques deviate from the true values for PLOAS by less than 1.1% in all cases. As was observed in the tests with linear data, the two MC techniques resulted in nearly identical performance for the high probability patterns (i.e., P2, P3, P4). MC1 performed slightly better than MC2 for the moderate probability pattern (P1). The two importance sampling techniques showed greater deviation from the true values for PLOAS. IMP1 showed small deviation (0.4%) for the moderate probability pattern (P1). IMP1 showed small deviation (0.4%) for the moderate probability pattern (P1). IMP2 under-predicted the PLOAS for pattern P1 by less than 7.9% in all cases.; however, IMP2 is near the limits of its applicability for pattern P1.

The performance of the quadrature, MC1, MC2, IMP1 and IMP2 techniques is acceptable for all test cases within their domains of applicability in this set of verification problems and performed similarly for all distribution forms of α and β. As indicated above, the two importance sampling procedures are inappropriate for P2, P3 and P4 and are so indicated in Table 5.

Table 5: Results from CPLOAS calculations for the system of 3 WLs, 9 SLs defined in Section 2.2. (Results highlighted in yellow obtained with procedures not appropriate for the problem under consideration)

	Analytical	Quadrature	MC1	MC2	IMP1	IMP2
Pattern 1						
Exact2a	1/220 (≈4.55E-03)	4.52E-03 (-0.5%)	4.55E-03 (0.1%)	4.56E-03 (0.3%)	4.56E-03 (0.4%)	4.42E-03 (-2.7%)
Exact2b	1/220 (≈4.55E-03)	4.52E-03 (-0.5%)	4.55E-03 (0.1%)	4.52E-03 (-0.5%)	4.56E-03 (0.4%)	4.19E-03 (-7.9%)
Exact2c	1/220 (≈4.55E-03)	4.52E-03 (-0.5%)	4.55E-03 (0.1%)	4.53E-03 (-0.3%)	4.56E-03 (0.4%)	4.41E-03 (-2.9%)
Exact2d	1/220 (≈4.55E-03)	4.51E-03 (-0.8%)	4.55E-03 (0.1%)	4.50E-03 (-1.1%)	4.56E-03 (0.4%)	4.39E-03 (-3.4%)
Exact2e	1/220 (≈4.55E-03)	4.52E-03 (-0.5%)	4.55E-03 (0.1%)	4.53E-03 (-0.3%)	4.56E-03 (0.4%)	4.27E-03 (-6.0%)
Pattern 2						
Exact2a	3/4 (=0.75)	0.751 (0.1%)	0.7495 (-0.1%)	0.7510 (0.1%)	0.7422 (-1.0%)	0.5786 (-22.9%)
Exact2b	3/4 (=0.75)	0.751 (0.1%)	0.7495 (-0.1%)	0.7509 (0.1%)	0.7422 (-1.0%)	0.5703 (-24.0%)
Exact2c	3/4 (=0.75)	0.751 (0.1%)	0.7495 (-0.1%)	0.7510 (0.1%)	0.7422 (-1.0%)	0.5621 (-25.1%)
Exact2d	3/4 (=0.75)	0.751 (0.1%)	0.7495 (-0.1%)	0.7511 (0.1%)	0.7422 (-1.0%)	0.5764 (-23.1%)
Exact2e	3/4 (=0.75)	0.751 (0.1%)	0.7495 (-0.1%)	0.7507 (0.1%)	0.7422 (-1.0%)	0.5660 (-24.5%)
Pattern 3						
Exact2a	1/4 (=0.25)	0.250 (-0.2%)	0.2505 (0.2%)	0.2503 (0.1%)	0.2706 (8.2%)	0.2149 (-14.0%)
Exact2b	1/4 (=0.25)	0.249 (-0.2%)	0.2505 (0.2%)	0.2503 (0.1%)	0.2706 (8.2%)	0.2280 (-8.8%)
Exact2c	1/4 (=0.25)	0.249 (-0.2%)	0.2505 (0.2%)	0.2503 (0.1%)	0.2706 (8.2%)	0.2164 (-13.4%)
Exact2d	1/4 (=0.25)	0.249 (-0.4%)	0.2505 (0.2%)	0.2503 (0.1%)	0.2706 (8.2%)	0.2192 (-12.3%)
Exact2e	1/4 (=0.25)	0.249 (-0.3%)	0.2505 (0.2%)	0.2504 (0.2%)	0.2706 (8.2%)	0.2149 (-14.0%)
Pattern 4						
Exact2a	219/220 (≈0.996)	0.997 (0.1%)	0.9954 (-0.0%)	0.9955 (0.0%)	1.0310 (3.6%)	0.6834 (-31.3%)
Exact2b	219/220 (≈0.996)	0.997 (0.2%)	0.9954 (-0.0%)	0.9955 (0.0%)	1.0310 (3.6%)	0.6834 (-31.3%)
Exact2c	219/220 (≈0.996)	0.997 (0.1%)	0.9954 (-0.0%)	0.9955 (0.0%)	1.0310 (3.6%)	0.6834 (-31.3%)
Exact2d	219/220 (≈0.996)	0.997 (0.2%)	0.9954 (-0.0%)	0.9955 (0.0%)	1.0310 (3.6%)	0.6834 (-31.3%)
Exact2e	219/220 (≈0.996)	0.997 (0.2%)	0.9954 (-0.0%)	0.9954 (-0.0%)	1.0310 (3.6%)	0.6834 (-31.3%)

2.4. Property-dependent *q*

2.4.1. Property-dependent *q*: Problem description

In this set of problems, $\bar{p}(t)$ is defined as an increasing non-linear function, and $\bar{q}(t)$ is defined as a function of the nominal property value $\bar{p}(t)$. For each link, $\bar{p}(t)$ is defined as in Section 2.2. The nominal failure value $\bar{q}(t)$ is defined by $f[\bar{p}(t)]$ in Figure 4. Although the defining relationships for $\bar{q}(t)$ in Figure 4 imply units of pressure for $\bar{q}(t)$, the resultant values for $\bar{q}(t)$ are assumed to be the same temperature units as $\bar{p}(t)$; this is a nonphysical use of the relationships in Figure 4 that nonetheless produces a workable test problem. The distributions for aleatory variables α and β associated with each of the five tests are shown in Table 2.

Temp (C)	Pressure (psi)
0.0	1000.0
100.0	975.0
200.0	950.0
300.0	890.0
400.0	750.0
500.0	550.0
600.0	375.0
700.0	200.0
800.0	150.0
900.0	125.0
1000.0	100.0

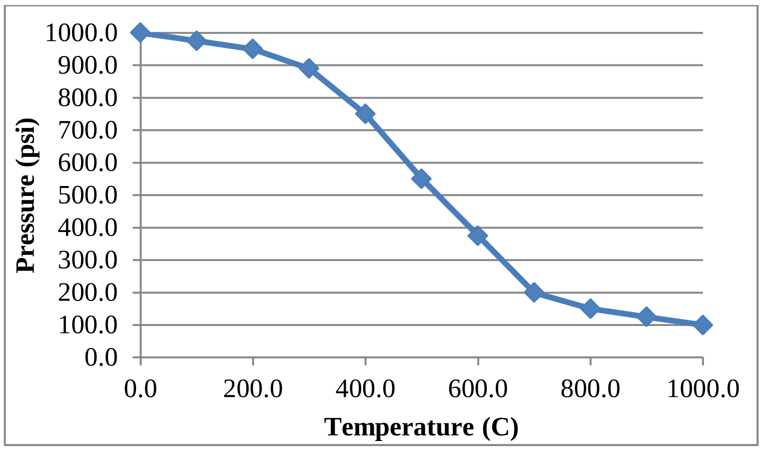

Figure 4: Failure pressure as a function of temperature.

For each test case, CPLOAS_2 executed five different numerical procedures: a quadrature procedure and four variants of sampling-based procedures. The test cases were run with the same parameters as the non-linear cases in Section 2.3: *nCDF* = 20,000; *nQUAD* = 10,000; *nFT* = 1,000,000; *nFT* = 1,000,000; *nSEED* = 32.

2.4.2. Property-dependent *q*: Results

Results of the test cases are presented in Table 6. Owing to the high PLOAS values involved, the two importance sampling procedures are not appropriate for test cases P2, P3 and P4; and the results of their application are highlighted in yellow to distinguish these results from results obtained with the other more appropriate procedures.

Each of the five techniques performed similarly for all distribution forms of α and β. The quadrature results deviate from the true values for PLOAS by less than 1.7% in all cases. MC1 predictions deviate from the true PLOAS by less than 0.2% in all cases. Both Monte Carlo sampling techniques deviate from the true values for PLOAS by less than 1.1% in all cases. As was observed in the tests with linear data, the two MC techniques resulted in nearly identical performance for the higher probability patterns (i.e., P2, P3, P4). MC1 performed slightly better than MC2 for the low probability pattern (P1). The two importance sampling techniques showed greater deviation from the true values for PLOAS. IMP1 showed small deviation (0.4%) for the moderate probability pattern (P1). IMP2 under-predicted the PLOAS for pattern P1 by less than 7.9% in all cases.; however, IMP2 is near the limits of its applicability for pattern P1.

The performance of the quadrature, MC1, MC2, IMP1 and IMP2 techniques is acceptable for all test cases within their domains of applicability in this set of verification problems and performed similarly for all distribution forms of α and β. As indicated above, the two importance sampling procedures are inappropriate for P2, P3 and P4 and are so indicated in Table 6.

Table 6: Results from CPLOAS_2 calculations for the system of 3 WLs, 9 SLs defined in Section 2.3. (Results highlighted in yellow obtained with procedures not appropriate for the problem under consideration)

	Analytical	Quadrature	MC1	MC2	IMP1	IMP2
Pattern 1						
Exact3a	1/220 (≈4.55E-03)	4.49E-03 (-1.2%)	4.55E-03 (0.1%)	4.56E-03 (0.3%)	4.56E-03 (0.4%)	4.42E-03 (-2.7%)
Exact3b	1/220 (≈4.55E-03)	4.49E-03 (-1.2%)	4.55E-03 (0.1%)	4.52E-03 (-0.5%)	4.56E-03 (0.4%)	4.19E-03 (-7.9%)
Exact3c	1/220 (≈4.55E-03)	4.49E-03 (-1.2%)	4.55E-03 (0.1%)	4.53E-03 (-0.3%)	4.56E-03 (0.4%)	4.41E-03 (-2.9%)
Exact3d	1/220 (≈4.55E-03)	4.47E-03 (-1.7%)	4.55E-03 (0.1%)	4.50E-03 (-1.1%)	4.56E-03 (0.4%)	4.39E-03 (-3.4%)
Exact3e	1/220 (≈4.55E-03)	4.49E-03 (-1.2%)	4.55E-03 (0.1%)	4.53E-03 (-0.3%)	4.56E-03 (0.4%)	4.27E-03 (-6.0%)
Pattern 2						
Exact3a	3/4 (=0.75)	0.751 (0.1%)	0.7495 (-0.1%)	0.7510 (0.1%)	0.7422 (-1.0%)	0.5786 (-22.9%)
Exact3b	3/4 (=0.75)	0.751 (0.2%)	0.7495 (-0.1%)	0.7509 (0.1%)	0.7422 (-1.0%)	0.5703 (-24.0%)
Exact3c	3/4 (=0.75)	0.751 (0.1%)	0.7495 (-0.1%)	0.7510 (0.1%)	0.7422 (-1.0%)	0.5621 (-25.1%)
Exact3d	3/4 (=0.75)	0.752 (0.2%)	0.7495 (-0.1%)	0.7511 (0.1%)	0.7422 (-1.0%)	0.5764 (-23.1%)
Exact3e	3/4 (=0.75)	0.752 (0.2%)	0.7495 (-0.1%)	0.7507 (0.1%)	0.7422 (-1.0%)	0.5660 (-24.5%)
Pattern 3						
Exact3a	1/4 (=0.25)	0.249 (-0.5%)	0.2505 (0.2%)	0.2503 (0.1%)	0.2706 (8.2%)	0.2149 (-14.0%)
Exact3b	1/4 (=0.25)	0.248 (-0.6%)	0.2505 (0.2%)	0.2503 (0.1%)	0.2706 (8.2%)	0.2280 (-8.8%)
Exact3c	1/4 (=0.25)	0.249 (-0.6%)	0.2505 (0.2%)	0.2503 (0.1%)	0.2706 (8.2%)	0.2164 (-13.4%)
Exact3d	1/4 (=0.25)	0.248 (-0.8%)	0.2505 (0.2%)	0.2503 (0.1%)	0.2706 (8.2%)	0.2192 (-12.3%)
Exact3e	1/4 (=0.25)	0.248 (-0.7%)	0.2505 (0.2%)	0.2504 (0.2%)	0.2706 (8.2%)	0.2149 (-14.0%)
Pattern 4						
Exact3a	219/220 (≈0.996)	0.998 (0.3%)	0.9954 (-0.0%)	0.9955 (0.0%)	1.0310 (3.6%)	0.6834 (-31.3%)
Exact3b	219/220 (≈0.996)	0.998 (0.3%)	0.9954 (-0.0%)	0.9955 (0.0%)	1.0310 (3.6%)	0.6834 (-31.3%)
Exact3c	219/220 (≈0.996)	0.998 (0.3%)	0.9954 (-0.0%)	0.9955 (0.0%)	1.0310 (3.6%)	0.6834 (-31.3%)
Exact3d	219/220 (≈0.996)	0.999 (0.4%)	0.9954 (-0.0%)	0.9955 (0.0%)	1.0310 (3.6%)	0.6834 (-31.3%)
Exact3e	219/220 (≈0.996)	0.999 (0.4%)	0.9954 (-0.0%)	0.9954 (-0.0%)	1.0310 (3.6%)	0.6834 (-31.3%)

3. Verification of PLOAS calculations when links have different properties

In these test problems, the solutions do not have a known algebraic form and thus can be obtained only through numerical approximation. For these problems, the test cases exercise the five numerical procedures implemented in CPLOAS_2 (a quadrature procedure and four variants of sampling-based procedures) and verification is accomplished by determining PLOAS in all procedures.

3.1. Test cases from SAND2012-8248

3.1.1. Test cases from SAND2012-8248: Problem description

SAND2012-8248 [1] defines five test cases representing different configurations of 3 SLs and 2 WLs, and provides PLOAS values calculated with the quadrature procedure and two Monte Carlo procedures (i.e., MC1 and MC2). The test cases in this section repeat those test cases to verify the results of the two importance sampling procedures against the results derived in SAND2012-8248.

The five verification problems are summarized in Ref. [1] and presented in detail in Sects. 9.1 – 9.5 of Ref. [1].These problems use solutions of the system of differential equations in Table 5 of Ref. [1] to define properties $\overline{p}(t)$ and $\overline{q}(t)$ for the individual links; further, the α and β values for the individual links are defined in Table 7, which is a reproduction of Table 7 in Ref. [1].

Table 7: Test cases used for verification of PLOAS calculations ([1], Table 7)

Approx1_T1: Sect. 9.1: Loss of assured safety (LOAS) occurs if SL1, SL2 or SL3 fails before WL1 fails. Verification results shown in Eq. (9.11) of Ref. [1].

Approx1_T2: Sect. 9.2: LOAS occurs if SL4 and SL5 both fail before WL2 fails. Verification results shown in Eq. (9.22) of Ref. [1].

Approx1_T3: Sect. 9.3: LOAS occurs if T1 occurs or T2 occurs. Verification results calculated using Eq. (9.24) of Ref. [1] on results for T1 and T2.

Approx1_T4: Sect. 9.4: LOAS occurs if SL4 and SL5 both fail before WL1 or WL2 fails. Verification results shown in Eq. (9.25) of Ref. [1].

Approx1_T5: Sect 9.5: LOAS occurs if T1 occurs or if T4 occurs (ignoring dependency between T1 and T4). Verification results calculated using Eq. (9.24) of Ref.[1] on results for T1 and T4.

Approx1_T6: Sect 9.5: LOAS occurs if T1 occurs or if T4 occurs (including dependency between T1 and T4). Verification results shown in Eq. (9.35) of Ref. [1].

For each test case, CPLOAS_2 executed five different numerical procedures: a quadrature procedure and four variants of sampling-based procedures. The quadrature procedure was run with *nCDF* = 200,000 CDF discretization steps and *nQUAD* = 10,000 quadrature discretization steps. For the four sampling procedures, *nFT* = 100,000 and *nFTC* = 1,000,000 as described in Sect. 2.2.1. The random seed used was *nSEED* = 352.

3.1.2. Test cases from SAND2012-8248: Results

As shown in Table 8, results from the quadrature and two MC techniques showed acceptable agreement with results from Ref. [1]. Quadrature results differed from the report by less than 1.10%. MC1 results are within 1.44% for high probability cases, and all MC results match the order of magnitude of the results in the previous report. MC2 results are within 2.70% of the results in the report for all cases, including the low probability cases. MC2 results more closely match the report than do MC1 results, especially for the low probability cases. In all cases, the Monte Carlo PLOAS values from Ref. [1] fall within the corresponding 95% confidence intervals calculated as part of the current verification study.

Table 8: Comparison of PLOAS values from Ref. [1] with CPLOAS_2 results for the patterns in Table 7

	Value in Ref. [1]	CPLOAS_2	% change	95% Confidence Interval
Quadrature				
Approx1_T1	6.88E-05	6.91E-05	0.41%	N/A
Approx1_T2	1.56E-02	1.57E-02	0.71%	N/A
Approx1_T3	1.57E-02	1.58E-02	0.65%	N/A
Approx1_T4	1.27E-05	1.28E-05	1.10%	N/A
Approx1_T5	8.15E-05	8.19E-05	0.52%	N/A
Approx1_T6	8.18E-05	8.17E-05	-0.12%	N/A
Monte Carlo 1				
Approx1_T1	6.60E-05	7.20E-05	9.09%	(5.54E-05, 8.86E-05)
Approx1_T2	1.60E-02	1.58E-02	-1.44%	(1.55E-02, 1.60E-02)
Approx1_T3	1.61E-02	1.59E-02	-1.34%	(1.56E-02, 1.61E-02)
Approx1_T4	1.60E-05	1.00E-05	-37.50%	(3.81E-06, 1.62E-05)
Approx1_T5	8.20E-05	8.20E-05	0.00%	(6.43E-05, 9.97E-05)
Approx1_T6	8.20E-05	7.20E-05	-12.20%	(5.54E-05, 8.86E-05)
Monte Carlo 2				
Approx1_T1	7.40E-05	7.20E-05	-2.70%	(5.54E-05, 8.86E-05)
Approx1_T2	1.59E-02	1.59E-02	-0.25%	(1.56E-02, 1.61E-02)
Approx1_T3	1.60E-02	1.59E-02	-0.27%	(1.57E-02, 1.62E-02)
Approx1_T4	1.00E-05	1.00E-05	0.00%	(3.81E-06, 1.62E-05)
Approx1_T5	8.40E-05	8.20E-05	-2.38%	(6.43E-05, 9.97E-05)
Approx1_T6	8.20E-05	8.20E-05	0.00%	(6.43E-05, 9.97E-05)

As shown in Table 9, results of the four sampling techniques show acceptable agreement with the quadrature results. For both MC techniques, results for most tests are within 4.23% of the quadrature results, and for all cases are within 22.12% of the quadrature result). For the high probability cases, both MC techniques are within 1.01% of the quadrature results. The higher percent changes in MC results are seen in the low probability cases, where order-of-magnitude fit is considered acceptable due to the limits of sampling accuracy for low probabilities as discussed in Section 1.2. For the IMP1 technique, results for most cases (including low probability cases) are within 8.41% of the quadrature results; the low probability case T6 is within 23.39%, and exhibits an order of magnitude correspondence with the quadrature result. For IMP2, tests T1-T5 are all within 1.40% of the quadrature result; the low probability case T6 is within 105.51% and exhibits an order of magnitude correspondence with the quadrature result.

Table 9: PLOAS results for CPLOAS_2 calculations of the patterns in Table 7

	Quad.	MC1	MC2	IMP1	IMP2
Approx1_T1	6.91E-05	7.20E-05 (4.23%)	7.20E-05 (4.23%)	6.56E-05 (-4.99%)	6.81E-05 (-1.40%)
Approx1_T2	1.57E-02	1.58E-02 (0.38%)	1.59E-02 (0.95%)	1.59E-02 (1.08%)	1.57E-02 (-0.19%)
Approx1_T3	1.58E-02	1.59E-02 (0.51%)	1.59E-02 (1.01%)	1.60E-02 (1.14%)	1.57E-02 (-0.19%)
Approx1_T4	1.28E-05	1.00E-05 (-22.12%)	1.00E-05 (-22.12%)	1.18E-05 (-8.41%)	1.30E-05 (0.86%)
Approx1_T5	8.19E-05	8.20E-05 (0.10%)	8.20E-05 (0.10%)	7.74E-05 (-5.54%)	8.11E-05 (-1.05%)
Approx1_T6	8.17E-05	7.20E-05 (-11.87%)	8.20E-05 (0.37%)	6.26E-05 (-23.39%)	1.68E-04 (105.51%)

Only two results that fall outside the 95% confidence intervals in Table 10. First, although the difference from the quadrature result is only 1.08% for the estimate of T2 with IMP1, the quadrature result of 1.57E-02 falls just outside of the 95% confidence interval of (1.58E-02,1.60E-02) for T2 obtained with IMP1. Second, although the difference from the quadrature result is 23.39% for the estimate of T6 with IMP1, the quadrature result of 8.17E-05 falls just outside of the 95% confidence interval of (4.91E-05, 7.61E-05) for T6 obtained with IMP1.

Table 10: LOAS results for CPLOAS_2 calculations of the patterns in Table 7 shown as 95% confidence intervals for the four sampling-based procedures for the estimation of PLOAS

	Quad.	MC1	MC2	IMP1	IMP2
Approx1_T1	6.91E-05	(5.54E-05, 8.86E-05)	(5.54E-05, 8.86E-05)	(5.22E-05, 7.91E-05)	(5.16E-05, 8.46E-05)
Approx1_T2	1.57E-02	(1.55E-02, 1.60E-02)	(1.56E-02,1.61E-02)	(1.58E-02,1.60E-02)	(1.55E-02, 1.58E-02)
Approx1_T3	1.58E-02	(1.56E-02, 1.61E-02)	(1.57E-02,1.62E-02)	(1.58E-02,1.61E-02)	(1.56E-02, 1.59E-02)
Approx1_T4	1.28E-05	(3.81E-06, 1.62E-05)	(3.81E-06, 1.62E-05)	(9.71E-06, 1.38E-05)	(1.16E-05, 1.43E-05)
Approx1_T5	8.19E-05	(6.43E-05, 9.97E-05)	(6.43E-05, 9.97E-05)	(6.38E-05, 9.10E-05)	(6.45E-05, 9.76E-05)
Approx1_T6	8.17E-05	(5.54E-05, 8.86E-05)	(6.43E-05, 9.97E-05)	(4.91E-05, 7.61E-05)	(1.06E-05, 3.25E-04)

Although the difference from the quadrature result is 105.51% for the estimate of T6 with IMP2, the quadrature result of 8.17E-05 falls within the confidence interval of (1.06E-05, 3.25E-04) for T6 obtained with IMP2. As a test, the calculation of T6 with IMP2 was rerun with a different initiating random seed (i.e., *nSEED*=35), which produced an estimate of 6.70E-05 for

T6 with a corresponding confidence interval of (4.78E-05, 8.63E-05) that contains the quadrature result of 8.17E-05. With this random seed, the confidence interval is smaller and the difference from the quadrature result is -18.0%. As illustrated here, there is always a certain amount of random variability in sampling-based calculations.

3.2. Test cases derived from Section 2

3.2.1. Test cases derived from Section 2: Problem description

In this set of test problems, five tests are run to exercise the use of each aleatory distribution type. In each test, there are 3 WLs and 9 SLs. As specified in Table 11, each link is assigned nominal property and failure values from one of the three sets of data used in Section 2. Each test case evaluated the four failure patterns handled by CPLOAS_2.

Table 11: Property and failure data values for the test cases in Section 3.2

Link	$\bar{p}(t)$	$\bar{q}(t)$
SL1	As defined in Section 2.1.1	As defined in Section 2.1.1
SL2	As defined in Section 2.1.1	As defined in Section 2.2.1
SL3	As defined in Section 2.1.1	As defined in Section 2.1.1
SL4	As defined in Section 2.1.1	As defined in Section 2.2.1
SL5	As defined in Section 2.2.1	As defined in Section 2.1.1
SL6	As defined in Section 2.2.1	As defined in Section 2.2.1
SL7	As defined in Section 2.2.1	As defined in Section 2.3.1
SL8	As defined in Section 2.2.1	As defined in Section 2.1.1
SL9	As defined in Section 2.2.1	As defined in Section 2.2.1
WL1	As defined in Section 2.1.1	As defined in Section 2.1.1
WL2	As defined in Section 2.1.1	As defined in Section 2.2.1
WL3	As defined in Section 2.2.1	As defined in Section 2.3.1

The five test cases are designed to exercise each of the six uncertainty distribution forms available in CPLOAS_2. The distributions for aleatory variables α and β associated with each of the five tests are defined in Table 2.

For each test case, CPLOAS_2 executed five different numerical procedures: a quadrature procedure and four variants of sampling-based procedures. The quadrature procedure was run with *nCDF* = 20,000 CDF discretization steps and *nQUAD* = 10,000 quadrature discretization steps. For the four sampling procedures, *nFT* = 1,000,000 and *nFTC* = 1,000,000 were used as described in Section 2.2.1. The random seed used to initiate each set of samplings was *nSEED* = 32.

3.2.2. Test cases derived from Section 2: Results

Results of the test cases are presented in Table 12. Owing to the high PLOAS values involved, the two importance sampling procedures are not appropriate for test cases P2, P3 and

P4, and the results of their application are highlighted in yellow to distinguish these results from results obtained with the other more appropriate procedures.

There is acceptable agreement between quadrature, MC1, MC2, and IMP1 techniques for all patterns.

Table 12: Results from CPLOAS calculations for the system of 3 WLs, 9 SLs defined in Section 3.2 (Results highlighted in yellow obtained with procedures not appropriate for the problem under consideration)

	Quad.	MC1	MC2	IMP1	IMP2
Pattern 1					
Approx2a	7.53E-04	7.63E-04 (1.3%)	7.42E-04 (-1.5%)	7.49E-04 (-0.6%)	7.38E-04 (-2.1%)
Approx2b	4.58E-04	4.81E-04 (5.1%)	4.51E-04 (-1.4%)	4.65E-04 (1.6%)	4.53E-04 (-1.0%)
Approx2c	6.60E-04	6.58E-04 (-0.3%)	6.63E-04 (0.4%)	6.61E-04 (0.2%)	6.43E-04 (-2.6%)
Approx2d	8.52E-05	8.30E-05 (-2.6%)	9.50E-05 (11.5%)	8.59E-05 (0.8%)	7.90E-05 (-7.3%)
Approx2e	4.21E-04	4.52E-04 (7.4%)	4.12E-04 (-2.1%)	4.26E-04 (1.2%)	4.35E-04 (3.4%)
Pattern 2					
Approx2a	0.958	0.957 (-0.1%)	0.957 (-0.0%)	0.976 (1.9%)	0.679 (-29.1%)
Approx2b	0.965	0.964 (-0.1%)	0.964 (-0.1%)	0.979 (1.5%)	0.682 (-29.3%)
Approx2c	0.951	0.951 (-0.1%)	0.951 (-0.0%)	0.971 (2.0%)	0.662 (-30.4%)
Approx2d	0.983	0.982 (-0.1%)	0.983 (-0.1%)	0.993 (1.0%)	0.683 (-30.5%)
Approx2e	0.976	0.975 (-0.1%)	0.975 (-0.1%)	0.990 (1.4%)	0.683 (-30.1%)
Pattern 3					
Approx2a	0.297	0.299 (0.6%)	0.298 (0.3%)	0.302 (1.7%)	0.243 (-18.0%)
Approx2b	0.315	0.317 (0.7%)	0.315 (0.2%)	0.317 (0.9%)	0.269 (-14.4%)
Approx2c	0.304	0.306 (0.6%)	0.305 (0.3%)	0.308 (1.2%)	0.260 (-14.4%)
Approx2d	0.328	0.330 (0.7%)	0.329 (0.3%)	0.339 (3.4%)	0.276 (-15.7%)
Approx2e	0.322	0.324 (0.7%)	0.322 (0.2%)	0.332 (3.1%)	0.273 (-15.1%)
Pattern 4					
Approx2a	1.001	0.9998 (-0.1%)	0.9998 (-0.1%)	1.0350 (3.4%)	0.684 (-31.7%)
Approx2b	1.001	0.9998 (-0.1%)	0.9998 (-0.1%)	1.0350 (3.4%)	0.684 (-31.7%)
Approx2c	1.001	0.9998 (-0.1%)	0.9998 (-0.1%)	1.0350 (3.4%)	0.684 (-31.7%)
Approx2d	1.001	1.0000 (-0.1%)	1.0000 (-0.1%)	1.0350 (3.4%)	0.684 (-31.7%)
Approx2e	1.001	0.9999 (-0.1%)	0.9999 (-0.1%)	1.0350 (3.4%)	0.684 (-31.7%)

The MC1 technique deviates from the quadrature result by less than 7.4% for all cases. MC1 performed within 0.7% for the high probability patterns. The MC2 technique deviates from the quadrature result by less than 2.1% for 19 of the 20 cases. For the remaining test case, MC2 deviates by 11.5%; this is a low probability test case, so results are considered acceptable due to inclusion of the PLOAS quadrature value of 8.52E-05 in the corresponding 95% confidence interval of (7.59E-05, 1.14E-04) for PLOAS obtained with MC2. IMP1 results deviate from quadrature by less than 1.6% for P1. IMP2 deviates by less than 7.3% from quadrature for P1. IMP1 and IMP2 results are not compared to quadrature for the high probability patterns (P2, P3, P4).

The performance of the quadrature, MC1, MC2, IMP1 and IMP2 techniques is acceptable for all test cases within their domains of applicability in this set of verification problems and performed similarly for all distribution forms of α and β. As indicated above, the two importance sampling procedures are inappropriate for P2, P3 and P4 and are so indicated in Table 12.

4. Verification of time and environmental margin calculations

4.1. Verification of time and environmental margin calculations: Problem description

In WL/SL systems with extremely low PLOAS values, time and environmental margins are more informative results than the PLOAS values. Time margin is the difference between the time at which SL failure would result in LOAS and the time at which WL failure would prevent LOAS. Environmental margin is the difference between the property value of the last SL to fail at the time that this SL failure would result in LOAS and the property value of this SL at the time that WL failure would prevent LOAS. A positive margin corresponds to LOAS being prevented by WL failure, and a negative margin corresponds to LOAS due to SL failure before deactivation of the system by WL failure.

For this series of tests, the goal is to verify the correct implementation of the time and environmental margin calculations in CPLOAS_2. In this set of problems, four test sets are run to verify CPLOAS_2 results against expected values from the “RaceMetrics_V2” macro for Excel [3], which calls an earlier version of CPLOAS_2 simply named CPLOAS1 ([4], App. III) for this document.

The test problems involve 2 SLs and 1WL. Further, time dependent property and failure data are provided by the two sets of simulations displayed in Figure 5 and Figure 6. Two test sets were run with each set of data, for a total of four test cases, as defined in [1]. Each test set is evaluated for each of the four SL/WL configurations handled by CPLOAS_2 (although results for cases 3 and 4 are omitted from this document due to the presence of a single WL, which results in case 3 being identical to case 1 and case 4 being identical to case 2).

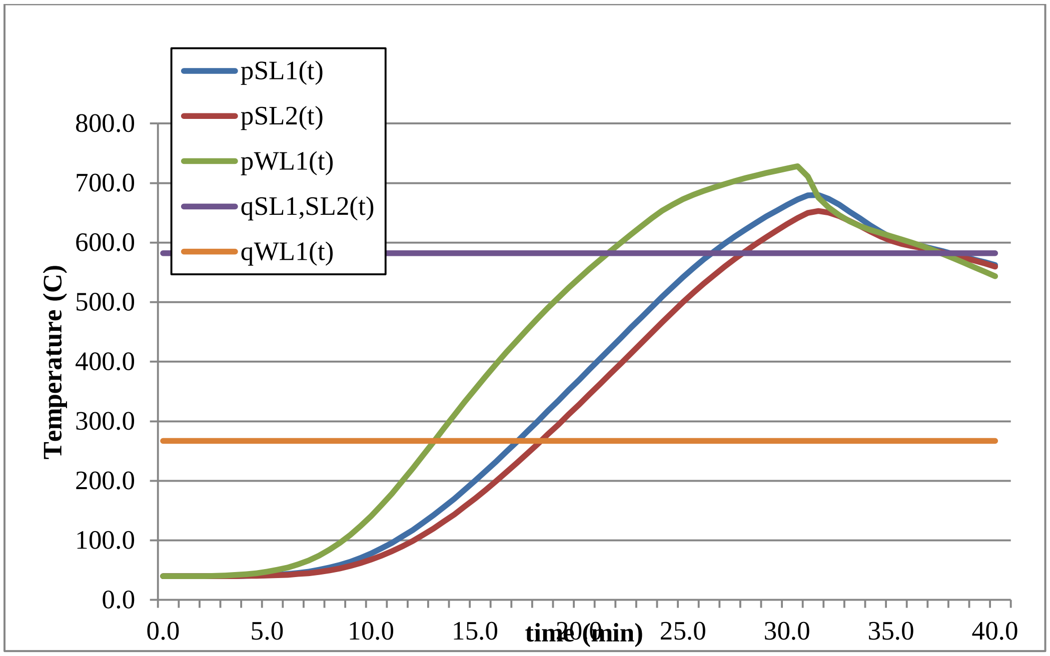

Figure 5: Property and failure values used in test cases labeled Margin 1.

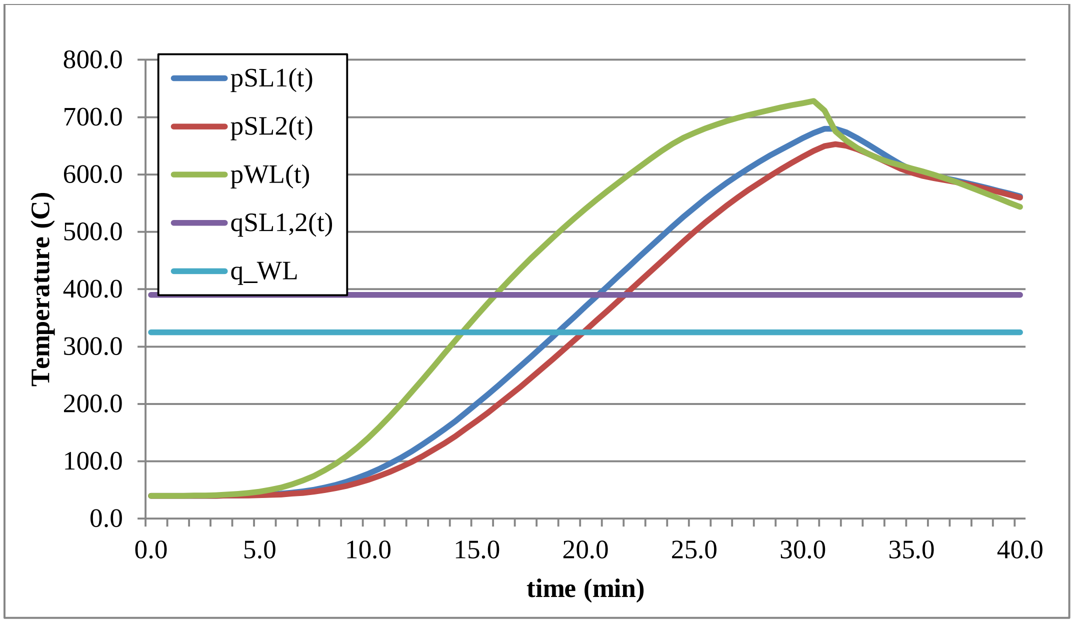

Figure 6: Property and failure values used in test cases labeled Margin 2.

Table 13: Test set configurations for time and thermal margin calculations

Test	Data		Distribution on alpha	Distribution on beta
Margi n1_cst	See Fig. 5	SL1, SL2	Constant(1)	Normal(mean=1.0,stdev=0.0172,trunc.=0.01)
		WL1	Constant(1)	Normal(mean=1.0,stdev=0.0749,trunc.=0.01)
Margi n1_n	See Fig. 5	SL1, SL2	Normal(mean=1.0, stdev=0.1, trunc.=0.01)	Normal(mean=1.0,stdev=0.0172,trunc.=0.01)
		WL1	Normal(mean=1.0, stdev=0.1, trunc.=0.01)	Normal(mean=1.0,stdev=0.0749,trunc.=0.01)
Margi n2_cst	See Fig. 6	SL1, SL2	Constant(1)	Normal(mean=1.0,stdev=0.0641,trunc.=0.01)
		WL1	Constant(1)	Normal(mean=1.0,stdev=0.0615,trunc.=0.01)
Margi n2_n	See Fig. 6	SL1, SL2	Normal(mean=1.0, stdev=0.1, trunc.=0.01)	Normal(mean=1.0,stdev=0.0641,trunc.=0.01)
		WL1	Normal(mean=1.0, stdev=0.1, trunc.=0.01)	Normal(mean=1.0,stdev=0.0615,trunc.=0.01)

Two of the test cases were run with constant α (i.e., $\alpha = 1$). These cases are identical to the set-up of the test in CPLOAS1 (which did not allow uncertainty on property values) and RaceMetrics_V2. The cases with normal α are used to verify that time and environmental margins are reasonably consistent with the constant α cases in CPLOAS_2. Future test cases should be designed to include a greater range of distribution forms.

For each test case, CPLOAS_2 executed five different numerical procedures: a quadrature procedure and four variants of sampling-based procedures. The quadrature procedure was run with *nCDF* = 20,000 CDF discretization steps and *nQUAD* = 10,000 quadrature discretization steps. For the four sampling procedures, *nFT*=10,000 and *nFTC*=100,000 as described in Section 2.2.1. The random seed used for each sampling was *nSEED* = 42.

4.2. Verification of time and environmental margin calculations: Results

The performance of all five CPLOAS_2 techniques is acceptable for all PLOAS calculations in this set. As shown in Table 14, all CPLOAS_2 techniques predict PLOAS values of 0.0 for (i) both patterns and both sets of data with constant α, and (ii) the first set of data with normally distribution α. For the remaining case, quadrature predicts very low PLOAS values (~10^{-11} and ~10^{-6}); and MC1, MC2, and IMP1 predict PLOAS values of 0.0, which is to be expected with a sample size of 100,000. IMP2 also predicts PLOAS value of 0.0 for the ~10^{-11} case. IMP2 is the only sampling technique that is able to calculate a probability for the ~10^{-6} case with a sample size of 100,000; the resultant value of 9.58E-07 is in general agreement with the quadrature value of 3.50E-06. However, the quadrature value is outside the confidence interval of (0, 3.26E-06) for the IMP2 result.

Table 14: PLOAS calculations for the system of 2 SLs, 1 WL defined in Section 4

	CPLOAS1 (Quad. only)	Quad.	MC1	MC2	IMP1	IMP2
Pattern 1						
Margin1_cst	~0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
Margin1_nrm	--	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
Margin2_cst	~0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
Margin2_nrm	--	6.10E-11	0.00E+00	0.00E+00	0.00E+00	0.00E+00
Pattern 2						
Margin1_cst	~0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
Margin1_nrm	--	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
Margin2_cst	~0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
Margin2_nrm	--	3.50E-06	0.00E+00	0.00E+00	0.00E+00	9.58E-07

As shown in Table 15, the CPLOAS_2 mean time margins matches the RaceMetrics_V2 mean values within 0.68% for the cases with constant α. For the cases with normal α, the CPLOAS_2 mean time margins are within 4.17% of the RaceMetrics_V2 result.

As shown in Table 16, the CPLOAS_2 mean environmental margins match the RaceMetrics_V2 mean values within 0.68% for the cases with constant α. For the cases with normal α, the CPLOAS_2 mean time margins are within 4.17% of the RaceMetrics_V2 result.

Table 15: Time margins for system of 2 SLs, 1 WL defined in Section 4, where (i) μ is the mean time margin of the 100,000 samples (with infinity values omitted from calculation of the mean) and (ii) also provided are the 5th, 25th, 50th, 75th and 95th percentiles.

Time (min)	μ (RaceMetrics)	μ (CPLOAS)	5%	25%	50%	75%	95%
Pattern 1							
Margin1_cst	14.82	14.81 (0.06%)	13.78	14.38	14.81	15.23	15.83
Margin1_nrm	14.82	15.01 (-1.29%)	12.37	14.10	15.58	17.52	Infinity
Margin2_cst	7.46	7.52 (-0.68%)	6.21	6.95	7.50	8.07	8.87
Margin2_nrm	7.46	7.78 (-4.17%)	5.42	6.76	7.73	8.75	10.28
Pattern 2							
Margin1_cst	13.34	13.34 (0.01%)	12.32	12.91	13.34	13.76	14.35
Margin1_nrm	13.34	12.93 (3.05%)	10.29	11.73	12.83	14.06	16.18
Margin2_cst	6.24	6.20 (0.65%)	4.87	5.66	6.21	6.75	7.50
Margin2_nrm	6.24	6.01 (3.68%)	3.78	5.11	6.02	6.92	8.21

Table 16: Environmental margins for system of 2 SLs, 1 WL defined in Section 4 where (i) μ is the mean environmental margins of the 100,000 samples (with infinity values omitted from calculation of the mean) and (ii) also provided are the 5th, 25th, 50th, 75th and 95th percentiles

Temperature (C)	μ (RaceMetrics)	μ (CPLOAS)	5%	25%	50%	75%	95%
Pattern 1							
Margin1_cnst	461.29	461.12 (0.04%)	439.30	452.10	461.10	470.20	482.80
Margin1_nrm	461.29	455.41 (1.27%)	419.60	446.20	463.60	484.60	Infinity
Margin2_cnst	237.57	239.87 (-0.68%)	200.30	223.20	239.80	256.40	280.10
Margin2_nrm	237.57	239.15 (-4.17%)	181.70	217.60	240.50	262.30	292.00
Pattern 2							
Margin1_cnst	438.84	438.53 (0.07%)	414.30	428.40	438.60	448.70	462.40
Margin1_nrm	438.84	436.35 (0.57%)	391.30	420.60	438.70	455.00	476.30
Margin2_cnst	210.09	206.39 (1.76%)	165.30	189.50	206.60	223.40	246.70
Margin2_nrm	210.09	201.76 (3.96%)	138.20	178.50	203.90	227.20	258.30

CPLOAS_2 calculates a distribution of time and environmental margin values. The RaceMetrics_V2 program only calculates mean values. Therefore, the RaceMetrics_V2 program can only be used to verify the mean values produced by CPLOAS_2. To demonstrate the additional functionality of CPLOAS_2 results, CDFs of the margin results are shown in Figure 7 and Figure 8, and some common distribution percentiles are included in Table 15 and Table 16.

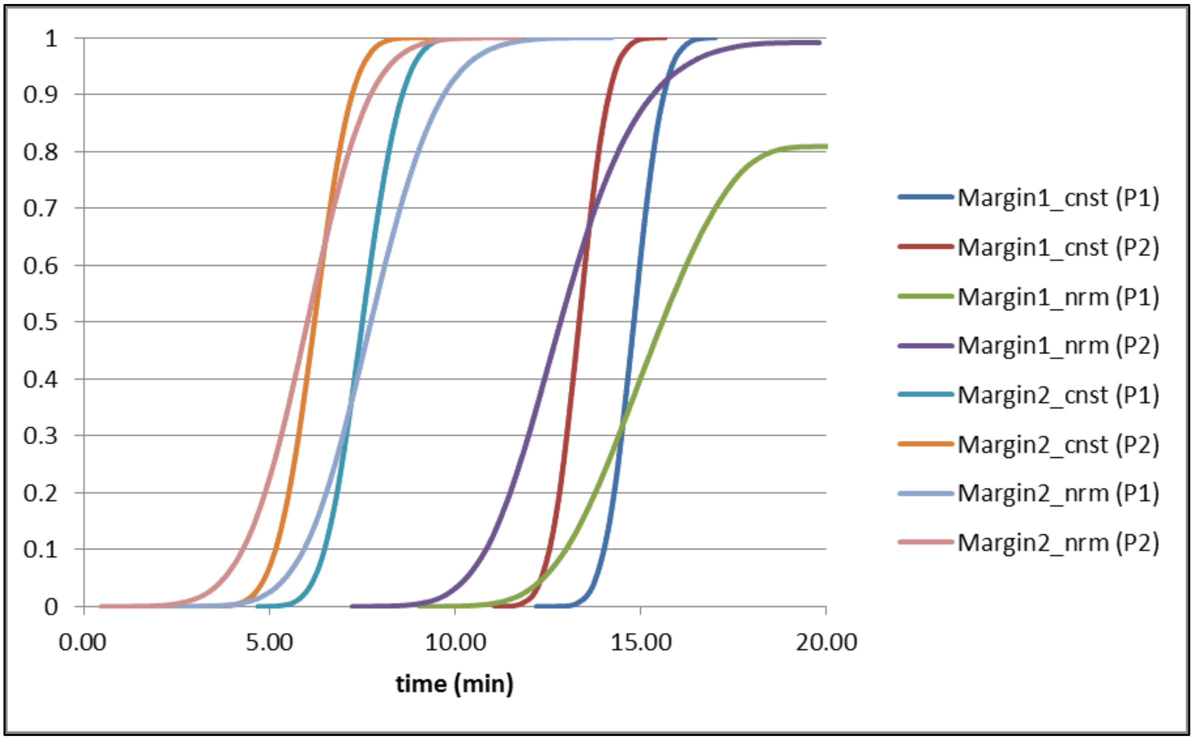

Figure 7: CDFs (cumulative distribution functions) for time margin for the test cases. (Note: Both Margin1_nrm cases returned some "Infinity" values, which means that the corresponding time margins cannot be calculated because one or more links do not fail. These infinity values are omitted from the plot.).

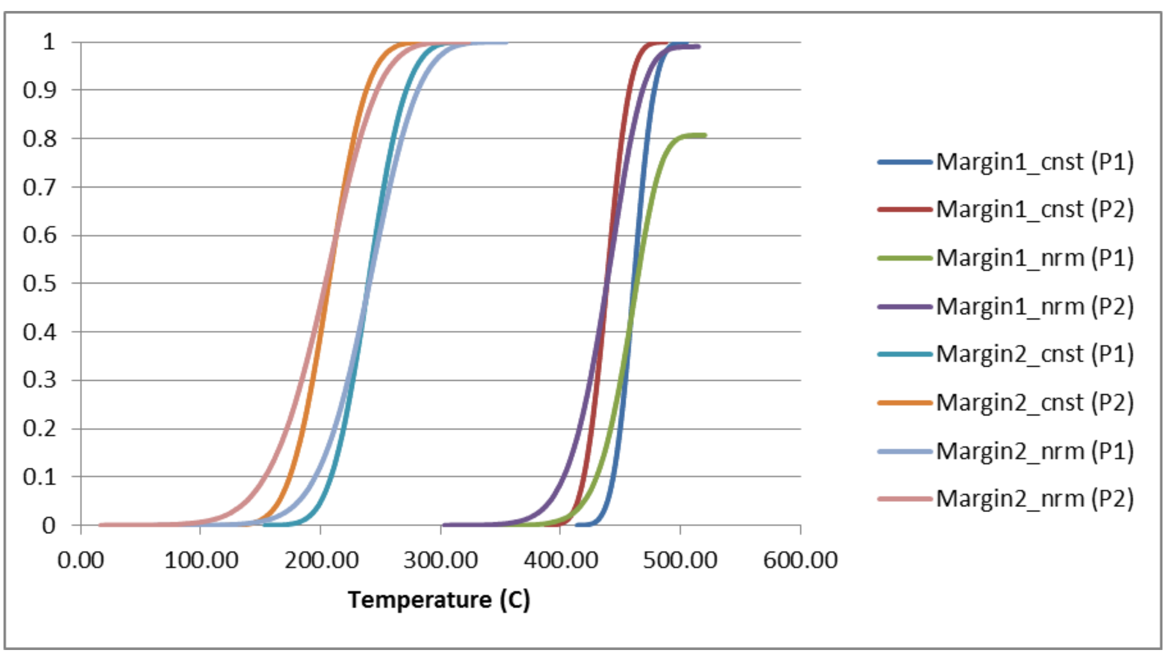

Figure 8: CDFs (cumulative distribution functions) for temperature margin for the test cases. (Note: Both Margin1_nrm cases returned some "Infinity" values, which are omitted in this plot).

5. Full scale problem

5.1. Full scale problem: Description

A multi-compartment system in a fire is now defined for use in testing the procedures implemented in CPLOAS_2. Specifically, the fourteen regions associated with the system are shown in Figure 9, and the system of differential equations defining time-dependent temperature and pressure for the individual regions is shown in Table 17. Region 1 corresponds to the fire encompassing the system. Values for the coefficients appearing in the system of equations were chosen to produce interesting results for the purpose of demonstrating the numerical procedures implemented in CPLOAS_2. The resulting analyses are not intended to be representative of any specific system of interest to Sandia. The relationship between link temperature and link failure pressure for SL 1 is defined in Figure 10.

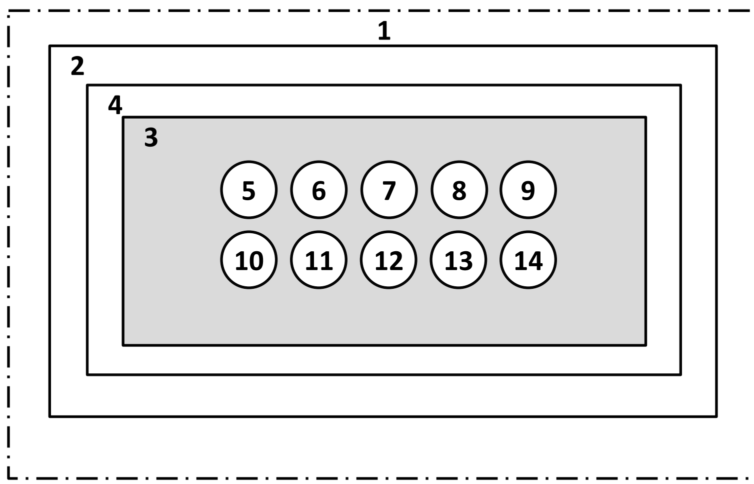

Figure 9: Multi-compartment system exposed to fire in region 1.

The purpose of this exercise is to verify the results of the five numerical procedures on a full scale problem. There are two test sets in this section: one test set involving aleatory uncertainty on failure values only (comparable to CPLOAS1), and one test set involving aleatory uncertainty on both parameter and failure values. In each case, all four PLOAS patterns are executed.

For each test case, CPLOAS_2 executed five different numerical procedures: a quadrature procedure and four variants of sampling-based procedures. The quadrature procedure was run with *nCDF* = 1,000,000 CDF discretization steps and *nQUAD* = 10,000 quadrature discretization steps. For the four sampling procedures, *nFT* = 1,000,000 and *nFTC* = 1,000,000 as described in Section 2.2.1. The random seed used was *nSEED* = 32.

Table 17: System of differential equations defining time-dependent temperature for the individual compartments shown in the multi-compartment system in Figure 9

$t = [0, 200; 0.1]$

$T_1(t) = 1273$ for all t

$T_i(0) = 273$ for $i = 2, 3, 4 \ldots, 14$

$$\frac{dT_2}{dt} = \frac{T_1 - T_2}{\tau_{1,2}} - \frac{T_2 - T_4}{\tau_{2,4}}$$

$$\frac{dT_3}{dt} = \frac{T_4 - T_3}{\tau_{4,3}} - \sum_{i=5}^{11} \frac{T_3 - T_i}{\tau_{3,i}} - \sum_{i=12}^{14} \frac{T_3 - T_i}{\tau_{3,i}}$$

$$\frac{dT_4}{dt} = \frac{T_2 - T_4}{\tau_{2,4}} - \frac{T_4 - T_3}{\tau_{4,3}}$$

$$\frac{dT_5}{dt} = \frac{T_3 - T_5}{\tau_{3,5}}$$

$$\frac{dT_6}{dt} = \frac{T_3 - T_6}{\tau_{3,6}}$$

$$\frac{dT_7}{dt} = \frac{T_3 - T_7}{\tau_{3,7}}$$

$$\frac{dT_8}{dt} = \frac{T_3 - T_8}{\tau_{3,8}}$$

$$\frac{dT_9}{dt} = \frac{T_3 - T_9}{\tau_{3,9}}$$

$$\frac{dT_{10}}{dt} = \frac{T_3 - T_{10}}{\tau_{3,10}}$$

$$\frac{dT_{11}}{dt} = \frac{T_3 - T_{11}}{\tau_{3,11}}$$

$$\frac{dT_{12}}{dt} = \frac{T_3 - T_{12}}{\tau_{3,12}}$$

$$\frac{dT_{13}}{dt} = \frac{T_3 - T_{13}}{\tau_{3,13}}$$

$$\frac{dT_{14}}{dt} = \frac{T_3 - T_{14}}{\tau_{3,14}}$$

$P_4(0) = 0$

$$\frac{dP_4}{dt} = 0.5\frac{dT_3}{dt} + (1.0E - 5)\,T_3^{\,2}$$

Where $\tau_{1,2}$=1.5, $\tau_{4,3}$=2, $\tau_{2,4}$=2, $\tau_{3,5}$=50, $\tau_{3,6}$=52, $\tau_{3,7}$=55, $\tau_{3,8}$=70, $\tau_{3,9}$=72, $\tau_{3,10}$=75, $\tau_{3,11}$=77, $\tau_{3,12}$=4, $\tau_{3,13}$=2, $\tau_{3,14}$=25.

Temperature (K)	Pcrit (PSI)
273	940
297	930
673	900
873	850
977	740
1050	550
1100	400
1150	220
1200	105
1300	63
1373	47.5
1700	0

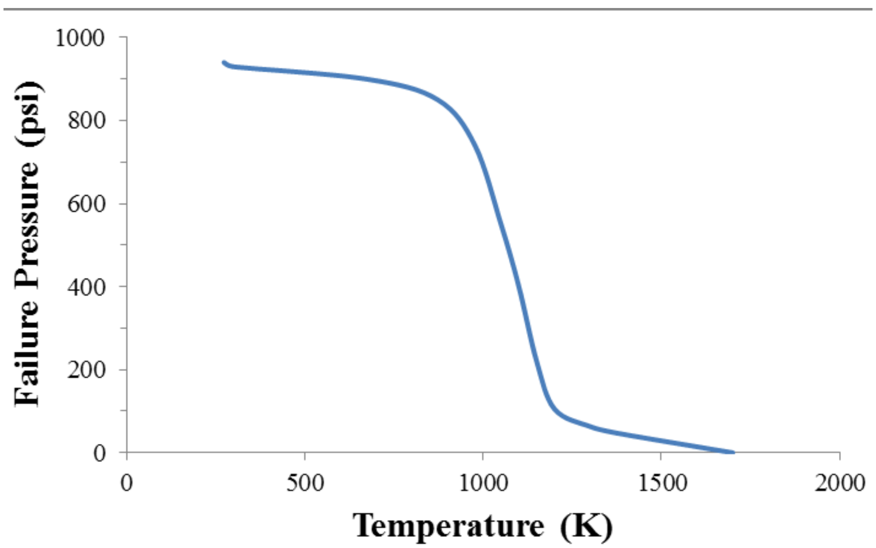

Figure 10: Failure pressure as a function of temperature for SL 1.

5.2. Full scale problem: Aleatory uncertainty on *q* only

The test cases in this section are designed to resemble test cases performed with CPLOAS1 (for which aleatory uncertainty was only included on the failure values). For the test cases in this section, the distributions for β are provided in Table 18 (next page) with $\alpha = 1$.

Results of the test cases are presented in Table 19. Owing to the high PLOAS values involved, the two importance sampling procedures are not appropriate for test cases P2 and P4; and the results of their application are highlighted in yellow to distinguish these results from results obtained with the other more appropriate procedures.

Table 18: PLOAS calculations for full scope test (with aleatory uncertainty on *q* only) defined in Section 5.1 (results highlighted in yellow obtained with procedures not appropriate for the problem under consideration)

	Quad.	**MC1**	**MC2**	**IMP1**	**IMP2**
Pattern 1	0.00E+00	0.00E+00 (--)	0.00E+00(--)	0.00E+00 (--)	0.00E+00(--)
Pattern 2	3.96E-01	3.96E-01 (0.1%)	3.96E-01 (0.0%)	4.04E-01 (2.1%)	2.89E-01 (-27.0%)
Pattern 3	1.63E-06	1.00E-06 (-38.5%)	0.00E+00 (--)	2.18E-06 (33.8%)	6.55E-06 (303.0%)
Pattern 4	8.32E-01	8.32E-01 (-0.1%)	8.32E-01 (-0.1%)	8.02E-01 (-3.6%)	6.48E-01 (-22.1%)

There is acceptable agreement between quadrature and all sampling techniques for all patterns. All 5 techniques predict PLOAS of 0.0 for pattern 1. For high probability patterns 2 and 4, MC1 deviates from quadrature by less than 0.1%. For the low probability pattern 3, the MC1 technique deviates by 38.5%, but contains the quadrature result within the corresponding 95% confidence interval of [0.00E+00, 2.96E-06]. For high probability patterns 2 and 4, MC2 deviates from quadrature by less than 0.1%. For the low probability pattern 3 the MC2 technique predicts PLOAS of 0.0. Since the sample size used is on the same order of magnitude as the PLOAS value for pattern 3, this result is considered to be acceptable. For the low probability pattern 3, the IMP1 technique deviates by 33.8%, but contains the quadrature result within the corresponding 95% confidence interval of [1.19E-06, 3.16E-06]. For the low probability pattern 3, the IMP2 technique deviates by 303.0%, but contains the quadrature result within the corresponding 95% confidence interval of [0.00E+00, 1.66E-05].

The performance of the four sampling techniques is in acceptable agreement with quadrature for all test cases in this set.

Table 19: Summary of WL-SL properties and associated distributions

WL1: Associated with region 11 in Figure 9. Link failure based on temperature, with (i) nominal temperature $\bar{p}_{WL1}(t)$ defined by function $T_{11}(t)$ in Table 17, with no aleatory uncertainty ($\alpha_{WL1} = 1$) and (ii) nominal failure temperature $\bar{q}_{WL1}(t) = 500$ K with aleatory variable β_{WL1} assigned a normal distribution with μ = 1.0, σ = 0.05, and q = 0.01.

WL2: Associated with region 10 in Figure 9. Link failure based on temperature, with (i) nominal temperature $\bar{p}_{WL2}(t)$ defined by function $T_{10}(t)$ in Table 17, with no aleatory uncertainty ($\alpha_{WL2} = 1$) and (ii) nominal failure temperature $\bar{q}_{WL2}(t) = 480$ K with aleatory variable β_{WL2} assigned a triangular distribution on [0.95, 1.05] with mode 1.0.

WL3: Associated with region 9 in Figure 9. Link failure based on temperature, with (i) nominal temperature $\bar{p}_{WL3}(t)$ defined by function $T_9(t)$ in Table 17, with no aleatory uncertainty ($\alpha_{WL3} = 1$) and (ii) nominal failure temperature $\bar{q}_{WL3}(t) = 550$ K with aleatory variable β_{WL3} assigned a triangular distribution on [0.95, 1.05] with mode 1.0.

SL1: Associated with region 4 in Figure 9. Link failure based on pressure, with (i) nominal temperature defined by function $T_4(t)$ in , (ii) nominal pressure $\bar{p}_{SL1}(t)$ defined by function $P_9(t)$ in Table 17, with no aleatory uncertainty ($\alpha_{SL1} = 1$) and (ii) nominal failure pressure $\bar{q}_{SL1}(t)$ defined by $f_{SL1}[T_4(t)]$ in Figure 10, with aleatory variable β_{SL3} assigned a normal distribution with μ = 1.0, σ = 0.05, and q = 0.001.

SL2: Associated with region 5 in Figure 9. Link failure based on temperature, with (i) nominal temperature $\bar{p}_{SL2}(t)$ defined by function $T_5(t)$ in Table 17, with no aleatory uncertainty ($\alpha_{SL2} = 1$) and (ii) nominal failure temperature $\bar{q}_{SL2}(t) = 675$ K with aleatory variable β_{SL2} assigned a normal distribution with μ = 1.0, σ = 0.1, and q = 0.001.

SL3: Associated with region 6 in Figure 9. Link failure based on temperature, with (i) nominal temperature $\bar{p}_{SL3}(t)$ defined by function $T_6(t)$ in Table 17, with no aleatory uncertainty ($\alpha_{SL3} = 1$) and (ii) nominal failure temperature $\bar{q}_{SL3}(t) = 675$ K with aleatory variable β_{SL3} assigned a uniform distribution on [0.9, 1.1].

SL4: Associated with region 7 in Figure 9. Link failure based on temperature, with (i) nominal temperature $\bar{p}_{SL4}(t)$ defined by function $T_7(t)$ in Table 17, with no aleatory uncertainty ($\alpha_{SL4} = 1$) and (ii) nominal failure temperature $\bar{q}_{SL4}(t) = 675$ K with aleatory variable β_{SL4} assigned a normal distribution with μ = 1.0, σ = 0.2, and q = 0.01.

SL5: Associated with region 12 in Figure 9. Link failure based on temperature, with (i) nominal temperature $\bar{p}_{SL5}(t)$ defined by function $T_{12}(t)$ in Table 17, with no aleatory uncertainty ($\alpha_{SL5} = 1$) and (ii) nominal failure temperature $\bar{q}_{SL5}(t) = 975$ K with aleatory variable β_{SL5} assigned a normal distribution with μ = 1.0, σ = 0.1, and q = 0.001.

SL6: Associated with region 13 in Figure 9. Link failure based on temperature, with (i) nominal temperature $\bar{p}_{SL6}(t)$ defined by function $T_{13}(t)$ in Table 17, with no aleatory uncertainty ($\alpha_{SL6} = 1$) and (ii) nominal failure temperature $\bar{q}_{SL6}(t) = 975$ K with aleatory variable β_{SL6} assigned a uniform distribution on [0.9, 1.1].

SL7: Associated with region 14 in Figure 9. Link failure based on temperature, with (i) nominal temperature $\bar{p}_{SL7}(t)$ defined by function $T_{14}(t)$ in Table 17, with no aleatory uncertainty ($\alpha_{SL7} = 1$) and (ii) nominal failure temperature $\bar{q}_{SL7}(t) = 975$ K with aleatory variable β_{SL7} assigned a normal distribution with μ = 1.0, σ = 0.2, and q = 0.001.

SL8: Associated with region 8 in Figure 9. Link failure based on temperature, with (i) nominal temperature $\bar{p}_{SL8}(t)$ defined by function $T_8(t)$ in Table 17, with no aleatory uncertainty ($\alpha_{SL8} = 1$) and (ii) nominal failure temperature $\bar{q}_{SL8}(t) = 600$ K with aleatory variable β_{SL8} assigned a triangular distribution on [0.95, 1.05] with mode 1.0.

5.3. Full scale problem: Aleatory uncertainty on *p* and *q*

The test cases in this section are designed to extend the test case in to include aleatory uncertainty on both property and failure values. For the test cases in this section, test cases use the distributions for β provided in Table 18. Distributions for α are provided in Table 20.

Table 20: Aleatory distributions forms for α for the full scale test case

Link	Distribution on alpha
SL 1	Constant (1)
SL 2	Normal(mean=1.0,stdev=0.05,trunc.=0.01)
SL 3	Uniform(min=0.95,max=1.05)
SL 4	Triangular(min=0.95,mode=1.0,max=1.05)
SL 5	Lognormal(mean=0.0, stdev=0.05, trunc=0.01)
SL 6	Log Uniform(min=0.95,max=1.05)
SL 7	Log triangular(min=0.95 mode=1.0, max=1.05)
SL 8	Constant (1)
WL 1	Uniform(min=0.9,max=1.1)
WL 2	Normal(mean=1.0,stdev=0.1,trunc.=0.01)
WL3	Lognormal(mean=0.0, stdev=0.1, trunc=0.01)

Results of the test cases are presented in Table 21. Owing to the high PLOAS values involved, the two importance sampling procedures are not appropriate for test cases P2 and P4; and the results of their application are highlighted in yellow to distinguish these results from results obtained with the other more appropriate procedures.

Table 21: PLOAS calculations for full scope test with aleatory uncertainty on both p and q defined in Section 5.2 (Results highlighted in yellow obtained with procedures not appropriate for the problem under consideration)

	Quad.	MC1	MC2	IMP1	IMP2
Pattern 1	1.19E-06	1.00E-06 (-16.1%)	1.00E-06 (-16.1%)	1.20E-06 (0.6%)	7.70E-07 (-35.4%)
Pattern 2	3.78E-01	3.78E-01 (-0.1%)	3.79E-01 (0.1%)	3.73E-01 (-1.3%)	3.28E-01 (-13.3%)
Pattern 3	6.79E-03	6.78E-03 (-0.1%)	6.69E-03 (-1.5%)	6.70E-03 (-1.3%)	6.90E-03 (1.6%)
Pattern 4	8.68E-01	8.67E-01 (-0.1%)	8.67E-01 (-0.1%)	8.42E-01 (-2.9%)	6.95E-01 (-19.9%)

There is acceptable agreement between quadrature, and the four sampling techniques for all patterns. For the high and moderate probability patterns (2, 3, 4), MC1 deviates from the quadrature results by less than 0.1%, and MC2 deviates by less than 1.5%. For the low probability pattern 1, both MC1 and MC2 have confidence intervals (i.e. [0.00E+00, 2.96E-06] for both MC1 and MC2) that include the quadrature value. For the low probability patterns (2, 4), IMP1 deviates from quadrature by less than 2.9%. IMP1 performs best for the low probability pattern, deviating from quadrature by only 0.6%. For the moderate probability pattern, IMP2 deviates by less than 1.6%. As expected, IMP2 performs less well for the low probability patterns (2, 4).

The performance of the four sampling techniques is in acceptable agreement with the quadrature results for all test cases in this set for which their use is appropriate.

6. Bugs and Errors identified during testing

6.1. Corrected in Version 2.10 or earlier

Early testing revealed some bugs which were corrected by version 2.10:

- Error in implementation of the normal distribution. In early testing, test cases in Section 2.1 using normal distribution produced results that did not match the analytical solution. This problem was traced to a coding error in implementation of the normal distribution. The error was corrected in version 2.8, and all test cases were rerun.

- Error in Monte Carlo time module. The module could not produce a solution when failures happen at time 0 (either due to misspecification of the problem, or due to problems with immediate link failure). This is traceable to a bug in interpolation at the first time step. This was corrected in Version 2.9.

- Seg Fault traceable to number used for link/property failure. During testing, due to misspecification of a test problem, a seg fault error was received when a column was specified in the link file that does not exist in the CPCDF.DAT file. To help prevent this error in the future, an informative error message was added to Version 2.10.

- Error in calculating case "T6" with IMP2. This is traceable to a coding error wherein PLOAS results were inadvertently normalized twice. This bug was identified with test case Approx1_T6, and was corrected in Version 2.9.

During CPLOAS_2 testing, several additional features were added in subsequent versions. All tests were re-run on each new version.

- The option to output confidence intervals or mean values was added in version 2.9.

- The option to normalize the importance measures by sample size or by sum of weights was added in v2.9.

- Version 2.9 also includes a summary file output that summarizes PLOAS results and confidence intervals for each circuit.

- In v2.10 the - change of formula in inverse normal distribution (distributions.f90) so that logloc is always strictly positive. Similarly, a change in CDF (quadrature.f90) the calculation of MAX_ALPHA and MAX_BETA for both normal and lognormal distribution such that the error due to a quantile =1 does not occur anymore.

- Changed implementation for confidence intervals/mean and standard deviation. If a confidence level is selected outside of the range [0, 1], then mean and standard deviation are displayed in the summary and the outputs when sampling methods are used.

6.2. Future additions to CPLOAS_2

The two importance sampling procedures in CPLOAS_2 are very rigid in that they apply importance sampling to all links and use only left and right importance sampling distributions. A possible enhancement to CPLOAS_ 2 would be to implement an enhancement to IMP1 and IMP2 that allowed the user flexibility in the implementation of importance sampling that is not currently present. Specifically, this enhancement would have two components: (i) Specification of which links would be subject to importance sampling, and (ii) Definition of the distributions used for importance sampling. With respect to (ii), a workable implementation would be to assume that importance sampling would be from CDFs for failure times or α's and β's as specified by the user with the importance sampling distributions being beta distributions with defining parameters again specified by the user. Given that CDFs are being sampled, beta distributions are a good choice as they are two parameter distributions defined on the interval [0, 1] that can be made to have almost any shape by the appropriate specification of the two defining parameters (e.g., left and right triangular distributions are examples of beta distributions). The indicated enhancements would support more effective importance analyses with CPLOAS_2 than is now possible, but would also require knowledge and insight on the part of the user with respect to what is and is not important in the determination of PLOAS in the analysis under consideration in in order to appropriately define the importance sampling strategy to be used.

7. Conclusions

The test problems included in this report can be stored for use in periodically checking a stored or updated version of CPLOAS_2. In addition, it is important for the users of CPLOAS_2 to recognize that the five implemented numerical procedures for the determination of PLOAS provide a way to perform verification tests for any problem under analysis.

Future test suites should include additional tests for low probability cases. To achieve this, it may be necessary to increase the number of links used in cases with analytical solutions. Additional testing should be done to verify correct implementation of time and environmental margins. It may also become necessary to develop additional analytical solutions to specific test set-ups to provide enhanced verification of solutions.

8. References:

1. Helton JC, Pilch M, Sallaberry CJ. Probability of Loss of Assured Safety in Systems with Multiple Time-Dependent Failure Modes. SAND2012-8248. Albuquerque, NM: Sandia National Laboratories; 2013.
2. Sallaberry CJ, Helton JC. CPLOAS_2 User Manual. SAND2012-9305. Albuquerque, NM: Sandia National Laboratories; 2013.
3. Dobranich D. Race Metrics V2 (Exel macro). Albuquerque, NM: Sandia National Laboratories; 2013.
4. Helton JC, Johnson JD, Oberkampf WL. Probability of Loss of Assured Safety in Temperature Dependent Systems with Multiple Weak and Strong Links. SAND2004-5216. Albuquerque, NM: Sandia National Laboratories; 2004.

Appendix A: Test Case Summary

Table 22: Test case summary

Test name	Description	# test scripts executed	Patterns executed (each test)
Exact	Tests with analytical solutions	15	4
Approx	Tests without analytical solutions	6	4
Margin	Margin tests	4	4
Full	Full scale test	2	4

Table 23: CPLOAS_2 parameters for each test set

Test set	nSL, nWL	Data	Min PLOAS in set (analytical or quad)	Max PLOAS in set (analytical or quad)	nCDF	nQUAD	nFT	nFTC	nSEED
Exact1(a-e)	9SL, 3WL	Linear p, constant q	4.55E-03	0.9955	20,000	10,000	1,000,000	1,000,000	32
Exact2(a-e)	9SL, 3WL	Non-linear p and q	4.55E-03	0.9955	20,000	10,000	1,000,000	1,000,000	32
Exact3(a-e)	9SL, 3WL	Non-linear p, property-dependent q	4.55E-03	0.9955	20,000	10,000	1,000,000	1,000,000	32
Approx1 (T1-T6)	Various combinations of 5SL, 2WL	Mixed	1.28E-05	1.58E-02	200,000	10,000	100,000	1,000,000	352
Approx2(a-e)	9SL, 3WL	Mixed	4.21E-04	1.001	20,000	10,000	1,000,000	1,000,000	32
Margin1	2SL, 1WL	Mixed	0.0	0.0	20,000	10,000	10,000	100,000	42
Margin2	2SL, 1WL	Mixed	0.0	3.50E-06	20,000	10,000	10,000	100,000	42
Full1	8SL, 3WL	Mixed	0.0	0.832	1,000,000	10,000	1,000,000	1,000,000	32
Full2	8SL, 3WL	Mixed	1.19E-06	0.868	1,000,000	10,000	1,000,000	1,000,000	32

Table 24: Comparison of performance (percent change) for all five numerical techniques to analytical solutions

Test name	Pattern	Analytical PLOAS	Quad.	MC1	MC2	IMP1	IMP2
Exact1(a-e)	1	4.55E-03	-0.6% to -0.4%	0.1%	-1.1% to 0.3%	0.4%	-7.9% to -2.7%
	2	0.75	0.0% to 0.1%	-0.1%	0.1%	-1.0%	-25.1% to -22.9%
	3	0.25	-0.3% to -0.2%	0.2%	0.1% to 0.2%	8.2%	-14.0% to -8.8%
	4	0.9955	0.1%	0.0%	0.0%	3.6%	-31.3%
Exact2(a-e)	1	4.55E-03	-0.8% to -0.5%	0.1%	-1.1% to 0.3%	0.4%	-7.9% to -2.7%
	2	0.75	0.1%	-0.1%	0.1%	-1.0%	-25.1% to -22.9%
	3	0.25	-0.4% to -0.2%	0.2%	0.1% to 0.2%	8.2%	-14.0% to -8.8%
	4	0.9955	0.1% to 0.2%	0.0%	0.0%	3.6%	-31.3%
Exact3(a-e)	1	4.55E-03	-1.7% to -1.2%	0.1%	-1.1% to 0.3%	0.4%	-7.9% to -2.7%
	2	0.75	0.1% to 0.2%	-0.1%	0.1%	-1.0%	-25.1% to -22.9%
	3	0.25	-0.8% to -0.5%	0.2%	0.1% to 0.2%	8.2%	-14.0% to -8.8%
	4	0.9955	0.3% to 0.4%	0.0%	0.0%	3.6%	-31.3%

Table 25: Comparison of performance (percent change) of sampling techniques to quadrature results

Test name	Pattern	Quad PLOAS	MC1	MC2	IMP1	IMP2
Approx2 (a-e)	1	8.52E-05 to7.53E-04	-2.6% to 7.4%	-2.1% to 11.5%	-0.6% to 1.6%	-7.3% to 3.4%
	2	0.951 to 0.983	-0.1%	-0.1% to 0.0%	1.0% to 2.0%	-30.5% to -29.1%
	3	0.297 to 0.328	0.6% to 0.7%	0.2% to 0.3%	0.9% to 3.4%	-18.0% to -14.4%
	4	1.001	-0.1%	-0.1%	3.4%	-31.7%
Full1	1	0.00E+00	--	--	--	--
	2	3.96E-01	0.1%	0.0%	2.1%	-27.0%
	3	1.63E-06	-38.5%	--	33.8%	303.0%
	4	8.32E-01	-0.1%	-0.1%	-3.6%	-22.1%
Full2	1	1.19E-06	-16.1%	-16.1%	0.6%	-35.4%
	2	3.78E-01	-0.1%	-0.1%	-1.3%	-13.3%
	3	6.79E-03	-0.1%	-1.5%	-1.3%	1.6%
	4	8.68E-01	-0.1%	-0.1%	-2.9%	-19.9%

Appendix B: Confidence Intervals

Table 26: Summary of 95% confidence intervals calculated within the domains of applicability for the individual sampling methods

		Quad	**MC1**	**MC2**	**IMP1**	**IMP2**				
			(2.50%, 97.50%)	**(2.50%, 97.50%)**	**(2.50%, 97.50%)**	**(2.50%, 97.50%)**	MC1	MC2	IMP1	IMP2
Approx 1	T1	6.91E-05	(5.54E-05, 8.86E-05)	(5.54E-05, 8.86E-05)	(5.22E-05, 7.91E-05)	(5.16E-05, 8.46E-05)	Y	Y	Y	Y
Approx 1	T2	1.57E-02	(1.55E-02, 1.60E-02)	(1.56E-02,1.61E-02)	(1.58E-02,1.60E-02)	(1.55E-02, 1.58E-02)	Y	Y	N	Y
Approx 1	T3	1.58E-02	(1.56E-02, 1.61E-02)	(1.57E-02,1.62E-02)	(1.58E-02,1.61E-02)	(1.56E-02, 1.59E-02)	Y	Y	Y	Y
Approx 1	T4	1.28E-05	(3.81E-06, 1.62E-05)	(3.81E-06, 1.62E-05)	(9.71E-06, 1.38E-05)	(1.16E-05, 1.43E-05)	Y	Y	Y	Y
Approx 1	T5	8.19E-05	(6.43E-05, 9.97E-05)	(6.43E-05, 9.97E-05)	(6.38E-05, 9.10E-05)	(6.45E-05, 9.76E-05)	Y	Y	Y	Y
Approx 1	T6	8.17E-05	(5.54E-05, 8.86E-05)	(6.43E-05, 9.97E-05)	(4.91E-05, 7.61E-05)	(1.06E-05, 3.25E-04)	Y	Y	N	Y
Approx 2a	P1	7.53E-04	(7.09E-04, 8.17E-04)	(6.89E-04, 7.95E-04)	(7.28E-04, 7.69E-04)	(6.39E-04, 8.36E-04)	Y	Y	Y	Y
Approx 2b	P1	4.58E-04	(4.38E-04, 5.24E-04)	(4.09E-04, 4.93E-04)	(4.52E-04, 4.78E-04)	(4.08E-04, 4.98E-04)	Y	Y	Y	Y
Approx 2c	P1	6.60E-04	(6.08E-04, 7.08E-04)	(6.13E-04, 7.13E-04)	(6.42E-04, 6.80E-04)	(5.58E-04, 7.28E-04)	Y	Y	Y	Y
Approx 2d	P1	8.52E-05	(6.52E-05, 1.01E-04)	(7.59E-05, 1.14E-04)	(7.70E-05, 9.48E-05)	(6.52E-05, 9.28E-05)	Y	Y	Y	Y
Approx 2e	P1	4.21E-04	(4.10E-04, 4.94E-04)	(3.72E-04, 4.52E-04)	(4.14E-04, 4.39E-04)	(3.61E-04, 5.09E-04)	Y	Y	Y	Y
Exact 1a	P1	4.53E-03	(4.42E-03, 4.68E-03)	(4.43E-03, 4.69E-03)	(4.48E-03, 4.64E-03)	(4.12E-03, 4.72E-03)	Y	Y	Y	Y
Exact 1b	P1	4.53E-03	(4.42E-03, 4.68E-03)	(4.39E-03, 4.65E-03)	(4.48E-03, 4.64E-03)	(3.95E-03, 4.42E-03)	Y	Y	Y	N
Exact 1c	P1	4.53E-03	(4.42E-03, 4.68E-03)	(4.40E-03, 4.66E-03)	(4.48E-03, 4.64E-03)	(4.12E-03, 4.71E-03)	Y	Y	Y	Y
Exact 1d	P1	4.52E-03	(4.42E-03, 4.68E-03)	(4.37E-03, 4.63E-03)	(4.48E-03, 4.64E-03)	(4.12E-03, 4.66E-03)	Y	Y	Y	Y
Exact 1e	P1	4.53E-03	(4.42E-03, 4.68E-03)	(4.40E-03, 4.66E-03)	(4.48E-03, 4.64E-03)	(3.98E-03, 4.56E-03)	Y	Y	Y	Y
Exact 2a	P1	4.52E-03	(4.42E-03, 4.68E-03)	(4.43E-03, 4.69E-03)	(4.48E-03, 4.64E-03)	(4.12E-03, 4.72E-03)	Y	Y	Y	Y
Exact 2b	P1	4.52E-03	(4.42E-03, 4.68E-03)	(4.39E-03, 4.65E-03)	(4.48E-03, 4.64E-03)	(3.95E-03, 4.42E-03)	Y	Y	Y	N
Exact 2c	P1	4.52E-03	(4.42E-03, 4.68E-03)	(4.40E-03, 4.66E-03)	(4.48E-03, 4.64E-03)	(4.12E-03, 4.71E-03)	Y	Y	Y	Y
Exact 2d	P1	4.51E-03	(4.42E-03, 4.68E-03)	(4.37E-03, 4.63E-03)	(4.48E-03, 4.64E-03)	(4.12E-03, 4.66E-03)	Y	Y	Y	Y
Exact 2e	P1	4.52E-03	(4.42E-03, 4.68E-03)	(4.40E-03, 4.66E-03)	(4.48E-03, 4.64E-03)	(3.98E-03, 4.56E-03)	Y	Y	Y	Y
Exact 3a	P1	4.49E-03	(4.42E-03, 4.68E-03)	(4.43E-03, 4.69E-03)	(4.48E-03, 4.64E-03)	(4.12E-03, 4.72E-03)	Y	Y	Y	Y
Exact 3b	P1	4.49E-03	(4.42E-03, 4.68E-03)	(4.39E-03, 4.65E-03)	(4.48E-03, 4.64E-03)	(3.95E-03, 4.42E-03)	Y	Y	Y	N
Exact 3c	P1	4.49E-03	(4.42E-03, 4.68E-03)	(4.40E-03, 4.66E-03)	(4.48E-03, 4.64E-03)	(4.12E-03, 4.71E-03)	Y	Y	Y	Y
Exact 3d	P1	4.47E-03	(4.42E-03, 4.68E-03)	(4.37E-03, 4.63E-03)	(4.48E-03, 4.64E-03)	(4.12E-03, 4.66E-03)	Y	Y	N	Y

Exact 3e	P 1	4.49E-03	(4.42E-03, 4.68E-03)	(4.40E-03, 4.66E-03)	(4.48E-03, 4.64E-03)	(3.98E-03, 4.56E-03)	Y	Y	Y	Y
Full1	P 3	1.63E-06	(0.00E+00, 2.96E-06)	(0.00E+00, 0.00E+00)	(1.19E-06, 3.16E-06)	(0.00E+00, 1.66E-05)	Y	N	Y	Y
Full2	P 1	1.19E-06	(0.00E+00, 2.96E-06)	(0.00E+00, 2.96E-06)	(9.92E-07, 1.41E-06)	(5.73E-07, 9.68E-07	Y	Y	Y	N
Full2	P 3	6.79E-03	(6.62E-03, 6.94E-03)	(6.53E-03, 6.85E-03)	(6.44E-03, 6.97E-03)	(6.05E-03, 7.75E-03)	Y	Y	Y	Y

Grey areas denote results that are possibly outside of the applicable range of the techniques. The Y's and N's indicate whether or not corresponding 95% confidence interval for a procedure includes the quadrature-calculated PLOAS value, with Y indicating inclusion and N indicating non-inclusion. For easy inspection, the N's are highlighted in yellow.

If the quadrature procedure is correctly calculating PLOAS and the four sampling-based procedures are performing their calculations correctly, then approximately 95% of the confidence intervals should contain the corresponding quadrature-calculated PLOAS values. The above table contains 116 confidence intervals, of which 108 contain the corresponding quadrature-calculated PLOAS values. Thus, (108/116) × 100% ≅ 93% of the 95% confidence intervals contain the corresponding quadrature-calculated PLOAS values, which provides a strong indication that both the quadrature procedure and the sampling-based procedures are operating correctly.

Appendix C: Summary of Numerical Procedures Used in CPLOAS_2 to Calculate Probability of Loss of Assured Safety

Author: Jon C. Helton

C.1 Introduction

This appendix provides a technical summary of the numerical procedures being tested in this report for the calculation of PLOAS. Specifically, the following numerical procedures are described: (i) quadrature procedure (Sect. C.2), (ii) random sampling procedure 1 (MC1) (Sect. C.3), (iii) random sampling procedure 2 (MC2) (Sect. C.4), (iv) importance sampling procedure 1 (IMP1) (Sect. C.4), and (v) importance sampling procedure 2 (IMP2) (Sect. C.5). Additional information and illustrations associated with these procedures are available in Ref. [1].

C.2 Quadrature Procedure

The defining integrals for PLOAS implemented in CPLOAS_2 are defined as shown in Table 27 with $CDF_{WL,j}(\tau)$ and $CDF_{SL,k}(\tau)$ representing the cumulative distribution functions (CDFs) for WL failure time and SL failure time, respectively. As described in Sect. 2 of Ref. [1], the failure time CDF for a single WL or SL is based on the following assumed properties of that link for a time interval $t_{mn} \le t \le t_{mx}$:

$$\bar{p}(t) = \text{nondecreasing function defining nominal link property for } t_{mn} \le t \le t_{mx}, \qquad \text{(C.1)}$$

$$\bar{q}(t) = \begin{array}{l}\text{nonincreasing function defining nominal failure value for link property}\\ \text{for } t_{mn} \le t \le t_{mx},\end{array} \qquad \text{(C.2)}$$

$$d_\alpha(\alpha) = \begin{array}{l}\text{density function for variable } \alpha \text{ used to characterize aleatory uncertainty}\\ \text{in link property,}\end{array} \qquad \text{(C.3)}$$

$$d_\beta(\beta) = \begin{array}{l}\text{density function for variable } \beta \text{ used to characterize aleatory uncertainty}\\ \text{in link failure value,}\end{array} \qquad \text{(C.4)}$$

$$p(t\,|\,\alpha) = \alpha\bar{p}(t) = \text{link property for } t_{mn} \le t \le t_{mx} \text{ given } \alpha, \qquad \text{(C.5)}$$

and

$$q(t\,|\,\beta) = \beta\bar{q}(t) = \text{link failure value for } t_{mn} \le t \le t_{mx} \text{ given } \beta. \qquad \text{(C.6)}$$

Further, $d_\alpha(\alpha)$ and $d_\beta(\beta)$ are assumed to be defined on intervals $[\alpha_{mn}, \alpha_{mx}]$ and $[\beta_{mn}, \beta_{mx}]$ and to equal zero outside these intervals.

Table 27: Representation of time-dependent values $pF_i(t)$, i = 1, 2, 3, 4, for PLOAS and associated verification tests for alternate definitions of LOAS for WL/SL Systems with (i) nWL WLs and nSL SLs and (ii) independent distributions for link failure time ([2], Table 10)

Case 1: Failure of all SLs before failure of any WL (Eqs. (2.1) and (2.5), Ref. [3])
$pF_1(t) = \sum_{k=1}^{nSL} \left(\int_0^t \left\{ \prod_{\substack{l=1 \\ l \neq k}}^{nSL} CDF_{SL,l}(\tau) \right\} \left\{ \prod_{j=1}^{nWL} \left[1 - CDF_{WL,j}(\tau) \right] \right\} \mathrm{d}CDF_{SL,k}(\tau) \right)$ Verification test: $pF_1(\infty) = nWL!\,nSL! / (nWL + nSL)!$
Case 2: Failure of any SL before failure of any WL (Eqs. (3.1) and (3.4), Ref. [3])
$pF_2(t) = \sum_{k=1}^{nSL} \left(\int_0^t \left\{ \prod_{\substack{l=1 \\ l \neq k}}^{nSL} \left[1 - CDF_{SL,l}(\tau) \right] \right\} \left\{ \prod_{j=1}^{nWL} \left[1 - CDF_{WL,j}(\tau) \right] \right\} \mathrm{d}CDF_{SL,k}(\tau) \right)$ Verification test: $pF_2(\infty) = nSL / (nWL + nSL)$
Case 3: Failure of all SLs before failure of all WLs (Eqs. (4.1) and (4.4), Ref. [3])
$pF_3(t) = \sum_{k=1}^{nSL} \left(\int_0^t \left\{ \prod_{\substack{l=1 \\ l \neq k}}^{nSL} CDF_{SL,l}(\tau) \right\} \left\{ 1 - \prod_{j=1}^{nWL} CDF_{WL,j}(\tau) \right\} \mathrm{d}CDF_{SL,k}(\tau) \right)$ Verification test: $pF_3(\infty) = nWL / (nWL + nSL)$
Case 4: Failure of any SL before failure of all WLs (Eqs. (5.1) and (5.4), Ref. [3])
$pF_4(t) = \sum_{k=1}^{nSL} \left(\int_0^t \left\{ \prod_{\substack{l=1 \\ l \neq k}}^{nSL} \left[1 - CDF_{SL,l}(\tau) \right] \right\} \left\{ 1 - \prod_{j=1}^{nWL} CDF_{WL,j}(\tau) \right\} \mathrm{d}CDF_{SL,k}(\tau) \right)$ Verification test: $pF_4(\infty) = 1 - \left[nWL!\,nSL! / (nWL + nSL)! \right]$

Once $CDF(t)$ and $CCDF(t) = 1 - CDF(t)$ are evaluated for individual links, the representations for PLOAS in Table 27 can be numerically evaluated with a quadrature procedure. Specifically, the probability $pF_1(t)$ for the failure all SLs before the failure of any WL defined as Case 1 in Table 27 is approximated in CPLOAS_2 by

$$\begin{aligned} pF_1(t) &\cong \sum_{k=1}^{nSL}\left(\sum_{i=1}^{n}\left\{\prod_{l=1,l\neq k}^{nSL} CDF_{SL,l}\left(t_{i-1}\right)\right\}\right. \\ &\qquad \left.\times\left\{\prod_{j=1}^{nWL}\left[1-CDF_{WL,j}\left(t_i\right)\right]\right\}\left\{CDF_{SL,k}\left(t_i\right)-CDF_{SL,k}\left(t_{i-1}\right)\right\}\right) \\ &= \sum_{k=1}^{nSL}\left(\sum_{i=1}^{n}\left\{\prod_{l=1,l\neq k}^{nSL} CDF_{SL,l}\left(t_{i-1}\right)\right\}\right. \\ &\qquad \left.\times\left\{\prod_{j=1}^{nWL}\left[CCDF_{WL,j}\left(t_i\right)\right]\right\}\left\{CDF_{SL,k}\left(t_i\right)-CDF_{SL,k}\left(t_{i-1}\right)\right\}\right) \end{aligned} \tag{C.7}$$

for subdivisions $0 = t_0 < t_1 < \ldots < t_n = t$ of $[0, t]$. As shown below, similar approximations are also used in CPLOAS_2 for the other three failure cases defined in Table 27. In the preceding approximation for $pF_1(t)$, left and right evaluations are indicated for SLs (i.e., $CDF_{SL,l}(t_{i-1})$ and $CCDF_{SL,l}(t_{i-1})$) and WLs (i.e., $CDF_{WL,j}(t_i)$ and $CCDF_{WL,j}(t_i)$), respectively, as the underlying assumption is that all SLs except for SL k have failed before time t_{i-1} and all WLs fail after time t_i. If the CDFs and CCDFs are continuous in time, this specification of evaluation times does not affect the limiting value for $pF_1(t)$ as Δt_i goes to zero.

Similarly, the representations $pF_2(t)$, $pF_3(t)$ and $pF_4(t)$ for PLOAS in Table 27 are approximated in CPLOAS_2 by

$$
\begin{aligned}
pF_2(t) &\cong \sum_{k=1}^{nSL}\left(\sum_{i=1}^{n}\left\{\prod_{l=1,l\neq k}^{nSL}\left[1-CDF_{SL,l}\left(t_{i-1}\right)\right]\right\}\right.\\
&\quad\left.\times\left\{\prod_{j=1}^{nWL}\left[1-CDF_{WL,j}\left(t_i\right)\right]\right\}\left\{CDF_{SL,k}\left(t_i\right)-CDF_{SL,k}\left(t_{i-1}\right)\right\}\right)\\
&=\sum_{k=1}^{nSL}\left(\sum_{i=1}^{n}\left\{\prod_{l=1,l\neq k}^{nSL}CCDF_{SL,l}\left(t_{i-1}\right)\right\}\right.\\
&\quad\left.\times\left\{\prod_{j=1}^{nWL}CCDF_{WL,j}\left(t_i\right)\right\}\left\{CDF_{SL,k}\left(t_i\right)-CDF_{SL,k}\left(t_{i-1}\right)\right\}\right),
\end{aligned}
\tag{C.8}
$$

$$
\begin{aligned}
pF_3(t) &\cong \sum_{k=1}^{nSL}\left(\sum_{i=1}^{n}\left\{\prod_{l=1,l\neq k}^{nSL}CDF_{SL,l}\left(t_{i-1}\right)\right\}\right.\\
&\quad\left.\times\left\{1-\prod_{j=1}^{nWL}CDF_{WL,j}\left(t_i\right)\right\}\left\{CDF_{SL,k}\left(t_i\right)-CDF_{SL,k}\left(t_{i-1}\right)\right\}\right)
\end{aligned}
\tag{C.9}
$$

and

$$
\begin{aligned}
pF_4(t) &\cong \sum_{k=1}^{nSL}\left(\sum_{i=1}^{n}\left\{\prod_{l=1,l\neq k}^{nSL}\left[1-CDF_{SL,l}\left(t_{i-1}\right)\right]\right\}\right.\\
&\quad\left.\times\left\{1-\prod_{j=1}^{nWL}CDF_{WL,j}\left(t_i\right)\right\}\left\{CDF_{SL,k}\left(t_i\right)-CDF_{SL,k}\left(t_{i-1}\right)\right\}\right)\\
&=\sum_{k=1}^{nSL}\left(\sum_{i=1}^{n}\left\{\prod_{l=1,l\neq k}^{nSL}CCDF_{SL,l}\left(t_{i-1}\right)\right\}\right.\\
&\quad\left.\times\left\{1-\prod_{j=1}^{nWL}CDF_{WL,j}\left(t_i\right)\right\}\left\{CDF_{SL,k}\left(t_i\right)-CDF_{SL,k}\left(t_{i-1}\right)\right\}\right)
\end{aligned}
\tag{C.10}
$$

for subdivisions $0 = t_0 < t_1 < \ldots < t_n = t$ of $[0, t]$.

In the numerical evaluation of the indicated integrals in CPLOAS_2, the individual link failure time CDFs are discretized by dividing time interval under consideration into *nCDF* equally-spaced discretization steps and the integrals are approximated by dividing time interval under consideration into *nQUAD* equally-spaced discretization steps. As needed, linear interpolation is used to incorporate the link failure CDFs into the quadrature approximations to PLOAS.

C.3 Sampling Procedure 1 (MC1)

Sampling Procedure 1 (MC1) is based to estimate the expected values of functions $\delta_i(t\,|\,\mathbf{t})$, i = 1, 2, 3, 4, where

$$t = \text{ time at which PLOAS (i.e., } pF_i(t) \text{ in Table 1) is to be determined,} \tag{C.11}$$

$$tWL_j = \text{ time at which WL } j \text{ fails, } j = 1, 2, ..., nWL, \tag{C.12}$$

$$tSL_j = \text{ time at which SL } j \text{ fails, } j = 1, 2, ..., nSL, \tag{C.13}$$

$$\mathbf{t} = \left[tWL_1, tWL_2, ..., tWL_{nWL}, tSL_1, tSL_2, ..., tSL_{nSL}\right], \tag{C.14}$$

$$\delta_1(t\,|\,\mathbf{t}) = \begin{cases} 1 \text{ if } \max\left\{tSL_1, tSL_2, ..., tSL_{nSL}\right\} \le \min\left\{t, tWL_1, tWL_2, ..., tWL_{nSL}\right\} \\ 0 \text{ otherwise,} \end{cases} \tag{C.15}$$

$$\delta_2(t\,|\,\mathbf{t}) = \begin{cases} 1 \text{ if } \min\left\{tSL_1, tSL_2, ..., tSL_{nSL}\right\} \le \min\left\{t, tWL_1, tWL_2, ..., tWL_{nSL}\right\} \\ 0 \text{ otherwise,} \end{cases} \tag{C.16}$$

$$\delta_3(t\,|\,\mathbf{t}) = \begin{cases} 1 \text{ if } \max\left\{tSL_1, tSL_2, ..., tSL_{nSL}\right\} \le \min\left\{t, \max\left\{tWL_1, tWL_2, ..., tWL_{nSL}\right\}\right\} \\ 0 \text{ otherwise,} \end{cases} \tag{C.17}$$

and

$$\delta_4(t\,|\,\mathbf{t}) = \begin{cases} 1 \text{ if } \min\left\{tSL_1, tSL_2, ..., tSL_{nSL}\right\} \le \min\left\{t, \max\left\{tWL_1, tWL_2, ..., tWL_{nSL}\right\}\right\} \\ 0 \text{ otherwise.} \end{cases} \tag{C.18}$$

In words, $\delta_1(t\,|\,\mathbf{t}) = 1$ corresponds to all SLs failing before time t and also before any WL fails (i.e., Case 1 in Table 27); $\delta_2(t\,|\,\mathbf{t}) = 1$ corresponds to any SL failing before time t and also before any WL fails (i.e., Case 2 in Table 27); $\delta_3(t\,|\,\mathbf{t}) = 1$ corresponds to all SLs failing before time t and also before all WLs fail (i.e., Case 3 in Table 27); and $\delta_4(t\,|\,\mathbf{t}) = 1$ corresponds to any SL failing before time t and also before all WLs fail (i.e., Case 4 in Table 27). If a time interval $[t_{mn}, t_{mx}]$ is under consideration, the possible failure time t is assumed to be contained in $[t_{mn}, t_{mx}]$; further, if a link has not failed within $[t_{mn}, t_{mx}]$, its failure time is set to a value greater than t_{mx} for use with the indicator functions $\delta_i(t\,|\,\mathbf{t})$ defined in Eqs. (C.15)-(C.18).

The expected value $E[\delta_i(t\,|\,\mathbf{t})]$, i = 1, 2, 3, 4, for $\delta_i(t\,|\,\mathbf{t})$ corresponds to the PLOAS value $pF_i(t)$ defined in Table 27. Approach MC1 uses random sampling from the CDFs $CDF_{WL,j}(\tau)$, j = 1, 2, …, *nWL*, and $CDF_{SL,j}(\tau)$, j = 1, 2, …, *nSL*, for link failure times in the estimation of PLOAS values. In this approach, $pF_i(t)$ is approximated by

$$\begin{aligned}
pF_i(t) &= \int_{I^{nL}} \delta_i\left[t\,|\,\mathbf{f}(\mathbf{r})\right]\prod_{k=1}^{nL} d_k\left(r_k\right)\prod_{k=1}^{nL}\mathrm{d}r_k \\
&= \int_{I^{nL}} \delta_i\left[t\,|\,\mathbf{f}(\mathbf{r})\right]\prod_{k=1}^{nL}\mathrm{d}r_k \\
&\cong \sum_{l=1}^{nR}\delta_i\left[t\,|\,\mathbf{f}(\mathbf{r}_l)\right]\Big/nR \\
&= \sum_{l=1}^{nR}\delta_i\left\{t\,|\left[tWL_{1l}, tWL_{2l}, \ldots, tWL_{nWL,l}, tSL_{1l}, tSL_{2l}, \ldots, tSL_{nSL,l}\right]\right\}\Big/nR,
\end{aligned} \tag{C.19}$$

where (i) $nL = nWL + nSL$, (ii) $d_k(r_k) = 1$ is the density function for a variable r_k with a uniform distribution on [0, 1], (iii) $I^{nL} = [0,1]^{nL}$ (i.e., the unit cube of dimension *nL*), (iv) $\mathbf{r} = [r_1, r_2, \ldots, r_{nL}] \in I^{nL}$, (v) the function $\mathbf{f}(\mathbf{r})$ is defined by

$$\begin{aligned}
\mathbf{f}(\mathbf{r}) = \Big[&CDF_{WL,1}^{-1}(r_1), CDF_{WL,2}^{-1}(r_2), \ldots, CDF_{WL,nWL}^{-1}(r_{nWL}), \\
&CDF_{SL,1}^{-1}(r_{nWL+1}), CDF_{SL,2}^{-1}(r_{nWL+2}), \ldots, CDF_{SL,nSL}^{-1}(r_{nL})\Big] \\
= \big[&tWL_1, tWL_2, \ldots, tWL_{nWL}, tSL_1, tSL_2, \ldots, tSL_{nSL}\big]
\end{aligned} \tag{C.20}$$

with $tWL_j = CDF_{WL,j}^{-1}(r_j)$ for j = 1, 2, …, *nWL* and $tSL_j = CDF_{SL,j}^{-1}(r_{nWL+j})$ for j = 1, 2, …, *nSL*, and (vi) $\mathbf{r}_l$, l = 1, 2, …, *nR*, is a random sample of size *nR* from a uniform distribution on I^{nL}. With respect to the approximation of $pF_i(t)$ in Eq. (C.19), the first equality defines $pF_i(t)$ as the expected value of $\delta_i[t\,|\,\mathbf{f}(\mathbf{r})]$; the second equality is a notational simplification based on the equalities $d_k(r_k) = 1$ for k = 1, 2, …, *nL*; the approximation at the third step is based on a random sample from the link failure times; and the final equality is a restatement of $\mathbf{f}(\mathbf{r})$ in terms of link failure times.

For computational efficiency, CPLOAS_2 implements MC1 as a two-step procedure. In the first step, *nFT* vectors of link failure times of the form indicated in Eq. (C.20) are generated. In the second step, *nFTC* vectors of link failure times of the form indicated in Eq. (C.20) are generated by randomly sampling from the link failure times generated in Step 1 (i.e., from the *nFT* failure times for each link). Then, the *nFTC* vectors of link failure times generated in Step 2 are used in Eq. (C.19) in the estimation of PLOAS. For this approach to be effective, *nFTC* must be significantly larger than *nFT*.

In addition, CPLOAS_2 also determines a variance and standard error for the estimate $\widehat{pF}_i(t)$ for $pF_i(t)$ in Eq. (C.19). Specifically, it follows from the Central Limit Theorem that

$$\frac{\widehat{pF}_i(t) - pF_i(t)}{s_i(t) \big/ \sqrt{nR}} \tag{C.21}$$

is approximately distributed as a normal distribution with mean 0 and variance 1 (i.e., is $N(0, 1)$), where

$$s_i(t) = \left[\sum_{l=1}^{nR} \left(\delta_i \left\{ t \mid \left[tWL_{1l}, tWL_{2l}, ..., tWL_{nWL,l}, tSL_{1l}, tSL_{2l}, ..., tSL_{nSL,l} \right] \right\} - \widehat{pF_i}(t) \right)^2 \Big/ nR \right]^{1/2} \tag{C.22}$$

and nR is sufficiently large ([4], p. 75). In turn, the quantity $s_i / \sqrt{nR}$ can be used to assess the potential error in the approximation $\widehat{pF}_i(t)$ for $pF_i(t)$ in Eq. (C.19). More specifically, if $z_{\alpha/2}$ is the $1-\alpha/2$ quantile of the unit normal distribution $N(0, 1)$, then

$$prob\left(\widehat{pF}_i(t) - z_{\alpha/2} s_i(t) \big/ \sqrt{nR} \le pF_i(t) \le \widehat{pF}_i(t) + z_{\alpha/2} s_i(t) \big/ \sqrt{nR} \right) = 1 - \alpha \tag{C.23}$$

where $prob(\sim)$ denotes probability ([5], pp. 168-169). In turn,

$$\left[\widehat{pF}_i(t) - z_{\alpha/2} s_i(t) \big/ \sqrt{nR}, \widehat{pF}_i(t) + z_{\alpha/2} s_i(t) \big/ \sqrt{nR} \right] \tag{C.24}$$

is a $100(1-\alpha)$ percent confidence interval for $pF_i(t)$. As an example, $z_{\alpha/2}$ = 1.96 for a 95% confidence interval. Because the sample size nR used in the estimation of $pF_i(t)$ in Eq. (C.19) will be a large integer, it is acceptable to use the unit normal distribution in the estimation of the confidence interval in Eq. (C.24) rather than the *t*-distribution, which would be used if *nR* was a very small integer.

For completeness, the computational implementation of the two-step sampling procedure used in CPLOAS_2 for MC1 is now described in more detail. In the first step, failure times for the individual links are randomly sampled *nFT* times from the CDFs for link failure time. This produces sets

$$\mathcal{WL}_j = \left\{ tWL_{jl} : l = 1, 2, ..., nFT \right\}, \ j = 1, 2, ..., nWL \tag{C.25}$$

and

$$\mathcal{SL}_j = \left\{ tSL_{jl} : l = 1, 2, ..., nFT \right\}, \ j = 1, 2, ..., nSL \tag{C.26}$$

of *nFT* randomly-sampled link failure times for each link, where (i) tWL_{jl} , l = 1, 2, …, *nFT*, are the sampled failure times for WL j, and (ii) tSL_{jl} , l = 1, 2, …, *nFT*, are the sampled failure times for SL j.

In the second step, failure times are randomly sampled *nFTC* times with replacement from the sets $\mathcal{WL}_j$ and $\mathcal{SL}_j$ to produce the following sequence of vectors

$$\mathbf{t}_l = \left[tWL_{1l}, tWL_{2l}, ..., tWL_{nWL,l}, tSL_{1l}, tSL_{2l}, ..., tSL_{nSL,l}\right], l = 1, 2, ..., nFTC, \tag{C.27}$$

of possible link failure times in Eq. (C.19) with *nFTC* corresponding to the sample size *nR* in Eq. (C.19). In turn, each vector $\mathbf{t}_l$ of link failure times results in a corresponding time tF_{il} at which LOAS occurs obtained in consistency with the definition of LOAS under consideration (i.e., i = 1, 2, 3, 4 as indicated in Eqs. (C.15)-(C.18)), which is the failure time used in the evaluation of the function $\delta_i[t \,|\, \mathbf{t}_l]$ in the final equality of Eq. (C.19).

Specifically, $pF_i(t)$ is approximated in CPLOAS_2 by

$$\widehat{pF_i}(t) = \sum_{l=1}^{nFTC} \delta(t \,|\, tF_{il}) \big/ nFTC \quad \text{with} \quad \delta(t \,|\, tF_{il}) = \begin{cases} 1 & \text{if } tF_{il} \le t \\ 0 & \text{otherwise.} \end{cases} \tag{C.28}$$

The computational procedure implemented within CPLOAS_2 does not save the times tF_{il} until the end of the calculation and then determine $\widehat{pF_i}(t)$ as indicated in Eq. (C.28). Rather, a running sum

$$S_{i0}(t) = 0, \quad S_{il}(t) = S_{i,l-1}(t) + \delta(t \,|\, tF_{il}), l = 1, 2, ..., nFTC, \tag{C.29}$$

of the functions $\delta(t \,|\, tF_{il})$ is performed that yields

$$\widehat{pF_i}(t) = S_{i,nFTC}(t) \big/ nFTC \tag{C.30}$$

at the end of the calculation but does not save the individual times tF_{il}.

The standard deviation associated with the estimate for $pF_i(t)$ in Eq. (C.28) is given by

$$
\begin{aligned}
s_i(t) &= \left\{ \sum_{l=1}^{nFTC} \left[\delta(t \mid tF_{il}) - \widehat{pF_i}(t) \right]^2 \Big/ nFTC \right\}^{1/2} \Big/ \sqrt{nFTC} \\
&= \left\{ \sum_{l=1}^{nFTC} \left[\delta^2(t \mid tF_{il}) / nFTC \right] - \widehat{pF_i}^{\,2}(t) \right\}^{1/2} \Big/ \sqrt{nFTC} \\
&= \left\{ \left[S_{i,nFTC}(t) / nFTC \right] - \widehat{pF_i}^{\,2}(t) \right\}^{1/2} \Big/ \sqrt{nFTC} \\
&= \left\{ \widehat{pF_i}(t) - \widehat{pF_i}^{\,2}(t) \right\}^{1/2} \Big/ \sqrt{nFTC}
\end{aligned}
\qquad \text{(C.31)}
$$

with the problem reformulation associated with the third equality possible because $\delta(t \mid tF_{il})$ is always either 0 or 1.

With respect to the two step sampling in use, (i) the first sample from the link failure CDFs is, in effect, a numerical procedure to facilitate the evaluation of the link failure CDFs, and (ii) the second sample corresponds to the sample used in the evaluation of the integral in Eq. (C.19).

C.4 Sampling Procedure 2 (MC2)

Sampling procedure 2 (MC2) is similar to sampling procedure 1 (MC1) but with use of the distributions for the variables $\alpha_{WL,j}, \beta_{WL,j}$, j = 1, 2, …, nWL, and $\alpha_{SL,j}, \beta_{SL,j}$, j = 1, 2, …, nSL, indicated in conjunction with Eqs. (C.1)-(C.6) that define properties and failure values for the individual links. The approximation to $pF_i(t)$ for MC2 is analogous to the approximation in Eq. (C.19) for MC1 but with changed definitions for **r**, $\mathbf{r}_l$, **f**(**r**) and $d_k(r_k)$. Specifically,

$$
\begin{aligned}
\mathbf{r} &= \left[r_1, r_2, ..., r_{2nL} \right] \text{ with } nL = nWL + nSL \\
&= \left[\alpha_{WL,1}, \alpha_{WL,2}, ..., \alpha_{WL,nWL}, \beta_{WL,1}, \beta_{WL,2}, ..., \beta_{WL,nWL}, \right. \\
&\qquad \left. \alpha_{SL,1}, \alpha_{SL,2}, ..., \alpha_{SL,nSL}, \beta_{SL,1}, \beta_{SL,2}, ..., \beta_{SL,nSL} \right],
\end{aligned}
\qquad \text{(C.32)}
$$

$$
\mathbf{p}_{WL,j} = \left[\alpha_{WL,j}, \beta_{WL,j} \right] = \left[r_j, r_{nWL+j} \right], \; j = 1, 2, ..., nWL, \qquad \text{(C.33)}
$$

$$
\mathbf{p}_{SL,j} = \left[\alpha_{SL,j}, \beta_{SL,j} \right] = \left[r_{2nWL+j}, r_{2nWL+nSL+j} \right], \; j = 1, 2, ..., nSL, \qquad \text{(C.34)}
$$

$$
\begin{aligned}
f_{WL,j}\left(\mathbf{p}_{WL,j}\right) &= \text{ time at which WL } j \text{ fails with } \mathbf{p}_{WL,j} = \left[\alpha_{WL,j}, \beta_{WL,j} \right] \\
&= tWL_j,
\end{aligned}
\qquad \text{(C.35)}
$$

$$
\begin{aligned}
f_{SL,j}\left(\mathbf{p}_{SL,j}\right) = & \text{ time at which SL } j \text{ fails with } \mathbf{p}_{SL,j} = \left[\alpha_{SL,j}, \beta_{SL,j}\right] \\
= & \, tSL_j,
\end{aligned}
\qquad (C.36)
$$

and

$$
\begin{aligned}
\mathbf{f}(\mathbf{r}) = \Big[& f_{WL,1}\left(\mathbf{p}_{WL,1}\right), f_{WL,2}\left(\mathbf{p}_{WL,2}\right), \ldots, f_{WL,nWL}\left(\mathbf{p}_{WL,nWL}\right) \\
& f_{SL,1}\left(\mathbf{p}_{SL,1}\right), f_{SL,2}\left(\mathbf{p}_{SL,2}\right), \ldots, f_{SL,nSL}\left(\mathbf{p}_{WL,nSL}\right)\Big] \\
= \big[& tWL_1, tWL_2, \ldots, tWL_{nWL}, tSL_1, tSL_2, \ldots, tSL_{nSL}\big].
\end{aligned}
\qquad (C.37)
$$

In turn,

$$
\begin{aligned}
pF_i(t) &= \int_{\mathcal{S}} \delta_i\left[t \mid \mathbf{f}(\mathbf{r})\right] \prod_{k=1}^{2nL} d_k\left(r_k\right) \prod_{k=1}^{2nL} \mathrm{d}r_k \\
&\cong \sum_{l=1}^{nR} \delta_i\left[t \mid \mathbf{f}\left(\mathbf{r}_l\right)\right] \Big/ nR \\
&= \sum_{l=1}^{nR} \delta_i\left\{t \mid \left[tWL_{1l}, tWL_{2l}, \ldots, tWL_{nWL,l}, tSL_{1l}, tSL_{2l}, \ldots, tSL_{nSL,l}\right]\right\} \Big/ nR,
\end{aligned}
\qquad (C.38)
$$

where (i) $d_k(r_k)$ is the density function for r_k defined on the set $\mathcal{S}_k$ of possible values for r_k (i.e., for $\alpha_{WL,j}$, $\beta_{WL,j}$, $\alpha_{SL,j}$ or $\beta_{SL,j}$ as appropriate; see Eq. (C.32)), (ii) $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots \times \mathcal{S}_{2nL}$, and (iii) $\mathbf{r}_l$, l = 1, 2, …, *nR*, is a random sample of size *nR* from $\mathcal{S}$ generated in consistency with the distributions defined by the density functions $d_k(r_k)$.

Unlike MC1, CPLOAS_2 implements MC2 with a single sampling step. Specifically, *nFTC* vectors of the form indicated in Eq. (C.32) are randomly generated and then used in the indicated sequence of calculations that lead to the approximation $\widehat{pF}_i(t)$ for $pF_i(t)$ in Eq. (C.38). As for MC1, confidence intervals for the approximation $\widehat{pF}_i(t)$ for $pF_i(t)$ obtained for MC2 are calculated as indicated in Eqs. (C.21)-(C.24).

C.5 Importance Sampling Procedure 1 (IMP1)

Importance sampling procedure 1 (IMP1) involves the use of importance sampling in the evaluation of the integral in Eq. (C.19) for the problem formulation described for MC1 (i.e., with sampling from the link failure time CDFs). Because the failure of SLs is less likely than the failure of WLs, the importance sampling procedure implemented in CPLOAS_2 for the evaluation of the integral in Eq. (C.19) uses right triangular importance sampling distributions for WLs and left triangular importance sampling distributions for SLs, which results in an overemphasis for large WL failure times and small SL failure times. Further, the importance sampling is performed on the cumulative probabilities associated with the link CDFs rather than directly on the link failure times. This choice was made as the calculation of the link CDFs in

CPLOAS_2 made these probabilities available. In contrast, importance sampling directly on the link failure times would have required the use of density functions for link time, which were not calculated.

Because the cumulative probabilities associated link failure probabilities are being sampled, the indicated right and left triangular importance sampling distributions $d_{IR}(r)$ and $d_{IL}(r)$ are defined on the interval [0, 1] by

$$d_{IR}(r) = 2r \text{ and } d_{IL}(r) = 2 - 2r \text{ for } 0 \le r \le 1. \tag{C.39}$$

Introduction of the importance sampling distributions $d_{IR}(r)$ and $d_{IL}(r)$ into Eq. (C.19) results in the following approximation for $pF_i(t)$:

$$\begin{aligned} pF_i(t) &= \int_{I^{nL}} \left\{ \delta_i \left[t \mid \mathbf{f}(\mathbf{r}) \right] \Big/ \prod_{k=1}^{nL} d_{I,k}(r_k) \right\} \prod_{k=1}^{nL} d_{I,k}(r_k) \prod_{k=1}^{nL} \mathrm{d}r_k \\ &\cong \sum_{l=1}^{nR} \left\{ \delta_i \left[t \mid \mathbf{f}(\mathbf{r}_l) \right] \Big/ \prod_{k=1}^{nL} d_{I,k}(r_{kl}) \right\} / \, nR \\ &= \sum_{l=1}^{nR} \frac{\delta_i \left\{ t \mid \left[tWL_{1l}, tWL_{2l}, \ldots, tWL_{nWL,l}, tSL_{1l}, tSL_{2l}, \ldots, tSL_{nSL,l} \right] \right\}}{nR \prod_{k=1}^{nL} d_{I,k}(r_{kl})}, \end{aligned} \tag{C.40}$$

where (i) the first equality derives from the introduction of the importance sampling distributions defined by the right and left density functions $d_{I,k}(r_k)$, k = 1, 2, …, nL, for WLs and SLs into the representation for $pF_i(t)$ in Eq. (C.19), (ii) the following approximation involves a random sample $\mathbf{r}_l$, l = 1, 2, …, nR, from I^{nL} generated in consistency with the distributions defined by the density functions $d_{I,k}(r_k)$, and (iii) the final equality is a restatement of $\mathbf{f}(\mathbf{r})$ in terms of link failure times.

Rather than use the result in Eq. (C.40) as the final approximation for $pF_i(t)$, IMP1 is implemented with a two-step sampling procedure that is analogous to the two-step sampling procedure used for MC1. The importance sampling that generates the link failure times in Eq. (C.40) corresponds to the first step in the two-step sampling procedure for IMP1. The second step in the two-step sampling procedure for IMP1 involves a sampling of the link failure times generated in the first step. For consistency in terminology with MC1, the first and second samplings of link failure time for IMP1 are referred to as being of size *nFT* and *nFTC*, respectively. As for MC1, this approach to the determination of IMP1 is only effective if *nFTC* is much larger than *nFTC*. If *nFTC* is not much larger than *nFTC*, then use of the initial approximation to $pF_i(t)$ in Eq. (C.40) will most likely be better than the result of approximation procedure that is now described.

The sampling procedure that generates the approximation in Eq. (C.40) corresponds to the first step in IMP1and produces the following set of results for $nR = nFT$:

$$\mathcal{L}_k = \left\{\left[tL_{kl}, wL_{kl}\right], l = 1, 2, ..., nFT\right\}, k = 1, 2, ..., nL, \tag{C.41}$$

where (i)

$$\mathbf{r}_l = \left[r_{1l}, r_{2l}, ..., r_{nL,l}\right], l = 1, 2, ..., nFT, \tag{C.42}$$

is a sample from I^{nL} generated in consistency with the importance sampling distributions defined by $d_{Ik}(r)$ for the individual links (i.e., r_{kl} is a random sample from [0, 1] generated in consistency with the importance sampling distribution defined by $d_{Ik}(r)$ for link k; see Eq. (C.39)), (ii)

$$tL_{kl} = \begin{cases} tWL_{kl} & \text{for } k = 1, 2, ..., nWL \\ tSL_{k-nWL,l} & \text{for } k = nWL+1, nWL+2, ..., nL \end{cases} \tag{C.43}$$

is the failure time for link k obtained with element r_{kl} of $\mathbf{r}_l$, and (iii)

$$wL_{kl} = 1/\left[nFT\ d_{Ik}\left(r_{kl}\right)\right] \tag{C.44}$$

is the importance sampling weight associated with the sampled link failure time tL_{kl}.

In concept, $pF_i(t)$ can be approximated by consideration of all possible combinations of the link failure times associated with the sets $\mathcal{L}_k$, k = 1, 2,…, nL, in Eq. (C.41) as indicated in the following summation:

$$pF_i(t) \cong \sum_{\mathbf{s}\in\mathcal{S}} \delta_i\left[t \mid \mathbf{t}(\mathbf{s})\right] w\left[\mathbf{t}(\mathbf{s})\right] \tag{C.45}$$

where

$$\mathcal{S} = \left\{\mathbf{s} : \mathbf{s} = \left[s(1), s(2), ..., s(nL)\right] \in \prod\nolimits_{i=1}^{nL}\{1, 2, ..., nL\}_i\right\}, \tag{C.46}$$

$$\mathbf{t}(\mathbf{s}) = \left[tL_{1,s(1)}, tL_{2,s(2)}, ..., tL_{nL,s(nL)}\right], \tag{C.47}$$

$$
\begin{aligned}
w[\mathbf{t}(\mathbf{s})] &= \prod_{k=1}^{nL} wL_{k,s(k)} \\
&= \prod_{k=1}^{nL} 1\Big/\left[nFT\ d_{Ik}\left(r_{k,s(k)}\right)\right] \qquad \text{(C.48)} \\
&= 1\Big/\left[nFT^{nL}\prod_{k=1}^{nL} d_{Ik}\left(r_{k,s(k)}\right)\right],
\end{aligned}
$$

and the summation in Eq. (C.45) involves nFT^{nL} terms (i.e., the number of elements in the set $\mathcal{S}$).

For IMP1, the summation in Eq. (C.45) is approximated by sampling from $\mathcal{S}$ with each element $\mathbf{s}$ of $\mathcal{S}$ assigned a probability of $1/nFT^{nL} = nFT^{-nL}$ .This produces the following approximation to $pF_i(t)$:

$$
\begin{aligned}
pF_i(t) &\cong \sum_{\mathbf{s}\in\mathcal{S}} \left\{\frac{\delta_i[t\,|\,\mathbf{t}(\mathbf{s})]\,w[\mathbf{t}(\mathbf{s})]}{1\big/nFT^{nL}}\right\}\left\{1\big/nFT^{nL}\right\} \\
&= \sum_{\mathbf{s}\in\mathcal{S}} \left[\frac{\delta_i[t\,|\,\mathbf{t}(\mathbf{s})]}{nFT^{nL}\prod_{k=1}^{nL} d_{Ik}\left(r_{k,s(k)}\right)\big/nFT^{nL}}\right]\left[\frac{1}{nFT^{nL}}\right] \\
&= \sum_{\mathbf{s}\in\mathcal{S}} \left[\frac{\delta_i[t\,|\,\mathbf{t}(\mathbf{s})]}{\prod_{k=1}^{nL} d_{Ik}\left(r_{k,s(k)}\right)}\right]\left[\frac{1}{nFT^{nL}}\right] \qquad \text{(C.49)} \\
&\cong \sum_{l=1}^{nFTC} \left[\frac{\delta_i[t\,|\,\mathbf{t}(\mathbf{s}_l)]}{\prod_{k=1}^{nL} d_{Ik}\left(r_{k,s(k,l)}\right)}\right]\left[\frac{1}{nFTC}\right],
\end{aligned}
$$

where

$$
\mathbf{s}_l = [s(1,l), s(2,l), \ldots, s(nL,l)],\ l = 1,2,\ldots,nFTC, \qquad \text{(C.50)}
$$

is a uniform random sample of size $nFTC$ from $\mathcal{S}$.

The final approximation $\widehat{pF}_i(t)$ for $pF_i(t)$ in Eq. (C.49) completes the second step of the two-step importance sampling procedure used for IMP1 in CPLOAS_2.

In addition, the standard deviation for $\widehat{pF}_i(t)$ is given by

$$
\begin{aligned}
s_i &= \left\{ \sum_{l=1}^{nFTC} \left[\frac{\delta_i\left[t \mid \mathbf{t}(\mathbf{s}_l)\right]}{\prod_{k=1}^{nL} d_{Ik}\left(r_{k,s(k,l)}\right)} - \widehat{pF}_i(t) \right]^2 \left[\frac{1}{nFTC}\right] \right\}^{1/2} \Bigg/ \sqrt{nFTC} \\
&= \left\{ \sum_{l=1}^{nFTC} \left[\frac{\delta_i\left[t \mid \mathbf{t}(\mathbf{s}_l)\right]}{\prod_{k=1}^{nL} d_{Ik}\left(r_{k,s(k,l)}\right)} \right]^2 \left[\frac{1}{nFTC}\right] - \widehat{pF}_i^{\,2}(t) \right\}^{1/2} \Bigg/ \sqrt{nFTC}.
\end{aligned}
\tag{C.51}
$$

In turn, s_i is used in the determination of confidence intervals for the approximation $\widehat{pF}_i(t)$ for $pF_i(t)$ obtained with IMP1 as indicated in Eqs. (C.21)-(C.24). Similarly to the procedure described in conjunction with Eqs. (C.28)-(C.30) for MC1, running sums are used in CPLOAS_2 in the calculation of $\widehat{pF}_i(t)$ and s_i.

C.6 Importance Sampling Procedure 2 (IMP2)

Importance sampling procedure 2 (IMP2) involves the use of importance sampling in the evaluation of the integral in Eq. (C.19) for the problem formulation described for MC2 (i.e., with sampling from the α’s and β’s for the individual links). Because the failure of SLs is less likely than the failure of WLs, the importance sampling procedure implemented in CPLOAS_2 for the evaluation of the integral in Eq. (C.19) uses right triangular importance sampling distributions for WL β’s and SL α’s and left triangular importance sampling distributions for WL α’s and SL β’s, which results in an overemphasis for large WL failure times and small SL failure times. Further, the importance sampling is performed on the cumulative probabilities associated with the α’s and β’s rather than directly on the α’s and β’s.

Introduction of the importance sampling distributions $d_{IR}(r)$ and $d_{IL}(r)$ defined in Eq. (C.39) into Eq. (C.19) results in the following approximations for $pF_i(t)$:

$$
\begin{aligned}
pF_i(t) &= \int_{I^{2nL}} \left\{ \delta_i\left[t \mid \mathbf{f}(\mathbf{r})\right] \Big/ \prod_{k=1}^{2nL} d_{I,k}(r_k) \right\} \prod_{k=1}^{2nL} d_{I,k}(r_k) \prod_{k=1}^{2nL} \mathrm{d}r_k \\
&\cong_1 \sum_{l=1}^{nR} \left\{ \delta_i\left[t \mid \mathbf{f}(\mathbf{r}_l)\right] \Big/ \prod_{k=1}^{2nL} d_{I,k}(r_{kl}) \right\} / \, nR \\
&= \sum_{l=1}^{nR} \left[\frac{\delta_i\left\{t \mid \left[tWL_{1l}, tWL_{2l}, ..., tWL_{nWL,l}, tSL_{1l}, tSL_{2l}, ..., tSL_{nSL,l}\right]\right\}}{\prod_{k=1}^{2nL} d_{I,k}(r_{kl})} \right] / \, nR \\
&\cong_2 \sum_{l=1}^{nR} \left[\frac{\delta_i\left\{t \mid \left[tWL_{1l}, tWL_{2l}, ..., tWL_{nWL,l}, tSL_{1l}, tSL_{2l}, ..., tSL_{nSL,l}\right]\right\}}{\prod_{k=1}^{2nL} d_{I,k}(r_{kl})} \right] \Bigg/ \sum_{l=1}^{nR} \prod_{k=1}^{2nL} d_{I,k}(r_{kl}) \\
&= \sum_{l=1}^{nR} \left\{ \delta_i\left[t \mid \mathbf{f}(\mathbf{r}_l)\right] \Big/ \prod_{k=1}^{2nL} d_{I,k}(r_{kl}) \right\} \Big/ \sum_{l=1}^{nR} \prod_{k=1}^{2nL} d_{I,k}(r_{kl}),
\end{aligned}
\tag{C.52}
$$

where (i) the first equality derives from the introduction of the importance sampling distributions defined by the right and left density functions $d_{I,k}(r_k)$, k = 1, 2, …, nL, for the α's and β's for the individual links into the representation for $pF_i(t)$ in Eq. (C.19), (ii) the following approximation (i.e., $\cong_1$) involves a random sample $\mathbf{r}_l$, l = 1, 2, …, nR, from I^{2nL} generated in consistency with the distributions defined by the density functions $d_{I,k}(r_k)$ with a sampling weight for each observation equal to the reciprocal of the sample size (i.e., 1/nR) , (iii) the next equality is a restatement of $\mathbf{f}(\mathbf{r})$ in terms of link failure times as indicated in Eqs. (C.32)-(C.37), (iv) the second approximation (i.e., $\cong_2$) results from replacing the reciprocal of the sample size (i.e., 1/nR) with the reciprocal of the sum weights from the importance sampling, and (v) the final equality results from a return to the previously used more compact representation for the indicator function for LOAs (i.e., $\delta_i[t \mid \mathbf{f}(\mathbf{r}_l)]$). For convenience, the first and second approximation procedures will be referred to as $\mathrm{IMP2}_1$ and $\mathrm{IMP2}_2$, respectively. The replacement of 1/nR in the definition of $\mathrm{IMP2}_1$ by

$$
1 \Big/ \sum_{l=1}^{nR} \prod_{k=1}^{2nL} d_{I,k}(r_{kl}) \tag{C.53}
$$

in the definition of $\mathrm{IMP2}_2$ is suggested by some authors (e.g., Refs. [6; 7]).

The use of $\mathrm{IMP2}_1$ or $\mathrm{IMP2}_2$ is possible in CPLOAS_2 as a user-specified option. At present, $\mathrm{IMP2}_1$ is the recommended option for use in the implementation of IMP2 as there is currently limited experience with the use of $\mathrm{IMP2}_2$ as an option for the implementation of IMP2. For consistency, CPLOAS_2 uses the same sample size *nFTC* for $\mathrm{IMP2}_1$ and $\mathrm{IMP2}_2$ as used for MC1, MC2 and IMP1; thus, the indicated sample size *nR* in Eq. (C.52) corresponds to the sample size *nFTC* in the notation used in CPLOAS_2. As for MC2, a single sample of size *nFTC* is used in CPLOAS_2 for $\mathrm{IMP2}_1$ and $\mathrm{IMP2}_2$.

For $\mathrm{IMP2}_1$, the standard deviation for $\widehat{pF}_i(t)$ is given by

$$\begin{aligned} s_i &= \left\{ \sum_{l=1}^{nFTC} \left\{ \delta_i\left[t \mid \mathbf{f}(\mathbf{r}_l)\right] \Big/ \prod_{k=1}^{2nL} d_{I,k}\left(r_{kl}\right) - \widehat{pF}_i(t) \right\}^2 / \, nFTC \right\}^{1/2} \Bigg/ \sqrt{nFTC} \\ &= \left\{ \sum_{l=1}^{nFTC} \left\{ \delta_i\left[t \mid \mathbf{f}(\mathbf{r}_l)\right] \Big/ \prod_{k=1}^{2nL} d_{I,k}\left(r_{kl}\right) \right\}^2 \Big/ nFTC - \widehat{pF}_i^{\,2}(t) \right\}^{1/2} \Bigg/ \sqrt{nFTC}. \end{aligned} \tag{C.54}$$

As suggested in Refs. [6; 7], the standard deviation for $\widehat{pF}_i(t)$ obtained for $\mathrm{IMP2}_2$ is defined in CPLOAS_2 by

$$\begin{aligned} s_i &= \left\{ \sum_{l=1}^{nFTC} w^2\left(\mathbf{r}_l\right) \left\{ \delta_i\left[t \mid \mathbf{f}(\mathbf{r}_l)\right] - \widehat{pF}_i(t) \right\}^2 \Big/ \left[\sum_{l=1}^{nFTC} w\left(\mathbf{r}_l\right) \right]^2 \right\}^{1/2} \\ &= \frac{\left(\sum_{l=1}^{nFTC} w^2\left(\mathbf{r}_l\right) \left\{ \delta_i\left[t \mid \mathbf{f}(\mathbf{r}_l)\right] - \widehat{pF}_i(t) \right\}^2 \right)^{1/2}}{\sum_{l=1}^{nFTC} w\left(\mathbf{r}_l\right)} \end{aligned} \tag{C.55}$$

with

$$w\left(\mathbf{r}_l\right) = 1 \Big/ \prod_{k=1}^{2nL} d_{I,k}\left(r_{kl}\right). \tag{C.56}$$

For both $\mathrm{IMP2}_1$ and $\mathrm{IMP2}_2$, CPLOAS_2 uses the indicated values for s_i to determine confidence intervals for $\widehat{pF}_i(t)$ as described in conjunction with Eqs. (C.21)-(C.24).

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