DYSON-SCHWINGER EQUATIONS AND THE QUARK-GLUON PLASMA

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We review applications of Dyson-Schwinger equations at nonzero temperature, T, and chemical potential, μ, touching topics such as: deconfinement and chiral symmetry restoration; the behaviour of bulk thermodynamic quantities; the (T,μ)-dependence of hadron properties; and the possibility of diquark condensation.

1 Introduction

Confinement and dynamical chiral symmetry breaking (DCSB) are consequences of the little-understood long-range behaviour of the QCD interaction, and developing a better understanding of that behaviour is a primary goal of contemporary nuclear physics. It is a prodigious problem whose solution admits many complementary strategies. Our approach is to apply a single phenomenological framework to many observables, thereby identifying the unifying qualitative features. Non-hadronic electroweak interactions are the best observables to study because the probes, the photon and W, Z bosons, are very well understood. Following such applications[1] we can infer consequences for QCD at extremes of temperature and chemical potential.

Our tools of choice are the Dyson-Schwinger equations[2] (DSEs), which at the simplest level provide a means of generating perturbation theory and are an invaluable aid in proving renormalisability. However, our interest stems from their essentially nonperturbative character. For example, the DSE for the quark propagator is the QCD *gap equation*. Its complete solution contains all that is necessary to describe DCSB and yields insights into confinement, both of which are absent at any finite order in perturbation theory. Further, the Bethe-Salpeter equations (BSEs) are just another form of DSE and these equations completely describe meson structure.

The formulation of the DSEs is straightforward but their solution is not. The equation for a particular propagator or vertex (n-point) function involves at least one $m > n$-point function; e.g., the gap equation whose solution is the dressed-quark propagator (2-point function) involves the dressed-gluon propagator, a 2-point function, and the dressed-quark-gluon vertex, a 3-point function. Thus in the DSEs we have a countable infinity of coupled equations and a tractable problem is only obtained if we truncate the system. This has been

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an impediment to their application: *a priori* it can be difficult to judge the fidelity of a particular truncation scheme. However, with expanding community involvement this barrier is being overcome as truncation schemes are explored and efficacious ones developed.

2 Gap equation

The gap equation in QCD is the DSE for the quark propagator:

$$S_f(p)^{-1} := i\gamma\cdot p\, A_f(p^2) + B_f(p^2) = A_f(p^2)\left(i\gamma\cdot p + M_f(p^2)\right) \quad (1)$$

$$= Z_2(i\gamma\cdot p + m_f^{\rm bm}) + Z_1\int_q^\Lambda g^2 D_{\mu\nu}(p-q)\frac{\lambda^a}{2}\gamma_\mu S_f(q)\Gamma_\nu^{fa}(q,p), \quad (2)$$

$$D_{\mu\nu}(k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2}\,\mathcal{P}(k^2)\,, \quad (3)$$

is the dressed-gluon propagator (in Landau gauge, just to be concrete), $\Gamma_\nu^{fa}(q,p)$ is the dressed-quark-gluon vertex, $m_f^{\rm bm}$ is the Λ-dependent bare f-quark current-mass and $\int_q^\Lambda := \int^\Lambda d^4q/(2\pi)^4$ represents mnemonically a translationally- invariant regularisation of the integral, with Λ the regularisation mass-scale. The renormalisation constants for the quark-gluon-vertex, quark wave function and mass: $Z_1(\zeta^2,\Lambda^2)$, $Z_2(\zeta^2,\Lambda^2)$ and $Z_m(\zeta^2,\Lambda^2) := Z_2(\zeta^2,\Lambda^2)^{-1}Z_4(\zeta^2,\Lambda^2)$, depend on the renormalisation point, ζ, and the regularisation mass-scale. (The renormalised current-quark mass is $m_f(\zeta) := Z_m^{-1}m_f^{\rm bm}$.)

The qualitative features of the QCD solution of Eq. (2) are known. The chiral limit is defined by $\hat{m}=0$, where $\hat{m}$ is the renormalisation-point-independent current-quark mass, and for $p^2 > 20\,{\rm GeV}^2$ the solution of Eq. (2) is[3]

$$M_0(p^2) \stackrel{\rm large-}{=}{}^{p^2} \frac{2\pi^2\gamma_m}{3}\,\frac{\left(-\langle\bar{q}q\rangle^0\right)}{p^2\left(\frac{1}{2}\ln\left[p^2/\Lambda^2_{\rm QCD}\right]\right)^{1-\gamma_m}}\,, \quad (4)$$

where $\gamma_m = 12/(33-2N_f)$ is the gauge-independent mass anomalous dimension and $\langle\bar{q}q\rangle^0$ is the renormalisation-point-independent vacuum quark condensate. The existence of DCSB means that $\langle\bar{q}q\rangle^0 \neq 0$, however, its actual value depends on the long-range behaviour of $D_{\mu\nu}(k)$ and $\Gamma_\nu^{0a}(q,p)$, which is modelled in contemporary DSE studies. Requiring a good description of light-meson observables necessitates $\langle\bar{q}q\rangle^0 \approx -(0.24\,{\rm GeV})^3$.

The momentum-dependence in Eq. (4) is a crucial, model-independent result because it is the *only* behaviour consistent with the definition of the

vacuum quark condensate as the trace of the chiral-limit quark propagator:[3]

$$-\langle\bar{q}q\rangle^0_\zeta = N_c \lim_{\Lambda\to\infty} Z_4(\zeta^2,\Lambda^2)\,\mathrm{tr}_D \int_k^\Lambda S_0(k). \tag{5}$$

Any model that generates

$$M_0(p^2) \sim p^{-2n},\ n>1 \tag{6}$$

will yield $\langle\bar{q}q\rangle^0_\zeta \equiv 0$ from the definition of the quark condensate.

Confinement is the absence of quark and gluon production thresholds in colour-singlet-to-singlet $\mathcal{S}$-matrix amplitudes. The absence of a Lehmann representation for dressed-quark and -gluon propagators is sufficient to ensure that.[2] Therefore the solution of Eq. (2) can also yield information about confinement, as shown clearly[4] for QED_3.

Studies of Eq. (2) that employ a dressed-gluon propagator with a strong infrared enhancement:[5] $\mathcal{P}(k^2) \sim 1/k^2$, and hence without a Lehmann representation, and $\Gamma^{f\,a}_\nu(q,p)$ regular in the infrared,[a] yield $S(p)$ that also does not have a Lehmann representation. Fine-tuning is not necessary. Such models also easily account for DCSB,[3] with the correct value of $\langle\bar{q}q\rangle^0$.

Contemporary DSE[6] and lattice[7] studies have reopened the possibility that $\mathcal{P}(k^2) \sim (k^2)^p$, $p \lesssim 1$, for $k^2 \simeq 0$; i.e., of an infrared *suppression*. The phenomenological consequences of this have been re-explored:[8] when that $\mathcal{P}(k^2)$ obtained in contemporary lattice simulations is used in Eq. (2) with an infrared-regular dressed-quark-gluon vertex, DCSB does *not* occur and $S(p)$ has a Lehmann representation; i.e, there is no signal of confinement. The $\mathcal{P}(k^2) \sim (k^2)^p$-form obtained in DSE studies can be made to support a nonzero condensate via Eq. (2), however, its value is typically[6,10] only 7-30% of that required to explain observed phenomena, and again $S(p)$ does not exhibit signs of confinement.[8]

3 Exploring QCD at nonzero T and μ

The dressed-quark propagator at nonzero-(T,μ) has the general form

$$S(\tilde{p}_k) = \frac{1}{i\vec{\gamma}\cdot\vec{p}\,A(\tilde{p}_k) + i\gamma_4\omega_{k+}\,C(\tilde{p}_k) + B(\tilde{p}_k)}, \tag{7}$$

$$= -i\vec{\gamma}\cdot\vec{p}\,\sigma_A(\tilde{p}_k) - i\gamma_4\omega_{k+}\,\sigma_C(\tilde{p}_k) + \sigma_B(\tilde{p}_k), \tag{8}$$

[a] It is difficult to interpret particle-like singularities in coloured Schwinger functions in a manner consistent with confinement.[8]

where we have omitted the flavour label, $\tilde{p}_k = (\vec{p}, \omega_{k+})$, $\omega_{k+} = \omega_k + i\mu$ and $\omega_k = (2k+1)\pi T$, with $k \in \mathbf{Z}$, is the quark's Matsubara frequency. The complex scalar functions: $A(\vec{p}, \tilde{\omega}_k)$, $B(\vec{p}, \tilde{\omega}_k)$ and $C(\vec{p}, \tilde{\omega}_k)$ satisfy: $\mathcal{F}(\vec{p}, \tilde{\omega}_k)^* = \mathcal{F}(\vec{p}, \tilde{\omega}_{-k-1})$, $\mathcal{F} = A, B, C$, and although not explicitly indicated they are functions only of $|\vec{p}|^2$ and $\tilde{\omega}_k^2$. The dependence of these functions on their arguments has important consequences in QCD: it can provide an understanding of quark confinement and is the reason why bulk thermodynamic quantities approach their ultrarelativistic limits slowly. The nonzero-(T, μ) gap equation is a straightforward generalisation[9] of Eq. (2) and the Landau gauge dressed-gluon propagator has the general form

$$g^2 D_{\mu\nu}(\vec{p}, \Omega) \;=\; P^L_{\mu\nu}(\vec{p}, \Omega)\, \Delta_F(\vec{p}, \Omega) + P^T_{\mu\nu}(\vec{p})\, \Delta_G(p, \Omega)\,, \tag{9}$$

$$P^T_{\mu\nu}(\vec{p}) := \begin{cases} 0; & \mu \text{ and/or } \nu = 4, \\ \delta_{ij} - \dfrac{p_i p_j}{|\vec{p}|^2}; & \mu, \nu = i, j = 1, 2, 3 \end{cases}, \tag{10}$$

with $P^T_{\mu\nu}(p) + P^L_{\mu\nu}(p, p_4) = \delta_{\mu\nu} - p_\mu p_\nu / (\sum_{\alpha=1}^4 p_\alpha p_\alpha)$; $\mu, \nu = 1, \ldots, 4$.

In studying the formation of a quark-gluon plasma, two transitions are important: deconfinement and chiral symmetry restoration. The simplest order parameter for the chiral transition is

$$\mathcal{X}(t, h) := \mathsf{Re}\, B_0(\vec{p} = 0, \tilde{\omega}_0)\,;\; t := \frac{T}{T_c} - 1\,,\; h := \frac{m^\zeta}{T}. \tag{11}$$

It is a general result that the zeroth Matsubara mode determines the character of the chiral phase transition.

An order parameter for the deconfinement transition is realised[9] via the Schwinger function:

$$\Delta_{B_0}(x, \tau = 0) := T \sum_{n=-\infty}^{\infty} \frac{1}{2\pi^2 x} \int_0^\infty dp\, p\, \sin(px)\, \sigma_{B_0}(p, \omega_n), \tag{12}$$

where we have set $\mu = 0$ for illustrative simplicity. If $\sigma_{B_0}(p, \omega_n)$ has complex conjugate poles, y_p, then: 1) it doesn't have a Lehmann representation; and 2) $\Delta_{B_0}(x, \tau = 0)$ has zeros. The position of the first zero, $r_0^{z1}(t)$, is inversely proportional to $\mathrm{Im}(y_p)$. Thus

$$\kappa_0(t) := 1/r_0^{z1}(t), \tag{13}$$

is a confinement order parameter because $\kappa_0(t) \to 0$ as $t \to 0^-$ indicates that a temperature has been reached at which the poles have migrated to the real-p^2 axis and the propagator has acquired a Lehmann representation. (This order parameter can be generalised to qualitatively different functional realisations of the absence of a Lehmann representation.)

4 Locating the phase boundary in the (T, μ)-plane

DSE models constrained at $T = 0 = \mu$ can be used to estimate the location of the phase boundary. The studies we review all use rainbow truncation: $\Gamma_\nu(q_{\omega_l}; p_{\omega_k}) = \gamma_\nu$, which is the leading term in a $1/N_c$-expansion of the vertex; and Landau gauge, with a dressed-gluon propagator characterised by

$$\Delta_F(p_{\Omega_k}) = \mathcal{D}(p_{\Omega_k}; m_g)\,, \; \Delta_G(p_{\Omega_k}) = \mathcal{D}(p_{\Omega_k}; 0)\,, \tag{14}$$

$$\mathcal{D}(p_{\Omega_k}; m_g) := 2\pi^2 D\, \frac{2\pi}{T} \delta_{0\,k}\, \delta^3(\vec{p}) + \mathcal{D}_{\rm M}(p_{\Omega_k}; m_g)\,, \tag{15}$$

$p_{\Omega_k} = (p, \Omega_k)$, where D is a mass-scale parameter and $\mathcal{D}_{\rm M}(p_{\Omega_k}; m_g)$ may be large in the vicinity of $p^2_{\Omega_k} = 0$ but must be finite.

4.1 $\mu = 0$, $T_c = ?$

The model obtained with $D = (8/9) m_t^2$ and

$$\mathcal{D}_{\rm M}(p_{\Omega_k}; m_g) = \frac{16}{9}\pi^2\, \frac{1 - {\rm e}^{-s_{\Omega_k}/(4m_t^2)}}{s_{\Omega_k}}\,, \tag{16}$$

where $s_{\Omega_k} := p^2_{\Omega_k} + m_g^2$ [$m_g^2 = 8\,\pi^2 T^2$ is a gauge boson Debye mass], yields a finite-T extension of a phenomenologically efficacious one-parameter model dressed-gluon propagator.[9] The mass-scale $m_t = 0.69\,{\rm GeV} = 1/0.29\,{\rm fm}$ was fixed by requiring a good description of π- and ρ-meson properties at $T = 0$. At a renormalisation point of $\zeta = 9.47\,{\rm GeV}$, $m_R = 1.1\,{\rm MeV}$ yields $m_\pi = 140\,{\rm MeV}$.

This model has *coincident* chiral symmetry restoration and deconfinement transitions at

$$T_c^\chi = 0.15\,{\rm GeV} = T_c^{\kappa 0} \tag{17}$$

with mean field critical exponents. Studies that employ the rainbow truncation *must* give mean field critical exponents[11] because contributions to the gap equation that describe the effects of mesonic correlations, which are expected to dominate near the transition temperature, can only arise as corrections to the vertex. The behaviour of m_π and f_π is depicted in Fig. 1.

As a *bona fide* order parameter $f_\pi \propto (-t)^{1/2}$, which is illustrated by the curve in Fig. 1. Hence, it follows from the pseudoscalar mass formula:[3] $f_\pi^2 m_\pi^2 = 2\, m_R(\zeta) \langle \bar{q} q \rangle^0_\zeta$, that m_π diverges at the critical temperature; i.e., $m_\pi \propto (-t)^{-1/4}$, as illustrated. Qualitatively, these two observations indicate that at $T = T_c^\chi$ there is insufficient attraction in the pseudoscalar channel for a bound state to form[9] and while correlations may persist above T_c^χ these are properly identified as a continuum contribution to the pseudoscalar vertex.

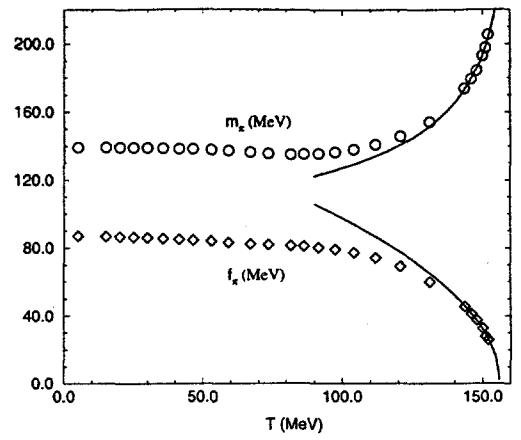

Figure 1: The pion mass and decay constant are independent of temperature for $T \lesssim 0.7 T_c^{\chi}$.

4.2 $T = 0$, $\mu_c = ?$

The difficulties encountered in numerical simulations of lattice-QCD at $\mu \neq 0$ are described in many contributions to this volume. In studies of the gap equation it only means that the self energies are complex-valued functions. The $T = 0$ version of the model in the previous section is obtained with

$$\frac{1}{k^2}\mathcal{P}(k^2) := \frac{16}{9}\pi^2 \left[4\pi^2 m_t^2 \delta^4(k) + \frac{1 - \mathrm{e}^{-[k^2/(4m_t^2)]}}{k^2} \right] \tag{18}$$

in Eq. (3). This model has[12] coincident, first order deconfinement and chiral symmetry restoring transitions at $\mu_c = 0.375\,\mathrm{GeV}$, as measured by the location of the zero in the μ-dependent "bag constant":[13] $\mathcal{B}(\mu)$. It is positive[b] when the Nambu-Goldstone phase is dynamically favoured; i.e., has the highest pressure, and becomes negative when the Wigner pressure becomes larger, which is why μ_c is the zero of $\mathcal{B}(\mu)$. To gauge the magnitude of μ_c we note that in a two-flavour free-quark gas the baryon number density $\rho_B = 2\mu^3/(3\pi^2)$ so

$$\mu_c = 0.375\,\mathrm{GeV} \;\Rightarrow\; \rho_B^{u_F + d_F} = 2.9\,\rho_0, \tag{19}$$

where $\rho_0 = 0.16\,\mathrm{fm}^{-3}$. This may be compared with the central core density of a $1.4 M_\odot$ neutron star: 3.6-4.1ρ_0, while $0.7\mu_c$ corresponds to ρ_0.

[b]The calculated value of $\mathcal{B}(0) = (0.104\,\mathrm{GeV})^4 = 15\,\mathrm{MeV/fm}^3$ is similar to that employed in bag-like models.

In this model m_π *decreases* slowly as μ increases, with $m_\pi(0.7\,\mu_c)/m_\pi(0) \approx 0.94$. At this point m_π begins to increase although, for $\mu < \mu_c$, $m_\pi(\mu)$ does not exceed $m_\pi(0)$, which precludes pion condensation. The behaviour of m_π results from mutually compensating increases in f_π^2 and $m(\zeta)\langle\bar{q}q\rangle_\zeta^\pi$. f_π is insensitive to μ until $\mu \approx 0.7\,\mu_c$, when it increases sharply so that $f_\pi(\mu_c^-)/f_\pi(\mu=0) \approx 1.25$. The relative insensitivity of m_π and f_π to changes in μ, until very near μ_c, mirrors the behaviour of these observables at finite-T.[9] This study reveals an anticorrelation between the μ-dependence of f_π and that of m_π.

4.3 $T \neq 0$, $\mu \neq 0$

This is a difficult problem and the most complete studies to date[14,15] employ the simple *Ansatz* for the dressed-gluon propagator obtained with $D = \eta^2/2$ and $\mathcal{D}(p_{\Omega_k}; m_g) \equiv 0$ in Eq. (15), and the mass-scale $\eta = 1.06\,\mathrm{GeV}$ fixed[16] by fitting π- and ρ-meson masses at $T = 0$. With this *Ansatz* the gap equation is

$$S^{-1}(\vec{p},\omega_k) = S_0^{-1}(\vec{p},\tilde{\omega}_k) + \tfrac{1}{4}\eta^2\gamma_\nu S(\vec{p},\tilde{\omega}_k)\gamma_\nu\,; \tag{20}$$

and an integral equation is reduced to an algebraic equation whose solution exhibits many of the qualitative features of more sophisticated models.

In the chiral limit Eq. (20) reduces to a quadratic equation for $B(\tilde{p}_k)$, which has two qualitatively distinct solutions. The Nambu-Goldstone solution, with

$$B(\tilde{p}_k) = \begin{cases} \sqrt{\eta^2 - 4\tilde{p}_k^2}\,, & \mathsf{Re}(\tilde{p}_k^2) < \frac{\eta^2}{4} \\ 0\,, & \text{otherwise} \end{cases}, \tag{21}$$

$$C(\tilde{p}_k) = \begin{cases} 2\,, & \mathsf{Re}(\tilde{p}_k^2) < \frac{\eta^2}{4} \\ \frac{1}{2}\left(1 + \sqrt{1 + \frac{2\eta^2}{\tilde{p}_k^2}}\right), & \text{otherwise} \end{cases}, \tag{22}$$

describes a phase of this model in which: 1) chiral symmetry is dynamically broken, because one has a nonzero quark mass-function, $B(\tilde{p}_k)$, in the absence of a current-quark mass; and 2) the dressed-quarks are confined, because the propagator described by these functions does not have a Lehmann representation. The alternative Wigner solution, for which

$$\hat{B}(\tilde{p}_k) \equiv 0\,,\ \hat{C}(\tilde{p}_k) = \frac{1}{2}\left(1 + \sqrt{1 + \frac{2\eta^2}{\tilde{p}_k^2}}\right), \tag{23}$$

describes a phase of the model with neither DCSB nor confinement.

Here the relative stability of the competing phases is measured by a (T,μ)-dependent bag constant:[14] $\mathcal{B}(T,\mu)$. The line $\mathcal{B}(T,\mu) = 0$ defines the phase

boundary, and the deconfinement and chiral symmetry restoration transitions are coincident. For $\mu = 0$ the transition is second order and the critical temperature is $T_c^0 = 0.159\,\eta = 0.17\,\mathrm{GeV}$, just 12% larger than the value reported in Sec. 4.1. For any $\mu \neq 0$ the transition is first-order and the $T = 0$ critical chemical potential is $\mu_c^0 = 0.3\,\mathrm{GeV}$, $\approx 30\%$ smaller than the result in Sec. 4.2.

The quark pressure, P_q, is easily calculated and $P_q \equiv 0$ in the confined domain.[14] However, this does *not* mean that the vacuum is unaffected by changes in (T, μ). On the contrary; e.g., in the models described above, the condensate evolves with these changes, as it must because it is a dynamical quantity. At each (T, μ) the properties of the hadronic excitations are calculated in the *evolved* vacuum and the modification of the quark-constituents' propagation characteristics, which the condensate's modification represents, makes a significant contribution to the (T, μ)-dependence of those properties.

In the deconfined domain, P_q slowly approaches the ultrarelativistic, free particle limit, P_{UR}, at large values of (T, μ); e.g., at $T \sim 0.3\,\eta \sim 2T_c^0$, or $\mu \sim \eta \sim 3\mu_c^0$, $P_q \simeq 0.5\,P_{\mathrm{UR}}$. This behaviour results from the persistence of momentum dependent modifications of the quark propagator into the deconfined domain, as evidenced by $C \not\equiv 1$ in Eq. (23), which also entails a "mirroring" of finite-T behaviour in the μ-dependence of the bulk thermodynamic quantities.

The (T, μ)-dependence of vacuum and meson properties is easily calculated in this model; e.g., the vacuum quark condensate is

$$-\langle\bar{q}q\rangle = \eta^3\,\frac{8N_c}{\pi^2}\bar{T}\sum_{l=0}^{l_{\max}}\int_0^{\bar{\Lambda}_l} dy\,y^2\,\mathrm{Re}\left(\sqrt{\tfrac{1}{4} - y^2 - \tilde{\omega}_l^2}\right)\,, \tag{24}$$

$\bar{T} = T/\eta$, $\bar{\mu} = \mu/\eta$; l_{max} is the largest value of l for which $\bar{\omega}^2_{l_{\max}} \leq (1/4) + \bar{\mu}^2$ and this also specifies $\omega_{l_{max}}$, $\bar{\Lambda}^2 = \bar{\omega}^2_{l_{\max}} - \bar{\omega}_l^2$, $\bar{p}_l = (\vec{y}, \bar{\omega}_l + i\bar{\mu})$. At $T = 0 = \mu$, $(-\langle\bar{q}q\rangle) = \eta^3/(80\,\pi^2) = (0.11\,\eta)^3$. Obvious from Eq. (24) is that $(-\langle\bar{q}q\rangle)$ decreases continuously to zero with T but *increases* with μ, up to a critical value of $\mu_c(T)$ when it drops discontinuously to zero: just the behaviour reported elsewhere.[9,12] That behaviour is a necessary consequence of the momentum-dependence of the quark self energy, with the finite-(T, μ) behaviour of observables determined by

$$\mathrm{Re}(\omega^2_{[\mu]})^d \sim [\pi^2 T^2 - \mu^2]^d\,, \tag{25}$$

where d is the observable's mass-dimension. This is confirmed in the chiral limit expression

$$f_\pi^2 = \eta^2\,\frac{16N_c}{\pi^2}\bar{T}\sum_{l=0}^{l_{\max}}\frac{\bar{\Lambda}_l^3}{3}\left(1 + 4\,\bar{\mu}^2 - 4\,\bar{\omega}_l^2 - \frac{8}{5}\,\bar{\Lambda}_l^2\right)\,. \tag{26}$$

The anticipated combination $\mu^2 - \omega_l^2$ appears and even without calculation it is clear that f_π will *decrease* with T and *increase* with μ.

The (T,μ)-response of meson masses is determined by the ladder BSE

$$\Gamma_M(\tilde{p}_k;\check{P}_\ell) = -\frac{\eta^2}{4}\,\mathsf{Re}\left\{\gamma_\mu\, S(\tilde{p}_i + \tfrac{1}{2}\check{P}_\ell)\,\Gamma_M(\tilde{p}_i;\check{P}_\ell)\, S(\tilde{p}_i - \tfrac{1}{2}\check{P}_\ell)\,\gamma_\mu\right\}, \quad (27)$$

where $\check{P}_\ell := (\vec{P},\Omega_\ell)$, with the bound state mass obtained by considering $\check{P}_{\ell=0}$. In this truncation the ω- and ρ-mesons are degenerate.

The pion solution of this equation is $\Gamma_\pi(P_0) = \gamma_5(i\theta_1 + \vec{\gamma}\cdot\vec{P}\,\theta_2)$ and, consistent with what we saw above, the mass is (T,μ)-independent, until very near the transition boundary.[15] For the ρ-meson the solution has two components: one longitudinal, $\theta_{\rho+}$, and one transverse, $\theta_{\rho-}$, to $\vec{P}$. Equation (27) yields an eigenvalue equation for the bound state mass, $M_{\rho\pm}$, and using the chiral-limit solutions, Eq. (21), one finds immediately that

$$M_{\rho-}^2 = \eta^2/2, \ \textit{independent} \text{ of } T \text{ and } \mu. \quad (28)$$

Even for $m \neq 0$, $M_{\rho-}$ changes by $< 1\%$ as (T,μ) are increased from zero toward their critical values. This insensitivity is consistent with the absence of a constant mass-shift in the transverse polarisation tensor for a gauge-boson.

For the longitudinal component one obtains in the chiral limit:

$$M_{\rho+}^2 = \tfrac{1}{2}\eta^2 - 4(\mu^2 - \pi^2 T^2)\,. \quad (29)$$

The combination $\mu^2 - \pi^2T^2$ again indicates the anticorrelation between the response of $M_{\rho+}$ to T and its response to μ, and, like a gauge-boson Debye mass, that $M_{\rho+}^2$ rises linearly with T^2 for $\mu = 0$. The $m \neq 0$ solution for the longitudinal component is semiquantitatively the same.

The BSE yields qualitatively the same behaviour for the ϕ-meson. The transverse component is insensitive to T and μ, and the longitudinal mass, $M_{\phi+}$, increases with T and decreases with μ. Using $\eta = 1.06\,\mathrm{GeV}$, $M_{\phi\pm} = 1.02\,\mathrm{GeV}$ for $m_s = 180\,\mathrm{MeV}$ at $T = 0 = \mu$.

In a 2-flavour, free-quark gas at $T = 0$ nuclear matter density corresponds to $\mu = \mu_0 := 260\,\mathrm{MeV} = 0.245\,\eta$ and the algebraic model yields

$$M_{\rho+}(\mu_0) \approx 0.75 M_{\rho+}(\mu = 0)\,,\ M_{\phi+}(\mu_0) \approx 0.85 M_{\phi+}(\mu = 0)\,. \quad (30)$$

Section 4.2 indicates that a better representation of the ultraviolet behaviour of $D_{\mu\nu}(k)$ increases the critical chemical potential by 25%. This suggests that a more realistic estimate is obtained by evaluating the mass at $\mu_0' = 0.20\,\eta$, which yields

$$M_{\rho+}(\mu_0') \approx 0.85 M_{\rho+}(\mu = 0)\,,\ M_{\phi+}(\mu_0') \approx 0.90 M_{\phi+}(\mu = 0)\,; \quad (31)$$

a small, quantitative modification. The difference between Eqs. (30) and (31) is a measure of the theoretical uncertainty in the estimates in each case. Pursuing this suggestion further, $\mu = \sqrt[3]{2}\,\mu_0'$, corresponds to $2\rho_0$, at which point $M_{\omega+} = M_{\rho+} \approx 0.72\,M_{\rho+}(\mu=0)$ and $M_{\phi+} \approx 0.85\,M_{\phi+}(\mu=0)$, while at the $T=0$ critical chemical potential, which corresponds to approximately $3\rho_0$ in Sec. 4.2, $M_{\omega+} = M_{\rho+} \approx 0.65\,M_{\rho+}(\mu=0)$ and $M_{\phi+} \approx 0.80\,M_{\phi+}(\mu=0)$. These are the maximum possible reductions in the meson masses.

5 Diquark condensation

A direct means of exploring the possibility that SU(N_c) gauge theories might support scalar diquark condensation is to study the gap equation satisfied by

$$\mathcal{S}(p)^{-1} := \begin{pmatrix} S(p)^{-1} & \Delta^i(p)\lambda^i_\Lambda\,\tau^2_f\,\gamma_5 \\ -\Delta^i(p)\lambda^i_\Lambda\,\tau^2_f\,\gamma_5 & C(S(-p)^{-1})^{\rm T}C^\dagger \end{pmatrix}, \tag{32}$$

where $S(p)^{-1} = i\gamma\cdot p\,A(p^2) + B(p^2)$, $\{\lambda^i_\Lambda,\, i=1\ldots d_c,\, d_c = N_c(N_c-1)/2\}$ are the antisymmetric generators of SU(N_c), $C=\gamma_2\gamma_4$ is the charge conjugation matrix, and here we consider SU($N_f=2$). $\mathcal{S}(p)^{-1}$ is a matrix in the space of quark bispinors:

$$Q(x) := \frac{1}{\sqrt{2}}\begin{pmatrix} q(x) \\ q_c(x)\end{pmatrix},\ \bar{Q}(x) := \frac{1}{\sqrt{2}}\left(\bar{q}(x)\ \bar{q}_c(x)\right), \tag{33}$$

$q_c := -\bar{q}\,C$, $\bar{q}_c := q^{\rm T}\,C$. The gap equation is

$$\mathcal{S}(p)^{-1} = \mathcal{S}_0(p)^{-1} + \begin{pmatrix} \Sigma(p)_{11} & \Sigma(p)_{12} \\ \Sigma(p)_{21} & C\Sigma(-p)^{\rm T}_{11}C^\dagger \end{pmatrix}, \tag{34}$$

where $\mathcal{S}_0(p)^{-1} = {\rm diag}(i\gamma\cdot p + m, C(-i\gamma\cdot p + m)^{\rm T}C^\dagger)$ and the form of $\Sigma(p)_{ij}$ specifies the theory and its truncation. This approach avoids a truncated bosonisation, which in all but the simplest models is a procedure difficult to improve systematically and prone to yielding misleading results.

SU($N_c=2$): Using the rainbow truncation and a Feynman-like gauge for illustrative simplicity, the $T=0=\mu$ gap equation in this theory yields

$$p^2(A(p^2)-1) = \frac{3}{2}\int_k^\Lambda g^2 D(p-k)\,p\cdot k\,\frac{A(k^2)}{d(k^2)}, \tag{35}$$

$$B(p^2) - m = \boxed{3}\int_k^\Lambda g^2 D(p-k)\,\frac{B(k^2)}{d(k^2)}, \tag{36}$$

$$\Delta(p^2) = \boxed{3}\int_k^\Lambda g^2 D(p-k)\,\frac{\Delta(k^2)}{d(k^2)}, \tag{37}$$

where $d(p^2) = p^2 A(p^2)^2 + B(p^2)^2 + \Delta(p^2)^2$, and the pseudo-reality of SU(2) is responsible for the identical couplings in Eqs. (36) and (37). Clearly, for $m = 0$ the theory admits degenerate and indistinguishable quark and (colour-singlet) diquark condensates. This result is valid independent of the truncation and gauge, and is just one of the manifestations of Pauli-Gürsey symmetry. Similarly, mesons and baryons, which are diquarks in $\mathrm{SU}(N_c = 2)$, are degenerate.[17] The phase structure of this theory at nonzero-(T, μ) can certainly be very rich.

SU($N_c = 3$): In this case the interesting possibility is the existence of a colour antitriplet diquark condensate: $\{\lambda^i_\Lambda\}_{i=1,2,3} = \{\lambda^2, \lambda^5, \lambda^7\}$. Choosing the condensate to point in the λ^1_Λ-direction, the bispinor propagator separates into two pieces, one parallel and the other perpendicular to the condensate's direction:

$$\mathcal{S}(p)^{-1} := \left(\begin{array}{ccc|c} S_\parallel(-p)^{-1} I_2^c & & 0 & \Delta^1(p)\lambda^1_\Lambda \tau_f^2 \gamma_5 & 0 \\ 0 & & S_\perp(-p)^{-1} & 0 & 0 \\ \hline -\Delta^1(p)\lambda^1_\Lambda \tau_f^2 \gamma_5 & & 0 & S_\parallel(-p)^{-1} I_2^c & 0 \\ 0 & & 0 & 0 & S_\perp(-p)^{-1} \end{array}\right). \quad (38)$$

Here, since

$$\begin{pmatrix} S_\parallel^{-1} I_2^c & 0 \\ 0 & S_\perp^{-1} \end{pmatrix} = I_3 \left(\tfrac{2}{3} S_\parallel^{-1} + \tfrac{1}{3} S_\perp^{-1}\right) + \frac{1}{\sqrt{3}} \lambda^8 \left(S_\parallel^{-1} - S_\perp^{-1}\right), \quad (39)$$

the $\lambda^a \mathcal{S} \lambda^a$ interaction in the gap equation *couples* the parallel and perpendicular components. In rainbow truncation and using a Feynman-like gauge, the gap equation yields

$$p^2(A_\parallel(p^2) - 1) = \int_k^\Lambda g^2 D(p-k)\, p\cdot k \left[\frac{A_\perp(k^2)}{d_\perp(k^2)} + \frac{5}{3}\frac{A_\parallel(k^2)}{d_\parallel(k^2)}\right], \quad (40)$$

$$p^2(A_\perp(p^2) - 1) = \int_k^\Lambda g^2 D(p-k)\, p\cdot k \left[\frac{2}{3}\frac{A_\perp(k^2)}{d_\perp(k^2)} + 2\frac{A_\parallel(k^2)}{d_\parallel(k^2)}\right], \quad (41)$$

$$B_\parallel(p^2) - m = \int_k^\Lambda g^2 D(p-k) \left[2\frac{B_\perp(k^2)}{d_\perp(k^2)} + \frac{10}{3}\frac{B_\parallel(k^2)}{d_\parallel(k^2)}\right], \quad (42)$$

$$B_\perp(p^2) - m = \int_k^\Lambda g^2 D(p-k) \left[\frac{4}{3}\frac{B_\perp(k^2)}{d_\perp(k^2)} + 4\frac{B_\parallel(k^2)}{d_\parallel(k^2)}\right], \quad (43)$$

$$\Delta^1(p^2) = \frac{8}{3}\int_k^\Lambda g^2 D(p-k)\, \frac{\Delta^1(k^2)}{d_\parallel(k^2)}, \quad (44)$$

$d_\parallel(p^2) = p^2 A_\parallel(p^2)^2 + B_\parallel(p^2)^2 + (\Delta^1(p^2))^2$ and $d_\perp(p^2) = p^2 A_\perp(p^2)^2 + B_\perp(p^2)^2$.

The class of models hitherto applied in exploring diquark condensation[18] can be characterised as those in which $\int d\Omega_4\, p\cdot k\, D(p-k) = 0$. In this class $A_{\|} = A_{\perp} \equiv 1$ and when $B_{\|} = B_{\perp} \equiv 0$, which is always a solution, $\Delta^1 \neq 0$ if the coupling is large enough. Hence such restricted models admit a rich phase structure at nonzero-(T,μ) because the $\| \leftrightarrow \perp$ coupling is eliminated.

However, if one includes the next order contribution to the kernel of the gap equation, the picture can change. One study[19] suggests that in that case, even without the $\| \leftrightarrow \perp$ coupling, $\Delta^1 \equiv 0$ is the only solution at $T = 0 = \mu$. The effect at nonzero-(T,μ) of correcting the kernel has yet to be investigated but this result signals a need for caution in making inferences about the phase structure of QCD based on the rainbow-like truncation of this class of models.

The more general class of models in which $\int d\Omega_4\, p\cdot k\, D(p-k) \neq 0$ can be exemplified by the confining model introduced in Sec. 4.3. In that case, if we consider $B_{\|} = B_{\perp} \equiv 0$, the gap equation is solved with

$$p^2 A_{\|}(p^2)^2 + (\Delta^1(p^2))^2 = \frac{1}{2}\eta^2 \tag{45}$$

and (setting $\eta^2 \to 1$)

$$A_{\|}(p^2) = \frac{1}{6}\left(7 + 3\sqrt{9 + 2/p^2}\right),\ A_{\perp}(p^2) = \frac{1}{2}\left(3 + \sqrt{9 + 2/p^2}\right). \tag{46}$$

However, inserting the result for $A_{\|}(p^2)$ into Eq. (45) yields $(\Delta^1(p^2))^2 \leq 0$ for all p^2; i.e., $\Delta^1(p^2) \equiv 0$, even in the rainbow truncation. Thus diquark condensation at $T = 0 = \mu$ is blocked by the $\| \leftrightarrow \perp$ coupling. We expect that this conclusion will be reinforced if the kernel is improved.[20] The effect of $\mu \neq 0$ has not yet been explored but this result too advocates caution in making inferences about the phase structure of QCD based on the simple models hitherto employed.

6 Epilogue

Hadron observables are insensitive to the behaviour of the interaction at $p^2 < \Lambda_{\rm QCD}^2$ and the rainbow truncation of the gap equation is quantitatively reliable for $p^2 \gtrsim 1\,{\rm GeV}^2$. Thus the model dependence in our approach is contained in an apparently small domain. However, as illustrated by the material presented in this volume, even that small domain of uncertainty admits a large variety of possibilities; although apparently distinct *Ansätze* may really be different realisations of the same phenomena. It is also a crucial domain, covering that in which mesonic correlations (vertex corrections) can influence quark propagation characteristics, an effect that may become qualitatively important

as the phase boundary is approached. Much has been achieved in localising the model-dependence but more must be done to further ameliorate it. Discussions of the type represented by this volume are crucial to that endeavour.

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