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Final Report for DE-FG02-99ER4110, May 15, 2011- October 15, 2014
Investigations of Beam Dynamics Issues at Current and Future Accelerators
Mathematics and Statistics, University of New Mexico

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1 Introduction

Beam Dynamics in modern particle accelerators is a rich source of deep and interesting problems in Applied Mathematics (AMa). The purview of AMa includes, but is not limited to, ordinary and partial differential equations and dynamical systems, numerical analysis and scientific computing, perturbation theory, probability and stochastic processes, mathematical statistics, and modern geometry. At one level AMa exists as a discipline which is the result of a cross fertilization between scientific fields. My own view of AMa is that of digging deeply into an area of science and then bringing the best mathematics to bear on significant problems.

In this final report we summarize our AMa work in Beam Dynamics since May 15, 2011. Our main emphasis has been three-fold, (1) Maxwell and Vlasov-Maxwell (VM) systems important in modern accelerators, (2) polarization physics and spin dynamics of relevance for HEP projects, and (3) mathematical problems for free electron lasers. These are discussed in Sections 2-4. In Section 5 we discuss our electron storage ring work, our work on quasiperiodic averaging and mention our small effort related to an experiment at FACET at SLAC. In addition to summarizing the work on this cycle we will point out things we will continue in the foreseeable future. Most written works from this last cycle are listed here as [GW1]-[GW12]. These, as well as drafts and notes mentioned here, can be accessed at `http://math.unm.edu/~ellison/doe_works.html`. This also contains DOE works from previous cycles.

The main players, Ellison, Heinemann, Lau and Bizzozero are listed above. Lau is an expert in Numerical Analysis and Scientific Computing which is invaluable to our work. He led the effort on the rapid integration over history, with very sophisticated tools, and was central to our introduction of the discontinuous Galerkin (DG) methods to the beam dynamics community. Heinemann has contributed to all aspects of the work, he is an expert in several areas of mathematics important to our research, e.g., abstract dynamical systems, modern geometry, mathematical statistics and distribution theory, and led our polarization effort. Ellison is an expert in many fields of AMa and has contributed to all aspects of the work. David Bizzozero joined us in the summer of 2012 and will complete his dissertation in the next few months. He is leading our DG effort and his DG simulations have made a significant contribution to understanding an experiment at the Canadian Light Source (CLS). Lau was supported for one month in each of the summers of 2011,12 and 13, Heinemann was supported full time, Ellison put in a full time effort without salary and Bizzozero was supported full time starting in summer 2012. Heinemann received his DOE funded Ph.D. with distinction in May 2010 and with DOE support. He received the 2012 departmental nomination for the UNM Popejoy Dissertation Prize for the best dissertation in Science and Mathematics over the 3 year period 2009-11, see `http://grad.unm.edu/funding/awards.html`, and was nominated for the beam physics dissertation award.

This work could not have evolved to its current stage without collaborators. D. Barber (DESY) brought his many years of experience in polarization physics, M. Vogt brought expertise in both polarization physics as well FEL physics in his senior position at the X-ray Free-Electron Laser FLASH at DESY. G. Bassi continued to collaborate with us on the VM work from his position at BNL where he has become an important contributor to the NSLS-II effort. Vogt and Bassi were postdocs under my prior DOE grants. Barber and Vogt are senior scientists at DESY. H.S. Dumas, a mathematician and dynamical systems expert at the University of Cincinnati, is working with us on two projects including a new approach to quasiperiodic averaging. Warnock (SLAC) led our CLS work and his CSR knowledge has been invaluable.

2 Maxwell and Vlasov-Maxwell

Most of our recent work has been on computational approaches to Maxwell equations including the importance of realistic boundary conditions. The underlying mathematical framework for our study is the coupled system consisting of the relativistic 6D Vlasov and 3D Maxwell equations, governing a relativistic particle bunch moving in a vacuum chamber and its associated electromagnetic self-field. We are interested in "small dense bunches" and realistic vacuum chamber (VC) boundary conditions for the Maxwell system such as found in HEP high luminosity machines. These problems are at the frontier of computational accelerator physics, so approximations must be made. We have focused on single pass systems with a planar design trajectory. While the design trajectory can be quite general, an important special case is that of arbitrary straights and bends, as in e.g. bunch compressors. Numerical integration is computationally intensive necessitating a combination of high-level analytical and computational work. Our goal is to develop algorithms and codes for efficient and accurate numerical integration of this VM system that can be used by the community for important applications. In a previous cycle we developed our 2D CSR VM Monte-Carlo algorithm and VM3@A code for bunches moving along arbitrary planar orbits between perfectly conducting parallel plates where each electromagnetic field component is represented by an integral over history (IOH), the most expensive part of our code.

In the present cycle our main work was two-fold. (1) We have worked on methods to speed up the IOH in the VM3@A algorithm, see §2.1. We have a detailed paper ready for submission [GW1]; preliminary results were presented [GW11]. (2) We have developed a new time-stepping approach, see §2.2. In addition, we sought improved techniques for solving and analyzing solutions of the Vlasov equation. Mainly, we have continued to develop a new density estimation procedure based on kernel smoothing which gives better control and studied a random number generator for pseudo-random numbers to speed up the Monte-Carlo aspect. While these Vlasov considerations have not been formalized, some initial studies were reported in [GW12] and ICAP and preliminary studies in VM3@A have been performed by Heinemann. We aim to prepare a documented and robust VM3@A code for use by the community.

2.1 VM3@A and Rapid Integration over History

We have developed two strategies for simplifying the IOH and have tested these in a simple model [GW1]. We will implement and test them in VM3@A, and Bassi has done some preliminary studies. A preliminary version of the first strategy was presented in [GW11] including its context within VM3@A. The first strategy, due to Lau, uses a representation of the Bessel function J_0 in terms of exponentials. The second relies on local sequences (developed recently by Hagstrom, Warburton, and Givoli, [1]) for radiation boundary conditions. Although motivated by practicality, both strategies involve interesting and rather deep analysis and approximation theory. To elaborate, the retarded-time integral is of the form

$$
\begin{aligned}
F(t,\mathbf{x}) &:= \frac{1}{2\pi}\int_{|\mathbf{x}-\mathbf{x}'|\le t} d\mathbf{x}' \frac{f(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \\
&= \frac{1}{2\pi}\int_0^t d\tau \int_0^{2\pi} d\phi f(\tau, x+(t-\tau)\cos\phi, y+(t-\tau)\sin\phi),
\end{aligned}
\tag{1}
$$

where the second expression for F is the singularity-free form used in VM3@A. It's the t in the integrand which slows down the code, as it requires the integral to be redone at every time step. In the first approach we take the spatial Fourier transform of (1) which yields a temporal convolution involving the Bessel function $J_0(t)$. We then approximate $J_0(t)$ by a sum of exponentials using the Alpert-Greengard-Hagstrom-Jiang algorithm with refinements due to Lau. This effectively localizes the convolution in time, thereby circumventing the issue. The second approach relates the two-dimensional integral (1) to a solution of the ordinary $3+1$ (three-space plus time) wave equation with a sheet source. We then evolve the wave equation in a neighborhood of the sheet $z=0$, with artificial planar boundaries above and below $z=0$. Placement of these boundaries at large distances from the sheet (such that they are causally disconnected from the sheet) would prove prohibitively costly. Therefore, we adopt radiation boundary conditions [1].

Lau led this effort, and the mathematical details for both approaches and simple model calculations appear in [GW1]. The details include Lau's J_0 approximation and the algorithm for the radiation boundary conditions. We also present new results on the spacetime form of the radiation boundary conditions.

2.2 Time-Stepping approach

As an alternative to VM3@A, we have a project to solve Maxwell's equations by time-stepping methods. We looked into several approaches (Finite difference (FD) and FD-Yee) and have decided to implement the Discontinuous Galerkin Finite Element Method (see e.g. [2]). DG is a generalization of the Finite Element and Finite Volume Methods. It is spectrally convergent for smooth problems unlike the commonly used Finite Difference methods, and can handle complicated vacuum chamber geometries. We experimented with the HEDGE software `http://mathema.tician.de/software/hedge` for DG, but we feel that our algorithms are easier to implement and run in standard programming languages such as MATLAB or FORTRAN. Our papers here are [GW4], [GW8] and a paper to be submitted to PRL. This work is also central to Bizzozero's dissertation.

Our DG approach may prove more efficient for the electromagnetic field calculation in VM3@A, but more importantly it is more flexible from the standpoint of geometry, coordinate systems, and the assumed spatial dimension and can handle more realistic boundary condition as we discuss below. The basic idea is to rely on a standard time-stepping of the discretized PDE (method of lines approach). However, to achieve high accuracy, we incorporate modern spectral approximations for the relevant spatial operators. Moreover, we envision some form of domain reduction, achieved via enforcement of radiation boundary conditions (see, for example, [GW1] which uses [1]). The idea here is to shrink the spatial domain around the sources, thereby minimizing the number of nodes or cells at which the fields must be maintained and evolved. Over the last decade, such domain reduction techniques have seen rapid development. We are content, for the moment, with our approach to integrating the Vlasov equation that we use in VM3@A, however in the future we may pursue the method of local characteristics that we used in previous work.

Bizzozero has spent considerable effort gaining experience with the DG method, and we have relied heavily on the computational and DG expertise of our co-PI Stephen Lau. We have applied the DG method, for 1, 2 and 3D elements, in four contexts which we now discuss.

1. In [GW8] we continued our study [3, GW11] of CSR from a bunch moving on an arbitrary curved trajectory. In [3] we developed an accurate 2D CSR VM code (VM3@A) and applied it to a four dipole chicane bunch compressor. Our starting here is the well-established paraxial approximation pioneered by Agoh, Yokoya, Stupakov and Zhou, with boundary conditions for a perfectly conducting vacuum chamber with uniform cross-section. This is considerably different from our VM3@A approach, using the IOH parallel plate boundary conditions. We use the DG method for the paraxial approximation equations. These are Schrödinger type equations with one "time" and two "spatial" dimensions and

thus require 2D triangular elements. Our basic tool is a MATLAB DG code on a GPU using MATLAB's `gpuArray`; the code was developed by Bizzozero. We discussed our results in the context of previous work and outlined future applications for DG. We used a fixed source appropriate for impedance studies. David's work attracted the attention of accomplished accelerator physicists who encouraged him to prepare a user friendly version of his DG and FD codes. Zhou referenced David's work in a recent seminar on SuperKEKB which he gave at SLAC.

2. Warnock suggested that we might obtain a faster algorithm if we used a Fourier series in y thus reducing the two spatial variable Schrödinger equation above to the one variable case. Of course one then needs to sum over Fourier modes, but nonetheless this is still a 3D Maxwell approach, albeit in the paraxial approximation. We pursued this idea and our work was presented in [GW4]. In summary, we continued our study in [3, GW8, GW11], but the vacuum chamber has rectangular cross section with possibly a mildly-varying horizontal width. We make a Fourier transform in $s - ct$ and use the slowly varying amplitude approximation. We invoke a Fourier series in the vertical coordinate y, which meet the boundary conditions on the top and bottom plates and makes contact with the Bessel equation of the frequency domain treatment. The fields are defined by a PDE in s and x, first order in s, which is discretized in x by finite differences (FD) or the DG. We compared results of FD and DG, and also compared the computation speeds to our earlier calculations in [GW8]. This approach provides more transparency in the physical description, and when only a few y-modes are needed, provides a large reduction in computation time.

   In the above two approaches we used a leap-frog scheme in FD and RK4 in DG for the s time step integration and have numerical evidence that this scheme is stable over the time intervals of interest. A von Neumann stability analysis shows there is a weak instability for leap-frog which affects the solution at large times. We will use RK4, which is von Neumann stable, instead if necessary.

3. For more general geometries, such as in flared vacuum chambers with varying horizontal width, we found that a paraxial approximation is not valid. We adopted a time-domain approach for the full Maxwell equations retaining the Fourier series expansion in y. This was incorporated in the simulation of an experiment at the Canadian Light Source (CLS). We initially began with a FDTD-Yee scheme but quickly encountered difficulties due to complicated geometries and discontinuous source terms. Bizzozero then created a code using DG, in collaboration with Warnock, which is more naturally suited for handling general geometry and discontinuities. Bizzozero has done an outstanding job here and this approach will comprise a key portion of his thesis. A paper entitled "Observation of wakefields and resonances in coherent synchrotron radiation" comparing the simulations and experiment is ready for submission to PRL, see GWDraft1 at `http://math.unm.edu/~ellison/doe_works.html`.

4. Another approach was investigated for the full 3D Maxwell system using DG. Here, the approach does not use a Fourier series in y but instead reduces the longitudinal size of the domain by means of a Galilean transformation in Frenet-Serret coordinates. This allows the computational mesh to follow the source and ignore fields which propagate too far from the source. A complicated aspect of this approach was the construction of the 3D tetrahedral elements; however, this has been accomplished. Bizzozero presented preliminary results at the 2014 International Conference On Spectral And High Order Methods in Salt Lake City, UT, attended by prominent PDE researchers in the field and results will appear in his dissertation. In this context, we are investigating the use of the radiation boundary conditions (RBCs) proposed in [GW1] to further reduce the size of the computational domain. We have done a careful study of RBCs in the context of our parallel-plate model and have prepared a detailed note.

2.3 Other features of the VM system

We have also looked into the relevance of our polarization work (see §3 below) for the VM system. In fact a basic feature of the former is that the particle motion affects the spin motion but, by neglecting the Stern-Gerlach force, the spin motion does not affect the particle motion. This gives the particle-spin dynamics a so-called skew-product structure where some degrees of freedom are spectators and in fact this skew product structure underlies the fibre bundle approach mentioned in §3. In constrast, in the VM system the beam acts on the self field via Maxwell's equation and the self field acts on the beam via Vlasov's equation. However since these two interaction effects happen in different equations, one can conveniently obtain a skew-product structure via recursion as follows. In the first recursion step the beam is only affected by the external field and the self field is zero. In the second step, the beam is as in the first step but it produces a nonzero self field via Maxwell's equation (this second step is also a mode of operation of VM3@A). In the third step the self field is as in the second step while the beam is affected via Vlasov's equation by the self field from the second step. It is clear how the recursion goes on. This recursion renders the nonlinear VM system of seven equations into a countable infinite set of linear equations and it allows one [4] to use some of the techniques mentioned in §3. Clearly it can be applied to the VM system in general as well to the sheet source model. In the latter case one use further techniques we developed over the years for the sheet source.

3 Polarization Physics and Spin Dynamics

We promised a paper related to the polarization part of the Heinemann dissertation [GW12], which was funded by HEP-DOE on a previous cycle. Before discussing the status of this work, [GW2], we want to point out there is much current and proposed activity using accelerators in polarization physics. Several rings for which spin polarization is crucial are either running, being built or having their feasibility seriously studied. These include LHeC (CERN), CEPC (China), FCC as well as RHIC, eRHIC (BNL), JPARC (Tokai), COSY (Juelich), the HESR (FAIR at Darmstadt), the updated muon g-2 ring (FNAL), MEIC (JLab), the rings for measuring the electric dipole moment (EDM) of the proton and deuteron (BNL (or FNAL) and Juelich), and damping rings for the ILC. Our colleague, Desmond Barber, who has worked on polarization in rings for many years, is a major inspiration for our mathematical work on polarization. His work (and work inspired by him) has contributed greatly over the last three decades to the clarification of tricky aspects of spin motion, and other practitioners continue to seek his advice (e.g., researchers at COSY, JLab, IHEP(Beijing), KEK, and the Tech-X Corporation). In addition to his impressive body of work (see his web site: `www.desy.de/~mpybar`), he introduced Heinemann to mathematical problems in spin dynamics, supervised the very nice thesis of M. Vogt, who was mentioned in the Introduction, and inspired G. Hoffstaetter, now at Cornell, to become involved in spin dynamics. Central to the contributions are the ISF (invariant spin field) and the ADST (amplitude dependent spin tune). The ISF and ADST were introduced in substance, but not in name, by Derbenev and Kondratenko in the 1970's. The ADST allows a proper definition of spin-orbit resonance (SOR) and needs special reference frames called invariant frame fields (IFF). In the meantime Barber has shown how ISF's can be applied to the so-called invariant tensor fields (ITF) (the latter being introduced for efficient handling of the spin motion of spin-1 particles, e.g., deuterons). We believe that it will be advantageous to the polarization physics community if the theory and phenomenology of spin dynamics, and of the ISF, ITF, IFF, and ADST in particular, continues to be developed, clarified and explained.

Barber cites several instances where theoretical issues have been or are important: (1) plans for polarization at RHIC; (2) calculation of corrections to the spin precession rate in the earlier g-2 experiments; (3) the EDM rings might run in an exotic mode in which the ISF will only be unique away from the closed orbit; (4) at the high electron/positron energies of the CEPC and FCC the Derbenev-Kondratenko picture of depolarization might be too pessimistic. Nevertheless it is recognized in the community that a special effort will be needed for getting good radiative polarization at those high energies and that in any case it would be good to have a new faster algorithm for finding the ISF; (5) a detailed understanding of spin motion in the COSY ring has been essential for interpreting some measurements; (6) tracking simulations and analysis for spin motion in damping rings have exposed fallacies resulting from superficial understanding.

Beyond these six items there are still questions about the existence of the ISF, especially in the presence of strong sextupoles so we will give a few more details. Polarization in storage rings is best systematized in terms of the ISF. It is essential for estimating the maximum proton polarization and the electron equilibrium polarization due to synchrotron radiation. For example, analytical estimates of equilibrium electron polarization rely on the "Derbenev-Kondratenko" formula which needs the ISF (we plan to deepen the understanding of this formula using our averaging expertise).

The detailed paper [GW2], which deals with the ISF and ADST, has been submitted to PRST-AB and is now on the arXiv. We were invited to submit this as a long version of our IPAC14 paper [GW5] for a special

IPAC14 Edition of PRST-AB. The paper [GW3] is a revision of [GW5] and Barber presented an informal summary at Spin2014: The 21st International Symposium on Spin Physics, Beijing, China, October 2014. The summary will appear in the International Journal of Modern Physics, Conference Series.

The paper [GW2] has been a major effort over the last three years. It beginnings were in [GW12] which evolved into an abstract version using the theory of principal bundles, ala Husemoller's book on fibre bundles. Because of the product structure of the underlying principal bundle we found a simpler approach, but the mathematics we learned on the way was important. Our mathematical framework had its origin in the work of R. Zimmer, who is now president of the University of Chicago and chairman of the board for FNAL and ANL. This may be the first connection between his mathematics and accelerators. The abstract follows.

We return to our study [5] of invariant spin fields and spin tunes for polarized beams in storage rings but in contrast to the continuous-time treatment in [5], we now employ a discrete-time formalism, beginning with the Poincaré maps of the continuous time formalism. We then substantially extend our toolset and generalize the notions of invariant spin field and invariant frame field. We revisit some old theorems and prove several theorems believed to be new. In particular we study two transformation rules, one of them known and the other new, where the former turns out to be an $SO(3)$-gauge transformation rule. We then apply the theory to the dynamics of spin-1/2 and spin-1 particle bunches and their density matrix functions, describing semiclassically the particle-spin content of bunches. Our approach thus unifies the spin-vector dynamics from the T-BMT equation with the spin-tensor dynamics and other dynamics. This unifying aspect of our approach relates the examples elegantly and uncovers relations between the various underlying dynamical systems in a transparent way. As in [5], the particle motion is integrable but we now allow for nonlinear particle motion on each torus. Since this work is inspired by notions from the theory of bundles, we also provide insight into the underlying bundle-theoretic aspects of the well-established concepts of invariant spin field, spin tune and invariant frame field. Thus the group theoretical notions hidden in [5] and the earlier work of Ya.S. Derbenev, A.M. Kondratenko and K.Yokoya will be exhibited. Since we neglect, as is usual, the Stern-Gerlach force, the underlying principal bundle is of product form so that we can present the theory in a fashion which does not use bundle theory at all. Nevertheless we occasionally mention the bundle-theoretic meaning of our concepts and we also mention the similarities with the geometrical approach to Yang-Mills Theory.

4 FEL

4.1 Noncollective: Averaging and the FEL Pendulum

Our paper [GW6], which was a major effort during the recent cycle, is published. Our effort here will be quite useful for our collective work and the abstract which follows gives a good overview.

We present a mathematical analysis of planar motion of energetic electrons moving through a planar dipole undulator, excited by a fixed planar polarized plane wave Maxwell field in the X-Ray FEL regime. Our starting point is the 6D Lorentz system, which allows planar motions, and we examine this dynamical system as the wave length λ of the traveling wave varies. By scalings and transformations the 6D system is reduced, without approximation, to a 2D system in a form for a rigorous asymptotic analysis using the Method of Averaging, a long time perturbation theory. The two dependent variables are a scaled energy deviation and a generalization of the so-called ponderomotive phase. As λ varies the system passes through resonant and NonResonant (NonR) intervals and we develop NonR and Near-to-resonant (NearR) Method of Averaging normal form approximations to the exact equations. The NearR normal forms contain a parameter which measures the distance from a resonance. For the planar motion, with the special initial condition that matches into the undulator design trajectory, and on resonance, the NearR normal form reduces to the well known FEL pendulum system. We then state and prove NonR and NearR first-order averaging theorems which give explicit error bounds for the normal form approximations. We prove the theorems in great detail, giving the interested reader a tutorial on mathematically rigorous perturbation theory in a context where the proofs are easily understood. The proofs are novel in that they do not use a near identity transformation and they use a *system* of differential inequalities. The NonR case is an example of quasiperiodic averaging where the small divisor problem enters in the simplest possible way. To our knowledge the planar problem has not been analyzed with the generality we aspire to here nor has the standard FEL pendulum system been derived with associated error bounds as we do here. We briefly discuss the low gain theory in light of our NearR normal form. Our mathematical treatment of the *noncollective* FEL beam dynamics problem in the framework of *dynamical systems theory* sets the stage for our mathematical investigation of the *collective* high gain regime.

4.2 Collective: A Klimontovich-Maxwell Approach

The 1D wave equation, $u_{tt} - c^2 u_{zz} = f$, with a Klimontovich source is often the starting point for the 1D FEL high gain theory [6]. We made progress on two fronts from this starting point.

First, we have a new representation of solutions of this 1D wave equation [7], which we have not seen in the FEL literature (although it's in many elementary PDE texts, see also Appendix G of [GW6]). We have been studying the consequences of this representation and it leads to a new derivation of the paraxial approximation. We show that the solution can be written as a sum of backward and forward moving waves. The evolution of each of these is governed by a 1D advection equation, $v_t \pm c v_z = g$. The forward moving wave is larger than the backward one by a factor of $O(\gamma^2)$ and the 1D advection equation for the forward moving wave is the paraxial approximation. Our next step is to couple this 1D advection equation with the the Lorentz equations of motion, which evolve the Klimontovich source, to see what new insights we find. For example, our solution form yields a new way to view the so-called slice average used in 1D high gain theories.

Second, we have interacted with Bob Warnock on his approach to the 1D theory based in Fourier space as presented in [8]. He makes an argument that the 1D theory can be done with a mean field Vlasov, rather than the Klimontovich form, and we're eager to see if this is the case. He integrates the Vlasov equation using a method of local characteristics, he and Ellison developed years ago, to which he has added some important fine points, and he already has some nice results. In addition he has developed a "wiggle average" approximation to the FEL Vlasov to make the size of the time step computationally feasible. We have done a detailed study of the wiggle average, but as yet have not been able to assess its accuracy. We are applying our many years of experience with the Method of Averaging (See §4.1 above and §5.2 below).

In the noncollective work of §4.1 above, we discovered that if we excite the bunch with a traveling wave with a continuous set of frequencies near a resonance, the effect of the resonance appears to be washed out in first order averaging. This is briefly discussed in Section 5 of [GW6] in the context of Eq. 5.1 and 5.2 therein and we will study this issue in the collective regime.

A future aim is a study of the FEL high gain regime, including a start up from shot noise. We will use our noncollective work as a basis. We will start with the full microscopic Maxwell equations with the Klimontovich source evolved by the Lorentz equations of motion. From there, we will study the approximations leading to the standard formulation in terms the 1D wave equation and the issues raised above. In the big picture we will follow the plan laid out in Section 4 of the proposal. One aspect we wish to mention here, which was discussed in the proposal, is an approximation central to Warnock's approach. Namely the approximation going from the microscopic Klimontovich-Maxwell system to the macroscopic VM system. We presented our work in a talk entitled: "From Microscopic Klimontovich-Maxwell (KM) to Macroscopic VM: Relativistic N-particle electron bunches in modern particle accelerator systems, for large N", the abstract follows.

We consider an N-particle electron bunch moving, at nearly the speed of light, through a particle accelerator system inside a vacuum chamber. Typically, N is of an order greater than 10^9 and the bunch is small relative to the vacuum chamber cross-section. We model the evolution of the bunch by a random initial boundary value problem with random, independent identically distributed (IID) initial conditions with a given density* and where the electron evolution is given in terms of the Lorentz force and the associated microscopic Maxwell fields. The electron phase space density is Klimontovich, i.e., a sum of delta functions. Taking expected value of the associated Klimontovich evolution equation (with respect to the random initial conditions) and making reasonable assumptions, we obtain the Vlasov equation with a correction term, for the expected value of the Klimontovich density, coupled to the macroscopic Maxwell equations. With this framework we then pose the important mathematical issues: (1) How well does the Vlasov density approximate a coarse-grained Klimontovich density when N is large? We imagine that the vast literature on probabilistic limit theorems will be relevant here, e.g. the Strong-Law of-Large-Numbers (SLLN). (2) The Vlasov equation without correction terms is the starting point for many beam dynamics calculations so it is important to estimate the size of the correction term (surely related to the correction term in the BBGKY hierarchy). In addition the correction term may shed light on FEL dynamics. These mathematical issues are likely difficult analysis issues. We begin the talk with the much simpler noncollective case, assuming the electrons do not radiate and thus ignoring the Maxwell self-fields, in order to set the stage for the more complex KM to VM case. The slides for the talk are in GWDraft2 at `http://math.unm.edu/~ellison/doe_works.html`.

*In a physical context the N initial conditions are impossible to know. One view is to think of them as a set of scattered data from which a density can be constructed using e.g., a density estimation procedure from Mathematical Statistics. The IID random initial conditions are then given in terms of such a density. In our work here we simply consider the initial density as given.

5 Smaller Projects on this cycle

5.1 Electron Storage Rings and Handbook article

Ellison and H. Mais completed an extensive revision of our article "Orbital Eigen-Analysis for Electron Storage Rings" in the first edition of *Handbook of Accelerator Physics and Engineering* edited by A.W. Chao and M. Tigner. It has been published in the second edition edited by K.H.Mess and F. Zimmermann in addition to Chao and Tigner. In this article a general 6D formalism is presented for the calculation of the bunch parameters for electron storage rings including radiation damping and quantum excitation. This builds on the early DESY work of Mais and Ripken. Basic to our approach is the orbital eigen-analysis first introduced by Chao which gives a framework more general than that of Courant-Synder. New aspects include: (1) a new improved notation; (2) a more precise discussion of linear Hamiltonian systems with periodic coefficients; (3) a formulation of an integral equation for the closed orbit; (4) a derivation of a differential equation for the moments from a linearized stochastic differential equation; and (5) an application of the averaging formalism to obtain an approximation to the beam moments. The article is terse so we have begun a long version giving details the five items mentioned above. In addition we are giving detailed discussions of the following: derivation of the equations of motion; the random bunch density's convergence to the single particle probability density using tools related to the central limit theorem and strong law of large numbers; linear Hamiltonian systems with periodic coefficients including stability and normal forms; a proof of existence and uniqueness of the closed orbit in item 3 above; stochastic differential equations, associated long time perturbation results (see [9]) and a comparison with item 4 above; a proof of the averaging theorem mentioned in item 4 above. Here we have dealt with the non-resonant case. Because of the progress in our averaging work discussed in §5.2 below, we are studying the resonant case and are comparing our rigorous results with the work in [10].

5.2 Quasiperiodic Averaging: A new approach and applications

We have been working on new results for quasiperiodic (QP) flows of importance in beam dynamics. We have a draft of our results in a very advanced form [11] which we plan to submit for publication. We could submit to PRSTAB, however in the spirit of spreading interest in beam dynamics outside our community we are considering a Dynamical Systems journal, e.g., the SIAM Journal of Applied Dynamical systems, a journal we have used before.

To be more specific, an important class of perturbation problems for ODEs can be reduced to

$$x' = \epsilon\, g(x, \omega t), \qquad x \in \mathbb{R}^d, \quad \omega \in \mathbb{R}^m, \quad t \in \mathbb{R}. \tag{2}$$

Here ϵ is a small parameter, ω is called the frequency vector of the QP flow, and the function $g(x, \omega t)$ is called a QP function of t where $g = g(x, \theta)$ is 2π-periodic in each component of $\theta \in \mathbb{R}^m$. For example, many beam dynamics problems for rings are given in the form of the perturbed linear system $u' = A(s)u + \epsilon h(u, s)$ as in the handbook work, mentioned above in Section 5.1, but without the stochastic term. This can be put into the form of (2) if the solutions for $\epsilon = 0$ are quasiperiodic and if $h(u, \cdot)$ is also quasiperiodic (i.e., let $u = \Psi(s)x$, where Ψ is a fundamental solution matrix for A, then x satisfies an equation of the form (2) with the arc length s replaced by t).

It is well known that solutions of (2) depend sensitively on ω and the goal of long time perturbation theory is to find "normal form" approximations

$$y' = \epsilon g_{NF}(y, \omega t), \ y(0) = \xi, \quad \text{such that} \quad |\mathrm{x(t) - y(t)}| \le \delta(\epsilon)\mathrm{K(T)}, \quad \text{for} \quad 0 \le \mathrm{t} \le \mathrm{T}/\epsilon, \tag{3}$$

where $\delta(\epsilon)$ goes to zero as ϵ goes to zero. Standard results, which do not take into account the important quasiperiodic structure, give a crude error bound $\delta(\epsilon) = \sqrt{\epsilon}$ and lead to normal form approximations which are not very useful. However, one can do much better in this QP case under a non-resonance condition and obtain the error in (3) with $\delta(\epsilon) = \epsilon$, see e.g. [12]. However the ω in [12] are restricted to a Cantor type set which is unsatisfactory for applications. In our approach we eliminate this unphysical condition and allow ω to be far from low order resonance defined by a "cut-off Diophantine condition". More significantly, we extend Sáenz' results by exploring neighborhoods of low-order resonances which were originally excluded. Overall, we believe our methods provide significant advantages over previous methods, especially in applications. Furthermore it fills a significant gap in the mathematical averaging/long-time-pertubation literature.

Our FEL averaging paper is an elementary application of the above. In addition we now have a collection of problems in beam dynamics which can be treated using the results of the above paper. This includes the following: (1) problems discussed in [12]; (2) the resonant and nonresonant cases in §5.1 above; (3) problems from spin dynamics including a rigorous derivation derivation of the single resonance model which was introduced in the early work of Courant and Ruth (but still plays a significant role today) and a perturbation theory for the ISF and ADST (see §3 above). We will prepare a paper on these examples and submit to PRSTAB.

5.3 Theory for Experiment at SLAC's FACET facility

Ellison is participating in a project to study radiation from GeV electrons and positrons in crystals at FACET at SLAC. The ultimate goal of the project is to produce a gamma ray laser, using the channeling effect in a so-called crystalline undulator. Ellison was involved in particle channeling in crystals, including channeling radiation, for many years before becoming involved in beam dynamics at the SSC. He helped write a proposal for beam time at FACET and the proposal entitled "Radiation from GeV electrons in diamond-with intensities approaching the amplified radiation regime" was approved and a very short test (3 times 8-10 hours) was completed in November; the data is being analyzed. Ellison's previous channeling radiation work [13] is relevant. He discovered a special effect and developed much of the theory. There is an overlap with our DOE FEL work.

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