LA-UR-15-20372

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Intended for: ARXIV and IEEE Transactions on Network Controls

Issued: 2015-01-22

Convexity of Energy-Like Functions: Theoretical Results and Applications to Power System Operations

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January 12, 2015

Abstract

Power systems are undergoing unprecedented transformations with increased adoption of renewables and distributed generation, as well as the adoption of demand response programs. All of these changes, while making the grid more responsive and potentially more efficient, pose significant challenges for power systems operators. Conventional operational paradigms are no longer sufficient as the power system may no longer have big dispatchable generators with sufficient positive and negative reserves. This increases the need for tools and algorithms that can efficiently predict safe regions of operation of the power system. In this paper, we study energy functions as a tool to design algorithms for various operational problems in power systems. These have a long history in power systems and have been primarily applied to transient stability problems. In this paper, we take a new look at power systems, focusing on an aspect that has previously received little attention: *Convexity*. We characterize the domain of voltage magnitudes and phases within which the energy function is convex in these variables. We show that this corresponds naturally with standard operational constraints imposed in power systems. We show that power flow equations can be solved using this approach, as long as the solution lies within the convexity domain. We outline various desirable properties of solutions in the convexity domain and present simple numerical illustrations supporting our results.

1 Introduction

Power systems are experiencing unprecedented changes. Increased adoption of renewable sources of energy, combined with smart metering infrastructure, direct or price-based load-control capabilities and distributed generation resources have contributed to a transformation that is making the power grid increasingly heterogeneous and subject to stochastic disturbances. While many of these changes are positive and should contribute to the power grid sustainable and economically efficient, they pose significant challenges for power systems operators. With these changes, system operators no longer have the luxury of large positive and negative reserves or predictability in power flow patterns and directions. Modern power system operation will require computationally efficient tools for power systems analysis that can handle these changes. More specifically, these tools will need to go beyond linearized analysis methods and provide tools to guarantee that power systems can be operated in a stable and reliable manner in the presence of large disturbances. In this paper, we study energy functions for power systems and argue that several power systems operations problems can be cast naturally in terms of the energy function.

Energy functions have been studied for over 50 years in the literature on direct methods for transient stability analysis of power systems, beginning with the work of Aylett [Ayl58]. In the 70s and 80s, a lot of progress was made, extending results to multimachine systems, first via Kron-reduction techniques and ultimately through the paradigm of structure-preserving energy functions, beginning with the seminal work of Bergen and Hill [BH81]. The developments of this period are summarized in [VWC85, Pai89]. Since this period, much of the work has been on effective heuristics for enabling these methods to produce non-conservative estimates of the region of stability [Chi11]. Also, with the advent of faster computers and the ability to do real-time dynamic simulations with more realistic models than those handled by direct methods, interest in these approaches declined in the 90s and 2000s.

Another line of work relevant to the results in this paper are on conditions for existence of solutions to power flow equations [LSP99, BZ14a]. Recent interest has also seen the development of fixed point characterizations of the power flow equations and analytical approaximation schemes for power flow solutions. Also of relevance are recent results on the exactness of convex relaxations to the optimal power flow problem [LTZ13] and heuristics to extend these results where one cannot guarantee exactness [MH14][MAL14].

In this work, we take a fresh perspective on energy functions: We argue that energy functions can be viewed as a unifying tool for several operational problems in power systems. This is based on the fact that stationary points of energy functions correspond to solutions of the power flow equations. Hence, any operational analysis that is based on power flow equations (ie static analysis): Power Flow Analysis, Critical Loadability studies, Optimal Power Flow can be studied using the energy function. This viewpoint was first presented in [BV12, BBC13], but our work significantly expands on these initial results, both in terms of theory and in outlining various potential applications. This viewpoint even works in more general cases where the energy-like functions are not true Lyapunov functions, but simply functions whose stationary points correspond to power flow solutions. This extension will allows us to deal with lossy systems that have fixed resistance to reactance ratio for all transmission lines.

In order to obtain formulations with efficiency and optimality guarantees, we look at a new property of energy functions that, to the best of our knowledge, has previously received little attention: *Convexity*. Since power systems are designed to operate at a stable equilibrium point, the energy (Lyapunov) functions characterizing their behavior are obviously convex at least in a small vicinity of the stable equilibrium. However, to the best of our knowledge, the exact domain of convexity has never been characterized. The classical papers by Bergen and Hill [BH81] on the structure preserving model of power systems, extended in [NM84, CRP85] to account for effects of the reactive power flows and variable voltages, form the solid ground for our analysis. We study various assumptions under which the energy function is *provably convex*. It is well-known that when all nodes are (P, V) nodes, the energy function is convex [BBC13][BV12] as long as all phase differences are smaller than 90 degrees. In this paper, we generalize this result to the networks with (P, Q) buses (ie variable voltage magnitudes) as well. Our first result applies to a special network structure where (P, Q) nodes are only connected to (P, V) nodes. In this setting, we can prove that the energy function is convex over a reasonable convex domain imposing limits on phase differences and lower bounds on voltage magnitudes. We then generalize to arbitrary topologies, where we provide a nonlinear but convex matrix inequality condition on the voltage phases and magnitudes such that the energy function is convex whenever the condition is satisfied. This condition is an *exact* characterization of the domain of convexity of the energy function for tree-like networks, but in general is an inner approximation of the convexity region for arbitrary networks. It captures the intuition that if voltage magnitudes and voltage phases at neigbhboring nodes are "not too far" from each other, the energy function is convex.

The rest of this paper is organized as follows: In Section 2, we describe the mathematical background and dynamical models. Section 3 is devoted to our (main) analysis of convexity. We first analyze convexity of the energy function jointly in voltage magnitudes and phases. We then describe how the convexity can be used to design a provably convergent algorithm solving the PF equations in Section 3.3. Finally, in section 6,we present conclusions and discuss directions for future work.

2 Modeling Power Systems

The structure preserving model [BH81] is the standard model for analyzing stability of power systems at the transmission (high voltage) level. The model admits rigorous analysis using a Lyapunov function (or energy function) approach under the following realistic assumptions:

(1) All transmission lines are purely reactive (zero resistance), i.e., the network resistive power losses are considered small and thus ignored.

(2) Active and Reactive power loads are constant.

2.1 Notation

The transmission network is modeled as an undirected graph $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ is the set of edges. In power systems parlance, the nodes are called buses and the edges are called lines (transmission lines). We shall use these terms interchangeably in this paper.

The set of vertices includes all load buses as well as buses representing generators (a single bus, or terminal and internal bus per generator). This allows us to develop a unifying framework that works for different load and generator models. Edges correspond to transmission lines. Each transmission line is characterized by its admittance $G_{ij} + \mathbf{j}B_{ij}$. However, since we assume that transmission lines are lossless, we set $G_{ij} = 0$ and each line is characterized by a single parameter B_{ij}, its susceptance.

If there is an edge between buses i and j, we write $i \sim j, j \sim i$.

Let V_i, θ_i, P_i and Q_i denote respectively voltage (magnitude), phase, active and reactive injection (with the convention that consumption is positive) at the bus i. Let $\rho_i = \log(V_i)$. Buses are of three types:

- (P, V) buses where active power injection and voltage are held constant, while voltage phase and reactive power adjust as conditions (e.g. power flows) change. The set of (P, V) buses is denoted by pv.
- (P, Q) buses where active and reactive powers are held constant, while voltage phase and magnitude are variable. The set of (P, Q) buses is denoted by pq.
- Slack bus, a reference bus at which the voltage magnitude and phase are fixed, and the active and reactive power injections are free variables. The slack bus is denoted by $\mathcal{S}$.

θ and V (with no subscript) denote the vectors of phases and voltages at all the buses. v denotes the vector of voltage setpoints at (P, V) buses and V_{pq} is the vector of voltage magnitudes at the (P, Q) buses. For submatrices and subvectors, we will use the same indices as the original vectors, that is $(V_{\text{pq}})_i = V_i$ for $i \in \text{pq}$. We will also work with $\rho_i = \log(V_i)$. Let $A \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{E}|}$ be defined as follows:

$$A_{ik} = \begin{cases} 1 \text{ if } k = (i, j), j \in \mathcal{V} \\ -1 \text{ if } k = (j, i), j \in \mathcal{V} \\ 0 \text{ otherwise} \end{cases}$$

Further, we assume that an arbitrary orientation is picked so that only one of $(i, j), (j, i)$ is in $\mathcal{E}$. For every edge between $i \in \text{pq}$, $j \in \text{pv}$, we assume that $(i, j) \in \mathcal{E}$ (so edges are always from (P, Q) to (P, V) nodes). For every bus i, define

$$B_i = \sum_{j \neq i} B_{ij} \tag{1}$$

If $k = (i, j)$ is an edge, we will also denote by $\rho_{k1} = \rho_i$, $\rho_{k2} = \rho_j$, the log-voltage magnitudes at the two ends of the line k. Let S be an ordered set of indices. We use the notation

$$M = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]_{ij}^{S}$$

where $i, j \in S$ to denote an $|S| \times |S|$ matrix whose rows and columns are indexed by the entries of S, with $M_{ii} = a, M_{ij} = b, M_{ji} = c, M_{jj} = d$ and all other entries equal to 0. Similarly, we use the notation $M = [a]_{ii}^{S}$ to denote an $|S| \times |S|$ matrix with the $(i, i) - th$ entry equal to a and all other entries equal to 0.

Given a vector x with indices $i \in S$, denote by $\text{diag}(x)$ the $|S| \times |S|$ matrix with diagonal entries x_i. We also denote the vector x by $\{x_i\}$.

2.2 Background

In this section, we introduce two power systems concepts that are critical to this paper: Power Flow Equations and Energy functions.

2.2.1 Power Flow Equations

Power Flow Equations model the flow of power in a power system. They are a set of coupled nonlinear equations that follow from Kirchoff's laws applied to the AC power network. Circuit elements in the standard power systems models are all linear, if one ignores discrete elements like phase shifters and tap-changing transformers. However, the power flow equations differ from standard circuit equations in one important way: Power Injections, rather than current injections, are fixed in the power flow equations. Since power is voltage times current, the resulting equations are nonlinear.

Power Flow Equations assume sinusoidal voltage profiles with the same frequency at all nodes in the network. This relies on the assumption that at the time-scale at which power flow equations are solved (every 5 minutes or so), the system is in quasi-steady state (the dynamic disturbances have been resolved by the AVRs and primary and secondary frequency control).

Since our main concern here is to study properties of the power flow equations, we simply state them without derivation. At each node in the power system, there are four variables: Voltage magnitude (V_i), Voltage phase (θ_i), Active Power Injection (P_i), Reactive Power Injection(Q_i). When solving power flow equations, at each node, two of these variables are fixed and the other two are free. This leads to $2n$ equations in $2n$ variables. Typically, power systems have two types of nodes: (P, V) nodes and (P, Q) nodes.

At (P, V) buses, the active power injection P_i and the voltage magnitude $V_i = \exp(\rho_i)$ are fixed. The fixed voltage magnitude is denoted v_i, to distinguish it from variable voltages at other nodes. Generators are typically modeled as (P, V) buses, since they have voltage regulators that inject reactive power to maintain voltages at fixed values. At (P, V) buses, we write the following equations:

$$P_i = \sum_{j \sim i} B_{ij} \exp(\rho_i + \rho_j) \sin(\theta_i - \theta_j) \ \text{(Active Power Balance)}$$

$$\rho_i = \log(v_i) \ \text{(Voltage Setpoint)}$$

Further, we can choose $v_i = 1$ for each $i \in \text{pv}$, since any non-unit voltage set-point can be absorbed into the susceptances B_{ij} in the active and reactive power flow equations.

At (P, Q) buses, the active power injection P_i and reactive power injection Q_i are held fixed. Loads are typically (P, Q) nodes, where P_i and Q_i represent the active and reactive power demands (assumed constant). At (P, Q) buses, we write the following equations:

$$P_i = \sum_{j \sim i} B_{ij} \exp(\rho_i + \rho_j) \sin(\theta_i - \theta_j) \ \text{(Active Power Balance)}$$

$$Q_i = \sum_{j \sim i} B_{ij} \left(\exp(\rho_i + \rho_j) \cos(\theta_i - \theta_j) - \exp(2\rho_i)\right) \ \text{(Reactive Power Balance)}$$

The variable Q_i at the (P, V) buses is usually not considered explicitly, they can be obtained simply via equation (2c) by plugging in the values of (ρ, θ). Finally, we have a special bus (slack bus) which absorbs all the power imbalances in the network:

$$\theta_{\mathcal{S}} = 0, \rho_{\mathcal{S}} = 0$$

At this node, P_i, Q_i are free variables whose values are adjusted to absorb the net imbalance in the network.

All of these equations taken together constitute the power flow equations:

$$P_i = \sum_{j \sim i} B_{ij} \exp(\rho_i + \rho_j) \sin(\theta_i - \theta_j) \quad \forall i \in \text{pv} \cup \text{pq} \tag{2a}$$

$$\rho_i = 0 \quad \forall i \in \text{pq} \tag{2b}$$

$$Q_i = \sum_{j \sim i} B_{ij} \left(\exp(2\rho_i) - \exp(\rho_i + \rho_j) \cos(\theta_i - \theta_j)\right) \quad \forall i \in \text{pq} \tag{2c}$$

$$\theta_{\mathcal{S}} = 0, \rho_{\mathcal{S}} = 0 \tag{2d}$$

The equations equations (2a–2d) are together called the power flow equations, and constitute a set of coupled nonlinear equations in (ρ, θ).

2.2.2 Energy Functions

Energy functions were first derived based on a first integral analysis of the power system swing dynamics [BH81][CRP85], which are essentially dynamical versions of the power flow equations. They consist of two terms: The Kinetic and Potential terms, the sum of which is a Lyapunov function for the swing dynamics, and can be used for transient stability analysis. In this paper, we are not explicitly concerned with dynamics, so the most useful property of energy functions is that stationary points of the potential energy function are power flow solutions. More concretely, the equations equations (2a–2d) can be re-written in the following variational form

$$(2a) : 0 = \frac{\partial E}{\partial \theta_i} \quad \forall i \in \text{pq} \cup \text{pv} \tag{3a}$$

$$(2c) : 0 = \frac{\partial E}{\partial \rho_i} \quad \forall i \in \text{pq} \tag{3b}$$

$$\rho_i = 0 \quad \forall i \in \text{pv}, \rho_{\mathcal{S}} = \theta_{\mathcal{S}} = 0 \tag{3c}$$

where E is called the energy function and is given by:

$$E(\rho, \theta) = -\sum_{i \in \text{pv} \cup \text{pq}} P_i \theta_i - \sum_{i \in \text{pq}} Q_i \rho_i + \frac{1}{2} \sum_{i \in \mathcal{V}, j \neq i} B_{ij} \left(\frac{\exp(2\rho_i) + \exp(2\rho_j)}{2} - \exp(\rho_i + \rho_j) \cos(\theta_i - \theta_j) \right) \tag{4}$$

Note that ρ_i is a variable for $i \in \text{pq}$ and $\rho_i = 0$ for $i \in \text{pv}$, $\rho_{\mathcal{S}} = 0$.

3 Energy Function and Convexity

In this section, we look at convexity of the energy function (4). We first discuss why convexity is an interesting and useful property, and then describe our results characterizing the domain of convexity of the energy function (4). Before we go further, we define the domain of convexity precisely.

Definition 1. The domain of convexity of the energy function is defined as:

$$\mathcal{D} = \left\{ (\rho, \theta) : \nabla^2 E(\alpha\rho, \alpha\theta) \succeq 0 \quad \forall \alpha \in [0, 1] \right\}$$

Remark 1. In general, there could be several (disconnected) regions over which the energy function is convex. We require that any $(\rho, \theta) \in \mathcal{D}$, the energy function is convex at every point on the line segment connecting the origin to (ρ, θ). This ensures that we pick the region of convexity that includes the origin.

3.1 Why is Convexity Interesting?

The major contribution of this paper is a characterization of the domain of convexity of the energy function (4). We now discuss why this characterization is important and interesting. The principal reason for this is that solutions to power flow equations are stationary points of the energy function (3b). We will show, in section 3.3, that we can construct a convex optimization problem that will either find a solution of the power flow equations within the convexity domain, or certify that there are no solutions within the convexity domain. In this section, we justify restricting ourselves to power flow solutions contained in $\mathcal{D}$, as solutions in $\mathcal{D}$ have some additional desirable properties.

3.1.1 Asymptotic Stability

The dynamics of the power system, under certain reasonable assumptions, reduce to the so-called "swing dynamics", a subject of study of several papers (see [VWC85]). In this model, each generator is modeled as a constant voltage source behind a transient reactance. For each generator, there is an "internal" node (a constant voltage source with second order angular dynamics) and a "terminal" node (a node with 0 active and reactive power injections). We denote the set of internal nodes by pv_0. At the internal nodes $i \in \text{pv}_0$, we have:

$$M_i \ddot{\theta}_i + D_i \dot{\theta}_i = -\frac{\partial E(\rho, \theta)}{\partial \theta_i} \tag{5}$$

At all other nodes $i \in \text{pv} \cup \text{pq}$, we have the *algebraic* active and reactive power balance equations equations (2a–2c). The dynamic state variables are $\{\dot{\theta}_i : i \in \text{pv}_0\} \cup \{\theta_i : i \in \text{pv}_0\}$.

Theorem 3.1. *Let (ρ,θ) be an equilibrium of the swing dynamics such that $(\rho,\theta)\in\mathcal{D}$ and satisfy all the algebraic equations* (2a),(2b) *and* (2c). *Then, (ρ,θ) is asymptotically stable, that is, the linearization of the dynamics* (5) *around (ρ,θ) is a stable linear system.*

Proof. This follows from of lemma 5.1 in [TAV85], which is a stronger result. □

3.1.2 Existence of Solutions

In a recent unpublished manuscript [DM], we have shown the following result:

Theorem 3.2. *Suppose that the network is a tree. Then, there is a solution to the power flow equations* (2a),(2b) *and* (2c) *if and only if there is a solution within $\mathcal{D}$.*

Thus, at least for trees, it is sufficient to look for solutions to the power flow equations contained in $\mathcal{D}$. Further, as we shall illustrate in the numerical results section 5, the solutions within $\mathcal{D}$ are the "desirable" solutions, since all other solutions have voltage magnitudes at one or more of the (P,Q) buses to be very low. Further, preliminary numerical evidence seems to suggest that this is true for non-tree networks as well.

3.2 Characterization of Domain of Convexity of the Energy Function

We now derive our results on convexity of energy functions. Since stationary points of energy functions correspond to solutions of the power flow equations and power flow equations are known to have multiple isolated solutions, the energy function is not globally convex. However, there are domains over which the energy function is convex. We choose to look for a domain such that the phase differences are smaller that 90 deg, since this is almost always true in practical power systems operations. This is sufficient for convexity if all buses are (P,V) buses, as observed in [BBC13]. When there are (P,Q) buses as well, additional conditions are required.

We now derive a nonlinear convex semidefinite constraint on (ρ,θ) that characterizes the domain over which the energy function is convex.

Theorem 3.3. *The energy function $E(\rho,\theta)$ is jointly convex in (ρ,θ) over the convex domain $\mathcal{C}\subset\mathcal{D}$ given by*

$$|\theta_i-\theta_j|\leq\frac{\pi}{2}\quad\forall(i,j)\in\mathcal{E} \tag{6}$$

$$\sum_{i\in\mathrm{pq}}\left[2B_i-\sum_{j\in\mathrm{pv}\cup\mathcal{S}}B_{ij}\frac{\exp(\rho_j-\rho_i)}{\cos(\theta_i-\theta_j)}\right]_{ii}^{\mathrm{pq}}-\sum_{(i,j)\in\mathcal{E},i,j\in\mathrm{pq}}\frac{B_{ij}}{\cos(\theta_i-\theta_j)}\left[\begin{pmatrix}\exp(\rho_j-\rho_i) & 1\\ 1 & \exp(\rho_i-\rho_j)\end{pmatrix}\right]_{ij}^{\mathrm{pq}}\succeq 0 \tag{7}$$

Further, suppose that the network has a tree topology. Then, the domain of convexity of the energy function $\mathcal{D}$ equals $\mathcal{C}$.

Remark 2. *Simple special cases*
When no (P,Q) nodes are connected to each other, the second term of the semidefinite inequality (7) is an empty sum. Thus, the semidefinite inequality is equivalent to

$$2B_i=2\left(\sum_{j\in\mathrm{pv}}B_{ij}\right)\geq\sum_{j\in\mathrm{pv}}B_{ij}\frac{v_j\exp(-\rho_i)}{\cos(\theta_i-\theta_j)}$$

Since $B_i=\sum_j B_{ij}$, it suffices that $|v_i|\exp(-\rho_j)\leq 2\cos(\theta_i-\theta_j)$ for this to be true, which is satisfied if $V_j\geq .7v_i$, $|\theta_i-\theta_j|\leq 45\,\mathrm{deg}$. This is a fairly reasonable restriction in practice. A similar condition was discussed in [TAV85]. However, it was imposed only at the equilibrium point where the energy function reaches its minimum, as a sufficient condition to guarantee (small deviation) asymptotic stability. The asymptotic stability follows from our results as well, which guarantees that the equilibrium point lies within the domain of convexity of the energy function and it is hence asymptotically stable.

Also, if there are no (P,Q) nodes, the matrix inequality condition in (7) is empty and hence the convexity condition reduces to requiring all phases differences to be smaller than $\frac{\pi}{2}$.

Remark 3. *Conservatism*
For tree networks, we have shown that are conditions for convexity are necessary and sufficient (assuming we are interested only in the domain with phases differences on all lines smaller than $\frac{\pi}{2}$). For meshed networks, we our conditions are sufficient but may not be necessary. This is essentially because our analysis of convexity treated the edge variables θ_{ij} as independent for each edge $(i, j) \in \mathcal{E}$ while in reality they are coupled since $\theta_{ij} = \theta_i - \theta_j$. Thus, we obtain a sufficient, but not necessary condition for convexity. Investigation of the gap is left for future work.

3.3 Solving Power Flow Equations via Convex Optimization

Although energy function was developed initially as a tool to assess dynamic stability, the convexity results can also be used in a number of other important power system applications. We choose to emphasize here one of these applications: Solution of the power flow equations.

As we have already seen, the energy function approaches provide a variational characterization of the power flow equations equations (3a–3b). Thus, finding a solution to the power flow equations is equivalent to finding a stationary point of the energy function.

Corollary 1. *Let $S \subset \mathrm{Int}\mathcal{C}$. Then, the power flow equations Equations (2a–2d) have a solution $(\rho^*, \theta^*) \in S$ if and only if the following optimization problem has its (unique) optimal solution in S:*

$$\min_{(\rho,\theta)\in\mathcal{C},\theta_{\mathcal{S}}=0,\rho_{\mathcal{S}}=0} E(\rho, \theta) \tag{8}$$

Proof. If $(\rho^*, \theta^*) \in S$ solves equations (2a–2d), it satisfies $\nabla_\rho E(\rho^*, \theta^*) = 0, \nabla_\theta E(\rho^*, \theta^*) = 0$. Thus, it also satisfies the KKT conditions for the optimization problem (8), choosing the respective Lagrange multipliers to be 0. Hence, the solution is optimal for (8).

Conversely, suppose that $(\rho^*, \theta^*) \in S \subset \mathrm{int}(\mathcal{C})$ solves (8). Then, by complementary slackness, the Lagrange multipliers are 0 and hence the KKT conditions reduce to $\nabla_\rho E(\rho, \theta) = 0, \nabla_\theta E(\rho, \theta) = 0$. Thus, $(\rho^*, \theta^*) \in \mathcal{C}$ also solves equations (2a–2d).

Further, by strict convexity of the energy function $E(\rho, \theta)$ on $\mathrm{Int}(\mathcal{C})$, there can be at most one stationary point of E in $\mathrm{Int}(\mathcal{C})$, and hence, the power flow solution, if it exists, is unique. □

4 Related Work

In this section, we discuss and contrast our work with related previous work.

4.1 Principal Singular Surfaces

The work closest in spirit to our approach is the series of papers [TS72c][TS72b][TS72a]. The authors in those papers consider lossless networks where all buses are (P, V) buses. They define Singular surfaces, which are closed surfaces on which $\det(\nabla^2 E) = 0$. The Principal Singular Surface is the unique such surface that encloses the origin $\theta = 0$ and the Principal Region is the region enclosed by it. The principal region coincides with our definition of $\mathcal{D}$ when all nodes are (P, V) nodes, apart from the additional requirement that phase differences are smaller than $\frac{\pi}{2}$. Our results characterizing $\mathcal{C}$ can be seen as constructing convex inner approximations to the principal region.

Several conjectures were made in these works regarding the nature of principal regions and the set of active and reactive power injections that correspond to power flow solutions in these set: Specifically that both these sets are convex. It would be interesting to see if these conjectures are true in our setting as well, with (P, Q) buses.

4.2 Convex Relaxations of OPF

A series of recent works [LTZ13][ZT13][LTZ13][Low14b][Low14a] have studied *convex relaxations* of Optimal Power Flow for tree networks. They show various conditions under which the convex relaxations are exact, in the sense that an optimal solution of the original OPF problem can be recovered from the solution to the convex relaxation. In this work, we primarily deal with the power flow equations, not optimal power flow. However, for lossless networks, we can deal with arbitrary topologies. Thus, our results are more applicable to transmission networks with mesh topologies and higher grade lines with low resistance to inductance ratios. Our approach deals primarily with the power flow equations and energy function-based descriptions of these. However, the approach can potentially be extended to optimal power flow, in the following manner:

$$\max_{P,Q\in T}\min_{(\rho,\theta)\in S} -\sum_{i\in\text{pv}\cup\text{pq}} P_i\theta_i - \sum_{i\in\text{pq}} Q_i\rho_i + \frac{1}{2}\sum_{i\in\mathcal{V},j\neq i} B_{ij}\left(\frac{\exp(2\rho_i)+\exp(2\rho_j)}{2} - \exp(\rho_i+\rho_j)\cos(\theta_i-\theta_j)\right) - \lambda\ell(P,Q)$$

where $\ell_t(P,Q)$ is a convex cost on the injections and T, S are convex operational constraint sets on the injections and voltages, respectively and $\lambda > 0$ is a positive scaling factor. This is a convex-concave saddle point problem, as long as $S \subset \mathcal{C}$. If the constraints in S are not binding at the optimal solution, the inner minimization effectively enforces the power flow equations equations (2a–2d), since the minimization corresponds to setting the gradient of the energy function to 0. The outer maximization then maximizes the negative injection cost plus the minimized energy function value. We can choose λ large enough so that the the energy function term can be neglected for the outer maximization, so that we are effectively solving the OPF problem:

$$\begin{gathered}\min_{P,Q,\rho,\theta} \ell(P,Q) \\ \text{Subject to equations (2a–2d)} \\ (P,Q)\in T \\ (\rho,\theta)\in S\end{gathered}$$

Further developments on this line of work, along with conditions when our minimax formulation is equivalent to the OPF problem above, will be pursued in subsequent work.

4.3 Conditions on Existence of solutions to PF equations

Several papers have studied conditions for existence of solutions to the Power Flow Equations [BZ14b][LSP99][WK82]. In [MLD12], the authors propose a sufficient condition for the insolvability of power flow equations based on a convex relaxation. However, our approach differs from these in the following important ways:

- For solutions to the power flow equations in $\mathcal{C}$, we provide necessary and sufficient conditions, that is, our approach finds a solution in $\mathcal{C}$ if and only if there exists a solution in $\mathcal{C}$.
- Our approach is algorithmic, that is, we provide an algorithm (based on convex optimization) that is guaranteed to find the solution efficiently (in polynomial time).
- If there are additional operational constraints $(\rho,\theta) \in S$, with $S \subset \mathcal{C}$, we can additionally answer the question of whether there exists a power flow solution with $(\rho,\theta) \in S$. This is an important contribution, since most of the time system operators are interested in finding power flow solutions that satisfy additional operational constraints.

In [BZ14b], the authors also propose an algorithm based on a contraction mapping result. However, the algorithm only works in a small ball around the 0 in the (P,Q) space. Our results are stated in terms of a nonlinear convex constraint in (ρ,θ) space. Precisely understanding the set of (P,Q) for which the solution $(\rho,\theta) \in \mathcal{C}$, and conversely the set of (ρ,θ) for which (P,Q) lies in a certain ball, is a subject of future work. It was conjectured in [TS72b] that the set of (P,Q) corresponding to solutions $(\rho,\theta) \in \mathcal{D}$ is a convex set on the basis of 3 and 4 bus examples. However, it remains to be seen if this is true in general. This is an interesting direction for future work.

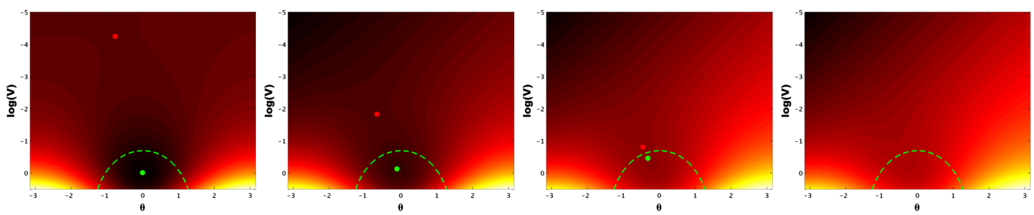
5 Numerical Illustrations

We present numerical illustrations of the results of this paper in this section.

5.1 Existence of Solutions: 2-bus Network

In this section, we study a 2 bus network with one generator and one load. The generator also serves as the slack bus. Thus, the only variables are the voltage phase θ and voltage magnitude $\exp(\rho)$ at the load bus, and the parameters are the active power demand P and reactive power demand Q at the load bus. The results, plotted in figures figs. 1a to 1d show the power flow solutions plotted on a heat map of the energy function. There are two solutions for small loads. The convexity region us plotted as a dashed green arc (the convexity region is the region inside the arc). The solution plotted as a green dot is a dynamically stable solution in the interior of the convexity region. The solution plotted as a red dot is a dynamically unstable solution outside the convexity region. As the loads are increased, the two solutions move closer to each other. At a certain critical value, they hit the boundary of the convexity region and then vanish, that is, beyond this loading level, there are no longer any solutions to the power flow equations.

This supports our conjecture that in general, if there is a solution to the power flow equations, there must be one within the convexity region. As stated earlier, we have proven this result for networks with a tree topology [DM]. For the special case when all nodes are (P, V) buses, again, this was stated without proof in [TS72b]. Verification of this conjecture is a direction of future work for us.



(a) $P = Q = .01$ (b) $P = Q = .1$ (c) $P = Q = .2$ (d) $P = Q = .25$

Figure 1: Solutions to Power Flow Equations, as Load Increases are plotted on a Heat Map of the Energy Function. Two distinct solutions exist initially, and they move closer as the load increases. They approach the boundary of the convexity region from opposite sides and disappear once they hit the boundary. The green dot is a dynamically stable solution inside the convexity region, and the red dot is a dynamically unstable solution outside the convexity region.

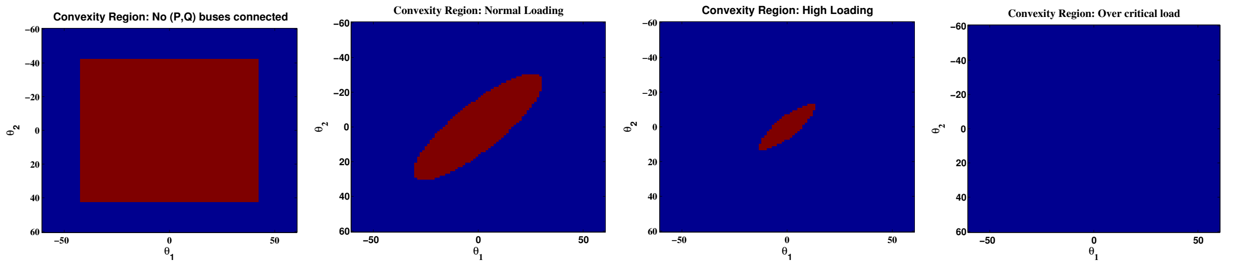
5.2 Theoretical versus Actual Region of Convexity

In this section, we compare numerically these different regions of energy function and reduced energy function convexity. In order to be able to visualize these regions, we restrict ourselves to a 3 bus system with one PV bus and two (P, Q) buses. Bus 1 is the PV bus (also the reference bus): Its voltage magnitude V_1 is fixed to 1 pu and its voltage phase is taken as a reference $\theta_1 = 0$. Buses 2 and 3 are (P, Q) buses with variable voltage magnitude $V_2 = \exp(\rho_2), V_3 = \exp(\rho_3)$ and voltage phase θ_2, θ_3. V_2, V_3 are determined by solving the reactive power flow equations

$$Q_2 = B_{12}\left(V_2\cos(\theta_2) - V_2^2\right) + B_{23}\left(V_2V_3\cos(\theta_2-\theta_3) - V_2^2\right)$$
$$Q_3 = B_{13}\left(V_3\cos(\theta_3) - V_3^2\right) + B_{23}\left(V_2V_3\cos(\theta_2-\theta_3) - V_3^2\right)$$

We choose reasonable values for all the parameters involved. The active power injections are $P_1 = 3.7, P_2 = -1.7, P_3 = -2$ (these numbers are normalized by the baseMVA so that they can directly be plugged in to the equations) and the reactive power demands are $Q_2^0 = 1.05, Q_3^0 = 1.24$. The susceptances on the lines are $B_{12} = 26.88, B_{13} = 26.88$. We consider two cases for B_{23} that lead to different network topologies: In the first case buses 2, 3 are not connected so $B_{23} = 0$ and in the second case we choose $B_{23} = 16.67$.

For different values of $\theta_2, \theta_3 \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$, we compute the value of the energy function E by solving the reactive power flow equations for V_2, V_3 (using Newton Raphson) and then plug the results into $E(V, \theta)$. For any value of θ_2, θ_3, we also estimate the Hessian $\nabla_\theta^2 E(V(\theta), \theta)$ through numerical differentiation. This gives us the exact region over which the reduced energy function, $E(V(\theta), \theta)$, is a convex function of θ. We compare this to the domain $\mathcal{C}$ of the energy function (and thus reduced energy function) convexity predicted by our theorems. Since the theorems only predict joint convexity of $E(\rho, \theta)$, we deduce reduced convexity by finding the set of θ such that $(\rho(\theta), \theta) \in \mathcal{C}$.



(a) No (P,Q) connected (b) Regular Load (c) High Load (d) Loading over critical

Figure 2: Actual vs Estimated Regions of Convexity. The dark blue represents the region of non-convexity, the brown represents the region of predicted convexity (which coincides with actual convexity for all cases). One can see that as loading increases, the region of convexity shrinks until it vanishes, meaning that there is no stable power flow solution within the specified domain.

Fig. 2 show the actual and predicted regions of the reduced energy function convexity. For the 3-bus case considered here, our theoretical results predict exactly the region of convexity. In figure 2a, we plot the convexity region for the case $B_{23} = 0$. In all the other figures, we have $B_{23} > 0$ but we scale the injections up. In figure 2b, we scale the injections (active and reactive) by a factor of 3. In figure 2c, we scale the injections by a factor 5.5. In figure 2d. we scale the injections by a factor 6. As the loading increases, the region of convexity shrinks and eventually we reach a critical load at which the region of convexity becomes empty. At this point, there is no longer a stable power flow solution within the given domain, since the energy function must be convex around a stable equilibrium point. These results show that, at least for the 3-bus case, our theorems give an exact description of the region of convexity.

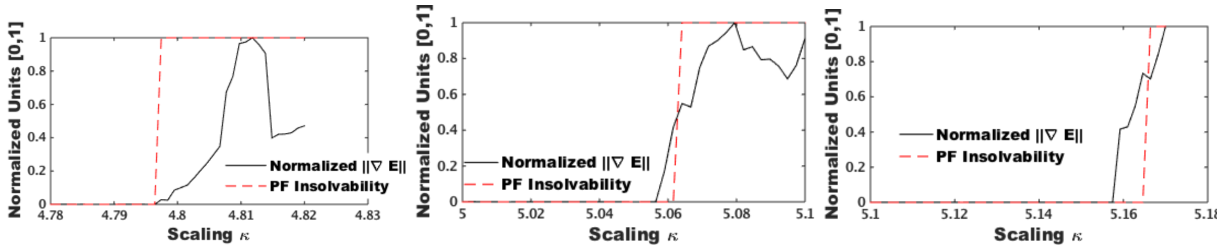
5.3 Power Flow In Larger Networks: Existence of Solutions

In section 3.1.2, we argued that one justification for seeking solutions only in the domain of convexity of the energy function is that there is evidence to suggest that if there are no solutions in the Convexity Domain, then there are no solutions anywhere. We have a proof of this for the tree case [DM] but we believe that the result is possibly true more generally. In order to examine this hypothesis, we studied power flow solutions for two IEEE test networks: The IEEE 14 bus network and the IEEE 118 bus network. We first modify these networks to be lossless (setting conductances to 0 for all transmission lines). We start with the nominal base load profiles for these networks and gradually scale up the active power injections by a factor κ at all nodes (except the slack bus) and the reactive power injections by a factor $\delta\kappa$ at all (P, Q) buses.

In order to verify insolvability of power flow equations, we use the technique developed in [MLD12] along with the code implementing this technique in the MATPOWER package [ZMST11]. For a fixed value of δ, we increase κ until the technique from [MLD12] detects insolvability of the power flow equation. Our convex solver fails to find a power flow solution when the optimal solution to (8) lies on the boundary of $\mathcal{C}$ or the norm of the gradient of energy function with respect to (ρ, θ) is non-zero at the optimum. As κ increases, we plot the norm of the gradient of the energy function at optimum (a non-zero value indicates that there are no solutions in the convexity domain) and the value of the insolvability test (1 if insolvability detected and 0 otherwise).

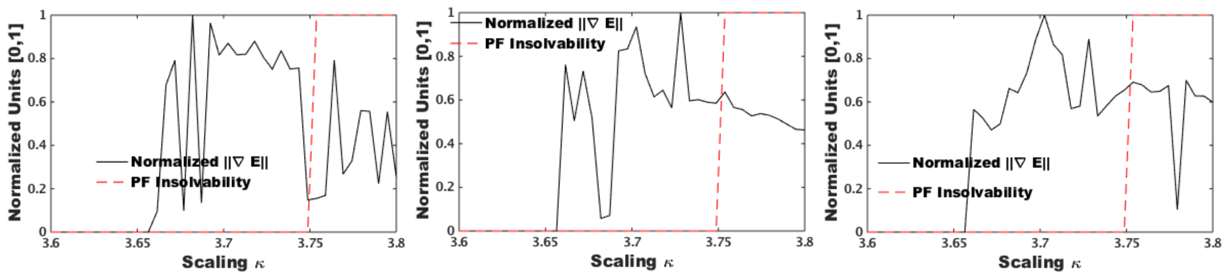
For the IEEE 14 bus test case, results are plotted in figures 3a,3b and 3c. One can observe that for $\delta = 1$, the scaling at which our convex approach fails to find a solution is exactly the point at which the technique from [MLD12] detects insolvability. For smaller values of δ, there is a slight gap between the two, but the values are still fairly close.

For the IEEE 118 bus test case, results are plotted in figures 3a,3b and 3c. Once gain, we can see that the point at which [MLD12] detects insolvability is close to the point at which our convex approach fails to



(a) $\delta = 1$ (b) $\delta = .5$ (c) $\delta = .1$

Figure 3: IEEE 14 Bus Case: Existence of Solutions as Load Scales



(a) $\delta = 1$ (b) $\delta = .5$ (c) $\delta = .1$

Figure 4: IEEE 118 Bus Case: Existence of Solutions as Load Scales

find a power flow solution. The gap here is larger than the 14 bus case, but still quite marginal (difference of .1 in κ, which amounts to a 2.5% difference in the actual injections).

The results overall show that our convex approach finds solutions to the power flow equations in almost all cases when a solution exists. There is a potential gap close to the boundary of insolvability of the power flow equations. However, in all the cases we tested, Newton-Raphson based approaches implemented in MATPOWER also failed to find a solution in these test cases. We conjecture that the gap is due to the fact that the test from [MLD12] is only a sufficient condition for insolvability but not necessary.

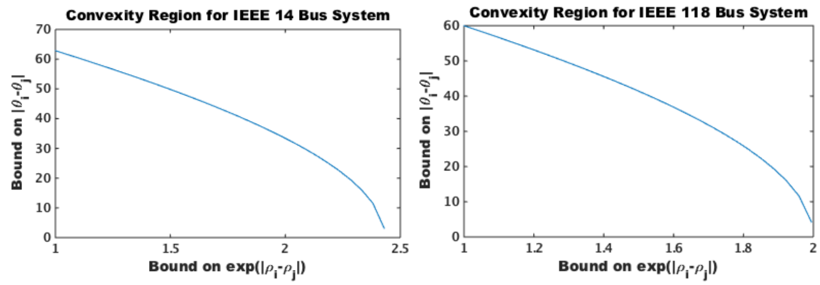
5.4 Operational Constraints and Convexity

The convexity condition (6)(7) is a complicated nonlinear matrix inequality. In this section, we describe simplified versions of the restrictions placed by this inequality on phase differences and differences in log-voltages across lines. We fix a bound on the log-voltage differences $\exp(|\rho_i - \rho_j|)$ and study how large the phase differences can be. Its easy to see that phase differences may be different for different lines, so we look at the tightest bound.

For the 14 bus case, the results are plotted in figure 5a. The results show that if we allow voltage deviations across lines to be $\exp(\rho_i - \rho_j) = \frac{V_i}{V_j} \leq 1.5$, the phase differences can still be as large as 50 deg. This is very reasonable for practical power systems, where these bounds are seldom exceeded.

For the 118 bus case, the results are plotted in 5b. Again, if we allow for voltage differences of upto $\frac{V_i}{V_j} \leq 1.5$, phase differences can still be as large as 45 deg.

The results of these sections show that with fairly lax constraints on the voltage magnitude ratios and voltage phase differences across neighboring buses, we are guaranteed to lie in the domain of convexity. In other words, most practical solutions to the power flow equations would lie inside the domain of convexity.



(a) IEEE 14 Bus Case (b)

Figure 5: Convexity Region

6 Conclusions and Path Forward

In this manuscript we have presented novel descriptions of the domain of convexity of the energy function. Since the energy function provides a variational description of the power flow equations, this allows us to exploit develop a convex optimization based algorithm to solve the power flow equations. This enables various applications, including detecting the existence of solutions to the power flow equations with a given injection profile. There are several other applications we have not discussed in detail here: Optimal Power Flow, State and Topology Estimation, Distance to Insolvability, Optimal Load Shedding etc. All of these applications along with efficient algorithms to solve the result.

Another direction for future work is to extend these results to lossy networks. As one concrete extension, we describe a result for lossy networks where all transmission lines have a fixed resistance to reactance ratio (fixed $\frac{R}{X}$). Once this is done, one can exploit continuity arguments to establish existence of solutions for networks with R, X values in a neighborhood of these networks with fixed $\frac{R}{X}$ values. It should also be possible to quantify the degree of deviations that can be tolerated, and this is a promising direction that will be pursued in future work.

7 Acknowledgments

This work has emerged from discussions in July of 2014 at Los Alamos with Scott Backhaus and Ian Hiskens, whom the authors are thankful for guidance, comments, encouragement and criticism. We thank Dan Molzahn and Enrique Mallada for helpful comments on various aspects of this work. The work at LANL was carried out under the auspices of the National Nuclear Security Administration of the U.S. Department of Energy at Los Alamos National Laboratory under Contract No. DE-AC52-06NA25396. This material is based upon work partially supported by the National Science Foundation award # 1128501, EECS Collaborative Research "Power Grid Spectroscopy" under NMC.

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Appendix

Proof of Theorem 3.3

Proof. We introduce an independent edge variable θ_k for each edge $k \in \mathcal{E}$. Let $\theta_{\mathcal{E}} = \{\theta_k : k \in \mathcal{E}\}$. We then have $\theta_{\mathcal{E}} = A\theta$. If the energy function is convex jointly in $(\rho, \theta, \theta_{\mathcal{E}})$, then it must be jointly convex in (ρ, θ). Writing the energy function in terms of $(\rho, \theta, \theta_{\mathcal{E}})$, we get

$$E(\rho, \theta, \theta_{\mathcal{E}}) = -\sum_{i \in \text{pq} \cup \text{pv}} P_i \theta_i - \sum_{i \in \text{pq}} Q_i \rho_i + \sum_k B_k \left(\frac{1}{2} \left(\exp(2\rho_{k1}) + \exp(2\rho_{k2}) \right) - \exp(\rho_{k1} + \rho_{k2}) \cos(\theta_k) \right)$$

Only the first term depends explicitly on θ and is linear and hence convex. Hence, convexity of the energy function reduces to convexity of the second and third terms. To study this, we study the Hessian which can be broken into 4 sub-matrices:

$$\begin{pmatrix} \nabla^2_{\rho} E(\rho, \theta_{\mathcal{E}}) & \nabla^2_{\rho, \theta_{\mathcal{E}}} E(\rho, \theta_{\mathcal{E}}) \\ \left(\nabla^2_{\rho, \theta_{\mathcal{E}}} E(\rho, \theta_{\mathcal{E}}) \right)^T & \nabla^2_{\theta_{\mathcal{E}}} E(\rho, \theta_{\mathcal{E}}) \end{pmatrix} = \begin{pmatrix} M & N \\ N^T & R \end{pmatrix}$$

where $M \in \mathbb{R}^{|\text{pq}| \times |\text{pq}|}, N \in \mathbb{R}^{|\text{pq}| \times |\mathcal{E}|}, R \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{E}|}$. For this to be positive semidefinite, we require that $R \succeq 0, M - NR^{-1}N^T \succeq 0$. It is easy to see that R is diagonal, with $R_{kk} = B_k \exp(\rho_{k1} + \rho_{k2}) \cos(\theta_k)$. Further,

$$N_{i,k} = |A_{ik}| B_k \exp(\rho_{k_1} + \rho_{k_2}) \sin(\theta_k), i \in \text{pq}, k \in \mathcal{E}$$

$$M_{i,j} = -\sum_k |A_{ik}||A_{jk}| B_k \exp(\rho_{k_1} + \rho_{k_2}) \cos(\theta_k), i, j \in \text{pq}, i \neq j$$

$$M_{i,i} = 2B_i \exp(2\rho_i) - \sum_k |A_{ik}| B_k \exp(\rho_{k_1} + \rho_{k_2}) \cos(\theta_k), i \in \text{pq}$$

Let $O = M - NR^{-1}N^T \in \mathbb{R}^{|\text{pq}| \times |\text{pq}|}$. Then,

$$\begin{aligned} O_{ij} &= -\sum_k |A_{ik}||A_{jk}| B_k \exp(\rho_{k_1} + \rho_{k_2}) \left(\cos(\theta_k) + \frac{\sin^2(\theta_k)}{\cos(\theta_k)} \right) \\ &= -\sum_k |A_{ik}||A_{jk}| B_k \exp(\rho_{k_1} + \rho_{k_2}) \frac{1}{\cos(\theta_k)} \end{aligned}$$

for $i \neq j$ and

$$\begin{aligned} O_{ii} &= 2B_i \exp(2\rho_i) + Q_i'(\exp(\rho_i)) \exp(\rho_i) - \sum_k |A_{ik}| B_k \exp(\rho_{k_1} + \rho_{k_2}) \left(\cos(\theta_k) + \frac{\sin^2(\theta_k)}{\cos(\theta_k)} \right) \\ &= 2B_i \exp(2\rho_i) - \sum_k |A_{ik}| B_k \exp(\rho_{k_1} + \rho_{k_2}) \frac{1}{\cos(\theta_k)} \end{aligned}$$

Note also that

$$L_{ij} = \frac{O_{ij}}{\exp(\rho_i)\exp(\rho_j)} = -\sum_k |A_{ik}||A_{jk}|B_k \frac{1}{\cos(\theta_k)} \quad \text{for} i \neq j$$
$$L_{ii} = \frac{O_{ii}}{\exp(2\rho_i)} = 2B_i - \sum_k |A_{ik}| \exp(\rho_{k_1} + \rho_{k_2} - 2\rho_i) B_k \frac{1}{\cos(\theta_k)}$$

Note that $L \succeq 0$ if and only if $O \succeq 0$ since $L = \operatorname{diag}(\exp(-\rho)) O \operatorname{diag}(\exp(-\rho))$. Finally, note that $L \succeq 0$ if and only if

$$\left[2B_i - \sum_{j\in \text{pv},(i,j)\in\mathcal{E}} B_{ij}\exp(\rho_j-\rho_i)\frac{1}{\cos(\theta_{ij})}\right]_{ii}^{\text{pq}} - \sum_{(i,j)\in\mathcal{E},i,j\in\text{pq}} \frac{B_{ij}}{\cos(\theta_{ij})}\left[\begin{pmatrix}\exp(\rho_j-\rho_i) & 1\\ 1 & \exp(\rho_i-\rho_j)\end{pmatrix}\right]_{ij}^{\text{pq}} \succeq 0$$

To see that this is convex constraint, we just need to note that

$$\frac{1}{\cos(\theta_{ij})}\left[\begin{pmatrix}\exp(\rho_j-\rho_i) & 1\\ 1 & \exp(\rho_i-\rho_j)\end{pmatrix}\right]_{ij}$$

is $\succeq$-convex in $(\rho_j - \rho_i, \theta_{ij})$ (lemma 5). The terms in the diagonal matrix are all concave functions of (ρ, θ) and are hence $\succeq$-concave. Finally, plugging in $\theta_{ij} = \theta_i - \theta_j$, one preserves convexity and hence the energy function is convex over $\mathcal{C}$. Further, at $(\rho, \theta) = 0$, the condition boils down to:

$$M \succeq 0, M_{ii} = 2B_i - \sum_{j\sim i} B_{ij} = \sum_{j\sim i} B_{ij}, M_{ij} = -B_{ij}$$

Since M is symmetric and diagonally dominant, it must be positive semidefinite. Hence, $0 \in \mathcal{C}$. Further, since $\mathcal{C}$ is convex $(\alpha\rho, \alpha\theta) \in \mathcal{C} \quad \forall \alpha \in [0,1]$. Hence, $\mathcal{C} \subset \mathcal{D}$.

For the converse, we look at tree networks. We assume that the tree is connected, else the energy function can be decomposed into a sum of independent functions on each connected component and the same arguments as below apply. First note that for a connected tree with n buses, we have $n-1$ edges. Further, note that $\theta_{\mathcal{S}} = 0$. Let $\tilde{\theta}$ denote the vector of phases at all buses except the slack bus. Then, there is a one-one correspondence between $\tilde{\theta}$ and $\theta_{\mathcal{E}}$, ie, there exists an $(n-1)\times(n-1)$ matrix $\tilde{A}$ (submatrix of A formed by deleting the column corresponding to the the slack bus) such that $\theta_{\mathcal{E}} = \tilde{A}\tilde{\theta}$. Further, note that $\tilde{A}$ is invertible. Hence, $\theta = \left(\tilde{A}\right)^{-1}\theta_{\mathcal{E}}$ and the Hessian of E wrt (ρ, θ) is equal to

$$\begin{pmatrix} \nabla^2_\rho E(\rho,\theta) & \nabla^2_{\rho,\theta_{\mathcal{E}}=\tilde{A}\theta} E(\rho,\theta_{\mathcal{E}})\tilde{A}^{-1} \\ \left(\tilde{A}^{-1}\right)^T \left(\nabla^2_{\rho,\theta_{\mathcal{E}}} E(\rho,\theta_{\mathcal{E}})\right)^T & \left(\tilde{A}^{-1}\right)^T \nabla^2_{\theta_{\mathcal{E}}=\tilde{A}\theta} E(\rho,\theta,\theta_{\mathcal{E}})\tilde{A}^{-1} \end{pmatrix}$$

Using Schur-complements and the invertibility of $\tilde{A}$, it is easy to see that the postive semidefiniteness of this matrix is equivalent to that of the matrix in the first part. The positive semidefinite-ness of the bottom right block requires $|\theta_{\mathcal{E}}| \leq \frac{\pi}{2}$. If this inequality is strict, this block is positive definite and the analysis in the first part is necessary and sufficient. The cases where this block is singular can be handled by a continuity argument. Hence, in this case, $\mathcal{C} \supset \mathcal{D}$, and combining the result from the first part, we get $\mathcal{D} = \mathcal{C}$. □

lemma 1. *The function* $\frac{1}{2}\exp(2x) - c\exp(x)\cos(y)$ *is jointly convex in* (x, y) *over the convex domain*

$$\left\{(x,y) : c\exp(-x) \leq 2, |y| \leq \arccos\left(\frac{c\exp(-x)}{2}\right)\right\}$$

Proof. It suffices to prove that the second derivative of the objective wrt y is non negative, and the determinant of the Hessian wrt (x, y) is non negative. The first statement is true over the domain $\cos(y) \geq 0$ and hence over the domain

$$2\cos(y) \geq c\exp(-x).$$

The determinant of the Hessian is given by $c\exp(x)(2\cos(y) - c\exp(-x))$, which is non-negative precisely over the given domain. □

lemma 2. *For $a, b > 0$, the function $\frac{1}{2}(a\exp(2x) + b\exp(2y)) - \exp(x+y)\cos(z)$ is jointly convex in (x, y, z) on the domain*

$$|z| \leq \arccos\left(\frac{a\exp(x-y) + b\exp(y-x)}{2}\right), a\exp(x-y) + b\exp(y-x) \leq 2$$

Proof. The proof follows by writing down the Hessian and writing conditions for its principal minors to be non-negative. □

lemma 3. *The function $-\cos(x)$ is convex over the domain $|x| \leq \frac{\pi}{2}$*

Proof. The second derivative is $\cos(x)$ and is non-negative over the domain. □

lemma 4. *The function $\frac{\exp(x)}{\cos(y)}$ is convex over the domain $|y| \leq \frac{\pi}{2}$*

Proof. The Hessian is given by

$$\begin{pmatrix} \frac{\exp(x)}{\cos(y)} & \frac{\exp(x)\sin(y)}{\cos^2(y)} \\ \frac{\exp(x)\sin(y)}{\cos^2(y)} & \frac{\exp(x)\left(1+\sin^2(y)\right)}{\cos^3(y)} \end{pmatrix}$$

It is easy to check that this is positive semidefinite. □

lemma 5. *The matrix-valued function*

$$f(x, y) = \frac{1}{\cos(y)}\begin{pmatrix} \exp(x) & 1 \\ 1 & \exp(-x) \end{pmatrix}$$

is $\succeq$-convex, that is,

$$f(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \preceq \lambda f(x_1, y_1) + (1-\lambda)f(x_2, y_2)$$

Proof. f is $\succeq$-convex if and only if $\mathrm{tr}(f(x,y)R)$ is convex for all $R \succeq 0$ [BV09]. Let $R = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \succeq 0$ be an arbitrary 2×2 positive semidefinite matrix. Then, we have

$$\begin{aligned} \mathrm{tr}(f(x,y)R) &= \frac{a\exp(x) + c\exp(-x) + 2b}{\cos(y)} \\ &= \frac{a\exp(x) + c\exp(-x) - 2\sqrt{ac}}{\cos(y)} + 2\frac{b + \sqrt{ac}}{\cos(y)} \end{aligned}$$

Since $R \succeq 0$, $|b| \leq \sqrt{ac}$ and hence the second term is convex (it is the inverse of a positive concave function, cos). The first term can be rewritten as $\frac{\left(|\sqrt{a}\exp(x/2) - \sqrt{c}\exp(-x/2)|\right)^2}{\cos(y)}$. It is easy to check that $|\sqrt{a}\exp(x/2) - \sqrt{c}\exp(-x/2)|$ is twice differentiable at all values of x and its second derivative is equal to $|\sqrt{a}\exp(x/2) - \sqrt{c}\exp(-x/2)|$. Hence, this function is convex. Thus, the function $|\sqrt{a}\exp(x/2) - \sqrt{c}\exp(-x/2)|^2/\cos(y)$ is convex by composition rules (since x^2/t is convex in x, t, increasing in x and decreasing in t). Thus, f is $\succeq$-convex as long as $|y| \leq \frac{\pi}{2}$. □

7.1 Lossy Networks

In this section, we show that all our results extend to special classes of lossy networks, where all transmission lines have fixed the same value of ressistance to reactance ratio: $\frac{X}{R} = \frac{G}{B} = \kappa$. Further, we assume that all buses are (P, Q) buses, except for the slack bus.

In this case, we can write the power balance equations as follows by combining the active and reactive power balance equations linearly:

$$P_i + \kappa Q_i = \sum_{j \neq i} \left(\kappa^2 + 1\right) \exp(\rho_i + \rho_j) B_{ij} \sin(\theta_i - \theta_j) \quad \forall i \in \mathrm{pq} \tag{9a}$$

$$\kappa P_i - Q_i = \sum_{j \neq i} \left(\kappa^2 + 1\right) \exp(\rho_i + \rho_j) B_{ij} \cos(\theta_i - \theta_j) \quad \forall i \in \mathrm{pq} \tag{9b}$$

$$\theta_{\mathcal{S}} = \rho_{\mathcal{S}} = 0 \tag{9c}$$

The power flow equations equations (9a–9c) can be rewritten in a variational form:

$$E(\rho,\theta) = \sum_{i\in \mathrm{pq}} -\left(\left(P_i+\kappa Q_i\right)\theta_i + \left(\kappa P_i - Q_i\right)\rho_i\right) + \sum_{(i,j)\in\mathcal{E}} B_{ij}\left(\kappa^2+1\right)\left(\frac{\exp\left(2\rho_i\right)+\exp\left(2\rho_j\right)}{2} - \exp\left(\rho_i+\rho_j\right)\cos\left(\theta_i-\theta_j\right)\right) \tag{10a}$$

$$\text{(9a)} \equiv \frac{\partial E}{\partial \theta_i} = 0, \text{(9b)} \equiv \frac{\partial E}{\partial \rho_i} = 0 \tag{10b}$$

Theorem 7.1. *The energy function for the lossy system with constant* $\frac{R}{X}$ *ratio* (10a) *is convex over the domain:*

$$|\theta_i - \theta_j| \leq \frac{\pi}{2} \quad \forall (i,j)\in\mathcal{E} \tag{11}$$

$$\sum_{i\in\mathrm{pq}}\left[2B_i - \sum_{j\in\{S\}} B_{ij}\frac{\exp\left(\rho_j-\rho_i\right)}{\cos\left(\theta_i-\theta_j\right)}\right]_{ii}^{\mathrm{pq}} - \sum_{(i,j)\in\mathcal{E},i,j\in\mathrm{pq}} \frac{B_{ij}}{\cos\left(\theta_i-\theta_j\right)}\left[\begin{pmatrix}\exp\left(\rho_j-\rho_i\right) & 1\\ 1 & \exp\left(\rho_i-\rho_j\right)\end{pmatrix}\right]_{ij}^{\mathrm{pq}} \succeq 0 \tag{12}$$

Proof. Almost identical to the proof of theorem 3.3. □