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Three-Dimensional Electromagnetic High Frequency Axisymmetric Cavity Scars

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Abstract

This report examines the localization of high frequency electromagnetic fields in three-dimensional axisymmetric cavities along periodic paths between opposing sides of the cavity. The cases where these orbits lead to unstable localized modes are known as scars. This report treats both the case where the opposing sides, or mirrors, are convex, where there are no interior foci, and the case where they are concave, leading to interior foci. The scalar problem is treated first but the approximations required to treat the vector field components are also examined. Particular attention is focused on the normalization through the electromagnetic energy theorem. Both projections of the field along the scarred orbit as well as point statistics are examined. Statistical comparisons are made with a numerical calculation of the scars run with an axisymmetric simulation. This axisymmetric case forms the opposite extreme (where the two mirror radii at each end of the ray orbit are equal) from the two-dimensional solution examined previously (where one mirror radius is vastly different from the other). The enhancement of the field on the orbit axis can be larger here than in the two-dimensional case.

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1 SUMMARY

Cavity modes at high frequencies tend to exhibit statistical homogeneity except in regions where periodic modes are supported by boundary topology. These enhanced regions can take the form of localized stable modes such as laser-type "bouncing-ball modes" or "whispering-gallery modes", which do not interact with the remainder of the cavity due to confinement by the caustic surfaces associated with these types of modes. Enhancements can alternatively be associated with unstable periodic ray orbits. These "scars", as they have been called, are local enhancements of cavity fields over the chaotic background along unstable periodic ray trajectories. These enhancements do interact with the remainder of the cavity, and hence can be excited by sources anywhere in the cavity.

These unstable scars are the subject of this report. Unlike in the case of stable modes, which are confined by caustics, the unstable modes have propagation transverse to, as well as along, the orbit direction. Hence a reflection with a random phase coefficient is used to capture the return from the outer chaotic region. The eigenvalue density in the cavity, which depends on the overall cavity volume, is used to normalize the statistics of this random reflection phase.

Previously, we have investigated these scars in two-dimensional convex and concave walled cavities. The present report discusses scar effects in a three-dimensional axisymmetric cavity. This arrangement captures the scarring when the mirrors at the ends of the orbit have the same radius of curvature in both azimuthal orientations; this case thus compliments the two-dimensional case, where one of the mirror radii of curvature is infinite.

In three-dimensions the electromagnetic problem also requires treatment of the vector problem. We begin each investigation with the scalar acoustic problem and then introduce the vector case. A quasi-rectangular coordinate system is introduced to treat the vector problem.

To make clear the enhancements resulting from the scar effects comparisons are done with purely chaotic fields represented by collections of random plane waves. These are constructed for both the scalar and the vector cases, for both the full three-dimensional case as well as the axisymmetric geometry.

It is crucial in these investigations to compare theoretical results with rigorous simulations of the cavity modal solutions. This was done by modification of a body of revolution scattering code. Cavity modes were identified by examining the solution condition number as a function of frequency. Normalization of the modes was carried out by integration of the field square over the cavity volume. Projections of the field along the orbit, or the field square at a point, were examined by gathering the results at the various modes and binning them as a function of the separation between modal frequency and scar frequency; the scar frequency is associated with condition that the free space wavenumber times the orbit length is a multiple of pi divided by two. These numerical comparisons help verify the vector scar constructions and the underlying assumptions made regarding the chaotic reflection phase from the outer chaotic region of

the cavity.

The end goal of these investigations is to enable an interior high frequency ray method to be used in order to model the problem in an efficient manner. Because, the fields in the cavity at high frequencies can vary due to small perturbations of the cavity geometry, or with small frequency shifts, it is believed that a method which captures the statistical properties of the field in the cavity would give results that could be most useful in describing the shielding. For our purpose it is important that such a technique capture the extreme statistics that are dominated by the presence enhanced field regions (hot spots) created by periodic orbits supported by the boundary topology.

1.1 Convex Walls

The cavity with convex walls does not generate interior focal regions and the high frequency scar construction holds over the entire orbit across the cavity. The source free electromagnetic energy theorem is used to normalize the scar amplitude, such that the integral of the square of the field over the entire cavity has a chosen value, even though the construction is only valid in the vicinity of the scarred orbit.

Fourier and other types of projections of the field along the orbit are taken which illustrate scar effects versus the random plane wave chaotic field. Comparisons with the numerical simulations verify the behavior of these modes.

The field squared statistics at points along the orbit are also examined. In the convex cavity these statistics are similar to the chaotic random plane wave statistics. Nevertheless, there is a large enhancement of the field along the central orbit that results from the axisymmetric nature of the cavity mode (when this mode is normalized throughout the cavity volume). This field enhancement is of a similar nature to the factor of two enhancements that were observed in two-dimensions along the symmetry planes of those cavities (when either the even modes were normalized over the cavity area).

1.2 Concave Walls

The cavity with concave walls generates interior focal regions and the high frequency scar construction treats the orbit over several separate intervals about and in the focal region. These are asymptotically matched so that a single scar amplitude determines the levels in all three regions.

The asymptotic matching of the solution phases actually requires a subwavelength shift of the focal region position with respect to the other two regions. An alternative view of this shift is that it arises from a refinement of the elliptical mirror radius used in the scar construction to better fit to the constant radius spherical mirrors at the short but finite wavelength involved. This view provides additional insight into the value that must be chosen for the subwavelength shift.

The electromagnetic energy theorem is again used to normalize the results, but it must be applied over the entire orbit encompassing the three regions of scar construction. In the high frequency limit the energy theorem integration can be taken in the form of a principal value, including only the two regions about the focal region.

The projections of the field along the orbit exhibit a peak when the scar frequency and modal frequency align, which is missing in the random plane wave representation. The field squared statistics in the focal region also exhibit this peak which is well above the level of the random plane wave statistics; this is on top of the symmetry enhancement on the central axis of the cavity.

Comparisons with numerically generated histograms show reasonable agreement with these theoretical statistics for the field projection and the field squared in the focal region. In addition, a comparison of the deterministic field behavior along the orbit from the scar construction and the numerical simulation confirm the matched functional form of the scar; this single mode comparison is made by normalizing the scar construction at a single point along the orbit to the simulation.

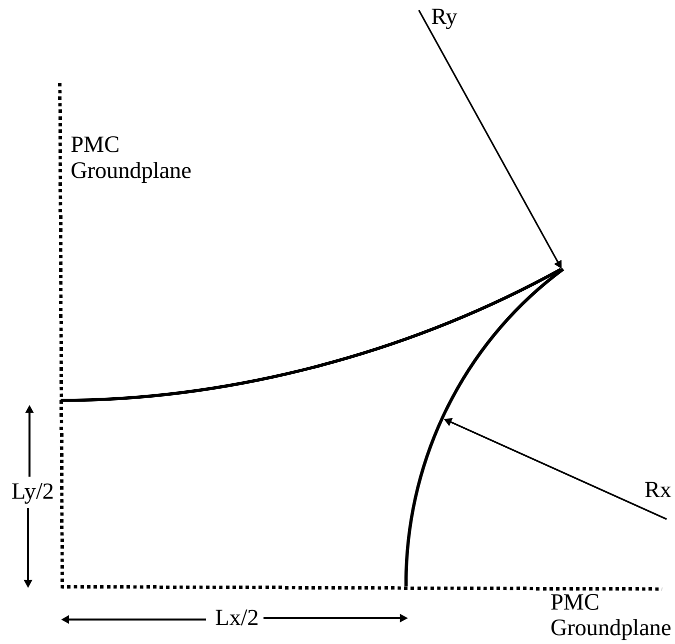

Figure 1. Quarter bowtie cavity geometry

2 INTRODUCTION

This report is directed at understanding the high frequency behavior of modal fields in axisymmetric three dimensional cavities. The idea is to capture how the modal fields can depart from statistical homogeneity because of certain types of boundary topologies. In particular, the localization of the eigenfunctions about unstable periodic orbits, known as scarring, is investigated. The approach used by Antonsen [1], on convex mirror geometries in two dimensions, is generalized [2], [3], [4] by introducing the curved ray path formalism, used previously by Vaynshteyn [5] on stable orbits. This combined approach was used recently to investigate scars in two-dimensional geometry [6], [7].

This report explores both the scalar (acoustic) [8], [9] and vector (electromagnetic) problems in three-dimensional axisymmetric geometry, first with convex walls, and second with concave walls supporting interior foci. The mirror boundaries at the ends of the orbit each have two equal radii of curvature in this axisymmetric geometry, which is the other distinct limit from the two-dimensional case, where one of the radii becomes infinite. This feature leads to differences in field enhancement along the orbit relative to the two-dimensional case.

This is a fairly long report so redundancy in derivations and definitions of quantities for the various sections have not been eliminated; hopefully the structure of the sections (as well as the preceding summary) make it possible to read without this repetition being a distraction.

3 CONVEX MIRRORS AND ROTATIONAL BOWTIE CAVITY

The bowtie cavity has been used as a canonical shape for a cavity with convex shaped mirrors. One quarter of this cavity is shown in Figure 1. Here we consider rotating this about the x axis (and renaming the axes as z and r) to form the rotationally symmetric cavity shown in Figure 2.

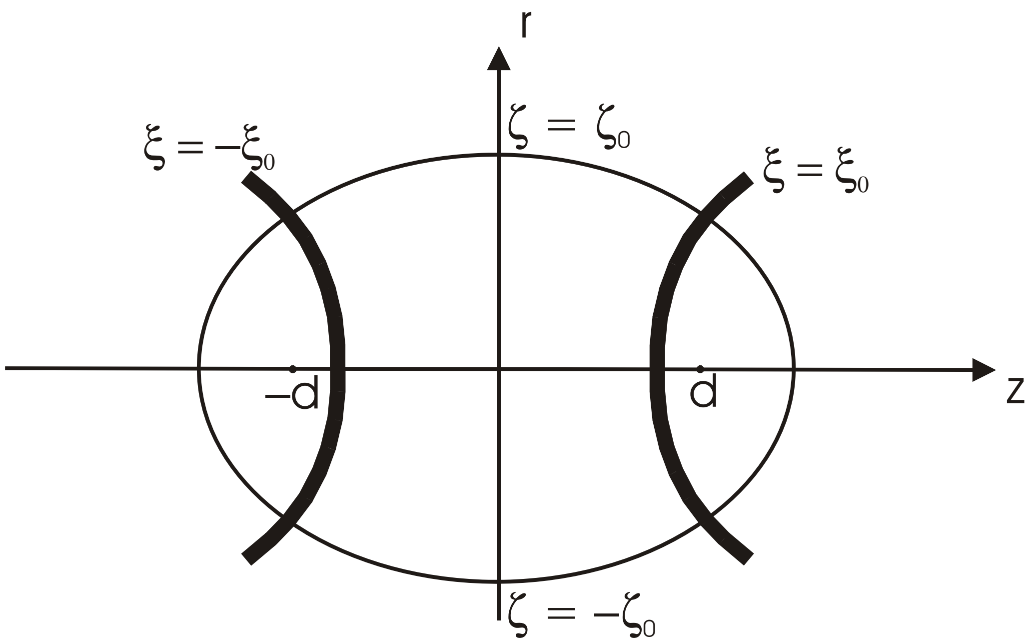

Figure 2. Axisymmetric three dimensional bowtie cavity.

3.1 Curved Trajectory Analysis

Vaynshteyn [5] has treatments for stable modes between concave mirrors. Here we wish to consider the generalization to unstable modes between convex mirrors. For three dimensions and axisymmetric geometry the prolate spheroid is fit to the local boundaries at the ends of the unstable orbit $\xi = \xi_0$ as shown in Figure 3. The prolate spheroidal coordinates (ζ, φ, ξ) are related to the cylindrical system (r, φ, z) by means of

$$r = d \sinh\zeta \cos\xi \tag{1}$$

$$z = d \cosh\zeta \sin\xi \tag{2}$$

where

$$0 < \zeta < \infty \tag{3}$$

$$-\pi/2 < \xi < \pi/2\,,\; 0 < \varphi < 2\pi \tag{4}$$

To match the local radius of curvature R at the ends of the orbit we take the focal positions (which in this case are exterior to the region) to be

$$d = \ell\sqrt{1 + R/\ell} \tag{5}$$

Thus the orbit extends over the range $-\xi_0 < \xi < \xi_0$ with $\zeta \to 0$ and

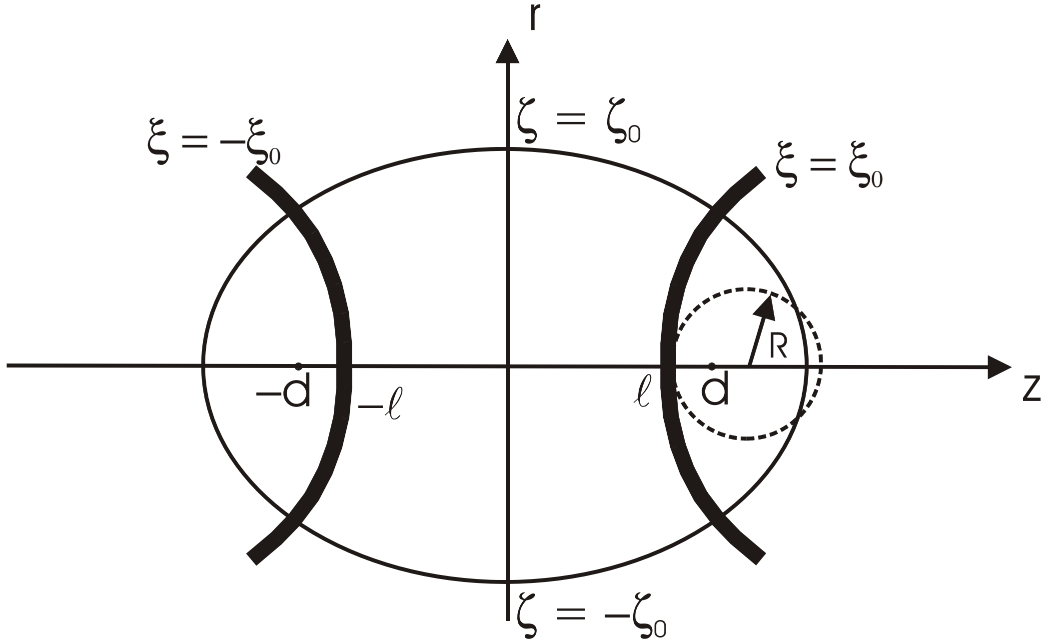

Figure 3. Fitting radius of curvature of the end walls (or mirrors) to the prolate spheroidal coordinates.

$$\ell = d \sin \xi_0 \tag{6}$$

$$L = 2\ell \tag{7}$$

or

$$\sin \xi_0 = 1/\sqrt{1 + R/\ell} \tag{8}$$

3.2 Scalar Field High Frequency Approximation

The modes of the scalar Helmholtz equation

$$\nabla^2 u + k^2 u = 0 \tag{9}$$

are first investigated. The metric coefficients in the prolate spheroidal system of coordinates (ζ, φ, ξ) are [10]

$$h_\zeta = h_\xi = d\sqrt{\sinh^2 \zeta + \cos^2 \xi} = d\sqrt{\cosh^2 \zeta - \sin^2 \xi} \tag{10}$$

$$h_\varphi = d \sinh \zeta \cos \xi \tag{11}$$

The Helmholtz equation can then be written in these three-dimensions as [9]

$$\frac{1}{\cos \xi} \frac{\partial}{\partial \xi} \left(\cos \xi \frac{\partial u}{\partial \xi} \right) + \left(\frac{1}{\cos^2 \xi} + \frac{1}{\sinh^2 \zeta} \right) \frac{\partial^2 u}{\partial \varphi^2}$$

$$+ \frac{1}{\sinh \zeta} \frac{\partial}{\partial \zeta} \left(\sinh \zeta \frac{\partial u}{\partial \zeta} \right) + \gamma^2 \left(\sinh^2 \zeta + \cos^2 \xi \right) u = 0 \tag{12}$$

or

$$\frac{1}{\cos \xi} \frac{\partial}{\partial \xi} \left(\cos \xi \frac{\partial u}{\partial \xi} \right) + \left(\frac{1}{\cos^2 \xi} + \frac{1}{\sinh^2 \zeta} \right) \frac{\partial^2 u}{\partial \varphi^2}$$

$$+ \frac{1}{\sinh \zeta} \frac{\partial}{\partial \zeta} \left(\sinh \zeta \frac{\partial u}{\partial \zeta} \right) + \gamma^2 \left(\cosh^2 \zeta - \sin^2 \xi \right) u = 0$$

where

$$\gamma = kd = k\ell \sqrt{1 + R/\ell} \tag{13}$$

On the mirror we want

$$u = 0 \,,\ \xi = \pm \xi_0 \,,\ -\zeta_0 < \zeta < \zeta_0 \tag{14}$$

We assume $\gamma >> 1$ and that $\sinh^2 \zeta << 1$. We take the function u to be even about the z axis. We seek a solution of the form [5]

$$u = W(\zeta, \varphi, \xi)\, e^{i\gamma \sin\xi} + W(\zeta, \varphi, -\xi)\, e^{-i\gamma \sin\xi} \tag{15}$$

Substituting into the Helmholtz equation gives

$$\frac{1}{\cos\xi}\frac{\partial}{\partial\xi}\left(\cos\xi\frac{\partial W}{\partial\xi}\right) + \left(i2\gamma\cos\xi\frac{\partial W}{\partial\xi}\right) + \left(\frac{1}{\cos^2\xi} + \frac{1}{\sinh^2\zeta}\right)\frac{\partial^2 W}{\partial\varphi^2}$$

$$+\frac{1}{\sinh\zeta}\frac{\partial}{\partial\zeta}\left(\sinh\zeta\frac{\partial W}{\partial\zeta}\right) + \gamma\left(\gamma\sinh^2\zeta - i2\sin\xi\right)W = 0$$

Now for high frequencies $\gamma >> 1$ we ignore the term [5]

$$\frac{1}{\cos\xi}\frac{\partial}{\partial\xi}\left(\cos\xi\frac{\partial W}{\partial\xi}\right)$$

and taking $\sinh^2\zeta << 1$ we can replace $\sinh\zeta$ by ζ and neglect the term

$$\frac{1}{\cos^2\xi}\frac{\partial^2 W}{\partial\varphi^2}$$

Then

$$\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial W}{\partial\zeta}\right) + i2\gamma\cos\xi\frac{\partial W}{\partial\xi}$$

$$+\frac{1}{\zeta^2}\frac{\partial^2 W}{\partial\varphi^2} + \left(\gamma^2\zeta^2 - i2\gamma\sin\xi\right)W = 0 \tag{16}$$

Next taking

$$W = \frac{1}{\cos\xi}\Psi \tag{17}$$

$$\tau = \sqrt{2\gamma}\zeta \tag{18}$$

$$\sigma = \int_0^\xi \frac{d\xi}{\cos\xi} = \text{arcsinh}\left(\tan\xi\right) \tag{19}$$

$$\sigma_0 = \text{arcsinh}\left(\tan\xi_0\right) \tag{20}$$

$$\ell = d\sin\xi_0$$

$$\tan\xi_0 = \frac{\ell}{\sqrt{d^2 - \ell^2}}$$

$$\sigma_0 = \frac{1}{2}\ln\left(\frac{d+\ell}{d-\ell}\right) = \text{Arcsinh}\left(\sqrt{\ell/R}\right) \tag{21}$$

where the two exact stability exponents [12] can be written as

$$\Lambda_{\pm} = \left(\frac{d+\ell}{d-\ell}\right)^{\pm 2} \tag{22}$$

The separation constant s is then

$$s_p = (k - k_p)\,\ell/\sigma_0 = \frac{(k-k_p)\,L/2}{\mathrm{Arcsinh}\left[\sqrt{L/(2R)}\right]} = 2\,(k-k_p)\,L/\ln(\Lambda_+) \tag{23}$$

These give

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial W}{\partial\tau}\right) + i\cos\xi\frac{\partial\sigma}{\partial\xi}\frac{\partial W}{\partial\sigma}$$

$$+\frac{1}{\tau^2}\frac{\partial^2 W}{\partial\varphi^2} + \left(\tau^2/4 - i\sin\xi\right)W = 0$$

or

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial\Psi}{\partial\tau}\right) + \frac{1}{\tau^2}\frac{\partial^2\Psi}{\partial\varphi^2} + i\frac{\partial\Psi}{\partial\sigma} + \frac{\tau^2}{4}\Psi = 0$$

The boundary conditions on the mirror result in

$$\Psi(\tau,\varphi,-\sigma_0) = e^{i2k\ell - i(2p-1)\pi}\Psi(\tau,\varphi,\sigma_0)$$

Taking

$$\Psi = \Psi_m(\tau,\sigma)\cos(m\varphi) \tag{24}$$

or

$$\Psi = \Psi_m(\tau,\sigma)\sin(m\varphi) \tag{25}$$

gives

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial\Psi_m}{\partial\tau}\right) + i\frac{\partial\Psi_m}{\partial\sigma} + \left(\frac{\tau^2}{4} - \frac{m^2}{\tau^2}\right)\Psi_m = 0$$

Letting

$$\Psi_m(\tau,\sigma) = e^{-is\sigma}\psi_m(\tau,s) \tag{26}$$

gives

$$1 = e^{i2k\ell - i(2p-1)\pi - i2s\sigma_0}$$

and

$$k\ell - (p - 1/2)\,\pi = s_p\sigma_0$$

or

$$k\ell - s_p\sigma_0 = (p - 1/2)\,\pi = k_p\ell\,,\ p = 1, 2, 3, ... \tag{27}$$

with

$$\frac{1}{\tau}\frac{d}{d\tau}\left(\tau\frac{d\psi_m}{d\tau}\right) + \left(\frac{\tau^2}{4} + s - \frac{m^2}{\tau^2}\right)\psi_m = 0 \tag{28}$$

If we let

$$\tau^2 = 2v \tag{29}$$

then

$$\frac{1}{\tau}\frac{d}{d\tau}\left(\tau\frac{d\psi_m}{d\tau}\right) = \frac{1}{\tau}\frac{dv}{d\tau}\frac{d}{dv}\left(\tau\frac{dv}{d\tau}\frac{d\psi_m}{dv}\right) = 2\frac{d}{dv}\left(v\frac{d\psi_m}{dv}\right)$$

and

$$\frac{d}{dv}\left(v\frac{d\psi_m}{dv}\right) + \left(\frac{v}{4} + \frac{s}{2} - \frac{m^2}{4v}\right)\psi_m = 0$$

Now taking

$$\psi_m = \frac{1}{\sqrt{v}}w_m \tag{30}$$

gives

$$v\frac{d\psi_m}{dv} = \sqrt{v}\left(\frac{dw_m}{dv} - \frac{w_m}{2v}\right)$$

$$\frac{d}{dv}\left(v\frac{d\psi_m}{dv}\right) = \sqrt{v}\frac{d^2w_m}{dv^2} + \sqrt{v}\left(\frac{w_m}{4v^2}\right)$$

$$\frac{d^2w_m}{dv^2} + \left(\frac{1}{4} + \frac{s/2}{v} + \frac{1-m^2}{4v^2}\right)w_m = 0$$

This is a form of Whittaker's equation [11]

$$\frac{d^2w}{dz^2} + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{1-4\mu^2}{4z^2}\right)w = 0 \tag{31}$$

$$w_m = M_{\kappa,\mu}\left(z\right), W_{\kappa,\mu}\left(z\right) \tag{32}$$

Therefore in our equation with $z = -iv$

$$\frac{d^2w_m}{dz^2} + \left(-\frac{1}{4} + \frac{is/2}{z} + \frac{1-m^2}{4z^2}\right)w_m = 0$$

and

$$\kappa = is/2$$

$$\pm m = 2\mu$$

we obtain solutions [11]

$$w_m = M_{\kappa,\mu}\left(-iv\right), W_{\kappa,\mu}\left(-iv\right) \tag{33}$$

Noting that [11]

$$W_{\kappa,\mu}\left(z\right) \sim z^{\kappa}e^{-z/2}\ ,\ \ |z| \to \infty \tag{34}$$

let us take the outward going wave to be

$$W_+\left(v, m, s\right) = W_{\kappa,\mu}\left(-iv\right) \tag{35}$$

The total solution with a random phase reflection coefficient from the outer region of the high frequency cavity [1], [2] is

$$w_m = c_m \operatorname{Re}\left[W_+ + e^{i\Phi_0}W_+^*\right] \tag{36}$$

In summary the potential is then

$$u = \frac{2}{\cos\xi}\psi_m\left(\tau, s\right)\cos\left(\gamma\sin\xi - s\sigma\right)\left\{\begin{array}{c}\cos\left(m\varphi\right)\\ \sin\left(m\varphi\right)\end{array}\right\} \tag{37}$$

where

$$\psi_m = c_m\frac{\sqrt{2}}{\tau}\operatorname{Re}\left[W_+ + e^{i\Phi_0}W_+^*\right] \tag{38}$$

$$W_+\left(\tau, m, s\right) = W_{is/2,m/2}\left(-i\tau^2/2\right) \tag{39}$$

and

$$W_+\left(\tau, m, s\right) \sim \left(-i\tau^2/2\right)^{is/2}e^{i\tau^2/4} = \left(-iv\right)^{is/2}e^{iv/2}\ ,\ \tau \to \infty \tag{40}$$

$$\psi_m \sim c_m\frac{1}{\sqrt{v}}\operatorname{Re}\left[\left(-iv\right)^{is/2}e^{iv/2} + e^{i\Phi_0}\left(iv\right)^{-is/2}e^{-iv/2}\right]\ ,\ v \to \infty$$

with

$$k\ell - s_p\sigma_0 = \left(p - 1/2\right)\pi = k_p\ell\,,\ p = 1, 2, 3, ...$$

$$s_p = \left(k - k_p\right)\ell/\sigma_0 = \frac{\left(k - k_p\right)L/2}{\operatorname{Arcsinh}\left[\sqrt{L/\left(2R\right)}\right]} = 2\left(k - k_p\right)L/\ln\left(\Lambda_+\right)$$

$$\sigma_0 = \operatorname{arcsinh}(\tan\xi_0) = \frac{1}{2}\ln\left(\frac{d+\ell}{d-\ell}\right) = \frac{1}{4}\ln(\Lambda_+)$$

$$\Lambda_\pm = \left(\frac{d+\ell}{d-\ell}\right)^{\pm 2}$$

and on axis we can write

$$\sigma = \operatorname{arcsinh}(\tan\xi) = \operatorname{arcsinh}\left(\frac{z}{\sqrt{d^2-z^2}}\right) = \frac{1}{2}\ln\left(\frac{d+z}{d-z}\right) \tag{41}$$

3.2.1 Neumann Boundary Conditions

On the mirror we now want

$$\frac{\partial u}{\partial \xi} = 0\,,\ \xi = \pm\xi_0\,,\ -\zeta_0 < \zeta < \zeta_0 \tag{42}$$

The scalar field is

$$u = W(\zeta,\varphi,\xi)\,e^{i\gamma\sin\xi} + W(\zeta,\varphi,-\xi)\,e^{-i\gamma\sin\xi} \tag{43}$$

$$\gamma = kd = k\ell\sqrt{1+R/\ell}$$

Next taking

$$W = \frac{1}{\cos\xi}\Psi$$

$$\tau = \sqrt{2\gamma}\zeta$$

$$\sigma = \int_0^\xi \frac{d\xi}{\cos\xi} = \operatorname{arcsinh}(\tan\xi)$$

$$\sigma_0 = \operatorname{arcsinh}(\tan\xi_0)$$

$$\ell = d\sin\xi_0$$

$$\tan\xi_0 = \frac{\ell}{\sqrt{d^2-\ell^2}}$$

$$\sigma_0 = \frac{1}{2}\ln\left(\frac{d+\ell}{d-\ell}\right) = \operatorname{Arcsinh}\left(\sqrt{\ell/R}\right)$$

The separation constant s is then

$$s_p = (k - k_p)\,\ell/\sigma_0 = \frac{(k-k_p)\,L/2}{\text{Arcsinh}\left[\sqrt{L/(2R)}\right]} = 2\,(k-k_p)\,L/\ln(\Lambda_+)$$

The boundary conditions on the mirror result in

$$\Psi(\tau,\varphi,-\sigma_0) = e^{i2k\ell - i2p\pi}\Psi(\tau,\varphi,\sigma_0)$$

Taking

$$\Psi = \Psi_m(\tau,\sigma)\cos(m\varphi)$$

or

$$\Psi = \Psi_m(\tau,\sigma)\sin(m\varphi)$$

and letting

$$\Psi_m(\tau,\sigma) = e^{-is\sigma}\psi_m(\tau,s)$$

we see that

$$1 = e^{i2k\ell - i2p\pi - i2s\sigma_0}$$

$$k\ell - p\pi = s_p\sigma_0$$

$$k\ell - s_p\sigma_0 = p\pi = k_p\ell\,,\ p = 0,1,2,3,... \tag{44}$$

3.2.2 Odd Symmetry Along Orbit

In the case of odd symmetry along the orbit we take

$$iu = W(\zeta,\varphi,\xi)\,e^{i\gamma\sin\xi} - W(\zeta,\varphi,-\xi)\,e^{-i\gamma\sin\xi}$$

The boundary conditions at the mirrors imply that

$$\pm iu(\zeta,\varphi,\pm\xi_0) = W(\zeta,\varphi,\xi_0)\,e^{i\gamma\sin\xi_0} - W(\zeta,\varphi,-\xi_0)\,e^{-i\gamma\sin\xi_0} = 0$$

Now with

$$W = \frac{1}{\cos\xi}\Psi$$

$$\Psi = \Psi_m(\tau,\sigma)\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}$$

$$\Psi_m(\tau,\sigma) = e^{-is\sigma}\psi_m(\tau,s)$$

$$\tau = \sqrt{2\gamma}\zeta \tag{45}$$

$$\sigma = \int_0^{\xi} \frac{d\xi}{\cos\xi} = \operatorname{arcsinh}(\tan\xi)$$

$$\sigma_0 = \operatorname{arcsinh}(\tan\xi_0)$$

$$\ell = d\sin\xi_0$$

we find

$$1 = e^{i2k\ell - i2p\pi - i2s\sigma_0}$$

or

$$k\ell - p\pi = s_p\sigma_0$$

$$k\ell - s_p\sigma_0 = p\pi = k_p\ell\,,\ p = 1, 2, 3, \ldots \tag{46}$$

Thus we have

$$u = \frac{2}{\cos\xi}\psi_m(\tau, s)\sin(\gamma\sin\xi - s\sigma)\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\} \tag{47}$$

where again we could take

$$\psi_m = c_m\frac{\sqrt{2}}{\tau}\operatorname{Re}\left[W_+ + e^{i\Phi_0}W_+^*\right]$$

$$W_+(\tau, m, s) = W_{is/2, m/2}\left(-i\tau^2/2\right) \tag{48}$$

3.2.3 Whittaker Function Calculation

We now want to summarize the generation of the Whittaker functions used here. We are interested in generating the function [11]

$$W_+(\tau, m, s) = W_{is/2, m/2}\left(-i\tau^2/2\right)$$

$$= e^{i\tau^2/4}\left(-i\tau^2/2\right)^{1/2+m/2} U\left(1/2 + m/2 - is/2, 1 + m, -i\tau^2/2\right) \tag{49}$$

The power series form can be found from [11]

$$U(a, m+1, z) = \frac{(-1)^{m+1}}{m!\Gamma(a-m)}\left[M(a, m+1, z)\ln z\right.$$

$$+\sum_{j=0}^{\infty}\frac{(a)_j\, z^j}{(m+1)_j\, j!}\left\{\psi\left(a+j\right)-\psi\left(1+j\right)-\psi\left(1+m+j\right)\right\}\Bigg]$$

$$+\frac{(m-1)!}{\Gamma\left(a\right)}z^{-m}M\left(a-m,1-m,z\right)_m \tag{50}$$

where

$$M\left(a-m,1-m,z\right)_m=\sum_{n=0}^{m-1}\frac{(a-m)_n}{(1-m)_n}\frac{z^n}{n!} \tag{51}$$

$$M\left(a,1,z\right)_0=0 \tag{52}$$

$$M\left(a,0,z\right)_1=1 \tag{53}$$

and

$$U\left(a,b,z\right)=\frac{\pi}{\sin\left(\pi b\right)}\left[\frac{M\left(a,b,z\right)}{\Gamma\left(1+a-b\right)\Gamma\left(b\right)}-z^{1-b}\frac{M\left(1+a-b,2-b,z\right)}{\Gamma\left(a\right)\Gamma\left(2-b\right)}\right] \tag{54}$$

$$M\left(a,b,z\right)=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{z^n}{n!} \tag{55}$$

$$M\left(a,m+1,z\right)=\sum_{n=0}^{\infty}\frac{(a)_n}{(m+1)_n}\frac{z^n}{n!} \tag{56}$$

The asymptotic form can be found from [11]

$$U\left(a,b,z\right)=z^{-a}\left[\sum_{n=0}^{R-1}\frac{(a)_n\left(1+a-b\right)_n}{n!}\left(-z\right)^{-n}+O\left(\left|z\right|^{-R}\right)\right],\ -\frac{3}{2}\pi<\arg\left(z\right)<\frac{3}{2}\pi \tag{57}$$

$$U\left(a,1+m,z\right)=z^{-a}\left[\sum_{n=0}^{R-1}\frac{(a)_n\left(a-m\right)_n}{n!}\left(-z\right)^{-n}+O\left(\left|z\right|^{-R}\right)\right],\ -\frac{3}{2}\pi<\arg\left(z\right)<\frac{3}{2}\pi \tag{58}$$

$$W_+\left(\tau,m,s\right)=W_{is/2,m/2}\left(-i\tau^2/2\right)$$

$$W_+\left(\tau,m,s\right)=e^{i\tau^2/4}\left(-i\tau^2/2\right)^{is/2}\left[1-i\frac{1}{2}\left\{\left(1-is\right)^2-m^2\right\}\tau^{-2}+O\left(\tau^{-4}\right)\right] \tag{59}$$

An integral representation is [11]

$$\Gamma\left(a\right)U\left(a,b,z\right)=\int_0^{\infty}e^{-zt}t^{a-1}\left(1+t\right)^{b-a-1}dt \tag{60}$$

$$= \int_0^{Re^{i\pi/2}} e^{-zt} t^{a-1} (1+t)^{b-a-1} \, dt + \int_{\pi/2}^0 e^{-zt} t^{a-1} (1+t)^{b-a-1} \, ie^{i\varphi} R d\varphi$$

$$\Gamma(a) U(a, m+1, -iz) \sim e^{im\pi/2} \int_0^R e^{-zt} t^{a-1} (t-i)^{m-a} \, dt + iR^m \int_{\pi/2}^0 e^{ie^{i\varphi} zR} e^{im\varphi} d\varphi$$

$$\int_0^{\pi/2} e^{i\left(e^{i\varphi} zR + m\varphi\right)} d\varphi = \int_0^{\pi/2} e^{(-\sin\varphi + i\cos\varphi) zR + im\varphi} d\varphi \sim \int_0^{\infty} e^{-\varphi zR} d\varphi = \frac{1}{zR}$$

$$\Gamma(a) U(a, m+1, -iz) = e^{im\pi/2} \int_0^R e^{-zt} t^{a-1} (t-i)^{m-a} \, dt - \frac{i}{z} R^{m-1}$$

If $m = 0$ the boundary term vanishes and if $m = 1$ it is a constant as $R \to \infty$. However, the asymptotic expansion of the integral is

$$\Gamma(a) U(a, m+1, -iz) \sim e^{ia\pi/2} \int_0^{\infty} e^{-zt} t^{a-1} dt = \Gamma(a) (-iz)^{-a} \tag{61}$$

which is the desired asymptotic form; thus this integral form is the analytic continuation of the function for this phase of argument $z = \tau^2/2$

$$\Gamma(a) U(a, m+1, -iz) = i^m \int_0^{\infty} e^{-zt} t^{a-1} (t-i)^{m-a} \, dt$$

Near the lower limit with $1/2 + m/2 - is/2 = a$

$$i^a \int_0^{\epsilon} e^{-zt} t^{a-1} (1+it)^{m-a} \, dt \sim i^a \int_0^{\epsilon} (1 - zt + \cdots) t^{a-1} (1 + it(m-a) + \cdots) \, dt$$

$$\sim (i\epsilon)^a \left[\frac{1}{a} + \frac{(i(m-a) - z)\epsilon}{a+1} + \cdots \right]$$

Near the upper limit

$$i^m \int_R^{\infty} e^{-zt} t^{m-1} (1 - i/t)^{m-a} \, dt \sim i^m \int_R^{\infty} e^{-zt} t^{m-1} (1 - i(m-a)/t + \cdots) \, dt$$

$$\sim i^m \left[\frac{1}{z} e^{-zR} R^{m-1} + \frac{m-1}{z} \int_R^{\infty} e^{-zt} t^{m-2} dt \right] - i^{m+1} (m-a) \left[\frac{1}{z} e^{-zR} R^{m-2} + \frac{m-2}{z} \int_R^{\infty} e^{-zt} t^{m-3} dt \right] + \cdots$$

$$\sim i^m \left[\frac{1}{z} e^{-zR} R^{m-1} + \frac{m-1}{z^2} e^{-zR} R^{m-2} + \frac{(m-1)(m-2)}{z^2} \int_R^{\infty} e^{-zt} t^{m-3} dt \right] - i^{m+1} \left(\frac{m-a}{z} \right) e^{-zR} R^{m-2} + \cdots$$

$$\sim \frac{1}{z} i^m e^{-zR} R^{m-1} \left[1 + \frac{(m-1)/z - i(m-a)}{R} \right] + \cdots$$

The Bessel series can also be used [11]

$$M\left(a,b,z\right)=e^{z/2}\Gamma\left(b-a-1/2\right)\left(z/4\right)^{a-b+1/2}$$

$$\sum_{n=0}^{\infty}\frac{\left(2b-2a-1\right)_n\left(b-2a\right)_n\left(b-a-1/2+n\right)}{n!\left(b\right)_n}\left(-1\right)^n I_{b-a-1/2+n}\left(z/2\right) \qquad (62)$$

3.2.4 Form For Vanishing Argument

The limit $s=0$ is of interest.

$$W_+\left(\tau,m,0\right)=W_{0,m/2}\left(-i\tau^2/2\right)$$

$$=e^{i\tau^2/4}\left(-i\tau^2/2\right)^{1/2+m/2}U\left(1/2+m/2,1+m,-i\tau^2/2\right)$$

$$=e^{i\tau^2/4}\left(-i\tau^2/2\right)^{1/2+m/2}\frac{\sqrt{\pi}}{2}e^{i(m+1)\pi/2-i\tau^2/4}\left(\tau^2/2\right)^{-m/2}H^{(1)}_{m/2}\left(\tau^2/4\right)$$

$$=\frac{1}{2}\tau\sqrt{\pi/2}e^{i(m+1)\pi/4}H^{(1)}_{m/2}\left(\tau^2/4\right) \qquad (63)$$

Now if we have $m=0$ in addition, this becomes

$$W_+\left(\tau,0,0\right)=\frac{1}{2}\tau\sqrt{\pi/2}e^{i\pi/4}H^{(1)}_0\left(\tau^2/4\right) \qquad (64)$$

3.2.5 Value Near Origin

The value of the Whittaker function and its derivative are now considered at the origin.

$$W_+\left(\tau,m,s\right)=W_{is/2,m/2}\left(-i\tau^2/2\right)=e^{i\tau^2/4}\left(-i\tau^2/2\right)^{1/2+m/2}U\left(1/2+m/2-is/2,1+m,-i\tau^2/2\right)$$

$$W'_+\left(0,m,s\right)=\lim_{\tau\to 0}\left[-i\tau W'_{is/2,m/2}\left(-i\tau^2/2\right)\right]$$

To get a feel for these let us expand the function in a power series about the origin.

$$W_+\left(\tau,m,s\right)=W_{is/2,m/2}\left(-i\tau^2/2\right)$$

$$=e^{i\tau^2/4}\left(-i\tau^2/2\right)^{1/2+m/2}U\left(1/2+m/2-is/2,1+m,-i\tau^2/2\right)$$

The power series form can be found from

$$U\left(a,m+1,z\right)=\frac{\left(-1\right)^{m+1}}{m!\Gamma\left(a-m\right)}\left[M\left(a,m+1,z\right)\ln z\right.$$

$$+\sum_{j=0}^{\infty}\frac{(a)_j\,z^j}{(m+1)_j\,j!}\left\{\psi\left(a+j\right)-\psi\left(1+j\right)-\psi\left(1+m+j\right)\right\}\Bigg]$$

$$+\frac{(m-1)!}{\Gamma\left(a\right)}z^{-m}M\left(a-m,1-m,z\right)_m$$

$$M\left(a-m,1-m,z\right)_m=\sum_{n=0}^{m-1}\frac{(a-m)_n}{(1-m)_n}\frac{z^n}{n!}$$

$$M\left(a,1,z\right)_0=0$$

$$U\left(a,b,z\right)=\frac{\pi}{\sin\left(\pi b\right)}\left[\frac{M\left(a,b,z\right)}{\Gamma\left(1+a-b\right)\Gamma\left(b\right)}-z^{1-b}\frac{M\left(1+a-b,2-b,z\right)}{\Gamma\left(a\right)\Gamma\left(2-b\right)}\right]$$

$$M\left(a,b,z\right)=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{z^n}{n!}$$

Thus

$$U\left(a,m+1,z\right)=\frac{(-1)^{m+1}}{m!\Gamma\left(a-m\right)}\Bigg[\ln z\sum_{n=0}^{\infty}\frac{(a)_n}{(m+1)_n}\frac{z^n}{n!}$$

$$+\sum_{j=0}^{\infty}\frac{(a)_j\,z^j}{(m+1)_j\,j!}\left\{\psi\left(a+j\right)-\psi\left(1+j\right)-\psi\left(1+m+j\right)\right\}\Bigg]$$

$$+\frac{(m-1)!}{\Gamma\left(a\right)}z^{-m}\sum_{n=0}^{m-1}\frac{(a-m)_n}{(1-m)_n}\frac{z^n}{n!}$$

$$W_+\left(\tau,m,s\right)=e^{i\tau^2/4}\left(-i\tau^2/2\right)^{1/2+m/2}U\left(1/2+m/2-is/2,1+m,-i\tau^2/2\right)$$

$$\sim\left(-i\tau^2/2\right)^{1/2-m/2}\frac{(m-1)!}{\Gamma\left(1/2+m/2-is/2\right)}\ ,\ \tau\to 0\ ,\ m\neq 0$$

$$\sim\frac{1}{\Gamma\left(1-is/2\right)}+O\left(\tau^2\ln\tau\right)\ ,\ \tau\to 0\ ,\ m=1$$

$$\sim-\left(-i\tau^2/2\right)^{1/2}\frac{1}{\Gamma\left(1/2-is/2\right)}\left[\ln\left(-i\tau^2/2\right)\right.$$

$$\left.+\left\{\psi\left(1/2-is/2\right)-2\psi\left(1\right)\right\}\right]\ ,\ \tau\to 0\ ,\ m=0 \tag{65}$$

and

$$W'_{+}(\tau,m,s) \sim -i\tau\left(-i\tau^2/2\right)^{-1/2-m/2}\frac{(1/2-m/2)\,(m-1)!}{\Gamma(1/2+m/2-is/2)}\ ,\ \tau\to 0\ ,\ m\neq 0$$

$$\sim i\tau\frac{1}{2}\left(-i\tau^2/2\right)^{-1/2}\frac{1}{\Gamma(1/2-is/2)}\left[\ln\left(-i\tau^2/2\right)+\{\psi(1/2-is/2)-2\psi(1)\}+2\right]\ ,\ \tau\to 0\ ,\ m=0 \quad (66)$$

The higher order singularities for $m>0$ are reminiscent of the Hankel function in cylindrical coordinates. The reflected phase would need to cancel these just as the addition of the Hankel function of the second kind leads to Bessel functions. The introduction of a multipolar source at the origin would generate such singularities away from resonance. Thus with the solution

$$\psi_m(\tau,s) = c_m\frac{\sqrt{2}}{\tau}\,\mathrm{Re}\left[W_{+}+e^{i\Phi_0}W_{+}^{*}\right] \quad (67)$$

if we impose

$$\tau\frac{\partial\psi_m}{\partial\tau} = c_m\sqrt{2}\,\mathrm{Re}\left[W'_{+}+e^{i\Phi_0}W'^{*}_{+}\right]-\psi_m\to 0\ ,\ \tau\to 0 \quad (68)$$

then

$$c_m\sqrt{2}\,\mathrm{Re}\left[\left(W'_{+}-\frac{1}{\tau}W_{+}\right)+e^{i\Phi_0}\left(W'_{+}-\frac{1}{\tau}W_{+}\right)^{*}\right]=0 \quad (69)$$

or by taking the sum of the function and its conjugate

$$\left(1+e^{-i\Phi_0}\right)\left(W'_{+}-\frac{1}{\tau}W_{+}\right)+\left(1+e^{i\Phi_0}\right)\left(W'_{+}-\frac{1}{\tau}W_{+}\right)^{*}=0$$

and

$$e^{i\Phi_0} = -\frac{\left(W'_{+}-\frac{1}{\tau}W_{+}\right)}{\left(W'_{+}-\frac{1}{\tau}W_{+}\right)^{*}} \quad (70)$$

Thus to leading order we expect the form of the reflection phase to be

$$e^{i\Phi_0}\sim e^{i\pi(1+m)/2}\frac{\Gamma(1/2+m/2+is/2)}{\Gamma(1/2+m/2-is/2)} \quad (71)$$

For general values of m a more accurate treatment of the asymptotic form of W_{+} is required to cancel all singular terms on the axis.

Value Near The Origin For m=0 Function Using the expansions for $m=0$ we can write

$$W_{+}(\tau,0,s) = -e^{i\tau^2/4}\left(-i\tau^2/2\right)^{1/2}\frac{1}{\Gamma(1/2-is/2)}\left[\sum_{n=0}^{\infty}\frac{(1/2-is/2)_n}{n!^2}\left(-i\tau^2/2\right)^n\ln\left(-i\tau^2/2\right)\right.$$

$$\left.+\sum_{j=0}^{\infty}\frac{(1/2-is/2)_j\left(-i\tau^2/2\right)^j}{j!^2}\{\psi(1/2-is/2+j)-2\psi(1+j)\}\right] \quad (72)$$

and then near the origin we have

$$W_+(\tau,0,s) \sim -\left(-i\tau^2/2\right)^{1/2} \frac{1}{\Gamma(1/2-is/2)} \left[\ln\left(-i\tau^2/2\right)\left(1-s\tau^2/4\right)\right.$$

$$\left.+\psi(1/2-is/2) - 2\psi(1) + \left(s\tau^2/4\right)\{-\psi(1/2-is/2)+2\psi(1)+2\} + O\left(\tau^4\ln\tau\right)\right] \tag{73}$$

$$W_+'(\tau,0,s) \sim i\tau\frac{1}{2}\left(-i\tau^2/2\right)^{-1/2}\frac{1}{\Gamma(1/2-is/2)}$$

$$\left[\ln\left(-i\tau^2/2\right) + \{\psi(1/2-is/2)-2\psi(1)\} + 2 + O\left(\tau^2\ln\tau\right)\right]\ ,\ \tau\to 0\ ,\ m=0 \tag{74}$$

3.2.6 Wronskian

The Wronskian for these functions is

$$W_+'W_+^* - W_+W_+^{*\prime} =$$

$$-i\tau\left(\frac{1/2+m/2}{-i\tau^2/2}-1/2\right)e^{i\tau^2/4}\left(-i\tau^2/2\right)^{1/2+m/2}U\left(1/2+m/2-is/2,1+m,-i\tau^2/2\right)$$

$$e^{-i\tau^2/4}\left(i\tau^2/2\right)^{1/2+m/2}U\left(1/2+m/2+is/2,1+m,i\tau^2/2\right)$$

$$-i\tau e^{i\tau^2/4}\left(-i\tau^2/2\right)^{1/2+m/2}U'\left(1/2+m/2-is/2,1+m,-i\tau^2/2\right)$$

$$e^{-i\tau^2/4}\left(i\tau^2/2\right)^{1/2+m/2}U\left(1/2+m/2+is/2,1+m,i\tau^2/2\right)$$

$$-i\tau\left(\frac{1/2+m/2}{i\tau^2/2}-1/2\right)e^{i\tau^2/4}\left(-i\tau^2/2\right)^{1/2+m/2}U\left(1/2+m/2-is/2,1+m,-i\tau^2/2\right)$$

$$e^{-i\tau^2/4}\left(i\tau^2/2\right)^{1/2+m/2}U\left(1/2+m/2+is/2,1+m,i\tau^2/2\right)$$

$$-i\tau e^{i\tau^2/4}\left(-i\tau^2/2\right)^{1/2+m/2}U\left(1/2+m/2-is/2,1+m,-i\tau^2/2\right)$$

$$e^{-i\tau^2/4}\left(i\tau^2/2\right)^{1/2+m/2}U'\left(1/2+m/2+is/2,1+m,i\tau^2/2\right)$$

$$= -i\tau\left(\tau^2/2\right)^{1+m}\left[-U\left(1/2+m/2-is/2,1+m,-i\tau^2/2\right)\right.$$

$$U\left(1/2+m/2+is/2,1+m,i\tau^2/2\right)$$

$$+U'\left(1/2+m/2-is/2,1+m,-i\tau^2/2\right)$$

$$U\left(1/2+m/2+is/2,1+m,i\tau^2/2\right)$$

$$+U\left(1/2+m/2-is/2,1+m,-i\tau^2/2\right)$$

$$U'\left(1/2+m/2+is/2,1+m,i\tau^2/2\right)\big]$$

Now noting that [11]

$$U\left(a,b,z\right)e^zU\left(b-a,b,-z\right)$$

$$-U\left(a,b,z\right)e^zU'\left(b-a.b.-z\right)-U'\left(a,b,z\right)e^zU\left(b-a,b,-z\right)$$

$$=e^{\varepsilon\pi i(b-a)}z^{-b}e^z\ ,\ \varepsilon=\mathrm{sgn}\left(\mathrm{Im}\left(z\right)\right)$$

we have

$$-UU^*+U'U^*+UU^{*\prime}=-e^{-\pi i(1/2+m/2+is/2)}\left(-i\tau^2/2\right)^{-1+m}$$

and thus the Wronskian of the solutions is

$$W_+'W_+^*-W_+W_+^{*\prime}=i\tau e^{-\pi i(m+is/2)}\left(\tau^2/2\right)^{2m} \tag{75}$$

3.2.7 The Axisymmetric Field

We will be focused on the axisymmetric case since we expect this will generate the largest values on axis. The case where $m=0$ results in

$$u=\frac{2}{\cos\xi}\psi_0\left(\tau,s\right)\cos\left(\gamma\sin\xi-s\sigma\right) \tag{76}$$

where

$$\psi_0=c_0\frac{\sqrt{2}}{\tau}\,\mathrm{Re}\left[W_++e^{i\Phi_0}W_+^*\right] \tag{77}$$

and

$$W_+\left(\tau,0,s\right)\sim-\left(-i\tau^2/2\right)^{1/2}\frac{1}{\Gamma\left(1/2-is/2\right)}\left[\ln\left(-i\tau^2/2\right)\right.$$

$$\left.+\left\{\psi\left(1/2-is/2\right)-2\psi\left(1\right)\right\}+O\left(\tau^2\ln\tau\right)\right]\ ,\ \tau\to0\ ,\ m=0$$

$$W_+'(\tau,0,s) \sim i\tau\frac{1}{2}\left(-i\tau^2/2\right)^{-1/2}\frac{1}{\Gamma(1/2-is/2)}$$

$$\left[\ln\left(-i\tau^2/2\right) + \{\psi(1/2-is/2) - 2\psi(1)\} + 2 + O\left(\tau^2\ln\tau\right)\right]\ ,\ \tau\to 0\ ,\ m=0$$

Because of the symmetry of the $m=0$ scalar field around the center of the orbit, we apply the condition

$$\tau\frac{\partial}{\partial\tau}\psi_0 = c_0\sqrt{2}\,\mathrm{Re}\left[W_+' + e^{i\Phi_0}W_+'^*\right] - \psi_0$$

$$= c_0\sqrt{2}\,\mathrm{Re}\left[\left(W_+' + e^{i\Phi_0}W_+'^*\right) - \frac{1}{\tau}\left(W_+ + e^{i\Phi_0}W_+^*\right)\right] \to 0 \tag{78}$$

which gives

$$e^{i\Phi_0} = -\frac{\left(W_+' - \frac{1}{\tau}W_+\right)}{\left(W_+' - \frac{1}{\tau}W_+\right)^*} =$$

$$\frac{\left[\ln\left(-i\tau^2/2\right) + \{\psi(1/2-is/2) - 2\psi(1)\} + 2\right] - \left[\ln\left(-i\tau^2/2\right) + \{\psi(1/2-is/2) - 2\psi(1)\}\right]}{\left[\ln\left(i\tau^2/2\right) + \{\psi(1/2+is/2) - 2\psi(1)\} + 2\right] - \left[\ln\left(i\tau^2/2\right) + \{\psi(1/2+is/2) - 2\psi(1)\}\right]}$$

$$i\frac{\Gamma(1/2+is/2)}{\Gamma(1/2-is/2)} + O\left(\tau^2\right)\ ,\ \tau\to 0$$

$$= i\frac{\Gamma(1/2+is/2)}{\Gamma(1/2-is/2)} + O\left(\tau^2\right) \tag{79}$$

It is interesting that if we had imposed the condition

$$\mathrm{Re}\left[W_+' + e^{i\Phi_0}W_+^{*\prime}\right] \to 0\,,\ \tau\to 0$$

we arrive at the same reflection phase in the limit (although the accuracy is not as great since it is only logarithmic here)

$$e^{i\Phi_0} = -\frac{W_+'(\tau,0,s)}{W_+^{*\prime}(\tau,0,s)} \to i\frac{\Gamma(1/2+is/2)}{\Gamma(1/2-is/2)}$$

Note also that because of the actual order of error $O\left(\tau^2\right)$ we actually have $\partial\psi_0/\partial\tau \to 0$ as $\tau\to 0$. The value of the function

$$\psi_0(\tau,s) \sim c_0\frac{\sqrt{2}}{\tau}\,\mathrm{Re}\left[W_+(\tau,0,s) + i\frac{\Gamma(1/2+is/2)}{\Gamma(1/2-is/2)}W_+^*(\tau,0,s)\right]$$

near $\tau=0$ is

$$\psi_0(\tau,s) \sim c_0\,\mathrm{Re}\left[\frac{e^{-i\pi/4}}{\Gamma(1/2-is/2)}\{i\pi + \psi(1/2+is/2) - \psi(1/2-is/2)\} + O\left(\tau^2\ln\tau\right)\right]\ ,\ \tau\to 0$$

$$= c_0 \operatorname{Re}\left[e^{-i\pi/4}\frac{1}{\Gamma(1/2 - is/2)}\{i\pi + \pi\cot\pi(1/2 - is/2)\} + O\left(\tau^2\ln\tau\right)\right]$$

$$= c_0 \operatorname{Re}\left[e^{-i\pi/4}\frac{1}{\Gamma(1/2 - is/2)}\{i\pi + \pi\tan(i\pi s/2)\} + O\left(\tau^2\ln\tau\right)\right]$$

$$= c_0 \operatorname{Re}\left[e^{-i\pi/4}\frac{i\pi}{\Gamma(1/2 - is/2)}\{1 + \tanh(\pi s/2)\} + O\left(\tau^2\ln\tau\right)\right]$$

$$= c_0 \operatorname{Re}\left[e^{-i\pi/4}\frac{i\pi}{\Gamma(1/2 - is/2)}\frac{e^{\pi s/2}}{\cosh(\pi s/2)} + O\left(\tau^2\ln\tau\right)\right]$$

$$= c_0 \operatorname{Re}\left[e^{i\pi/4}e^{\pi s/2}\Gamma(1/2 + is/2) + O\left(\tau^2\ln\tau\right)\right] \tag{80}$$

In addition we use

$$W_+'(\tau,0,s) - \frac{1}{\tau}W_+(\tau,0,s) =$$

$$-\frac{e^{-i\pi/4}\sqrt{2}}{\Gamma(1/2 - is/2)}\left[1 + O\left(\tau^2\ln\tau\right)\right] \ , \ \tau \to 0$$

and

$$\tau\frac{\partial}{\partial\tau}\psi_0 = c_0\sqrt{2}\operatorname{Re}\left[W_+' + e^{i\Phi_0}W_+'^*\right] - \psi_0 \tag{81}$$

to arrive at

$$\tau\frac{\partial^2}{\partial\omega\partial\tau}\psi_0 \sim c_0\sqrt{2}\operatorname{Re}\left[i\frac{\partial\Phi_0}{\partial\omega}e^{i\Phi_0}\left(W_+' - W_+/\tau\right)^*\right]$$

$$\sim c_0\sqrt{2}\frac{\partial\Phi_0}{\partial\omega}\operatorname{Re}\left[e^{i\Phi_0}\frac{e^{-i\pi/4}\sqrt{2}}{\Gamma(1/2 + is/2)} + O\left(\tau^2\right)\right] \tag{82}$$

For $\tau \to 0$ ($\tau^2 = 2v$) this also means that the energy flux from the center vanishes (see the intensity vector in the acoustic section below)

$$\tau u\frac{\partial u}{\partial\tau} \to 0 \ , \ \tau \to 0$$

In fact in this case $u\partial u/\partial\tau \to 0$.

3.2.8 Eigenvalue Density

The density of cavity eigenvalues will be needed in the normalization sections of this report. Here we simply review results for three-dimensions as well as the axisymmetric problem.

Three Dimensions From Courant [15] the number of eigenvalues is

$$N \sim k^3V/\left(6\pi^2\right) \tag{83}$$

and thus

$$\frac{dN}{dk} \sim k^2 V / \left(2\pi^2\right)$$

or

$$\frac{dk^2}{dN} = 2k\frac{\partial k}{\partial N} \sim 4\pi^2 / \left(kV\right) \tag{84}$$

In the vector case we have twice the number

$$N \sim k^3 V / \left(3\pi^2\right) \tag{85}$$

Axisymmetric Problem The number of eigenvalues in an axisymmetric 3D problem is now discussed. The differential equation

$$\left(\nabla^2 + k^2\right) u = 0 \tag{86}$$

in the cylindrical coordinate system is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\varphi^2} + k^2 u = 0 \tag{87}$$

Now take u to have dependence $\cos\left(m\varphi\right)$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) - \frac{m^2}{\rho^2}u + k^2 u = 0 \tag{88}$$

In the terminology of Courant and Hilbert [15] we can write this as

$$\frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(p\frac{\partial u}{\partial y}\right) - qu + \lambda\rho u = 0$$

where $(p > 0, \rho > 0)$

$$p = \rho = x$$

$$\lambda = k^2$$

$$q = m^2/\rho$$

with boundary condition

$$\frac{\partial u}{\partial n} + \sigma u = 0\ ,\ \ \sigma \geq 0$$

The number of eigenvalues is asymptotically equal to

$$\lim_{\lambda\to\infty}\frac{N(\lambda)}{\lambda}=\frac{1}{4\pi}\int\int_G\frac{\rho}{p}dxdy=\frac{1}{4\pi}\int\int_G dxdy=\frac{A_0}{4\pi}$$

where A_0 represents the area of the cavity in the original ρ, z space (with $\rho > 0$). Translating back to our original notation gives

$$N\sim\frac{k^2}{4\pi}A_0 \tag{89}$$

Now taking the derivative

$$\frac{dN}{dk}=\frac{dk^2}{dk}\frac{dN}{dk^2}=2k\frac{dN}{dk^2}\sim\frac{k}{2\pi}A_0$$

$$\frac{dN}{dk^2}\sim\frac{A_0}{4\pi} \tag{90}$$

Setting the total cross sectional area of the axisymmetric cavity to

$$A=2A_0 \tag{91}$$

we find

$$\frac{dk^2}{dN}=8\pi/A \tag{92}$$

We would have twice this spacing if we restrict attention to odd or even along the orbit and thus

$$\Delta k^2\sim 16\pi/A \tag{93}$$

In a vector problem we would expect twice the number

$$N\sim\frac{k^2}{2\pi}A_0=\frac{k^2}{4\pi}A \tag{94}$$

Now taking the derivative

$$\frac{dN}{dk}=\frac{dk^2}{dk}\frac{dN}{dk^2}=2k\frac{dN}{dk^2}\sim\frac{k}{\pi}A_0$$

$$\frac{dN}{dk^2}\sim\frac{A}{4\pi} \tag{95}$$

This is the same density as found in a Cartesian 2D cavity of area A in the scalar case. As to why this electromagnetic density is the same as the scalar case: evidently the number from Courant and Hilbert corresponds to only one azimuthal parity. This is actually what we desire in the axisymmetric geometry, since we will be focusing on only one parity. Evenness along the orbit would result in a further reduction by a factor of two

$$\frac{dN}{dk^2}\sim\frac{A}{8\pi} \tag{96}$$

3.2.9 Bowtie Area & Volume

The rotationally symmetric bowtie cavity area and volume are given here since they are frequently used.

Bowtie Cavity Cross Sectional Area The area of a bowtie cavity is now given. Referring to Figure 4, the area of the red right triangle is

$$A_r = \frac{1}{2}\left(L_x/2 + R_x\right)\left(L_y/2 + R_y\right) \tag{97}$$

The vertex of the green triangle must now be located. Letting

$$c^2 = \left(L_x/2 + R_x\right)^2 + \left(L_y/2 + R_y\right)^2 \tag{98}$$

and using the law of cosines

$$\frac{R_x^2 + R_y^2 - c^2}{2R_xR_y} = \cos\chi = \sin\left(\frac{\pi}{2} - \chi\right) \tag{99}$$

or

$$\frac{L_x^2/4 + R_xL_x + L_y^2/4 + R_yL_y}{2R_xR_y} = \sin\left(\chi - \frac{\pi}{2}\right) \tag{100}$$

Next the law of sines gives

$$\frac{\sin\alpha}{R_y} = \frac{\sin\beta}{R_x} = \frac{\sin\chi}{c} \tag{101}$$

The area of the green triangle is

$$A_g = \frac{1}{2}cR_x\sin\alpha = \frac{1}{2}cR_y\sin\beta = \frac{1}{2}R_xR_y\sin\chi \tag{102}$$

The area of the R_y circle included in the red triangle is

$$A_y = \frac{1}{2}R_y^2\left[\arcsin\left(\frac{L_x/2 + R_x}{c}\right) - \beta\right] \tag{103}$$

The area of the R_x circle included in the red triangle is

$$A_x = \frac{1}{2}R_x^2\left[\arcsin\left(\frac{L_y/2 + R_y}{c}\right) - \alpha\right] \tag{104}$$

One quarter area of the bowtie cavity is then

$$A/4 = A_r - A_g - A_x - A_y \tag{105}$$

As an example if we take $L_x = L_y = 2$ m, $R_x = 1.5$ m, and $R_y = 10$ m then $A_r = 55/4$ m^2, $c = 11.28051$ m, $\sin(\chi - \pi/2) = 5/6$, $\sin\chi = 0.5527708$, $A_g = 4.145781$ m^2, $\alpha = 0.5121158$, $A_x = 0.9396044$ m^2, $\beta = 0.07356975$, $A_y = 7.495343$ m^2, and

$$A = 4.677086 \text{ m}^2 \tag{106}$$

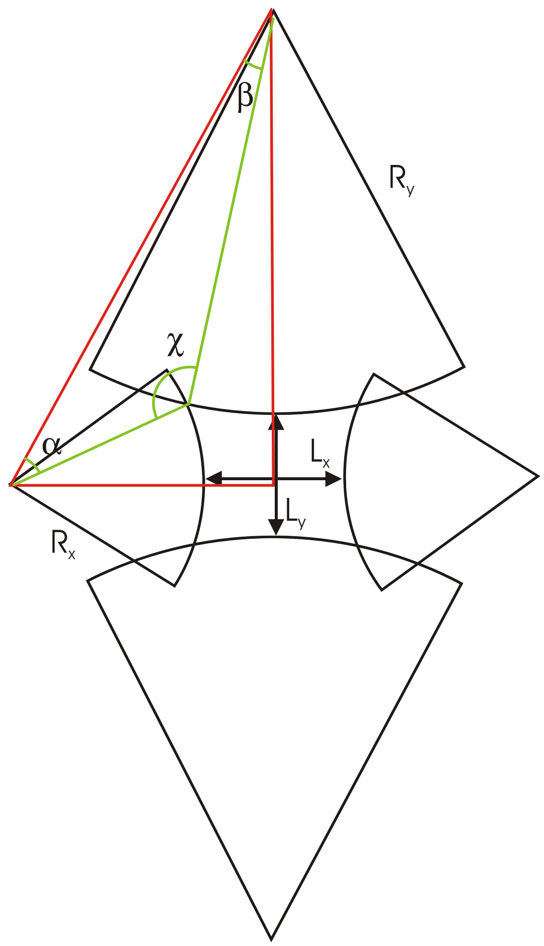

Figure 4. Geometry of bow tie cavity and circular walls used in the calculation of the interior cross sectional area.

Volume Of Rotationally Symmetric Bowtie The parameters are $R_x = 1.5$ m, $R_y = 10$ m, and $L_x = L_y = 2$ m. The volume of the rotationally symmetric bowtie is found by using the volume of a right angle cone

$$V = \frac{1}{3}\pi r^2 h$$

and that of a spherical cap

$$V = \frac{1}{3}\pi h^2 (3r - h)$$

The red cone has volume

$$V_{rc} = \frac{1}{3}\pi (R_x + L_x/2)^2 (R_y + L_y/2) \approx 71.9948\,(1)\ \text{m}^3 \tag{107}$$

The top angle of the red cone is

$$\theta = \arctan\left(\frac{R_x + L_x/2}{R_y + L_y/2}\right) \approx 0.223477\,(2) \tag{108}$$

The yellow right angle cone has volume

$$V_{yc} = \frac{1}{3}\pi R_y^2 \cos^2(\theta - \beta)\tan^2\theta R_y \cos(\theta - \beta) \approx 52.2912\,(3)\ \text{m}^3 \tag{109}$$

The spherical cap volume is ($\beta = 0.07356975$)

$$V_s = \frac{1}{3}\pi R_y^3 (1 - \cos(\theta - \beta))^2 [3 - (1 - \cos(\theta - \beta))] \approx 0.393661\,(4)\ \text{m}^3 \tag{110}$$

The volume of a rectangle of height h ,extending from r_0 to $r_0 + l$, spun around the y axis is

$$V_{rec} = 2\pi h \int_{r_0}^{r_0+l} r dr = \pi h \left[(r_0 + l)^2 - r_0^2\right] = \pi h l (2r_0 + l) \approx 14.0441\,(6)\ \text{m}^3 \tag{111}$$

where

$$r_0 = R_y \sin(\theta - \beta) \approx 1.49346\,(5)\ \text{m} \tag{112}$$

$$h = L_y/2 + R_y (1 - \cos(\theta - \beta)) \approx 1.11215\ \text{m} \tag{113}$$

$$l = R_x + L_x/2 - r_0 \approx 1.00654\ \text{m} \tag{114}$$

The rotation of a circular cap of height h, base position r_0, and radius r_1 gives

$$V_c = 2\pi \int_{r_0}^{r_0+h} r dr \int_0^{\sqrt{r_1^2 - (r_1 + r_0 - r)^2}} dy = 2\pi \int_{r_0}^{r_0+h} \sqrt{r_1^2 - (r_1 + r_0 - r)^2} r dr$$

$$= 2\pi \int_{r_1-h}^{r_1} \sqrt{r_1^2 - u^2}\,(r_1 + r_0 - u)\,du = 2\pi\,(r_1 + r_0) \int_{r_1-h}^{r_1} \sqrt{r_1^2 - u^2}du - 2\pi \int_{r_1-h}^{r_1} \sqrt{r_1^2 - u^2}udu$$

$$= 2\pi\,(r_1 + r_0) \left[-\frac{r_1 - h}{2} \sqrt{(2r_1 - h)\,h} - \frac{r_1^2}{2} \arcsin\left(\frac{r_1 - h}{r_1}\right) + \frac{\pi}{4} r_1^2 \right] - 2\pi \frac{1}{3} \left(2r_1 h - h^2\right)^{3/2}$$

$$\approx 3.08631\,(7)\ \ \mathrm{m}^3 \tag{115}$$

where

$$r_1 = R_x = 1.5\ \mathrm{m} \tag{116}$$

$$h = R_y \sin(\theta - \beta) - L_x/2 \approx 0.49346\ \mathrm{m} \tag{117}$$

$$r_0 = L_x/2 \approx 1\ \mathrm{m} \tag{118}$$

The bottom triangle can be handled by the following formula

$$V_t = 2\pi \int_{r_0}^{r_0+l} rdr \int_0^{h(r-r_0)/l} dy = \frac{2\pi h}{l} \int_{r_0}^{r_0+l} (r - r_0)\,rdr = \frac{2\pi h}{l} \left[\frac{1}{3} (r_0 + l)^3 - \frac{1}{3} r_0^3 - \frac{1}{2} r_0 (r_0 + l)^2 + \frac{1}{2} r_0^3 \right]$$

$$= \frac{\pi}{3} hl\,(3r_0 + 2l) \approx 2.13341\,(8)\ \ \mathrm{m}^3 \tag{119}$$

where

$$l = h \tan\theta \approx 0.252761\,(9)\ \ \mathrm{m} \tag{120}$$

$$h = R_y\,(1 - \cos(\theta - \beta)) + L_y/2 \approx 1.11215\ \mathrm{m} \tag{121}$$

$$r_0 = R_x + L_x/2 - l \approx 2.24724\ \mathrm{m} \tag{122}$$

Thus

$$V/2 = V_{rc} - V_{yc} - V_s - V_c - V_{rec} + V_t \approx 4.31296\ \mathrm{m}^3 \tag{123}$$

or

$$V \approx 8.625920602\ \mathrm{m}^3 \tag{124}$$

3.3 Acoustic Energy Normalization

A scalar physical wave problem of interest is acoustics. The equations are [14]

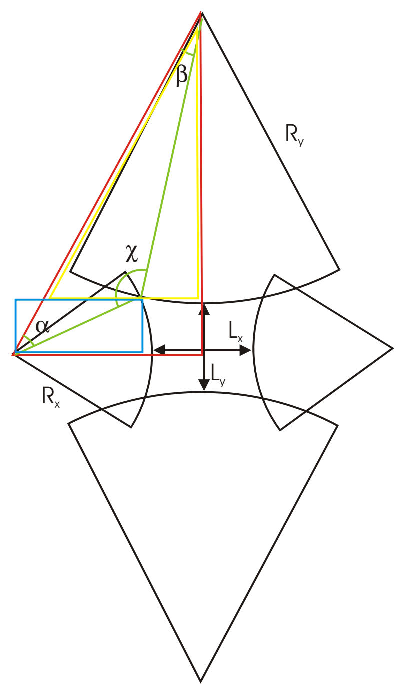

Figure 5. Geometry for the calculation of the rotationally symmetric bow tie volume.

$$\rho \frac{\partial}{\partial t} \underline{u} = -\nabla P \tag{125}$$

$$\kappa \frac{\partial}{\partial t} P = -\nabla \cdot \underline{u} \tag{126}$$

where $\underline{u}$ is the particle velocity, P is the pressure, κ is the compressibility, and ρ is the density. Our scalar function u corresponds to the pressure P in this subsection. Thus eliminating the velocity gives the scalar wave equation

$$\rho \frac{\partial}{\partial t} \left(\kappa \frac{\partial}{\partial t} P \right) = \nabla^2 P \tag{127}$$

Let us suppress time harmonic dependence $e^{-i\omega t}$

$$i\omega\rho\underline{u} = \nabla P \tag{128}$$

$$i\omega\kappa P = \nabla \cdot \underline{u} \tag{129}$$

and with P taking the place of the scalar u

$$\nabla^2 P + \omega^2 \kappa\rho P = \left(\nabla^2 + k^2\right) P = 0 \tag{130}$$

Note that the acoustic energy flux is represented by the intensity [14]

$$\underline{I} = P\underline{u} = P\nabla P / \left(i\omega\rho\right) \tag{131}$$

Now examine the quantity

$$\nabla \cdot \left(\frac{\partial P}{\partial \omega} \underline{u}^* + P^* \frac{\partial \underline{u}}{\partial \omega} \right) = \frac{\partial P}{\partial \omega} \nabla \cdot \underline{u}^* + \frac{\partial \nabla P}{\partial \omega} \cdot \underline{u}^* + \nabla P^* \cdot \frac{\partial \underline{u}}{\partial \omega} + P^* \frac{\partial \nabla \cdot \underline{u}}{\partial \omega}$$

$$= -\frac{\partial P}{\partial \omega} i\omega\kappa P^* + i\rho \left(\underline{u} + \omega \frac{\partial \underline{u}}{\partial \omega} \right) \cdot \underline{u}^* - i\omega\rho\underline{u}^* \cdot \frac{\partial \underline{u}}{\partial \omega} + i\kappa P^* \left(\omega \frac{\partial P}{\partial \omega} + P \right)$$

$$= i \left(\rho \underline{u} \cdot \underline{u}^* + \kappa P^* P \right) = i \left(\frac{1}{\omega^2 \rho} \left| \nabla P \right|^2 + \kappa \left| P \right|^2 \right) \tag{132}$$

or

$$\nabla \cdot \left[\frac{\partial P}{\partial \omega} \left(\frac{1}{\omega} \nabla P^* \right) - P^* \frac{\partial}{\partial \omega} \left(\frac{1}{\omega} \nabla P \right) \right] = \frac{1}{\omega^2} \left(\left| \nabla P \right|^2 + \omega^2 \rho\kappa \left| P \right|^2 \right)$$

$$= \frac{1}{\omega^2} \left(\left| \nabla P \right|^2 + k^2 \left| P \right|^2 \right) \tag{133}$$

where the wavenumber is

$$k^2 = \omega^2 \rho\kappa \tag{134}$$

Now using

$$|\nabla P|^2 = \nabla P \cdot \nabla P^* = \nabla \cdot (P^* \nabla P) - P^* \nabla^2 P = \nabla \cdot (P^* \nabla P) + k^2 |P|^2$$

for either the soft outer boundary with $P = 0$ or the hard outer boundary with $\partial P/\partial n = 0$ gives

$$\oint_{S_{scar}} \left[\frac{\partial P}{\partial \omega} \left(\frac{1}{\omega} \nabla P^* \right) - P^* \frac{\partial}{\partial \omega} \left(\frac{1}{\omega} \nabla P \right) \right] \cdot \underline{n} dS = 2 \frac{k^2}{\omega^2} \int_V |P|^2 \, dV$$

$$= \oint_{S_{scar}} \left[\frac{\partial P}{\partial \omega} \left(\frac{1}{\omega} \frac{\partial P^*}{\partial n} \right) - P^* \frac{\partial}{\partial \omega} \left(\frac{1}{\omega} \frac{\partial P}{\partial n} \right) \right] dS \tag{135}$$

If the pressure is taken to be real

$$\oint_{S_{scar}} \left[\frac{\partial P}{\partial \omega} \left(\frac{1}{\omega} \frac{\partial P}{\partial n} \right) - P \frac{\partial}{\partial \omega} \left(\frac{1}{\omega} \frac{\partial P}{\partial n} \right) \right] dS$$

$$= 2 \frac{k^2}{\omega^2} \int_V |P|^2 \, dV \tag{136}$$

In addition if we take

$$\frac{\partial P}{\partial n} = 0 \text{ , on } S_{scar} \tag{137}$$

then

$$- \oint_{S_{scar}} \left[P \frac{\partial^2 P}{\partial \omega \partial n} \right] dS = 2 \frac{k^2}{\omega} \int_V |P|^2 \, dV \tag{138}$$

where here n points into the scar region.

3.3.1 Application Of Scalar Field Normalization

The above acoustic case of scalar normalization gives the normalization condition

$$\oint_{S_{scar}} \left[\frac{\partial u}{\partial \omega} \left(\frac{1}{\omega} \frac{\partial u^*}{\partial n} \right) - u^* \frac{\partial}{\partial \omega} \left(\frac{1}{\omega} \frac{\partial u}{\partial n} \right) \right] dS$$

$$= 2 \frac{k^2}{\omega^2} \int_V |u|^2 \, dV \tag{139}$$

or for a real function

$$\oint_{S_{scar}} \left[\frac{\partial u}{\partial \omega} \left(\frac{1}{\omega} \frac{\partial u}{\partial n} \right) - u \frac{\partial}{\partial \omega} \left(\frac{1}{\omega} \frac{\partial u}{\partial n} \right) \right] dS = 2 \frac{k^2}{\omega} \int_V |u|^2 \, dV \tag{140}$$

If we set

$$\frac{\partial u}{\partial n} = 0 \text{ , on } S_{scar} \tag{141}$$

then

$$2\frac{k^2}{\omega}\int_V |u|^2\,dV = -\int_{S_{scar}} u^2\frac{\partial}{\partial\omega}\left(\frac{\partial u}{\partial n}/u\right)dS = -\int_{S_{scar}} u\frac{\partial^2 u}{\partial\omega\partial n}dS$$

$$= \int_{-\xi_0}^{\xi_0}\int_0^{2\pi} u\frac{\partial^2 u}{\partial\omega h_\zeta\partial\zeta}h_\varphi d\varphi h_\xi d\xi \sim d\int_{-\xi_0}^{\xi_0}\int_0^{2\pi} u\frac{\partial^2 u}{\partial\omega\partial\zeta}\zeta d\varphi\cos\xi d\xi = d\int_{-\xi_0}^{\xi_0}\int_0^{2\pi} u\frac{\partial^2 u}{\partial\omega\partial\tau}\tau d\varphi\cos\xi d\xi \quad (142)$$

with metric coefficients

$$h_\zeta = h_\xi = d\sqrt{\sinh^2\zeta + \cos^2\xi}$$

$$h_\varphi = d\sinh\zeta\cos\xi$$

where for the axisymmetric field

$$u = \frac{2}{\cos\xi}\psi_0(\tau,s)\cos(\gamma\sin\xi - s\sigma) \quad (143)$$

The normalization in this case, with a unit volume integral, is

$$2\frac{k^2}{\omega} = 2\pi d\int_{-\xi_0}^{\xi_0} u\frac{\partial^2 u}{\partial\omega\partial\tau}\tau\cos\xi d\xi$$

Thus noting

$$\psi_0(\tau,s) = c_0\frac{\sqrt{2}}{\tau}\,\mathrm{Re}\left[W_+ + e^{i\Phi_0}W_+^*\right]$$

$$\psi_0 \sim c_0\,\mathrm{Re}\left[e^{\pi s/2}e^{i\pi/4}\Gamma(1/2+is/2) + O\left(\tau^2\right)\right]\ ,\ \tau\to 0$$

$$\tau\frac{\partial^2}{\partial\omega\partial\tau}\psi_0 \sim c_0\sqrt{2}\frac{\partial\Phi_0}{\partial\omega}\,\mathrm{Re}\left[e^{i\Phi_0}\frac{e^{-i\pi/4}\sqrt{2}}{\Gamma(1/2+is/2)} + O\left(\tau^2\right)\right]\ ,\ \tau\to 0$$

where

$$k\ell - s_p\sigma_0 = (p-1/2)\pi = k_p\ell\,,\ p = 1,2,3,...$$

$$s_p = (k-k_p)\,\ell/\sigma_0 = \frac{(k-k_p)\,L/2}{\mathrm{Arcsinh}\left[\sqrt{L/(2R)}\right]} = 2(k-k_p)\,L/\ln(\Lambda_+)$$

and

$$\sigma = \mathrm{arcsinh}(\tan\xi) = \mathrm{arcsinh}\left(\frac{z}{\sqrt{d^2-z^2}}\right) = \frac{1}{2}\ln\left(\frac{d+z}{d-z}\right)$$

$$\sinh\sigma = \tan\xi$$

$$\cosh\sigma = \sec\xi$$

$$\tanh\sigma = \sin\xi$$

$$d\sigma = \sec\xi d\xi$$

$$\sigma_0 = \operatorname{arcsinh}(\tan\xi_0) = \frac{1}{2}\ln\left(\frac{d+\ell}{d-\ell}\right) = \frac{1}{4}\ln(\Lambda_+)$$

$$\Lambda_\pm = \left(\frac{d+\ell}{d-\ell}\right)^{\pm 2}$$

$$\ell = d\sin\xi_0$$

we find

$$2\frac{k^2}{\omega} = 2\pi d\int_{-\xi_0}^{\xi_0} u\frac{\partial^2 u}{\partial\omega\partial\tau}\tau\cos\xi d\xi$$

$$\sim 8\pi d\int_{-\xi_0}^{\xi_0}\psi_0(0,s_p)\,c_0\sqrt{2}\operatorname{Re}\left[ie^{i\Phi_0}\left(W_+' - W_+/\tau\right)^*\right]\frac{\partial\Phi_0}{\partial\omega}\cos^2(\gamma\sin\xi - s\sigma)\frac{d\xi}{\cos\xi}$$

$$\sim 16\pi d\psi_0(0,s_p)\,c_0\sqrt{2}\operatorname{Re}\left[ie^{i\Phi_0}\left(W_+' - W_+/\tau\right)^*\right]\frac{\partial\Phi_0}{\partial\omega}\int_0^{\sigma_0}\cos^2(\gamma\tanh\sigma - s\sigma)\,d\sigma$$

$$\sim -8\pi d\psi_0(0,s_p)\,c_0\sqrt{2}\operatorname{Re}\left[i\left(W_+' - \frac{1}{\tau}W_+\right)\right]\frac{\partial\Phi_0}{\partial\omega}\sigma_0$$

$$\sim 16\pi dc_0^2\operatorname{Re}\left[e^{\pi s_p/2}e^{i\pi/4}\Gamma(1/2 + is_p/2)\right]\operatorname{Re}\left[\frac{e^{i\pi/4}}{\Gamma(1/2 - is_p/2)}\right]\frac{\partial\Phi_0}{\partial\omega}\sigma_0$$

$$\sim 16\pi dc_0^2 e^{\pi s_p/2}\frac{\operatorname{Re}^2\left[e^{i\pi/4}\Gamma(1/2 + is_p/2)\right]}{|\Gamma(1/2 - is_p/2)|^2}\frac{\partial\Phi_0}{\partial\omega}\sigma_0$$

$$\sim 16\pi dc_0^2 e^{\pi s_p/2}\frac{\operatorname{Re}^2\left[e^{i\pi/4}\Gamma(1/2 + is_p/2)\right]}{|\Gamma(1/2 - is_p/2)|^2}\frac{\partial k^2}{\partial\omega}\frac{\partial\Phi_0}{\partial k^2}\sigma_0$$

$$\sim 32\frac{k^2}{\omega}\pi dc_0^2 e^{\pi s_p/2}\frac{\operatorname{Re}^2\left[e^{i\pi/4}\Gamma(1/2 + is_p/2)\right]}{|\Gamma(1/2 - is_p/2)|^2}\frac{\partial\Phi_0}{\partial k^2}\sigma_0$$

or

$$2\frac{k^2}{\omega}\int_V |u|^2\,dV \sim 32\frac{k^2}{\omega}\pi dc_0^2 e^{\pi s_p/2}\frac{\operatorname{Re}^2\left[e^{i\pi/4}\Gamma(1/2 + is_p/2)\right]}{|\Gamma(1/2 - is_p/2)|^2}\frac{\partial\Phi_0}{\partial k^2}\sigma_0 \tag{144}$$

The phase Φ_0 indicates the reflection phase of the pth component. Following Antonsen [1] the average derivative is set by taking $\Delta\Phi_0 = 2\pi$ and the spacing between eigenvalues to be given by the Weyl asymptotic result $\Delta k \sim \pi^2/\left(k^2V\right)$, or $\Delta k^2 = 2k\Delta k \sim 2\pi^2/(kV)$ [16]. Note in the acoustic case $\Delta k \sim 2\pi^2/\left(k^2V\right)$ [15] and $\Delta k^2 = 2k\Delta k \sim 4\pi^2/(Vk)$. In this case we are interested in the scar amplitude on axis (like the even-even eigenvalues in the 2D problem). These $m = 0$ eigenvalues ($m = 0$ is the only azimuthal mode giving values on axis) which have even symmetry along the orbit are spaced as [1] (this is twice the spacing discussed previously where the extra factor of two results from arbitrarily taking the eigenfunction to be even along the orbit)

$$\Delta k^2 \sim 16\pi/A \tag{145}$$

where the area is

$$\frac{1}{2}A = A_0 = \int_{-\infty}^{\infty} \rho(z)\, dz \tag{146}$$

where $\rho(z) > 0$ is the radius of the cavity. Thus we take [1]

$$\left(\frac{d\Phi_0}{dk^2}\right)^{-1} = \frac{8}{A}v^2 \tag{147}$$

where v is the Gaussian random variable with unit variance and density

$$f(v) = \frac{1}{\sqrt{2\pi}}e^{-v^2/2} \tag{148}$$

discussed previously. If we had used the eigenvalue spacing for the 3D acoustic cavity

$$\left(\frac{d\Phi_0}{dk^2}\right)^{-1} = \frac{2\pi}{Vk}v^2$$

We recall that in the 2D case we introduced a factor of two (we also used a factor of two for evenness along the orbit) to account for evenness about the normal to the orbit.

Now introducing this outer phase derivative gives the normalization constant

$$\frac{e^{-\pi s_p/2}\left|\Gamma(1/2 + is_p/2)\right|^2}{\mathrm{Re}^2\left[e^{i\pi/4}\Gamma(1/2 + is_p/2)\right]}\left(\frac{d\Phi_0}{dk^2}\right)^{-1} / \left[4\pi d \ln(\Lambda_+)\right] \sim c_0^2$$

or

$$c_0^2 = v^2\frac{2e^{-\pi s_p/2}\left|\Gamma(1/2 + is_p/2)\right|^2}{\mathrm{Re}^2\left[e^{i\pi/4}\Gamma(1/2 + is_p/2)\right]} / \left[A\pi d \ln(\Lambda_+)\right]$$

or

$$c_0 = v\frac{e^{-\pi s_p/4}\left|\Gamma(1/2 + is_p/2)\right|}{\sqrt{A(\pi/2)\, d \ln(\Lambda_+)}\,\mathrm{Re}\left[e^{i\pi/4}\Gamma(1/2 + is_p/2)\right]} \tag{149}$$

and

$$u_p = \frac{2}{\cos\xi}\psi_0\left(\tau, s_p\right)\cos\left(\gamma\sin\xi - s_p\sigma\right) \tag{150}$$

$$\psi_0\left(0, s_p\right) = c_0 \operatorname{Re}\left[e^{\pi s_p/2} e^{i\pi/4}\Gamma\left(1/2 + is_p/2\right)\right] \ , \ \tau \to 0 \tag{151}$$

or

$$u_p = \frac{2d}{\sqrt{d^2 - z^2}}\psi_0\left(0, s_p\right)\cos\left[kz - \left(k - k_p\right)\ell\ln\left(\frac{d+z}{d-z}\right)/\ln\left(\frac{d+\ell}{d-\ell}\right)\right]$$

This can also be written as

$$u_p = 2\psi_0\left(0, s_p\right)\cos\left[kz - \frac{1}{2}s_p\ln\left(\frac{d+z}{d-z}\right)\right]/\sqrt{1 - z^2/d^2}$$

or

$$u_p = 2\psi_0\left(0, s_p\right)\cos\left[k_p z + p_0\left(z\right)\right]/\sqrt{1 - z^2/d^2} \tag{152}$$

where

$$p_0\left(z\right) = \frac{1}{2}s_p\left\{\ln\left(\frac{d+\ell}{d-\ell}\right)\left(z/\ell\right) - \ln\left(\frac{d+z}{d-z}\right)\right\} \tag{153}$$

$$\psi_0\left(0, s_p\right) = c_0 \operatorname{Re}\left[e^{\pi s_p/2} e^{i\pi/4}\Gamma\left(1/2 + is_p/2\right)\right] \ , \ \tau \to 0 \tag{154}$$

Odd Symmetry Scalar Field Normalization Again for a real function with unit volume integral

$$\oint_{S_{scar}}\left[\frac{\partial u}{\partial\omega}\left(\frac{1}{\omega}\frac{\partial u}{\partial n}\right) - u\frac{\partial}{\partial\omega}\left(\frac{1}{\omega}\frac{\partial u}{\partial n}\right)\right]dS = 2\frac{k^2}{\omega^2}$$

If we again set

$$\frac{\partial u}{\partial n} = 0 \ , \ \text{on } S_{scar}$$

then

$$2\frac{k^2}{\omega} = -\int_{S_{scar}} u\frac{\partial^2 u}{\partial\omega\partial n}dS = \int_{-\xi_0}^{\xi_0}\int_0^{2\pi} u\frac{\partial^2 u}{\partial\omega h_\zeta\partial\zeta}h_\varphi d\varphi h_\xi d\xi$$

$$\sim d\int_{-\xi_0}^{\xi_0}\int_0^{2\pi} u\frac{\partial^2 u}{\partial\omega\partial\zeta}\zeta d\varphi\cos\xi d\xi = d\int_{-\xi_0}^{\xi_0}\int_0^{2\pi} u\frac{\partial^2 u}{\partial\omega\partial\tau}\tau d\varphi\cos\xi d\xi$$

The axisymmetric field with odd symmetry along the orbit is

$$u = \frac{2}{\cos\xi}\psi_0\left(\tau, s\right)\sin\left(\gamma\sin\xi - s\sigma\right) \tag{155}$$

The normalization in this case is

$$2\frac{k^2}{\omega} = 2\pi d \int_{-\xi_0}^{\xi_0} u \frac{\partial^2 u}{\partial\omega\partial\tau} \tau \cos\xi d\xi$$

$$\sim 8\pi d \int_{-\xi_0}^{\xi_0} \psi_0\left(0, s_p\right) c_0 \sqrt{2}\,\mathrm{Re}\left[ie^{i\Phi_0}\left(W_+' - W_+/\tau\right)^*\right] \frac{\partial\Phi_0}{\partial\omega} \sin^2\left(\gamma\sin\xi - s\sigma\right) \frac{d\xi}{\cos\xi}$$

$$\sim 16\pi d\psi_0\left(0, s_p\right) c_0\sqrt{2}\,\mathrm{Re}\left[ie^{i\Phi_0}\left(W_+' - W_+/\tau\right)^*\right] \frac{\partial\Phi_0}{\partial\omega} \int_0^{\sigma_0} \sin^2\left(\gamma\tanh\sigma - s\sigma\right) d\sigma$$

$$\sim -8\pi d\psi_0\left(0, s_p\right) c_0\sqrt{2}\,\mathrm{Re}\left[i\left(W_+' - \frac{1}{\tau}W_+\right)\right] \frac{\partial\Phi_0}{\partial\omega}\sigma_0$$

$$\sim 16\pi d c_0^2\,\mathrm{Re}\left[e^{\pi s_p/2} e^{i\pi/4}\Gamma\left(1/2 + is_p/2\right)\right] \mathrm{Re}\left[\frac{e^{i\pi/4}}{\Gamma\left(1/2 - is_p/2\right)}\right] \frac{\partial\Phi_0}{\partial\omega}\sigma_0$$

$$\sim 16\pi d c_0^2 e^{\pi s_p/2} \frac{\mathrm{Re}^2\left[e^{i\pi/4}\Gamma\left(1/2 + is_p/2\right)\right]}{\left|\Gamma\left(1/2 - is_p/2\right)\right|^2} \frac{\partial\Phi_0}{\partial\omega}\sigma_0$$

$$\sim 16\pi d c_0^2 e^{\pi s_p/2} \frac{\mathrm{Re}^2\left[e^{i\pi/4}\Gamma\left(1/2 + is_p/2\right)\right]}{\left|\Gamma\left(1/2 - is_p/2\right)\right|^2} \frac{\partial k^2}{\partial\omega} \frac{\partial\Phi_0}{\partial k^2}\sigma_0$$

$$\sim 32\frac{k^2}{\omega}\pi d c_0^2 e^{\pi s_p/2} \frac{\mathrm{Re}^2\left[e^{i\pi/4}\Gamma\left(1/2 + is_p/2\right)\right]}{\left|\Gamma\left(1/2 - is_p/2\right)\right|^2} \frac{\partial\Phi_0}{\partial k^2}\sigma_0$$

or

$$2\frac{k^2}{\omega}\int_V |u|^2\,dV \sim 32\frac{k^2}{\omega}\pi d c_0^2 e^{\pi s_p/2} \frac{\mathrm{Re}^2\left[e^{i\pi/4}\Gamma\left(1/2 + is_p/2\right)\right]}{\left|\Gamma\left(1/2 - is_p/2\right)\right|^2} \frac{\partial\Phi_0}{\partial k^2}\sigma_0$$

$$\int_V |u|^2\,dV \sim 4\pi d c_0^2 e^{\pi s_p/2} \frac{\mathrm{Re}^2\left[e^{i\pi/4}\Gamma\left(1/2 + is_p/2\right)\right]}{\left|\Gamma\left(1/2 - is_p/2\right)\right|^2} \frac{\partial\Phi_0}{\partial k^2} \ln\left(\Lambda_+\right)$$

These $m = 0$ eigenvalues ($m = 0$ is the only azimuthal mode giving values on axis) which have odd symmetry along the orbit are spaced as [1] (this is twice the spacing discussed previously where the extra factor of two results from arbitrarily taking the eigenfunction to be odd along the orbit)

$$\Delta k^2 \sim 16\pi/A$$

where $\rho\left(z\right) > 0$ is the radius of the cavity. Thus we again take [1]

$$\left(\frac{d\Phi_0}{dk^2}\right)^{-1} = \frac{8}{A}v^2$$

where v is the Gaussian random variable with unit variance and density. Now introducing this outer phase derivative gives the normalization constant

$$\frac{e^{-\pi s_p/2}\left|\Gamma\left(1/2+is_p/2\right)\right|^2}{\mathrm{Re}^2\left[e^{i\pi/4}\Gamma\left(1/2+is_p/2\right)\right]}\left(\frac{d\Phi_0}{dk^2}\right)^{-1}/\left[4\pi d\ln\left(\Lambda_+\right)\right]\sim c_0^2$$

or

$$c_0^2=v^2\frac{2e^{-\pi s_p/2}\left|\Gamma\left(1/2+is_p/2\right)\right|^2}{\mathrm{Re}^2\left[e^{i\pi/4}\Gamma\left(1/2+is_p/2\right)\right]}/\left[A\pi d\ln\left(\Lambda_+\right)\right]$$

or

$$c_0=v\frac{e^{-\pi s_p/4}\left|\Gamma\left(1/2+is_p/2\right)\right|}{\sqrt{A\left(\pi/2\right)d\ln\left(\Lambda_+\right)}\,\mathrm{Re}\left[e^{i\pi/4}\Gamma\left(1/2+is_p/2\right)\right]}$$

and

$$u_p=\frac{2}{\cos\xi}\psi_0\left(\tau,s_p\right)\sin\left(\gamma\sin\xi-s_p\sigma\right)$$

or

$$u_p=\frac{2d}{\sqrt{d^2-z^2}}\psi_0\left(0,s_p\right)\sin\left[kz-\left(k-k_p\right)\ell\ln\left(\frac{d+z}{d-z}\right)/\ln\left(\frac{d+\ell}{d-\ell}\right)\right]$$

This can also be written as

$$u_p=2\psi_0\left(0,s_p\right)\sin\left[kz-\frac{1}{2}s_p\ln\left(\frac{d+z}{d-z}\right)\right]/\sqrt{1-z^2/d^2}$$

or

$$u_p=2\psi_0\left(0,s_p\right)\sin\left[k_pz+p_0\left(z\right)\right]/\sqrt{1-z^2/d^2} \tag{156}$$

3.4 Projections Along Orbit (Even Symmetry)

Projections of the scarred eigenfunction along the orbit will now be discussed. In two dimensions [2], [6] we considered a trigonometric (or Fourier) projection as well as a scar (or Galerkin) projection of the eigenfunction. The first was useful to exhibit the Fourier decomposition of the eigenfunction and the second was useful because it restores certain approximate orthogonality properties among the scarred eigenfunctions. In the present section we will begin with the scar projection followed by the trigonometric projection and show the limiting connections between them. In three-dimensions there is also a reason to consider a third definition that avoids certain divergence issues that may arise with interior foci.

3.4.1 Scar (Galerkin) Projection

Instead of representing the eigenfunction in terms of Fourier components suppose we define the projection operator as (note here that we have taken the inverse square root in contrast to the 2D operator)

$$V_p=2\int_0^{\ell}\left(1-z^2/d^2\right)^{-1/2}\cos\left[k_pz+p_0\left(z\right)\right]u\left(0,z\right)dz \tag{157}$$

Because the pth components are asymptotically orthogonal at high frequencies, if the eigenfunction is made up of a sum

$$u\left(0,z\right)\sim\sum_{p}u_{p}\left(0,z\right) \tag{158}$$

this projection is

$$V_{p}\sim4\psi_{0}\left(0,s_{p}\right)\int_{0}^{\ell}\left(1-z^{2}/d^{2}\right)^{-1/2}\cos^{2}\left[k_{p}z+p_{0}\left(z\right)\right]dz \tag{159}$$

Now averaging over the rapidly varying $k_{p}x$ we find

$$V_{p}\sim2d\psi_{0}\left(0,s_{p}\right)\int_{0}^{\ell/d}\left(1-z^{2}\right)^{-1}dz=d\psi_{0}\left(0,s_{p}\right)\int_{0}^{\ell/d}\left(\frac{1}{1-z}+\frac{1}{1+z}\right)dz$$

$$\sim c_{0}\,\mathrm{Re}\left[e^{\pi s_{p}/2}e^{i\pi/4}\Gamma\left(1/2+is_{p}/2\right)\right]d\ln\left|\frac{1+\ell/d}{1-\ell/d}\right|$$

$$\sim v\frac{e^{\pi s_{p}/4}\left|\Gamma\left(1/2+is_{p}/2\right)\right|}{\sqrt{A\left(\pi/2\right)d\ln\left(\Lambda_{+}\right)}}d\ln\left|\frac{d+\ell}{d-\ell}\right|$$

$$\sim v\frac{e^{\pi s_{p}/4}\left|\Gamma\left(1/2+is_{p}/2\right)\right|}{\sqrt{A2\pi d/\ln\left(\Lambda_{+}\right)}}d$$

where

$$\ln\left(\frac{d+\ell}{d-\ell}\right)=\frac{1}{2}\ln\left(\Lambda_{+}\right)$$

Taking the average of the square yields

$$\left\langle kLV_{p}^{2}\right\rangle\sim L^{2}G_{1}\left(s_{p}\right)/A$$

where

$$G_{1}\left(s_{p}\right)=\frac{e^{\pi s_{p}/2}\left|\Gamma\left(1/2+is_{p}/2\right)\right|^{2}k/L}{2\pi}d\ln\left(\Lambda_{+}\right)$$

Recall that in 2D we had

$$\left\langle\sqrt{kL}V_{p}^{2}\right\rangle=L^{2}G_{1}\left(s_{p}\right)/A$$

where

$$G_{1}\left(s_{p}\right)=\frac{2\sqrt{L/d}\exp\left(\pi s_{p}/2\right)\left|\Gamma\left(1/4-is_{p}/2\right)\right|^{2}}{\left(\pi/2\right)\ln\left(\Lambda_{+}\right)}$$

and we set

$$\int_{A}\left|u\right|^{2}dS=1$$

In 3D if we had defined

$$\left\langle LV_p^2\right\rangle = L^2 G_1\left(s_p\right)/A \tag{160}$$

then

$$G_1\left(s_p\right) = \frac{(L/d)\exp\left(\pi s_p/2\right)\left|\Gamma\left(1/2 - is_p/2\right)\right|^2}{(\pi/2)\ln\left(\Lambda_+\right)} \tag{161}$$

Note that

$$\left|\Gamma\left(1/2 - is_p/2\right)\right|^2 = \frac{\pi}{\cosh\left(\pi s_p/2\right)} \tag{162}$$

and

$$G_1\left(s_p\right) = \frac{4\left(L/d\right)/\ln\left(\Lambda_+\right)}{1+\exp\left(-\pi s_p\right)} \tag{163}$$

The asymptotic forms are

$$G_1\left(s_p\right) \sim 4\left(L/d\right)e^{-\pi s_p}/\ln\left(\Lambda_+\right)\ ,\ s_p \to -\infty \tag{164}$$

$$G_1\left(0\right) \approx \frac{2\left(L/d\right)}{\ln\left(\Lambda_+\right)} \tag{165}$$

$$G_1\left(s_p\right) \sim 4\left(L/d\right)/\ln\left(\Lambda_+\right)\ ,\ s_p \to +\infty \tag{166}$$

where here we set

$$\int_V |u|^2\, dV = 1$$

where again

$$k\ell - s_p\sigma_0 = (p-1/2)\,\pi = k_p\ell\,,\ p = 1,2,3,...$$

$$s_p = \frac{\left(k-k_p\right)L/2}{\mathrm{Arcsinh}\left[\sqrt{L/\left(2R\right)}\right]} = 2\left(k-k_p\right)L/\ln\left(\Lambda_+\right)$$

3.4.2 Trigonometric (Fourier) Projection

The trigonometric projection is

$$V_p = 2\int_0^\ell \cos\left(k_p z\right)u\left(0,z\right)dz \tag{167}$$

$$u_p = 2\psi_0\left(0,s_p\right)\cos\left[k_p z + p_0\left(z\right)\right]/\sqrt{1-z^2/d^2} \tag{168}$$

Then neglecting the sum term we have

$$V_p \sim 2\psi_0\left(0, s_p\right) \int_0^{\ell} \cos\left[p_0\left(z\right)\right] \frac{dz}{\sqrt{1 - z^2/d^2}}$$

$$\sim 2\psi_0\left(0, s_p\right) \int_0^{\ell} \cos\left[\frac{1}{2} s_p \left\{\ln\left(\frac{d+\ell}{d-\ell}\right)\left(z/\ell\right) - \ln\left(\frac{d+z}{d-z}\right)\right\}\right] \frac{dz}{\sqrt{1 - z^2/d^2}}$$

If we let $d >> \ell$

$$V_p \sim 2\psi_0\left(0, s_p\right) \ell \sim 2\ell v \frac{e^{\pi s_p/4}\left|\Gamma\left(1/2 + i s_p/2\right)\right|}{\sqrt{A\left(\pi/2\right) d \ln\left(\Lambda_+\right)}}$$

$$\left\langle L V_p^2 \right\rangle = L^2 G_1\left(s_p\right)/A \tag{169}$$

$$G_1\left(s_p\right) = \frac{e^{\pi s_p/2}\left|\Gamma\left(1/2 + i s_p/2\right)\right|^2}{\left(\pi/2\right)} \left(L/d\right)/\ln\left(\Lambda_+\right)$$

$$= \frac{e^{\pi s_p/2}\Gamma\left(1/2 + i s_p/2\right)\Gamma\left(1/2 - i s_p/2\right)}{\left(\pi/2\right)} \left(L/d\right)/\ln\left(\Lambda_+\right)$$

$$= \frac{2e^{\pi s_p/2}}{\sin\pi\left(1/2 + i s_p/2\right)} \left(L/d\right)/\ln\left(\Lambda_+\right) = \frac{2e^{\pi s_p/2}}{\cosh\left(\pi s_p/2\right)} \left(L/d\right)/\ln\left(\Lambda_+\right)$$

$$= \frac{4}{1 + e^{-\pi s_p}} \left(L/d\right)/\ln\left(\Lambda_+\right)$$

If we let $R >> L$ and take $\Lambda_+ \to 1$ with

$$\left(L/d\right)/\ln\left(\Lambda_+\right) \sim 1/2 \tag{170}$$

then

$$G_1\left(s_p\right) \to \frac{2}{1 + \exp\left(-\pi s_p\right)} \tag{171}$$

which is the limiting case of the result from the preceding subsection.

3.4.3 Elliptic System Projection

Another way to define the projection is to carry out an integration around the scar in prolate spheroidal coordinates. The resulting projection is

$$\ell = d \sin\xi_0$$

$$V_p = \lim_{\zeta\to 0} \frac{1}{\zeta d} \int_{-\xi_0}^{\xi_0} \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos\left[k_p z + p_0\left(z\right)\right]}{\cos\xi} u\left(\zeta, \varphi, \xi\right) h_\varphi d\varphi h_\xi d\xi \tag{172}$$

$$V_p = \int_{-\ell}^{\ell} \cos\left[k_p z + p_0\left(z\right)\right] u\left(0,0,\xi\right) h_\xi \frac{d\xi}{dz} dz$$

where

$$p_0\left(z\right) = \frac{1}{2} s_p \left\{ \ln\left(\frac{d+\ell}{d-\ell}\right)\left(z/\ell\right) - \ln\left(\frac{d+z}{d-z}\right) \right\}$$

$$\frac{dz}{d\xi} = d\cos\xi = \sqrt{d^2 - z^2} = d\sqrt{1 - z^2/d^2}$$

Then we obtain

$$V_p = 2\int_0^{\ell} \cos\left[k_p z + p_0\left(z\right)\right] u\left(0,0,z\right) dz$$

$$r = d\sinh\zeta\cos\xi$$

$$z = d\cosh\zeta\sin\xi$$

$$h_\zeta = h_\xi = d\sqrt{\sinh^2\zeta + \cos^2\xi} \sim d\cos\xi = \sqrt{d^2 - z^2}$$

$$h_\varphi = r = d\sinh\zeta\cos\xi \sim d\zeta\cos\xi = \zeta\sqrt{d^2 - z^2}$$

Note that this limit definition of the integral around the scar eliminates the extra amplitude divergence factor from the kernel of the projection operator. This is a useful definition when interior foci are present since the projection operator then converges without the need for special treatment of the focal region. Now inserting u as

$$u_p = 2\psi_0\left(0, s_p\right) \cos\left[k_p z + p_0\left(z\right)\right] / \sqrt{1 - z^2/d^2}$$

$$c_0 = v \frac{e^{-\pi s_p/4}\left|\Gamma\left(1/2 + i s_p/2\right)\right|}{\sqrt{A\left(\pi/2\right) d \ln\left(\Lambda_+\right)}\,\mathrm{Re}\left[e^{i\pi/4}\Gamma\left(1/2 + i s_p/2\right)\right]}$$

$$\psi_0\left(0, s_p\right) = c_0\,\mathrm{Re}\left[e^{\pi s_p/2} e^{i\pi/4}\Gamma\left(1/2 + i s_p/2\right)\right]$$

$$= v \frac{e^{\pi s_p/4}\left|\Gamma\left(1/2 + i s_p/2\right)\right|}{\sqrt{A\left(\pi/2\right) d \ln\left(\Lambda_+\right)}}$$

gives

$$V_p = 4\psi_0\left(0, s_p\right) \int_0^{\ell} \cos^2\left[k_p z + p_0\left(z\right)\right] \frac{dz}{\sqrt{1 - z^2/d^2}}$$

$$\approx 2\psi_0(0,s_p)\int_0^{\ell}\frac{dz}{\sqrt{1-z^2/d^2}} = 2d\psi_0(0,s_p)\int_0^{\ell/d}\frac{dz}{\sqrt{1-z^2}} = 2d\psi_0(0,s_p)\arcsin(\ell/d)$$

$$= v2d\frac{e^{\pi s_p/4}\left|\Gamma(1/2+is_p/2)\right|}{\sqrt{A(\pi/2)\,d\ln(\Lambda_+)}}\arcsin(\ell/d)$$

and thus

$$\left\langle LV_p^2\right\rangle = L8d\frac{e^{\pi s_p/2}\left|\Gamma(1/2+is_p/2)\right|^2}{\pi A\ln(\Lambda_+)}\arcsin^2(\ell/d) = L^2G_1/A$$

$$\left|\Gamma(1/2+is_p/2)\right|^2 = \frac{\pi}{\cosh(\pi s_p/2)}$$

$$G_1 = 4\frac{e^{\pi s_p/2}\left|\Gamma(1/2+is_p/2)\right|^2}{\pi\ln(\Lambda_+)}(d/\ell)\arcsin^2(\ell/d) = \frac{4e^{\pi s_p/2}}{\ln(\Lambda_+)\cosh(\pi s_p/2)}(d/\ell)\arcsin^2(\ell/d)$$

$$\ell/d = \frac{\sqrt{\Lambda_+}-1}{\sqrt{\Lambda_+}+1} = \frac{\Lambda_+-1}{\left(\sqrt{\Lambda_+}+1\right)^2}$$

$$G_1 = \frac{2}{1+\exp(-\pi s_p)}\frac{4(d/\ell)\arcsin^2(\ell/d)}{\ln(\Lambda_+)} \tag{173}$$

3.5 Scalar Random Plane Wave

The random plane wave construction is now discussed in the three-dimensional scalar formulation. Initially this is not constructed in axisymmetric geometry but in full 3D. This is compared with the axisymmetric $m=0$ solution next because this is the solution that exists as the scar orbit is approached in the axisymmetric scalar acoustic case.

3.5.1 Scalar 3D Random Plane Wave Construction

Following [1]

$$u_r = \lim_{N\to\infty}\sqrt{2/(VN)}\,\mathrm{Re}\left[\sum_{j=1}^{N}a_je^{i\alpha_j+i\underline{k}_j\cdot\underline{r}}\right] \tag{174}$$

where a_j are real random numbers with $\left\langle a_j^2\right\rangle = 1$, $\left|\underline{k}_j\right| = k$ are random vectors uniformly distributed in angles, and the random phases α_j are uniformly distributed on a 2π interval.

Checking the normalization in 3D, we first average over the amplitudes

$$u_r^2 = \lim_{N\to\infty}\sqrt{2/(VN)}\,\mathrm{Re}\left[\sum_{j=1}^{N}a_je^{i\alpha_j+i\underline{k}_j\cdot\underline{r}}\right]\sqrt{2/(VN)}\,\mathrm{Re}\left[\sum_{j'=1}^{N}a_{j'}e^{i\alpha_{j'}+i\underline{k}_{j'}\cdot\underline{r}}\right]$$

and

$$\left\langle u_r^2 \right\rangle_{a_j} = \lim_{N\to\infty} \sqrt{2/\left(VN\right)} \sum_{j=1}^{N} \lim_{N'\to\infty} \sqrt{2/\left(VN'\right)} \sum_{j'=1}^{N'} \left\langle a_j a_{j'} \right\rangle_{a_j} \cos\left(\alpha_j + \underline{k}_j \cdot \underline{r}\right) \cos\left(\alpha_{j'} + \underline{k}_{j'} \cdot \underline{r}\right)$$

where the amplitudes a_j for different j values are regarded as independent and the cross terms vanish

$$\left\langle u_r^2 \right\rangle_{a_j} = \lim_{N\to\infty} \frac{2}{VN} \sum_{j=1}^{N} \cos^2\left(\alpha_j + \underline{k}_j \cdot \underline{r}\right)$$

Next, averaging over the phases

$$\left\langle \cos^2\left(\alpha_j + \underline{k}_j \cdot \underline{r}\right) \right\rangle_{\alpha_j} = \frac{1}{2\pi} \frac{1}{2} \int_0^{2\pi} \left[1 + \cos 2\left(\alpha_j + \underline{k}_j \cdot \underline{r}\right)\right] d\alpha_j = \frac{1}{2}$$

Thus

$$\left\langle u_r^2 \right\rangle_{a_j,\alpha_j} = \lim_{N\to\infty} \frac{2}{VN} \sum_{j=1}^{N} \frac{1}{2} = \frac{1}{V}$$

We also note, upon letting

$$\underline{k}_j = k\left(\underline{e}_x \sin\theta\cos\varphi + \underline{e}_y \sin\theta\sin\varphi + \underline{e}_z \cos\theta\right)$$

and carrying out the average over angles

$$\left\langle u_r^2 \right\rangle = \frac{1}{4\pi} \int_{4\pi} \left\langle u_r^2 \right\rangle_{a_j,\alpha_j} d\Omega = \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_0^{\pi} \left\langle u_r^2 \right\rangle_{a_j,\alpha_j} \sin\theta d\theta = \frac{1}{V}$$

3.5.2 Trigonometric (Fourier) 3D Projection

Let us first consider the random plane wave projection with the simplifying assumption of d large

$$V_{pr} = \int_{-\ell}^{\ell} \cos\left(k_p z\right) u_r\left(0, z\right) dz \tag{175}$$

Taking the projection

$$V_{pr} = \lim_{N\to\infty} \sqrt{2/\left(VN\right)} \sum_{j=1}^{N} a_j \int_{-\ell}^{\ell} \cos\left(k_p z\right) \cos\left(\alpha_j + kz\cos\theta_j\right) dz$$

and squaring

$$V_{pr}^2 = \lim_{N\to\infty} \sqrt{2/\left(VN\right)} \sum_{j=1}^{N}$$

$$\lim_{N'\to\infty} \sqrt{2/(VN')} \sum_{j'=1}^{N'} a_j a_{j'} \int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \cos(k_p z)\cos(k_p z')\cos(\alpha_j + kz\cos\theta_j)\cos(\alpha_{j'} + kz'\cos\theta_{j'})\, dzdz'$$

If we average over the amplitudes a_j and regard different j values as independent the cross terms vanish

$$\left\langle V_{pr}^2 \right\rangle_{a_j,\alpha_j} = \lim_{N\to\infty} \frac{2}{VN} \sum_{j=1}^{N} \int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \cos(k_p z)\cos(k_p z') \left\langle \cos(\alpha_j + kz\cos\theta_j)\cos(\alpha_j + kz'\cos\theta_j)\right\rangle_{\alpha_j} dzdz'$$

$$\left\langle \cos(\alpha_j + kz\cos\theta_j)\cos(\alpha_j + kz'\cos\theta_j)\right\rangle_{\alpha_j} = \frac{1}{2\pi}\frac{1}{2}\int_0^{2\pi} \left[\cos(k(z-z')\cos\theta_j) + \cos(2\alpha_j + k(z+z')\cos\theta_j)\right] d\alpha_j$$

$$= \frac{1}{2}\cos(k(z-z')\cos\theta_j) = \frac{1}{2}\cos(kz\cos\theta_j)\cos(kz'\cos\theta_j) + \frac{1}{2}\sin(kz\cos\theta_j)\sin(kz'\cos\theta_j)$$

$$\left\langle V_{pr}^2 \right\rangle_{a_j,\alpha_j} = \lim_{N\to\infty} \frac{1}{VN} \sum_{j=1}^{N} \int_{-\ell}^{\ell} \cos(k_p z)\cos(kz\cos\theta_j)\, dz \int_{-\ell}^{\ell} \cos(k_p z')\cos(kz'\cos\theta_j)\, dz'$$

$$= \lim_{N\to\infty} \frac{1/4}{VN} \sum_{j=1}^{N} \int_{-\ell}^{\ell} \left[\cos(k_p z - kz\cos\theta_j) + \cos(k_p z + kz\cos\theta_j)\right] dz$$

$$\int_{-\ell}^{\ell} \left[\cos(k_p z' - kz'\cos\theta_j) + \cos(k_p z' + kz'\cos\theta_j)\right] dz'$$

or

$$\left\langle V_{pr}^2 \right\rangle = \lim_{N\to\infty} \frac{1}{VN} \sum_{j=1}^{N} \left[\frac{\sin(k_p - k\cos\theta_j)\,\ell}{k_p - k\cos\theta_j} + \frac{\sin(k_p + k\cos\theta_j)\,\ell}{k_p + k\cos\theta_j}\right]^2$$

$$= \frac{1}{4\pi V}\int_0^{2\pi} d\varphi \int_0^{\pi} \left[\frac{\sin(k_p - k\cos\theta)\,\ell}{k_p - k\cos\theta} + \frac{\sin(k_p + k\cos\theta)\,\ell}{k_p + k\cos\theta}\right]^2 \sin\theta d\theta$$

$$= \frac{1}{2V}\int_0^{\pi} \left[\frac{\sin(k_p - k\cos\theta)\,\ell}{k_p - k\cos\theta} + \frac{\sin(k_p + k\cos\theta)\,\ell}{k_p + k\cos\theta}\right]^2 \sin\theta d\theta$$

Changing variables to $x = \cos\theta$ gives

$$\left\langle V_{pr}^2 \right\rangle = \frac{1}{2V}\int_{-1}^{1} \left[\frac{\sin(k_p - kx)\,\ell}{k_p - kx} + \frac{\sin(k_p + kx)\,\ell}{k_p + kx}\right]^2 dx$$

$$= \frac{1}{2V}\int_{-1}^{1} \left[\frac{\sin^2(k_p - kx)\,\ell}{(k_p - kx)^2} + 2\frac{\sin(k_p - kx)\,\ell}{(k_p - kx)}\frac{\sin(k_p + kx)\,\ell}{(k_p + kx)} + \frac{\sin^2(k_p + kx)\,\ell}{(k_p + kx)^2}\right] dx$$

$$= \frac{1}{2kV}\left[\ell\int_{(k_p-k)\ell}^{(k_p+k)\ell}\frac{\sin^2 x}{x^2}dx + \frac{1}{2k_p}\int_{-k\ell}^{k\ell}\left[\cos\left(2x\right)-\cos\left(2k_p\ell\right)\right]\left(\frac{1}{k_p\ell - x}+\frac{1}{k_p\ell + x}\right)dx + \ell\int_{(k_p-k)\ell}^{(k_p+k)\ell}\frac{\sin^2 x}{x^2}dx\right]$$

$$= \frac{\ell}{2kV}\left[\int_{(k_p-k)\ell}^{(k_p+k)\ell}\frac{\sin^2 x}{x^2}dx + \frac{1}{2k_p\ell}\int_{(k_p-k)\ell}^{(k_p+k)\ell}\left[\cos\left(2\left(k_p\ell - x\right)\right)-\cos\left(2k_p\ell\right)\right]\frac{dx}{x}\right.$$

$$\left. + \frac{1}{2k_p\ell}\int_{(k_p-k)\ell}^{(k_p+k)\ell}\left[\cos\left(2\left(x-k_p\ell\right)\right)-\cos\left(2k_p\ell\right)\right]\frac{dx}{x} + \int_{(k_p-k)\ell}^{(k_p+k)\ell}\frac{\sin^2 x}{x^2}dx\right]$$

$$= \frac{\ell}{kV}\left[\int_{(k_p-k)\ell}^{(k_p+k)\ell}\frac{\sin^2 x}{x^2}dx + \frac{1}{2k_p\ell}\int_{(k_p-k)\ell}^{(k_p+k)\ell}\left\{\sin\left(2k_p\ell\right)\sin\left(2x\right)-\cos\left(2k_p\ell\right)\left(1-\cos\left(2x\right)\right)\right\}\frac{dx}{x}\right]$$

$$= \frac{\ell}{kV}\left[-\frac{\sin^2\left(k_p+k\right)\ell}{\left(k_p+k\right)\ell}+\frac{\sin^2\left(k_p-k\right)\ell}{\left(k_p-k\right)\ell}+\mathrm{Si}\left(2\left(k_p+k\right)\ell\right)-\mathrm{Si}\left(2\left(k_p-k\right)\ell\right)\right]$$

$$+\frac{1}{2k_pkV}\sin\left(2k_p\ell\right)\left\{\mathrm{Si}\left(2\left(k_p+k\right)\ell\right)-\mathrm{Si}\left(2\left(k_p-k\right)\ell\right)\right\}$$

$$-\frac{1}{2k_pkV}\cos\left(2k_p\ell\right)\left\{\mathrm{Cin}\left(2\left(k_p+k\right)\ell\right)-\mathrm{Cin}\left(2\left(k_p-k\right)\ell\right)\right\}$$

or

$$\left\langle V_{pr}^2\right\rangle = \frac{\ell}{kV}\left[-\frac{\sin^2\left(k_p+k\right)\ell}{\left(k_p+k\right)\ell}+\frac{\sin^2\left(k_p-k\right)\ell}{\left(k_p-k\right)\ell}+\mathrm{si}\left(2\left(k_p+k\right)\ell\right)-\mathrm{si}\left(2\left(k_p-k\right)\ell\right)\right]$$

$$+\frac{1}{2k_pkV}\sin\left(2k_p\ell\right)\left\{\mathrm{si}\left(2\left(k_p+k\right)\ell\right)-\mathrm{si}\left(2\left(k_p-k\right)\ell\right)\right\}$$

$$-\frac{1}{2k_pkV}\cos\left(2k_p\ell\right)\left\{\mathrm{Ci}\left(2\left(k_p-k\right)\ell\right)-\mathrm{Ci}\left(2\left(k_p+k\right)\ell\right)+\ln\left(\frac{k_p+k}{k_p-k}\right)\right\}$$

or

$$\left\langle V_{pr}^2\right\rangle = \frac{\ell}{kV}\left[-\frac{\sin^2\left(k_p+k\right)\ell}{\left(k_p+k\right)\ell}+\frac{\sin^2\left(k_p-k\right)\ell}{\left(k_p-k\right)\ell}+\mathrm{si}\left(2\left(k_p+k\right)\ell\right)+\pi/2-\mathrm{Si}\left(2\left(k_p-k\right)\ell\right)\right]$$

$$+\frac{1}{2k_pkV}\sin\left(2k_p\ell\right)\left\{\mathrm{si}\left(2\left(k_p+k\right)\ell\right)+\pi/2-\mathrm{Si}\left(2\left(k_p-k\right)\ell\right)\right\}$$

$$-\frac{1}{2k_pkV}\cos\left(2k_p\ell\right)\left\{-\mathrm{Cin}\left(2\left(k_p-k\right)\ell\right)-\mathrm{Ci}\left(2\left(k_p+k\right)\ell\right)+\ln\left(k_p+k\right)+\gamma\right\}$$

Retaining only the $O\left(1/k\right)$ terms gives

$$\left\langle V_{pr}^2 \right\rangle = \frac{\ell}{kV}\left[\frac{\sin^2 (k_p - k)\,\ell}{(k_p - k)\,\ell} - \pi/2 - \mathrm{Si}\left(2\left(k_p - k\right)\ell\right)\right]$$

Returning to the preceding form with integrals in θ we note that when $k \to k_p$ the first term peaks for $\theta \to 0$ and the second term peaks for $\theta \to \pi$ with $\cos\theta = -\cos(\theta - \pi)$ and $\sin\theta = -\sin(\theta - \pi)$. Thus we find (and $\theta = \zeta/\sqrt{k\ell/2}$)

$$\left\langle V_{pr}^2 \right\rangle = \frac{1}{2V}\int_0^\infty \left[\frac{\sin\left(k_p - k + k\theta^2/2\right)\ell}{k_p - k + k\theta^2/2}\right]^2 \theta d\theta + \frac{1}{2V}\int_{-\infty}^{\pi}\left[\frac{\sin\left(k_p - k + k\left(\theta - \pi\right)^2/2\right)\ell}{k_p - k + k\left(\theta - \pi\right)^2/2}\right]^2 (\pi - \theta)\, d\theta$$

$$= \frac{1}{V}\int_0^\infty \left[\frac{\sin\left(k_p - k + k\theta^2/2\right)\ell}{k_p - k + k\theta^2/2}\right]^2 \theta d\theta$$

$$= \frac{\ell^2}{V\left(k\ell/2\right)}\int_0^\infty \left[\frac{\sin\left(\left(k_p - k\right)\ell + \zeta^2\right)}{\left(k_p - k\right)\ell + \zeta^2}\right]^2 \zeta d\zeta$$

Letting

$$\lambda = 2\left(k - k_p\right)L \tag{176}$$

this becomes

$$\left\langle V_{pr}^2 \right\rangle = \frac{L}{Vk}\int_0^\infty \frac{\sin^2\left(\lambda/4 - \zeta^2\right)}{\left(\lambda/4 - \zeta^2\right)^2}\zeta d\zeta$$

If we define

$$\left\langle kLV_p^2 \right\rangle_r = L^2 G\left(\lambda\right)/V \tag{177}$$

we obtain

$$G\left(\lambda\right) = \int_0^\infty \frac{\sin^2\left(\lambda/4 - \zeta^2\right)}{\left(\lambda/4 - \zeta^2\right)^2}\zeta d\zeta$$

$$= \frac{1}{2}\int_0^\infty \frac{\sin^2\left(\lambda/4 - \xi\right)}{\left(\lambda/4 - \xi\right)^2} d\xi = \frac{1}{2}\int_{-\lambda/4}^\infty \frac{\sin^2 \xi}{\xi^2} d\xi$$

$$= \frac{1}{2}\int_0^\infty \frac{\sin^2 \xi}{\xi^2} d\xi + \frac{1}{2}\int_{-\lambda/4}^0 \frac{\sin^2 \xi}{\xi^2} d\xi$$

$$= \frac{\pi}{4} - \frac{1}{2}\frac{\sin^2\left(\lambda/4\right)}{\lambda/4} + \frac{1}{2}\int_{-\lambda/4}^0 \frac{\sin\left(2\xi\right)}{\xi} d\xi$$

$$= \frac{\pi}{4} - \frac{\sin^2\left(\lambda/4\right)}{\lambda/2} + \frac{1}{2}\mathrm{Si}\left(\lambda/2\right) \tag{178}$$

The limiting cases are

$$G(\lambda) \sim -1/\lambda \to 0 \ , \ \ \lambda \to -\infty$$

$$G(0) = \pi/4$$

$$G(\infty) = \pi/2$$

Notice that the previous projection of the axisymmetric cavity scar field had the normalization

$$\left\langle LV_p^2 \right\rangle = L^2 G_1 / A$$

whereas this projection of the 3D random plane wave field has normalization

$$\left\langle kLV_{pr}^2 \right\rangle = G(\lambda) L^2 / V$$

These are not the same, and the difference argues that

$$\left\langle LV_p^2 \right\rangle / \left\langle LV_{pr}^2 \right\rangle = O(kV/A) >> 1 \tag{179}$$

To understand this enhancement in value due to the axisymmetric nature of the cavity scar field we next attempt to construct the analog of the axisymmetric random plane wave field. We should also note that this enhancement in value represents the enhancement due to the axisymmetry of the cavity field, analogous to the factor of two due to the even symmetry enhancement perpendicular to the orbit, observed previously in 2D [2], [6]. This factor of two enhancement persisted when the symmetry of the outer wall of the cavity was broken but the local even symmetry was maintained on the mirrors at the ends of the high frequency ray trajectory, as observed in the case of the asymmetric bowtie cavity [2], [6]. We might conjecture here that the key in the axisymmetric case is the local symmetry of the mirrors at the ends of the orbital ray trajectory.

3.5.3 Construction Of Scalar Axisymmetric Random Plane Wave Field

Using the plane wave cylindrical decomposition

$$e^{i\underline{k}_j \cdot \underline{r}} = e^{ikz\cos\theta_j} \sum_{m=-\infty}^{\infty} e^{im\left(\varphi - \varphi_j\right)} J_m\left(k\rho \sin\theta_j\right) \tag{180}$$

we take the $m = 0$ term only to construct the axisymmetric random plane wave field

$$u_r = u_0 \lim_{N \to \infty} \sqrt{2/(VN)}\, \mathrm{Re}\left[\sum_{j=1}^{N} a_j J_0\left(k\rho\sin\theta_j\right) e^{i\alpha_j + ikz\cos\theta_j}\right] \tag{181}$$

We will insert the constant u_0 to obtain the desired normalization. To check the normalization we first average over the amplitudes (without u_0)

$$u_r^2 = \lim_{N\to\infty} \sqrt{2/(VN)}\,\mathrm{Re}\left[\sum_{j=1}^{N} a_j J_0(k\rho\sin\theta_j)\, e^{i\alpha_j + ikz\cos\theta_j}\right]$$

$$\sqrt{2/(VN)}\,\mathrm{Re}\left[\sum_{j'=1}^{N} a_{j'} J_0(k\rho\sin\theta_{j'})\, e^{i\alpha_{j'} + ikz\cos\theta_{j'}}\right]$$

and

$$\left\langle u_r^2\right\rangle_{a_j} = \lim_{N\to\infty} \sqrt{2/(VN)} \sum_{j=1}^{N} \lim_{N'\to\infty} \sqrt{2/(VN')}$$

$$\sum_{j'=1}^{N'} \left\langle a_j a_{j'}\right\rangle_{a_j} J_0(k\rho\sin\theta_j)\, J_0(k\rho\sin\theta_{j'}) \cos(\alpha_j + kz\cos\theta_j)\cos(\alpha_{j'} + kz\cos\theta_{j'})$$

where the amplitudes a_j for different j values are regarded as independent and the cross terms vanish

$$\left\langle u_r^2\right\rangle_{a_j} = \lim_{N\to\infty} \frac{2}{VN} \sum_{j=1}^{N} J_0^2(k\rho\sin\theta_j)\cos^2(\alpha_j + kz\cos\theta_j)$$

Next, averaging over the phases

$$\left\langle \cos^2\left(\alpha_j + \underline{k}_j\cdot\underline{r}\right)\right\rangle_{\alpha_j} = \frac{1}{2\pi}\frac{1}{2}\int_0^{2\pi} \left[1 + \cos 2(\alpha_j + kz\cos\theta_j)\right] d\alpha_j = \frac{1}{2}$$

and

$$\left\langle u_r^2\right\rangle_{a_j,\alpha_j} = \lim_{N\to\infty} \frac{1}{VN} \sum_{j=1}^{N} J_0^2(k\rho\sin\theta_j)$$

Finally, averaging over θ_j

$$\left\langle u_r^2\right\rangle = \frac{1}{2V}\int_0^{\pi} J_0^2(k\rho\sin\theta_j)\sin\theta_j d\theta_j = \frac{1}{V}\int_0^{\pi/2} J_0^2(k\rho\sin\theta_j)\sin\theta_j d\theta_j$$

$$= \frac{1}{V}\int_0^1 J_0^2(k\rho x)\frac{xdx}{\sqrt{1-x^2}} = \frac{1}{Vk\rho}\int_0^{k\rho} J_0^2(x)\frac{xdx}{\sqrt{k^2\rho^2 - x^2}}$$

Inserting the asymptotic form for the Bessel function gives

$$\left\langle u_r^2\right\rangle \sim \frac{2}{V\pi k\rho}\int_0^{k\rho} \cos^2(x - \pi/4)\frac{dx}{\sqrt{k^2\rho^2 - x^2}} = \frac{1}{V\pi k\rho}\int_0^{k\rho} \left[1 + \sin(2x)\right]\frac{dx}{\sqrt{k^2\rho^2 - x^2}}$$

$$\sim \frac{1}{V\pi k\rho}\int_0^{k\rho} \frac{dx}{\sqrt{k^2\rho^2 - x^2}} = \frac{1}{V\pi k\rho}\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{1}{V2k\rho}\ ,\ k\rho >> 1$$

Alternatively for small values of $k\rho$ we set the Bessel function to unity

$$\left\langle u_r^2 \right\rangle = \frac{1}{2V}\int_0^\pi J_0^2\left(k\rho\sin\theta\right)\sin\theta d\theta = \frac{1}{2V}\int_0^\pi \sin\theta d\theta \sim \frac{1}{V}\ ,\ k\rho << 1$$

Note that if we integrate over ρ

$$2\pi\int_0^\rho \left\langle u_r^2\right\rangle \rho d\rho = 2\pi\frac{1}{2V}\int_0^\pi\int_0^\rho J_0^2\left(k\rho\sin\theta\right)\rho d\rho\sin\theta d\theta$$

$$= 2\pi\frac{1}{2V}\int_0^\pi \frac{\rho^2}{2}\left[J_0^2\left(k\rho\sin\theta\right) + J_1^2\left(k\rho\sin\theta\right)\right]\sin\theta d\theta$$

Now asymptotically expand for large $k\rho$

$$2\pi\int_0^\rho \left\langle u_r^2\right\rangle \rho d\rho \sim \frac{\rho}{kV}\int_0^\pi \left[\cos^2\left(k\rho\sin\theta - \pi/4\right) + \sin^2\left(k\rho\sin\theta - \pi/4\right)\right]d\theta = \frac{\pi\rho}{kV}$$

If we integrate this expression over the cross section we find

$$\int_{-\infty}^\infty 2\pi\int_0^{\rho(z)}\left\langle u_r^2\right\rangle \rho d\rho dz \sim \frac{\pi}{2kV}\int_{-\infty}^\infty \rho\left(z\right)dz = \frac{\pi A_0}{kV} = \frac{\pi A}{2kV}$$

Thus if we replace V by $\pi A/\left(2k\right)$ (or set $u_0^2 = 2kV/\left(\pi A\right)$) we will have the desired normalization

$$u_r = \lim_{N\to\infty}\sqrt{4k/\left(\pi AN\right)}\,\mathrm{Re}\left[\sum_{j=1}^N a_j J_0\left(k\rho\sin\theta_j\right)e^{i\alpha_j + ikz\cos\theta_j}\right] \tag{182}$$

3.5.4 Trigonometric (Fourier) Projection Of Axisymmetric Random Plane Wave Field

Taking the trigonometric projection on the axis ($\rho = 0$)

$$V_{pr} = \int_{-\ell}^\ell \cos\left(k_p z\right)u_r\left(0,z\right)dz$$

$$= \lim_{N\to\infty}\sqrt{4k/\left(\pi AN\right)}\sum_{j=1}^N a_j\int_{-\ell}^\ell \cos\left(k_p z\right)\cos\left(\alpha_j + kz\cos\theta_j\right)dz \tag{183}$$

Except for the normalizing factor $u_0 = \sqrt{2kV/\left(\pi A\right)}$ this is the same as in the preceding subsection. Therefore, multiplying by u_0^2 we find

$$\left\langle V_{pr}^2\right\rangle = \frac{2L}{\pi A}\int_0^\infty \frac{\sin^2\left(\lambda/4 - \zeta^2\right)}{\left(\lambda/4 - \zeta^2\right)^2}\zeta d\zeta$$

with

$$\lambda = 2\left(k - k_p\right)L$$

Defining

$$\left\langle LV_p^2\right\rangle_r = L^2 G\left(\lambda\right)/A \tag{184}$$

gives

$$G(\lambda) = \frac{2}{\pi}\int_0^\infty \frac{\sin^2\left(\lambda/4-\zeta^2\right)}{\left(\lambda/4-\zeta^2\right)^2}\zeta d\zeta$$

Note that we expect to have to multiply by a factor of two to account for the symmetry (say, even) assumed along the orbital direction (the perpendicular direction has already been accounted for by the axisymmetric construction)

$$G_s(\lambda) = 2G(\lambda) = 1 - \frac{4}{\pi}\frac{\sin^2(\lambda/4)}{\lambda/2} + \frac{2}{\pi}\,\mathrm{Si}(\lambda/2) \tag{185}$$

This symmetry along the orbit was not imposed on the random plane wave construction. Also this scaling is now consistent with the cavity scar field. Note that

$$G_s(\lambda) \sim -4/(\pi\lambda) \to 0\ ,\ \ \lambda \to -\infty \tag{186}$$

$$G_s(0) = 1 \tag{187}$$

$$G_s(\infty) = 2 \tag{188}$$

Axisymmetric Trigonometric Projection By Averaging It is interesting that we can also obtain this result easily from the 3D projection by averaging over azimuth

$$V_{pr}^{(0)} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_{-\ell}^{\ell} \cos(k_p z)\, u_r(\rho,\varphi,z)\, dz$$

$$= \lim_{N\to\infty}\sqrt{2/(VN)}\sum_{j=1}^N a_j \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_{-\ell}^{\ell} \cos(k_p z)\cos\left(\alpha_j + kz\cos\theta_j + kx\sin\theta_j\cos\varphi_j + ky\sin\theta_j\sin\varphi_j\right) dz$$

$$= \lim_{N\to\infty}\sqrt{2/(VN)}\sum_{j=1}^N a_j \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_{-\ell}^{\ell} \cos(k_p z)\cos\left[\alpha_j + kz\cos\theta_j + k\rho\sin\theta_j\cos\left(\varphi_j-\varphi\right)\right] dz$$

$$= \lim_{N\to\infty}\sqrt{2/(VN)}\sum_{j=1}^N a_j \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_{-\ell}^{\ell} \cos(k_p z)\cos\left[\alpha_j + kz\cos\theta_j + k\rho\sin\theta_j\cos(\varphi)\right] dz$$

$$= \lim_{N\to\infty}\sqrt{2/(VN)}\sum_{j=1}^N a_j \frac{1}{2\pi}\int_0^{\pi} d\varphi \int_{-\ell}^{\ell} \cos(k_p z)\cos\left(\alpha_j + kz\cos\theta_j + k\rho\sin\theta_j\cos\varphi\right) dz$$

$$+ \lim_{N\to\infty}\sqrt{2/(VN)}\sum_{j=1}^N a_j \frac{1}{2\pi}\int_0^{\pi} d\varphi \int_{-\ell}^{\ell} \cos(k_p z)\cos\left(\alpha_j + kz\cos\theta_j - k\rho\sin\theta_j\cos\varphi\right) dz$$

$$= \lim_{N\to\infty} \sqrt{2/(VN)} \sum_{j=1}^{N} a_j \frac{1}{\pi} \int_0^{\pi} d\varphi \int_{-\ell}^{\ell} \cos(k_p z) \cos(\alpha_j + kz\cos\theta_j) \cos(k\rho \sin\theta_j \cos\varphi)\, dz$$

$$= \lim_{N\to\infty} \sqrt{2/(VN)} \sum_{j=1}^{N} a_j J_0 (k\rho \sin\theta_j) \int_{-\ell}^{\ell} \cos(k_p z) \cos(\alpha_j + kz\cos\theta_j)\, dz$$

We have to renormalize by the factor we determined above $u_0^2 = 2kV/(\pi A)$ to obtain the preceding result. There is a question here about what choice of normalization is most useful. If the three-dimensional random plane wave normalization is used, and the $m = 0$ component is simply extracted, then this result for the projection is obtained. If, on the other hand, we normalize the $m = 0$ component alone, then preceding result with the u_0^2 factor is obtained.

3.5.5 Comparison Of Scar And Random Plane Wave Projections

Let us compare the flat limit $\Lambda_+ \to 1$ of the scar projection

$$G_1(s_p) \to \frac{2}{1 + \exp(-\pi s_p)} \tag{189}$$

with the preceding elliptic scar result

$$\ell/d = \frac{(\Lambda_+ - 1)}{\left(\sqrt{\Lambda_+} + 1\right)^2}$$

$$G_1(s_p) = \frac{2}{1 + \exp(-\pi s_p)} \frac{4(d/\ell)\arcsin^2(\ell/d)}{\ln(\Lambda_+)}$$

and with the trigonometric projection of the axisymmetric random plane wave result

$$G_s(\lambda) = 1 - \frac{4}{\pi} \frac{\sin^2(\lambda/4)}{\lambda/2} + \frac{2}{\pi} \mathrm{Si}(\lambda/2)$$

where

$$s_p = 2(k - k_p) L/\ln(\Lambda_+) = \lambda/\ln(\Lambda_+) \tag{190}$$

For example with $L = 2\ell = 2$ m and $R = 10$ m we find

$$d = \ell\sqrt{1 + R/\ell} \approx 3.3166 \text{ m}$$

$$\Lambda_+ = \left(\frac{d+\ell}{d-\ell}\right)^2 \approx 3.47198$$

This comparison is shown in Figure 6. Notice that there is a sharper cutoff for the cavity scar field than for the random plane wave field. However there is no peak near $s_p = 0$, where the cavity mode frequency aligns with the scar frequency $k \to k_p$. There is of course the common axisymmetric enhancement $u_0^2 = 2kV/(\pi A)$ (analogous to the factor of two increase in the 2D even bowtie).

Note that for $R/L = 1$ we find

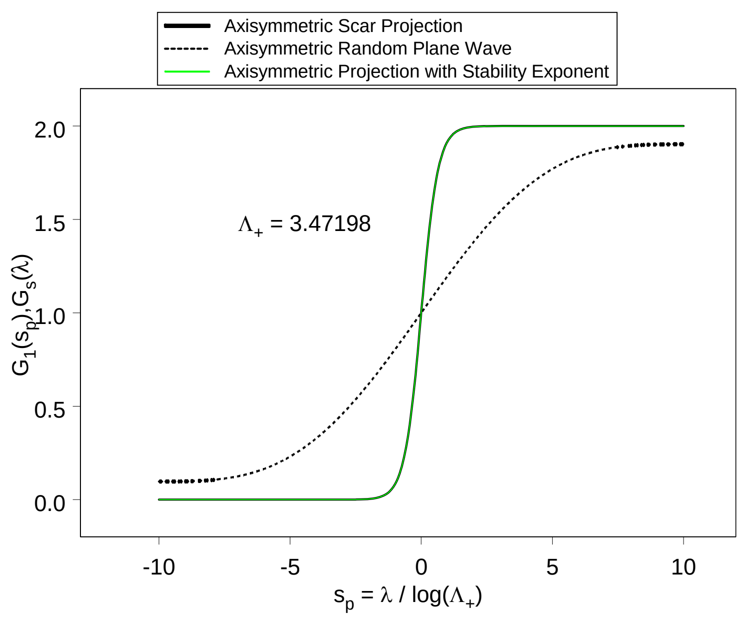

Figure 6. Comparison of projections of the random plane wave field and the cavity field as a function of the difference of cavity modal frequency and scar frequency.

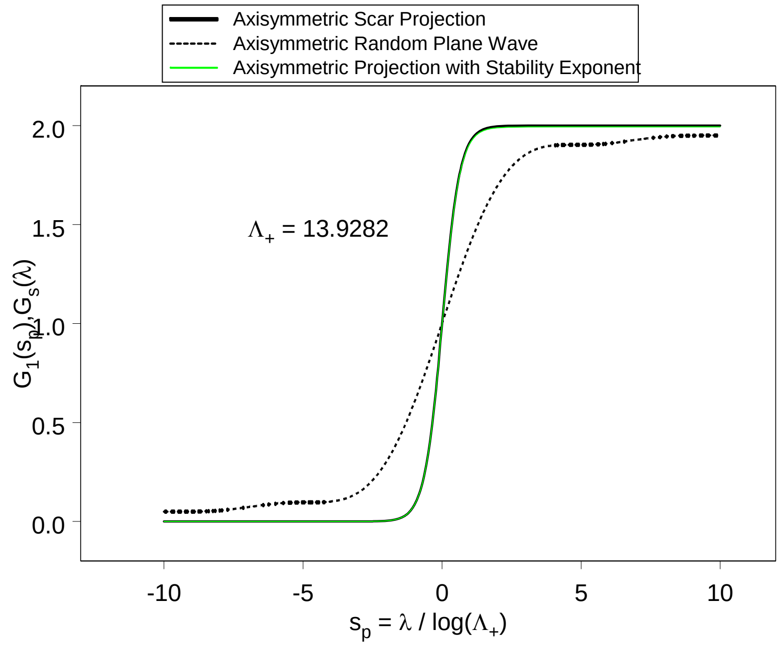

Figure 7. Comparison of projections of the random plane wave field and the cavity field as a function of the difference of cavity modal frequency and scar frequency for a smaller radius of curvature at the ends of the periodic ray trajectory.

$$\Lambda_+ = \left(\frac{d+\ell}{d-\ell}\right)^2 \approx 3.47198 \text{ and } 13.9282$$

the comparison of which is shown in Figure 7.

Somewhat surprisingly the elliptic projection and flat limit projections overlay. To understand why let us plot the quantity

$$F\left(\Lambda_+\right) = \frac{4\left(d/\ell\right)\arcsin^2\left(\ell/d\right)}{\ln\left(\Lambda_+\right)} \tag{191}$$

as a function of Λ_+. We see in Figure 8 that it is nearly unity for a very large range of stability exponents.

4 VECTOR TREATMENT OF AXISYMMETRIC BOWTIE

We now turn to the consideration of the vector electromagnetic problem. We first investigate the properties of a quasirectangular coordinate system [5] based on the prolate spheroidal system. We then construct the high frequency approximate solution for the Hertz potentials in this system. The vector electromagnetic energy theorem is used to normalize the eigenfunctions in this system. The vector random plane solution is constructed in three dimensions and for the axisymmetric case. Projections of the scar theory and random plane waves are compared to electromagnetic numerical simulations using an axisymmetric code. The field at various locations in the cavity is also compared to the random plane wave construction.

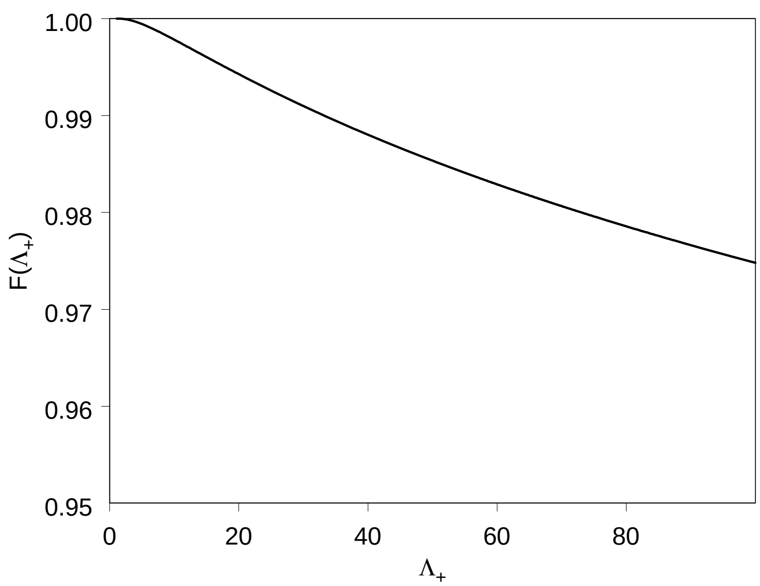

Figure 8. The quantity $F\left(\Lambda_{+}\right)$ is very close to unity.

4.1 Unit Vectors in Prolate Spheroidal Coordinates

It will be useful in the next subsection to have the relations between the prolate spheroidal unit vectors and the Cartesian unit vectors which are now given. The prolate spheroidal and Cartesian coordinate systems are related by

$$x = r\cos\varphi = d\sinh\zeta\cos\xi\cos\varphi$$

$$y = r\sin\varphi = d\sinh\zeta\cos\xi\sin\varphi$$

$$r = \sqrt{x^2+y^2} = d\sinh\zeta\cos\xi$$

$$z = d\cosh\zeta\sin\xi$$

The position vector in Cartesian coordinates is

$$\underline{r} = x\underline{e}_x + y\underline{e}_y + z\underline{e}_z \tag{192}$$

The unit vectors can be found by differentiation [10]

$$\left|\frac{\partial\underline{r}}{\partial\zeta}\right|\underline{e}_\zeta = \frac{\partial\underline{r}}{\partial\zeta} = \frac{\partial x}{\partial\zeta}\underline{e}_x + \frac{\partial y}{\partial\zeta}\underline{e}_y + \frac{\partial z}{\partial\zeta}\underline{e}_z$$

$$= d\cosh\zeta\cos\xi\cos\varphi\underline{e}_x + d\cosh\zeta\cos\xi\sin\varphi\underline{e}_y + d\sinh\zeta\sin\xi\underline{e}_z$$

where

$$\left|\frac{\partial\underline{r}}{\partial\zeta}\right| = d\sqrt{\cosh^2\zeta\cos^2\xi + \sinh^2\zeta\sin^2\xi} = h_\zeta$$

or

$$\underline{e}_\zeta = \frac{\cosh\zeta\cos\xi\cos\varphi\underline{e}_x + \cosh\zeta\cos\xi\sin\varphi\underline{e}_y + \sinh\zeta\sin\xi\underline{e}_z}{\sqrt{\cosh^2\zeta\cos^2\xi + \sinh^2\zeta\sin^2\xi}} \tag{193}$$

Similarly

$$\left|\frac{\partial\underline{r}}{\partial\xi}\right|\underline{e}_\xi = \frac{\partial\underline{r}}{\partial\xi} = \frac{\partial x}{\partial\xi}\underline{e}_x + \frac{\partial y}{\partial\xi}\underline{e}_y + \frac{\partial z}{\partial\xi}\underline{e}_z$$

$$= -d\sinh\zeta\sin\xi\cos\varphi\underline{e}_x - d\sinh\zeta\sin\xi\sin\varphi\underline{e}_y + d\cosh\zeta\cos\xi\underline{e}_z$$

$$\left|\frac{\partial\underline{r}}{\partial\xi}\right| = d\sqrt{\sinh^2\zeta\sin^2\xi + \cosh^2\zeta\cos^2\xi} = h_\xi$$

or

$$\underline{e}_\xi = \frac{-\sinh\zeta\sin\xi\cos\varphi\underline{e}_x - \sinh\zeta\sin\xi\sin\varphi\underline{e}_y + \cosh\zeta\cos\xi\underline{e}_z}{\sqrt{\sinh^2\zeta\sin^2\xi + \cosh^2\zeta\cos^2\xi}} \tag{194}$$

Finally

$$\left|\frac{\partial \underline{r}}{\partial\varphi}\right|\underline{e}_\varphi = \frac{\partial \underline{r}}{\partial\varphi} = \frac{\partial x}{\partial\varphi}\underline{e}_x + \frac{\partial y}{\partial\varphi}\underline{e}_y$$

$$= -d\sinh\zeta\cos\xi\sin\varphi\underline{e}_x + d\sinh\zeta\cos\xi\cos\varphi\underline{e}_y$$

$$\left|\frac{\partial \underline{r}}{\partial\varphi}\right| = d\sinh\zeta\cos\xi = h_\varphi$$

or

$$\underline{e}_\varphi = -\sin\varphi\underline{e}_x + \cos\varphi\underline{e}_y \tag{195}$$

Of course in this orthonormal system

$$\underline{e}_\zeta\cdot\underline{e}_\zeta = \underline{e}_\xi\cdot\underline{e}_\xi = \underline{e}_\varphi\cdot\underline{e}_\varphi = 1 \tag{196}$$

$$\underline{e}_\zeta\cdot\underline{e}_\xi = \underline{e}_\zeta\cdot\underline{e}_\varphi = \underline{e}_\varphi\cdot\underline{e}_\xi = 0 \tag{197}$$

4.2 Quasirectangular Coordinates

It will be useful in the vector problem to introduce a quasirectangular system of coordinates (υ, η, ξ) [5]

$$\upsilon = \zeta\cos\varphi \tag{198}$$

$$\eta = \zeta\sin\varphi \tag{199}$$

where

$$\zeta = \sqrt{\upsilon^2+\eta^2} \tag{200}$$

$$\cos\varphi = \upsilon/\sqrt{\upsilon^2+\eta^2} \tag{201}$$

$$\sin\varphi = \eta/\sqrt{\upsilon^2+\eta^2} \tag{202}$$

We can write

$$x = d\sinh\zeta\cos\xi\cos\varphi = d\cos\xi\frac{\sinh\sqrt{\upsilon^2+\eta^2}}{\sqrt{\upsilon^2+\eta^2}}\upsilon \tag{203}$$

$$y = d\sinh\zeta\cos\xi\sin\varphi = d\cos\xi\frac{\sinh\sqrt{\upsilon^2+\eta^2}}{\sqrt{\upsilon^2+\eta^2}}\eta \tag{204}$$

$$z = d\cosh\zeta\sin\xi = d\sin\xi\cosh\sqrt{\upsilon^2+\eta^2} \tag{205}$$

Thus the position vector

$$\underline{r} = x\underline{e}_x + y\underline{e}_y + z\underline{e}_z$$

can be used to define the unit vectors [10]

$$|\partial\underline{r}/\partial\upsilon|\,\underline{e}_\upsilon = \partial\underline{r}/\partial\upsilon =$$

$$\frac{d\cos\xi}{\sqrt{\upsilon^2+\eta^2}}\left[\frac{\upsilon^2}{\sqrt{\upsilon^2+\eta^2}}\cosh\sqrt{\upsilon^2+\eta^2} + \frac{\eta^2}{\upsilon^2+\eta^2}\sinh\sqrt{\upsilon^2+\eta^2}\right]\underline{e}_x$$

$$+\frac{d\cos\xi}{\sqrt{\upsilon^2+\eta^2}}\left[\frac{1}{\sqrt{\upsilon^2+\eta^2}}\cosh\sqrt{\upsilon^2+\eta^2} - \frac{1}{\upsilon^2+\eta^2}\sinh\sqrt{\upsilon^2+\eta^2}\right]\upsilon\eta\underline{e}_y$$

$$+\frac{d\sin\xi}{\sqrt{\upsilon^2+\eta^2}}\upsilon\sinh\sqrt{\upsilon^2+\eta^2}\underline{e}_z \tag{206}$$

$$|\partial\underline{r}/\partial\eta|\,\underline{e}_\eta = \partial\underline{r}/\partial\eta =$$

$$\frac{d\cos\xi}{\sqrt{\upsilon^2+\eta^2}}\left[\frac{1}{\sqrt{\upsilon^2+\eta^2}}\cosh\sqrt{\upsilon^2+\eta^2} - \frac{1}{\upsilon^2+\eta^2}\sinh\sqrt{\upsilon^2+\eta^2}\right]\upsilon\eta\underline{e}_x$$

$$+\frac{d\cos\xi}{\sqrt{\upsilon^2+\eta^2}}\left[\frac{\eta^2}{\sqrt{\upsilon^2+\eta^2}}\cosh\sqrt{\upsilon^2+\eta^2} + \frac{\upsilon^2}{\upsilon^2+\eta^2}\sinh\sqrt{\upsilon^2+\eta^2}\right]\underline{e}_y$$

$$+\frac{d\sin\xi}{\sqrt{\upsilon^2+\eta^2}}\eta\sinh\sqrt{\upsilon^2+\eta^2}\underline{e}_z \tag{207}$$

We can write that

$$\underline{e}_\upsilon\cdot\underline{e}_\upsilon = \underline{e}_\eta\cdot\underline{e}_\eta = 1 \tag{208}$$

and the cross term is

$$\frac{\partial\underline{r}}{\partial\upsilon}\cdot\frac{\partial\underline{r}}{\partial\eta} = \frac{d^2\upsilon\eta}{\upsilon^2+\eta^2}\left[\sin^2\xi\sinh^2\sqrt{\upsilon^2+\eta^2} + \cos^2\xi\left(\cosh^2\sqrt{\upsilon^2+\eta^2} - \frac{\sinh^2\sqrt{\upsilon^2+\eta^2}}{\upsilon^2+\eta^2}\right)\right]$$

$$= \frac{d^2\upsilon\eta}{\upsilon^2+\eta^2}\left[\sinh^2\sqrt{\upsilon^2+\eta^2} + \cos^2\xi\left(1 - \frac{\sinh^2\sqrt{\upsilon^2+\eta^2}}{\upsilon^2+\eta^2}\right)\right] \tag{209}$$

Alternatively we can begin in the prolate spheroidal system with metric coefficients

$$h_\zeta = h_\xi = d\sqrt{\sinh^2\zeta + \cos^2\xi} = d\sqrt{\cos^2\xi + \sinh^2\sqrt{\upsilon^2+\eta^2}}$$

$$h_\varphi = d\sinh\zeta\cos\xi = d\cos\xi\sinh\sqrt{\upsilon^2+\eta^2}$$

and differential position vector [10]

$$d\underline{r} = h_\xi d\xi \underline{e}_\xi + h_\varphi d\varphi \underline{e}_\varphi + h_\zeta d\zeta \underline{e}_\zeta \tag{210}$$

Therefore

$$\left|\frac{\partial\underline{r}}{\partial\upsilon}\right|\underline{e}_\upsilon = \frac{\partial\underline{r}}{\partial\upsilon} = h_\xi\frac{\partial\xi}{\partial\upsilon}\underline{e}_\xi + h_\varphi\frac{\partial\varphi}{\partial\upsilon}\underline{e}_\varphi + h_\zeta\frac{\partial\zeta}{\partial\upsilon}\underline{e}_\zeta = h_\varphi\frac{\partial\varphi}{\partial\upsilon}\underline{e}_\varphi + h_\zeta\frac{\partial\zeta}{\partial\upsilon}\underline{e}_\zeta$$

$$= -h_\varphi\frac{\eta}{\upsilon^2+\eta^2}\underline{e}_\varphi + h_\zeta\frac{\upsilon}{\sqrt{\upsilon^2+\eta^2}}\underline{e}_\zeta \tag{211}$$

$$\left|\frac{\partial\underline{r}}{\partial\upsilon}\right| = \frac{1}{\sqrt{\upsilon^2+\eta^2}}\sqrt{h_\varphi^2\frac{\eta^2}{\upsilon^2+\eta^2} + h_\zeta^2\upsilon^2} = \sqrt{g_{\upsilon\upsilon}}$$

$$= \frac{1}{\zeta}\sqrt{h_\varphi^2\sin^2\varphi + h_\zeta^2\zeta^2\cos^2\varphi} = \frac{d}{\zeta}\sqrt{\sinh^2\zeta\cos^2\xi\sin^2\varphi + \left(\sinh^2\zeta + \cos^2\xi\right)\zeta^2\cos^2\varphi}$$

$$= \frac{d}{\zeta}\sqrt{\sinh^2\zeta\sin^2\xi'\sin^2\varphi + \left(\sinh^2\zeta + \sin^2\xi'\right)\zeta^2\cos^2\varphi} \tag{212}$$

As a check, if we transform the metric coefficients and the prolate spheroidal unit vectors in this expression to the Cartesian system by means of the results in the preceding subsection we end up with the same unit vector as the preceding expression. Nevertheless these prolate spheroidal expressions are somewhat simpler. Thus the other direction becomes

$$\left|\frac{\partial\underline{r}}{\partial\eta}\right|\underline{e}_\eta = \frac{\partial\underline{r}}{\partial\eta} = h_\xi\frac{\partial\xi}{\partial\eta}\underline{e}_\xi + h_\varphi\frac{\partial\varphi}{\partial\eta}\underline{e}_\varphi + h_\zeta\frac{\partial\zeta}{\partial\eta}\underline{e}_\zeta = h_\varphi\frac{\partial\varphi}{\partial\eta}\underline{e}_\varphi + h_\zeta\frac{\partial\zeta}{\partial\eta}\underline{e}_\zeta$$

$$= h_\varphi\frac{\upsilon}{\upsilon^2+\eta^2}\underline{e}_\varphi + h_\zeta\frac{\eta}{\sqrt{\upsilon^2+\eta^2}}\underline{e}_\zeta \tag{213}$$

$$\left|\frac{\partial\underline{r}}{\partial\eta}\right| = \frac{1}{\sqrt{\upsilon^2+\eta^2}}\sqrt{h_\varphi^2\frac{\upsilon^2}{\upsilon^2+\eta^2} + h_\zeta^2\eta^2} = \sqrt{g_{\eta\eta}}$$

$$= \frac{1}{\zeta}\sqrt{h_\varphi^2\cos^2\varphi + h_\zeta^2\zeta^2\sin^2\varphi} = \frac{d}{\zeta}\sqrt{\sinh^2\zeta\cos^2\xi\cos^2\varphi + \left(\sinh^2\zeta + \cos^2\xi\right)\zeta^2\sin^2\varphi}$$

$$= \frac{d}{\zeta}\sqrt{\sinh^2\zeta\sin^2\xi'\cos^2\varphi + \left(\sinh^2\zeta + \sin^2\xi'\right)\zeta^2\sin^2\varphi} \tag{214}$$

Taking the dot product gives

$$\left|\frac{\partial \underline{r}}{\partial \upsilon}\right|\left|\frac{\partial \underline{r}}{\partial \eta}\right|\underline{e}_\upsilon \cdot \underline{e}_\eta = \frac{\partial \underline{r}}{\partial \upsilon}\cdot\frac{\partial \underline{r}}{\partial \eta}$$

$$= \frac{\upsilon\eta}{\upsilon^2+\eta^2}\left(h_\zeta^2 - \frac{h_\varphi^2}{\upsilon^2+\eta^2}\right) = \frac{\upsilon\eta}{\upsilon^2+\eta^2}d^2\left(\sinh^2\zeta + \cos^2\xi + \frac{\sinh^2\zeta\cos^2\xi}{\upsilon^2+\eta^2}\right) = \sqrt{g_{\upsilon\eta}} \tag{215}$$

or

$$\underline{e}_\upsilon \cdot \underline{e}_\eta = \frac{\upsilon\eta\zeta^2\left(h_\zeta^2 - h_\varphi^2/\zeta^2\right)}{\sqrt{h_\varphi^2\eta^2 + h_\zeta^2\upsilon^2\zeta^2}\sqrt{h_\varphi^2\upsilon^2 + h_\zeta^2\eta^2\zeta^2}} = \frac{\upsilon\eta\left(\upsilon^2+\eta^2\right)\left(h_\zeta^2 - h_\varphi^2/\zeta^2\right)}{\sqrt{h_\varphi^2\eta^2 + h_\zeta^2\upsilon^2\left(\upsilon^2+\eta^2\right)}\sqrt{h_\varphi^2\upsilon^2 + h_\zeta^2\eta^2\left(\upsilon^2+\eta^2\right)}}$$

$$= \frac{\zeta^2\left(h_\zeta^2 - h_\varphi^2/\zeta^2\right)\cos\varphi\sin\varphi}{\sqrt{h_\varphi^2\sin^2\varphi + h_\zeta^2\zeta^2\cos^2\varphi}\sqrt{h_\varphi^2\cos^2\varphi + h_\zeta^2\zeta^2\sin^2\varphi}}$$

$$= \frac{\zeta^2\left(\sinh^2\zeta + \cos^2\xi - \sinh^2\zeta\cos^2\xi/\zeta^2\right)\cos\varphi\sin\varphi}{\sqrt{\sinh^2\zeta\cos^2\xi\sin^2\varphi + \left(\sinh^2\zeta+\cos^2\xi\right)\zeta^2\cos^2\varphi}\sqrt{\sinh^2\zeta\cos^2\xi\cos^2\varphi + \left(\sinh^2\zeta+\cos^2\xi\right)\zeta^2\sin^2\varphi}}$$

$$= \frac{\zeta^2\left(\sinh^2\zeta + \sin^2\xi' - \sinh^2\zeta\sin^2\xi'/\zeta^2\right)\cos\varphi\sin\varphi}{\sqrt{\sinh^2\zeta\sin^2\xi'\sin^2\varphi + \left(\sinh^2\zeta+\sin^2\xi'\right)\zeta^2\cos^2\varphi}\sqrt{\sinh^2\zeta\sin^2\xi'\cos^2\varphi + \left(\sinh^2\zeta+\sin^2\xi'\right)\zeta^2\sin^2\varphi}}$$

$$\sqrt{g_{\upsilon\upsilon}g_{\eta\eta}}\underline{e}_\upsilon \cdot \underline{e}_\eta = d^2\left(\sinh^2\zeta + \sin^2\xi' - \sinh^2\zeta\sin^2\xi'/\zeta^2\right)\cos\varphi\sin\varphi \tag{216}$$

where the metric coefficients are [10]

$$g_{\upsilon\upsilon} = \frac{\partial x}{\partial \upsilon}\frac{\partial x}{\partial \upsilon} + \frac{\partial y}{\partial \upsilon}\frac{\partial y}{\partial \upsilon} + \frac{\partial z}{\partial \upsilon}\frac{\partial z}{\partial \upsilon} \tag{217}$$

$$g_{\upsilon\eta} = \frac{\partial x}{\partial \upsilon}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \upsilon}\frac{\partial y}{\partial \eta} + \frac{\partial z}{\partial \upsilon}\frac{\partial z}{\partial \eta} \tag{218}$$

$$g_{\eta\eta} = \frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \eta} + \frac{\partial z}{\partial \eta}\frac{\partial z}{\partial \eta} \tag{219}$$

Of course in the system (υ, η, ξ) it is immediately clear from these representations and the orthogonality of the prolate spheroidal system that

$$\underline{e}_\upsilon \cdot \underline{e}_\xi = \underline{e}_\eta \cdot \underline{e}_\xi = 0 \tag{220}$$

4.2.1 Near The Axis (Orbit)

Note that we expect ζ to have small values near the orbit. Orthogonality approximately holds in the coordinate system (υ, η, ξ) if $h_\zeta \approx h_\varphi/\zeta$. Note that $h_\zeta = d\sqrt{\sinh^2\zeta + \cos^2\xi} \sim h_\varphi/\zeta = (d/\zeta)\sinh\zeta\cos\xi$ where we must have $\zeta << 1$, consistent with $\zeta \to 0$ on the orbit.

Thus if orthogonality approximately holds, then we can write [10]

$$\underline{dr} = h_\zeta d\zeta \underline{e}_\zeta + h_\varphi d\varphi \underline{e}_\varphi + h_\xi d\xi \underline{e}_\xi \approx h_\upsilon d\upsilon \underline{e}_\upsilon + h_\eta d\eta \underline{e}_\eta + h_\xi d\xi \underline{e}_\xi \tag{221}$$

$$g_{\upsilon\upsilon} \approx h_\upsilon^2 \tag{222}$$

$$g_{\upsilon\eta} \approx 0 \tag{223}$$

$$g_{\eta\eta} \approx h_\eta^2 \tag{224}$$

The coordinate relations in this limit $\zeta^2 = \upsilon^2 + \eta^2 \to 0$ (with $-\pi/2 < \xi < \pi/2$) become

$$x = d \sinh\zeta \cos\xi \cos\varphi = d\cos\xi \frac{\sinh\sqrt{\upsilon^2+\eta^2}}{\sqrt{\upsilon^2+\eta^2}} \upsilon \sim \upsilon d\cos\xi \tag{225}$$

$$y = d \sinh\zeta \cos\xi \sin\varphi = d\cos\xi \frac{\sinh\sqrt{\upsilon^2+\eta^2}}{\sqrt{\upsilon^2+\eta^2}} \eta \sim \eta d\cos\xi \tag{226}$$

$$r = d\sinh\zeta\cos\xi \sim d\zeta\cos\xi \tag{227}$$

$$z = d\cosh\zeta\sin\xi = d\sin\xi\cosh\sqrt{\upsilon^2+\eta^2} \sim d\sin\xi \tag{228}$$

Notice that for $-\pi/2 < \xi < \pi/2$ the positive sign of υ points in the direction of positive x. The metric coefficients become

$$h_\upsilon \sim d\cos\xi \sim h_\eta \tag{229}$$

where

$$h_\zeta = h_\xi = d\sqrt{\sinh^2\zeta + \cos^2\xi} \sim d\sqrt{\cos^2\xi + \zeta^2} \sim d\cos\xi \tag{230}$$

$$h_\varphi = d\sinh\zeta\cos\xi \sim \zeta d\cos\xi \tag{231}$$

Thus in this approximate limit we can regard them as all equal (except h_φ)

$$h_\upsilon \sim h_\eta \sim h_\zeta = h_\xi \sim h = d\sqrt{\cos^2\xi + \zeta^2} \tag{232}$$

Note also that

$$\underline{e}_\upsilon \cdot \underline{e}_\zeta = \frac{h_\zeta \zeta \upsilon}{\sqrt{h_\varphi^2 \eta^2 + h_\zeta^2 \upsilon^2 \zeta^2}} \sim \cos\varphi \tag{233}$$

$$\underline{e}_\upsilon \cdot \underline{e}_\varphi = -\frac{h_\varphi \eta}{\sqrt{h_\varphi^2 \eta^2 + h_\zeta^2 \upsilon^2 \zeta^2}} \sim -\sin\varphi \tag{234}$$

$$\underline{e}_\eta \cdot \underline{e}_\zeta = \frac{h_\zeta \zeta \eta}{\sqrt{h_\varphi^2 \upsilon^2 + h_\zeta^2 \eta^2 \zeta^2}} \sim \sin\varphi \tag{235}$$

$$\underline{e}_\eta \cdot \underline{e}_\varphi = \frac{h_\varphi \upsilon}{\sqrt{h_\varphi^2 \upsilon^2 + h_\zeta^2 \eta^2 \zeta^2}} \sim \cos\varphi \tag{236}$$

$$\underline{e}_\upsilon \sim \underline{e}_\zeta \cos\varphi - \underline{e}_\varphi \sin\varphi \tag{237}$$

$$\underline{e}_\eta \sim \underline{e}_\zeta \sin\varphi + \underline{e}_\varphi \cos\varphi \tag{238}$$

$$\underline{e}_\zeta \sim \underline{e}_\upsilon \cos\varphi + \underline{e}_\eta \sin\varphi \tag{239}$$

$$\underline{e}_\varphi \sim -\underline{e}_\upsilon \sin\varphi + \underline{e}_\eta \cos\varphi \tag{240}$$

Cross Products From the preceding results the cross products of the prolate spheroidal unit vectors are

$$\underline{e}_\xi \times \underline{e}_\zeta = \underline{e}_\varphi \tag{241}$$

$$\underline{e}_\zeta \times \underline{e}_\varphi = \underline{e}_\xi \tag{242}$$

$$\underline{e}_\varphi \times \underline{e}_\xi = \underline{e}_\zeta \tag{243}$$

and the cross products of the quasirectangular unit vectors near the axis are

$$\underline{e}_\upsilon \times \underline{e}_\eta \sim \underline{e}_\xi \tag{244}$$

$$\underline{e}_\xi \times \underline{e}_\upsilon \sim \underline{e}_\eta \tag{245}$$

$$\underline{e}_\eta \times \underline{e}_\xi \sim \underline{e}_\upsilon \tag{246}$$

4.2.2 Transformation of Scalar Parabolic Equation to Quasirectangular System

The preceding parabolic equation in the scalar three-dimensional axisymmetric case was (16) or

$$\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial W}{\partial\zeta}\right)+i2\gamma\cos\xi\frac{\partial W}{\partial\xi}$$

$$+\frac{1}{\zeta^2}\frac{\partial^2 W}{\partial\varphi^2}+\left(\gamma^2\zeta^2-i2\gamma\sin\xi\right)W=0$$

This can be transformed from the prolate spheroidal system to this quasirectangular system using

$$\zeta\frac{\partial W}{\partial\zeta}\sim\zeta\frac{\partial W}{\partial\upsilon}\frac{\partial\upsilon}{\partial\zeta}+\zeta\frac{\partial W}{\partial\eta}\frac{\partial\eta}{\partial\zeta}=\frac{\partial W}{\partial\upsilon}\upsilon+\frac{\partial W}{\partial\eta}\eta \tag{247}$$

$$\frac{\partial W}{\partial\varphi}\sim\frac{\partial W}{\partial\upsilon}\frac{\partial\upsilon}{\partial\varphi}+\frac{\partial W}{\partial\eta}\frac{\partial\eta}{\partial\varphi}=-\frac{\partial W}{\partial\upsilon}\eta+\frac{\partial W}{\partial\eta}\upsilon \tag{248}$$

$$\zeta\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial W}{\partial\zeta}\right)\sim\frac{\partial}{\partial\upsilon}\left(\frac{\partial W}{\partial\upsilon}\upsilon+\frac{\partial W}{\partial\eta}\eta\right)\upsilon+\frac{\partial}{\partial\eta}\left(\frac{\partial W}{\partial\upsilon}\upsilon+\frac{\partial W}{\partial\eta}\eta\right)\eta$$

$$\sim\frac{\partial^2 W}{\partial\upsilon^2}\upsilon^2+\frac{\partial W}{\partial\upsilon}\upsilon+2\frac{\partial^2 W}{\partial\eta\partial\upsilon}\eta\upsilon+\frac{\partial W}{\partial\eta}\eta+\frac{\partial^2 W}{\partial\eta^2}\eta^2 \tag{249}$$

$$\frac{\partial^2 W}{\partial\varphi^2}\sim-\frac{\partial}{\partial\upsilon}\left(-\frac{\partial W}{\partial\upsilon}\eta+\frac{\partial W}{\partial\eta}\upsilon\right)\eta+\frac{\partial}{\partial\eta}\left(-\frac{\partial W}{\partial\upsilon}\eta+\frac{\partial W}{\partial\eta}\upsilon\right)\upsilon$$

$$\sim\frac{\partial^2 W}{\partial\upsilon^2}\eta^2-\frac{\partial W}{\partial\eta}\eta-2\frac{\partial^2 W}{\partial\eta\partial\upsilon}\upsilon\eta-\frac{\partial W}{\partial\upsilon}\upsilon+\frac{\partial^2 W}{\partial\eta^2}\upsilon^2 \tag{250}$$

and

$$\zeta\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial W}{\partial\zeta}\right)+\frac{\partial^2 W}{\partial\varphi^2}\sim\left(\frac{\partial^2 W}{\partial\upsilon^2}+\frac{\partial^2 W}{\partial\eta^2}\right)\left(\upsilon^2+\eta^2\right)=\zeta^2\left(\frac{\partial^2 W}{\partial\upsilon^2}+\frac{\partial^2 W}{\partial\eta^2}\right) \tag{251}$$

as

$$\frac{\partial^2 W}{\partial\upsilon^2}+\frac{\partial^2 W}{\partial\eta^2}+i2\gamma\cos\xi\frac{\partial W}{\partial\xi}$$

$$+\left[\gamma^2\left(\upsilon^2+\eta^2\right)-i2\gamma\sin\xi\right]W=0 \tag{252}$$

where W is now a function of υ, η, and ξ. This is the same leading order parabolic equation as we find in the vector case below.

Note that we will later need

$$\zeta\frac{\partial W}{\partial\zeta}\sim\frac{\partial W}{\partial\upsilon}\upsilon+\frac{\partial W}{\partial\eta}\eta=\frac{\partial W}{\partial\upsilon}\zeta\cos\varphi+\frac{\partial W}{\partial\eta}\zeta\sin\varphi \tag{253}$$

$$\frac{\partial W}{\partial\varphi}\sim-\frac{\partial W}{\partial\upsilon}\eta+\frac{\partial W}{\partial\eta}\upsilon=-\frac{\partial W}{\partial\upsilon}\zeta\sin\varphi+\frac{\partial W}{\partial\eta}\zeta\cos\varphi \tag{254}$$

or

$$\zeta\frac{\partial W}{\partial \upsilon}\sim\zeta\frac{\partial W}{\partial \zeta}\cos\varphi-\frac{\partial W}{\partial \varphi}\sin\varphi$$

$$\zeta\frac{\partial W}{\partial \eta}\sim\zeta\frac{\partial W}{\partial \zeta}\sin\varphi+\frac{\partial W}{\partial \varphi}\cos\varphi \qquad (255)$$

4.3 Hertz Potentials

The electromagnetic field is a vector field which in source free homogeneous regions satisfies

$$\nabla\times\underline{H}=-i\omega\varepsilon\underline{E} \qquad (256)$$

$$\nabla\times\underline{E}=i\omega\mu\underline{H} \qquad (257)$$

$$\nabla\cdot\underline{H}=0 \qquad (258)$$

$$\nabla\cdot\underline{E}=0 \qquad (259)$$

and the boundary conditions on the walls (in our case these hold on the end mirrors $\xi=\pm\xi_0$)

$$E_\zeta=E_\varphi=H_\xi=0 \qquad (260)$$

or in the quasi-rectangular system

$$E_\upsilon=E_\eta=H_\xi=0 \qquad (261)$$

We use the electric Hertz potential $\underline{\Pi}_e$ (we can alternatively use the magnetic Hertz potential $\underline{\Pi}_m$), with the fields given by

$$\underline{E}=\nabla\times\nabla\times\underline{\Pi}_e \qquad (262)$$

$$\underline{H}=-i\omega\varepsilon\nabla\times\underline{\Pi}_e$$

where the Hertz vector satisfies the vector wave equation

$$-\nabla\times\nabla\times\underline{\Pi}_e+\nabla\left(\nabla\cdot\underline{\Pi}_e\right)+k^2\underline{\Pi}_e=\left(\nabla^2+k^2\right)\underline{\Pi}_e=0 \qquad (263)$$

$$k^2=\omega^2\mu\varepsilon$$

and the electric field is thus given by either (262) or, using (263), by

$$\underline{E}=\nabla\left(\nabla\cdot\underline{\Pi}_e\right)+k^2\underline{\Pi}_e \qquad (264)$$

At high frequencies we can make one of the following sets of approximations [5]

$$\Pi_{e\upsilon} = \Phi \tag{265}$$

$$\Pi_{e\eta} = 0 = \Pi_{e\xi} \tag{266}$$

or

$$\Pi_{e\eta} = \Phi \tag{267}$$

$$\Pi_{e\upsilon} = 0 = \Pi_{e\xi} \tag{268}$$

We cannot satisfy (263) exactly with this approximate choice, so instead we require that the potential $\Pi_{e\upsilon}$ (or in the second case $\Pi_{e\eta}$) satisfies the equation resulting from equating E_υ (or in the second case E_η) from (262) and (264) [5]. Thus the equation for the potential is [5]

$$\left[-\nabla\times\nabla\times\left(\underline{\Pi}_{e\upsilon}\underline{e}_\upsilon\right)+\nabla\left\{\nabla\cdot\left(\underline{\Pi}_{e\upsilon}\underline{e}_\upsilon\right)\right\}+k^2\left(\Pi_{e\upsilon}\underline{e}_\upsilon\right)\right]\cdot\underline{e}_\upsilon=0 \tag{269}$$

or

$$\left[-\nabla\times\nabla\times\left(\Pi_{e\eta}\underline{e}_\eta\right)+\nabla\left\{\nabla\cdot\left(\Pi_{e\eta}\underline{e}_\eta\right)\right\}+k^2\left(\Pi_{e\eta}\underline{e}_\eta\right)\right]\cdot\underline{e}_\eta=0 \tag{270}$$

4.3.1 Approximate Orthogonality And Fields

When we assume approximate orthogonality the equations simplify (as given in Vaynshteyn [5]). We can derive the following equations by using the orthogonal curvilinear coordinate results for gradient, divergence and curl [10], using the same metric coefficient h for all three coordinates

$$h = d\sqrt{\cos^2\xi+\zeta^2} = d\sqrt{\cos^2\xi+\upsilon^2+\eta^2} \sim d\cos\xi \tag{271}$$

Note that $h_\varphi = d\sinh\zeta\cos\xi \sim \zeta d\cos\xi \sim h\zeta$. The potential $\Pi_{e\upsilon} = \Phi$ then satisfies [5]

$$\begin{aligned}&\left\{-\nabla\times\left[\frac{1}{h^2}\underline{e}_\eta\frac{\partial}{\partial\xi}\left(h\Pi_{e\upsilon}\right)-\frac{1}{h^2}\underline{e}_\xi\frac{\partial}{\partial\eta}\left(h\Pi_{e\upsilon}\right)\right]\right\}\cdot\underline{e}_\upsilon\\&+\left\{\nabla\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2\Pi_{e\upsilon}\right)\right]\right\}\cdot\underline{e}_\upsilon+k^2\Pi_{e\upsilon}=0\end{aligned} \tag{272}$$

or

$$\begin{aligned}&\frac{\partial}{\partial\xi}\left\{\frac{1}{h}\frac{\partial}{\partial\xi}\left(h\Pi_{e\upsilon}\right)\right\}+\frac{\partial}{\partial\eta}\left\{\frac{1}{h}\frac{\partial}{\partial\eta}\left(h\Pi_{e\upsilon}\right)\right\}\\&+h\frac{\partial}{\partial\upsilon}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2\Pi_{e\upsilon}\right)\right]+k^2h^2\Pi_{e\upsilon}=0\end{aligned} \tag{273}$$

or in spheroidal coordinates, using (255)

$$\zeta\frac{\partial W}{\partial\upsilon}\sim\zeta\frac{\partial W}{\partial\zeta}\cos\varphi-\frac{\partial W}{\partial\varphi}\sin\varphi$$

$$\zeta\frac{\partial W}{\partial\eta}\sim\zeta\frac{\partial W}{\partial\zeta}\sin\varphi+\frac{\partial W}{\partial\varphi}\cos\varphi$$

it satisfies

$$\frac{\partial}{\partial\xi}\left\{\frac{1}{h}\frac{\partial}{\partial\xi}\left(h\Pi_{ev}\right)\right\}+\frac{\partial}{\partial\zeta}\left\{\frac{1}{h}\frac{\partial}{\partial\zeta}\left(h\Pi_{ev}\right)\sin\varphi+\frac{1}{h}\frac{1}{\zeta}\frac{\partial}{\partial\varphi}\left(h\Pi_{ev}\right)\cos\varphi\right\}\sin\varphi$$

$$+\frac{1}{\zeta}\frac{\partial}{\partial\varphi}\left\{\frac{1}{h}\frac{\partial}{\partial\zeta}\left(h\Pi_{ev}\right)\sin\varphi+\frac{1}{h}\frac{1}{\zeta}\frac{\partial}{\partial\varphi}\left(h\Pi_{ev}\right)\cos\varphi\right\}\cos\varphi$$

$$+h\frac{\partial}{\partial\zeta}\left[\frac{1}{h^3}\frac{\partial}{\partial\zeta}\left(h^2\Pi_{ev}\right)\cos\varphi-\frac{1}{h^3}\frac{1}{\zeta}\frac{\partial}{\partial\varphi}\left(h^2\Pi_{ev}\right)\sin\varphi\right]\cos\varphi$$

$$-\frac{h}{\zeta}\frac{\partial}{\partial\varphi}\left[\frac{1}{h^3}\frac{\partial}{\partial\zeta}\left(h^2\Pi_{ev}\right)\cos\varphi-\frac{1}{h^3}\frac{1}{\zeta}\frac{\partial}{\partial\varphi}\left(h^2\Pi_{ev}\right)\sin\varphi\right]\sin\varphi+k^2h^2\Pi_{ev}=0$$

or

$$\frac{\partial}{\partial\xi}\left\{\frac{1}{h}\frac{\partial}{\partial\xi}\left(h\Pi_{ev}\right)\right\}+\frac{\partial}{\partial\zeta}\left\{\frac{1}{h}\frac{\partial}{\partial\zeta}\left(h\Pi_{ev}\right)\right\}+\frac{1}{\zeta}\left\{\frac{1}{h}\frac{\partial}{\partial\zeta}\left(h\Pi_{ev}\right)\right\}$$

$$+\frac{1}{h}\left[\frac{\partial^2 h}{\partial\zeta^2}-\frac{3}{h}\left(\frac{\partial h}{\partial\zeta}\right)^2\right]\Pi_{ev}\cos^2\varphi+\frac{1}{\zeta}\left(\frac{1}{h}\frac{\partial h}{\partial\zeta}\right)\Pi_{ev}\sin^2\varphi$$

$$+\frac{1}{\zeta^2}\frac{\partial^2\Pi_{ev}}{\partial\varphi^2}+k^2h^2\Pi_{ev}=0$$

or

$$\frac{\partial}{\partial\xi}\left\{\frac{1}{h}\frac{\partial}{\partial\xi}\left(h\Pi_{ev}\right)\right\}+\frac{\partial}{\partial\zeta}\left\{\frac{1}{h}\frac{\partial}{\partial\zeta}\left(h\Pi_{ev}\right)\right\}+\frac{1}{\zeta}\left\{\frac{1}{h}\frac{\partial}{\partial\zeta}\left(h\Pi_{ev}\right)\right\}$$

$$-\frac{1}{2}h^2\frac{\partial^2 h^{-2}}{\partial\zeta^2}\Pi_{ev}\cos^2\varphi+\frac{1}{\zeta}\left(\frac{1}{h}\frac{\partial h}{\partial\zeta}\right)\Pi_{ev}\sin^2\varphi$$

$$+\frac{1}{\zeta^2}\frac{\partial^2\Pi_{ev}}{\partial\varphi^2}+k^2h^2\Pi_{ev}=0 \tag{274}$$

Using the asymptotic metric coefficient form

$$h=d\sqrt{\cos^2\xi+\zeta^2}\sim d\cos\xi=\pm d\sin\xi'\sim\pm d\xi'$$

gives

$$\frac{\partial}{\partial\xi}\left\{\frac{1}{h}\frac{\partial}{\partial\xi}\left(h\Pi_{ev}\right)\right\}+\frac{\partial^2}{\partial\zeta^2}\Pi_{ev}+\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\Pi_{ev}$$

$$+\frac{1}{\zeta^2}\frac{\partial^2\Pi_{ev}}{\partial\varphi^2}+k^2h^2\Pi_{ev}=0 \tag{275}$$

The scalar Helmholtz equation (12) is

$$\frac{1}{\cos\xi}\frac{\partial}{\partial\xi}\left(\cos\xi\frac{\partial u}{\partial\xi}\right)+\left(\frac{1}{\cos^2\xi}+\frac{1}{\sinh^2\zeta}\right)\frac{\partial^2 u}{\partial\varphi^2}$$

$$+\frac{1}{\sinh\zeta}\frac{\partial}{\partial\zeta}\left(\sinh\zeta\frac{\partial u}{\partial\zeta}\right)+\gamma^2\left(\sinh^2\zeta+\cos^2\xi\right)u=0$$

and for $\zeta << 1$ it becomes

$$\frac{1}{\cos\xi}\frac{\partial}{\partial\xi}\left(\cos\xi\frac{\partial u}{\partial\xi}\right)+\frac{\partial^2 u}{\partial\zeta^2}+\frac{1}{\zeta}\frac{\partial u}{\partial\zeta}$$

$$+\left(\frac{1}{\cos^2\xi}+\frac{1}{\zeta^2}\right)\frac{\partial^2 u}{\partial\varphi^2}+\gamma^2\cos^2\xi u=0$$

or

$$\frac{\partial^2 u}{\partial\xi^2}-\tan\xi\frac{\partial u}{\partial\xi}+\frac{\partial^2 u}{\partial\zeta^2}+\frac{1}{\zeta}\frac{\partial u}{\partial\zeta}$$

$$+\gamma^2\left(\cos^2\xi-m^2\sec^2\xi-m^2/\zeta^2\right)u=0$$

The preceding equation for the Hertz potential can be written as

$$\frac{\partial^2}{\partial\xi^2}\Pi_{ev}-\tan\xi\frac{\partial}{\partial\xi}\Pi_{ev}+\frac{\partial^2}{\partial\zeta^2}\Pi_{ev}+\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\Pi_{ev}$$

$$+\left(\gamma^2\cos^2\xi-\left(m^2+1\right)\sec^2\xi-m^2/\zeta^2\right)\Pi_{ev}=0$$

Since our main focus is $m=0$, in the vector case

$$\frac{\partial^2}{\partial\xi^2}\Pi_{ev}-\tan\xi\frac{\partial}{\partial\xi}\Pi_{ev}+\frac{\partial^2}{\partial\zeta^2}\Pi_{ev}+\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\Pi_{ev}$$

$$+\left(\gamma^2\cos^2\xi-\sec^2\xi\right)\Pi_{ev}=0 \tag{276}$$

The $m=0$ scalar case is

$$\frac{\partial^2 u}{\partial\xi^2}-\tan\xi\frac{\partial u}{\partial\xi}+\frac{\partial^2 u}{\partial\zeta^2}+\frac{1}{\zeta}\frac{\partial u}{\partial\zeta}$$

$$+\gamma^2\cos^2\xi u=0 \tag{277}$$

As long as $\gamma^2\cos^4\xi >> 1$ these two are the same. As the focus is approached $\xi\to\pi/2$ this equivalence eventually fails; but we are also not sure about the accuracy of the equation for Π_{ev} in this limit since the asymptotic orthogonality of the quasirectangular coordinates breaks down there.

Note that the first and second terms of the quasirectangular form are from $(-\nabla \times \nabla \times \underline{\Pi}_e) \cdot \underline{e}_\upsilon$ and the third term results from $(\nabla\nabla \cdot \underline{\Pi}_e) \cdot \underline{e}_\upsilon$. The fields are

$$h^2 E_\upsilon = h \frac{\partial}{\partial \upsilon} \left[\frac{1}{h^3} \frac{\partial}{\partial \upsilon} \left(h^2 \Pi_{e\upsilon} \right) \right] + k^2 h^2 \Pi_{e\upsilon} \tag{278}$$

$$h^2 E_\eta = h \frac{\partial}{\partial \eta} \left[\frac{1}{h^3} \frac{\partial}{\partial \upsilon} \left(h^2 \Pi_{e\upsilon} \right) \right] \tag{279}$$

$$h^2 E_\xi = h \frac{\partial}{\partial \xi} \left[\frac{1}{h^3} \frac{\partial}{\partial \upsilon} \left(h^2 \Pi_{e\upsilon} \right) \right] \tag{280}$$

and

$$h^2 H_\xi = i\omega\varepsilon_0 \frac{\partial}{\partial \eta} \left(h \Pi_{e\upsilon} \right) \tag{281}$$

$$h^2 H_\eta = -i\omega\varepsilon_0 \frac{\partial}{\partial \xi} \left(h \Pi_{e\upsilon} \right) \tag{282}$$

Alternatively the potential $\Pi_{e\eta} = \Phi$ satisfies

$$\frac{\partial}{\partial \xi} \left\{ \frac{1}{h} \frac{\partial}{\partial \xi} \left(h \Pi_{e\eta} \right) \right\} + \frac{\partial}{\partial \upsilon} \left\{ \frac{1}{h} \frac{\partial}{\partial \upsilon} \left(h \Pi_{e\eta} \right) \right\}$$

$$+ h \frac{\partial}{\partial \eta} \left[\frac{1}{h^3} \frac{\partial}{\partial \eta} \left(h^2 \Pi_{e\eta} \right) \right] + k^2 h^2 \Pi_{e\eta} = 0 \tag{283}$$

The fields are

$$h^2 E_\upsilon = h \frac{\partial}{\partial \upsilon} \left[\frac{1}{h^3} \frac{\partial}{\partial \eta} \left(h^2 \Pi_{e\eta} \right) \right] \tag{284}$$

$$h^2 E_\eta = h \frac{\partial}{\partial \eta} \left[\frac{1}{h^3} \frac{\partial}{\partial \eta} \left(h^2 \Pi_{e\eta} \right) \right] + k^2 h^2 \Pi_{e\eta} \tag{285}$$

$$h^2 E_\xi = h \frac{\partial}{\partial \xi} \left[\frac{1}{h^3} \frac{\partial}{\partial \eta} \left(h^2 \Pi_{e\eta} \right) \right] \tag{286}$$

and

$$h^2 H_\xi = -i\omega\varepsilon_0 \frac{\partial}{\partial \upsilon} \left(h \Pi_{e\eta} \right) \tag{287}$$

$$h^2 H_\upsilon = i\omega\varepsilon_0 \frac{\partial}{\partial \xi} \left(h \Pi_{e\eta} \right) \tag{288}$$

4.3.2 Asymptotic Solution Of Quasirectangular Equations

Let us take

$$\Pi_{ev} = \Phi = W(\upsilon, \eta, \xi) e^{i\gamma \sin \xi} + W(\upsilon, \eta, -\xi) e^{-i\gamma \sin \xi} \tag{289}$$

and insert this into

$$\frac{\partial^2}{\partial \xi^2}\Pi_{ev} + \left(\frac{1}{h}\frac{\partial h}{\partial \xi}\right)\frac{\partial}{\partial \xi}\Pi_{ev} + \Pi_{ev}\frac{\partial}{\partial \xi}\left(\frac{1}{h}\frac{\partial h}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left\{\frac{1}{h}\frac{\partial}{\partial \eta}(h\Pi_{ev})\right\}$$

$$+h\frac{\partial}{\partial \upsilon}\left[\frac{1}{h^3}\frac{\partial}{\partial \upsilon}\left(h^2\Pi_{ev}\right)\right] + k^2h^2\Pi_{ev} = 0 \tag{290}$$

then W satisfies

$$\frac{\partial^2 W}{\partial \xi^2} + \left(i2\gamma\cos\xi + \frac{1}{h}\frac{\partial h}{\partial \xi}\right)\frac{\partial W}{\partial \xi}$$

$$+h\frac{\partial}{\partial \upsilon}\left[\frac{1}{h^3}\frac{\partial}{\partial \upsilon}\left(h^2 W\right)\right] + \frac{\partial}{\partial \eta}\left\{\frac{1}{h}\frac{\partial}{\partial \eta}(hW)\right\}$$

$$+\left[k^2h^2 - \gamma^2\cos^2\xi - i\gamma\sin\xi + \left(i\gamma\cos\xi + \frac{\partial}{\partial \xi}\right)\left(\frac{1}{h}\frac{\partial h}{\partial \xi}\right)\right]W = 0$$

or

$$\frac{1}{h}\frac{\partial}{\partial \xi}\left(h\frac{\partial W}{\partial \xi}\right) + i2\gamma\cos\xi\frac{\partial W}{\partial \xi}$$

$$+\frac{1}{h}\frac{\partial}{\partial \upsilon}\left(h\frac{\partial}{\partial \upsilon}W\right) + \frac{1}{h}\frac{\partial}{\partial \eta}\left(h\frac{\partial}{\partial \eta}W\right)$$

$$+\left[k^2h^2 - \gamma^2\cos^2\xi - i\gamma\sin\xi + \left(i\gamma\cos\xi + \frac{\partial}{\partial \xi}\right)\left(\frac{1}{h}\frac{\partial h}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{1}{h}\frac{\partial h}{\partial \eta}\right) + h\frac{\partial}{\partial \upsilon}\left(\frac{2}{h^2}\frac{\partial h}{\partial \upsilon}\right)\right]W = 0$$

or

$$\frac{1}{h}\frac{\partial}{\partial \xi}\left(h\frac{\partial W}{\partial \xi}\right) + i2\gamma\cos\xi\frac{\partial W}{\partial \xi}$$

$$+\frac{1}{h}\frac{\partial}{\partial \upsilon}\left(h\frac{\partial}{\partial \upsilon}W\right) + \frac{1}{h}\frac{\partial}{\partial \eta}\left(h\frac{\partial}{\partial \eta}W\right)$$

$$+\left[\gamma^2\zeta^2 - i\gamma\sin\xi + \left(i\gamma\cos\xi + \frac{\partial}{\partial \xi}\right)\left(\frac{1}{h}\frac{\partial h}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{1}{h}\frac{\partial h}{\partial \eta}\right) + h\frac{\partial}{\partial \upsilon}\left(\frac{2}{h^2}\frac{\partial h}{\partial \upsilon}\right)\right]W = 0$$

We now neglect the first term $\partial^2 W/\partial \xi^2$ and let (except in the k^2h^2 term)

$$h \sim d\cos\xi$$

giving

$$(i2\gamma\cos\xi - \tan\xi)\frac{\partial W}{\partial \xi} + \frac{\partial^2 W}{\partial \upsilon^2} + \frac{\partial^2 W}{\partial \eta^2}$$

$$+\left[\gamma^2\left(\upsilon^2+\eta^2\right)-i2\gamma\sin\xi-\sec^2\xi\right]W=0 \tag{291}$$

Now take

$$W=f(\xi)\,\Psi(\upsilon,\eta,\sigma) \tag{292}$$

where we want

$$\frac{f'}{f}=\frac{i2\gamma\sin\xi+\sec^2\xi}{i2\gamma\cos\xi-\tan\xi}$$

or

$$1/f=\cos\xi+\frac{i}{2\gamma}\tan\xi \tag{293}$$

Using this along with the transformation

$$\sigma=\int_0^{\xi}\frac{d\xi}{\cos\xi}=\operatorname{arcsinh}(\tan\xi)=\ln(\tan\xi+\sec\xi) \tag{294}$$

gives

$$\left(i2\gamma-\frac{\tan\xi}{\cos\xi}\right)\frac{\partial\Psi}{\partial\sigma}+\frac{\partial^2\Psi}{\partial\upsilon^2}+\frac{\partial^2\Psi}{\partial\eta^2}$$

$$+\left[\gamma^2\left(\upsilon^2+\eta^2\right)\right]\Psi=0 \tag{295}$$

We could also have dropped the terms $\tan\xi$ and $\sec^2\xi$ terms compared to the $\gamma\cos\xi$ and the $\gamma\sin\xi$ term (these terms could be accounted for by higher order terms in the asymptotic series just like the $\partial^2W/\partial\xi^2$ term and the higher order terms $\partial h/\partial\upsilon$ and $\partial h/\partial\eta$) to obtain directly

$$i2\gamma\cos\xi\frac{\partial W}{\partial\xi}+\frac{\partial^2W}{\partial\upsilon^2}+\frac{\partial^2W}{\partial\eta^2}$$

$$+\left[\gamma^2\left(\upsilon^2+\eta^2\right)-i2\gamma\sin\xi\right]W=0 \tag{296}$$

which is the same as the preceding scalar case. For this leading term approach we would take

$$W=\frac{1}{\cos\xi}\Psi(\upsilon,\eta,\sigma) \tag{297}$$

and obtain the same equation for Ψ. Scaling the independent variables by letting

$$\tau_x=\sqrt{2\gamma}\upsilon \tag{298}$$

$$\tau_y=\sqrt{2\gamma}\eta \tag{299}$$

gives

$$i\frac{\partial\Psi}{\partial\sigma}+\frac{\partial^2\Psi}{\partial\tau_x^2}+\frac{\partial^2\Psi}{\partial\tau_y^2}+\frac{1}{4}\left(\tau_x^2+\tau_y^2\right)\Psi=0 \tag{300}$$

Separating the variables

$$\Psi\left(\tau_x,\tau_y,\sigma\right)=\psi_a\left(\tau_x\right)\psi_b\left(\tau_y\right)\psi_c\left(\sigma\right) \tag{301}$$

gives

$$\left(\frac{i}{\psi_c}\frac{\partial\psi_c}{\partial\sigma}\right)+\left(\frac{1}{\psi_a}\frac{\partial^2\psi_a}{\partial\tau_x^2}+\tau_x^2/4\right)+\left(\frac{1}{\psi_b}\frac{\partial^2\psi_b}{\partial\tau_y^2}+\tau_y^2/4\right)=0$$

Each parenthetical term is independent of the variables associated with the others and therefore each equals a constant. We take the first term as s

$$\frac{\partial\psi_c}{\partial\sigma}=-is\psi_c$$

or

$$\psi_c=e^{-is\sigma} \tag{302}$$

Then we can write

$$\frac{\partial^2\psi_a}{\partial\tau_x^2}+\left(s_a+\tau_x^2/4\right)\psi_a=0 \tag{303}$$

$$\frac{\partial^2\psi_b}{\partial\tau_y^2}+\left(s_b+\tau_y^2/4\right)\psi_b=0 \tag{304}$$

where

$$s_a+s_b=s \tag{305}$$

and

$$\Psi\left(\tau_x,\tau_y,\sigma\right)=e^{-is_a\sigma}\psi_a\left(s_a,\tau_x\right)e^{-is_b\sigma}\psi_b\left(s_b,\tau_y\right)=e^{-is\sigma}\psi_a\left(s_a,\tau_x\right)\psi_b\left(s_b,\tau_y\right)$$

This is a form of the equation of the parabolic cylinder functions

$$\frac{\partial^2\psi}{\partial\tau^2}+\left(\frac{\tau^2}{4}+s\right)\psi=0 \tag{306}$$

The solution that is outgoing in τ is [1]

$$U_+\left(s,\tau\right)=e^{-\pi(s+i/2)/4}U\left(-is,\tau e^{-i\pi/4}\right) \tag{307}$$

where $U\left(a,z\right)$ is the standard solution [11]. Following [1] the total transverse solution is taken as the incident plus reflected form

$$\psi\left(s,\tau\right)=c\,\mathrm{Re}\left[U_+\left(s,\tau\right)+e^{i\Phi_0}U_+^*\left(s,\tau\right)\right] \tag{308}$$

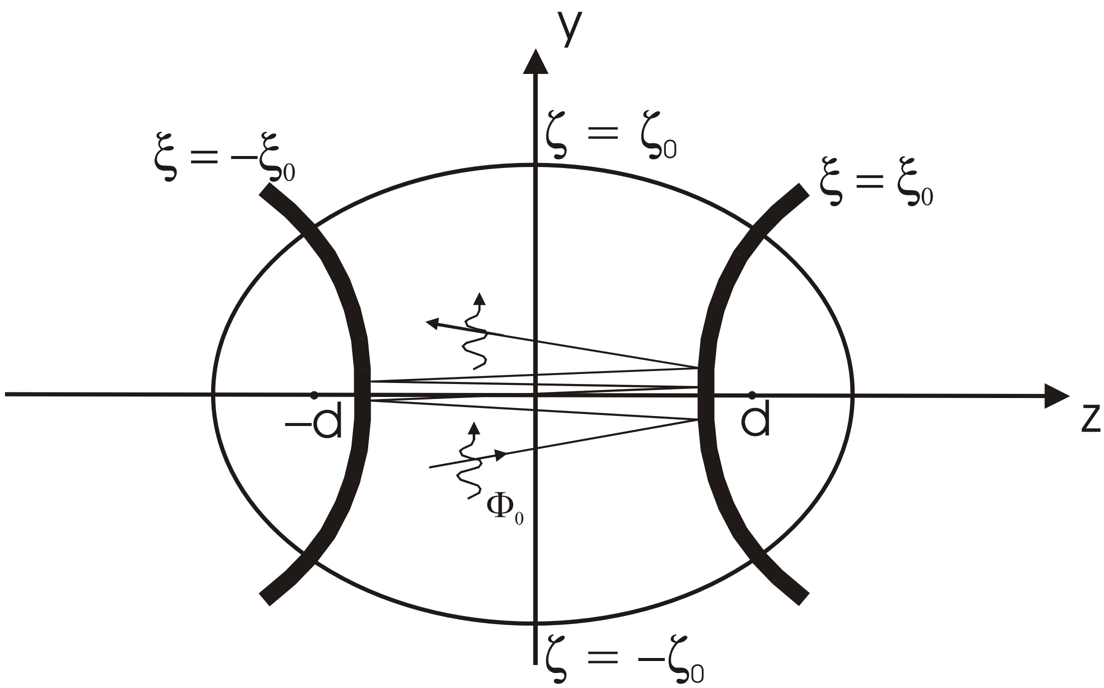

Figure 9. Illustration of reflected wave from outer region returning with phase Φ_0.

where the constant c is used for normalization. The transverse boundary condition in τ is a reflection with a random phase $\Phi_0\left(k^2\right)$ which was introduced by Antonsen to match to the chaotic region of the cavity; it describes the phase relation between a wave leaving the vicinity of the unstable periodic orbit and one returning [1] with the variation of the pth component along the orbit. Figure 9 schematically illustrates a wave bouncing back and forth between mirrors in the region of the scarred orbit; it leaves the vicinity of the orbit and eventually returns from the outer chaotic region with transverse reflection phase $\Phi_0\left(k^2\right)$. (For purposes of simplification this figure does not include the vertical evenness of the cavity, which confines the wave leaving and the wave reflected to either the upper half or lower half of the cavity.) Thus $\psi_a = \psi_a\left(s_a, \tau_x\right)$ and $\psi_b = \psi_b\left(s_b, \tau_y\right)$ are elliptic cylinder functions like in the two-dimensional case [6], [1].

The perfect conductor boundary conditions imply

$$E_\upsilon = E_\eta = H_\xi = 0\ ,\ \xi = \pm\xi_0 \tag{309}$$

or using the preceding relations for the field in terms of the Hertz potential

$$h^2 E_\upsilon = h\frac{\partial}{\partial\upsilon}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2\Pi_{e\upsilon}\right)\right] + k^2 h^2 \Pi_{e\upsilon}$$

$$h^2 E_\eta = h\frac{\partial}{\partial\eta}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2\Pi_{e\upsilon}\right)\right]$$

$$h^2 H_\xi = i\omega\varepsilon_0\frac{\partial}{\partial\eta}\left(h\Pi_{e\upsilon}\right)$$

or

$$h^2 E_\upsilon = h\frac{\partial}{\partial\upsilon}\left[\frac{1}{h^3}\frac{\partial}{\partial\eta}\left(h^2\Pi_{e\eta}\right)\right]$$

$$h^2 E_\eta = h\frac{\partial}{\partial\eta}\left[\frac{1}{h^3}\frac{\partial}{\partial\eta}\left(h^2\Pi_{e\eta}\right)\right] + k^2 h^2 \Pi_{e\eta}$$

$$h^2 H_\xi = -i\omega\varepsilon_0\frac{\partial}{\partial\upsilon}\left(h\Pi_{e\eta}\right)$$

We see that we must take the potential to vanish on the mirrors for the perfect conductor boundary conditions

$$\Pi_{e\upsilon} = \Pi_{e\eta} = 0 \ , \ \xi = \pm\xi_0 \tag{310}$$

Then in the present case

$$e^{-i\gamma\sin\xi_0}\Psi\left(\tau_x,\tau_y,-\sigma_0\right) = e^{i\gamma\sin\xi_0}e^{-i(2p-1)\pi}\Psi\left(\tau_x,\tau_y,\sigma_0\right)$$

or

$$\Psi\left(\tau_x,\tau_y,-\sigma_0\right) = e^{i2k\ell-i(2p-1)\pi}\Psi\left(\tau_x,\tau_y,\sigma_0\right)$$

or

$$1 = e^{i2k\ell-i(2p-1)\pi-i2s\sigma_0}$$

$$2k\ell = (2p-1)\,\pi + 2s\sigma_0 = (2p-1)\,\pi + 2\left(s_a+s_b\right)\sigma_0$$

or

$$\left(k-k_p\right)\ell = s\sigma_0 = s\ln\left(\tan\xi_0+\sec\xi_0\right) = s\ln\sqrt{\frac{d+\ell}{d-\ell}}$$

$$= s\frac{1}{4}\ln\left(\Lambda_+\right) \tag{311}$$

where

$$\Lambda_\pm = \left(\frac{d+\ell}{d-\ell}\right)^{\pm 2} = \left[1+L/R\pm\sqrt{\left(1+L/R\right)^2-1}\right]^2$$

are the stability exponents of the orbit and

$$k_p\ell = (p-1/2)\,\pi \ , \ p = 1,2,... \tag{312}$$

where p is a large integer representing the number of half wave variations along the z axis. The quantities s_a and s_b are the transverse numbers of half wave variations (not necessarily integers in this unstable case, and usually random functions versus eigenvalue), where $s = s_a + s_b$. The preceding result thus connects the eigenvalue k minus the wavenumber in z, or k_p, to the sum of the transverse variations.

$$\Pi_{e\upsilon} = 2\psi_a\left(s_a,\tau_x\right)\psi_b\left(s_b,\tau_y\right)\cos\left(\gamma\sin\xi - s\sigma\right)/\cos\xi$$

$$\psi_a\left(s_a,\tau_x\right) = c_a\,\mathrm{Re}\left[U_+\left(s_a,\tau_x\right) + e^{i\Phi_{0a}}U_+^*\left(s_a,\tau_x\right)\right]$$

$$\psi_b\left(s_b,\tau_y\right) = c_b\,\mathrm{Re}\left[U_+\left(s_b,\tau_y\right) + e^{i\Phi_{0b}}U_+^*\left(s_b,\tau_y\right)\right]$$

Note that if the eigenfunction $\Pi_{e\upsilon}$ is chosen to be even with respect to the normals of the orbit η and υ (say, perfect magnetic conductor or PMC at $\eta = 0$ and perfect electric conductor or PEC at $\upsilon = 0$, resulting in $E_\upsilon \neq 0$, $E_\eta = 0$, $E_\xi = 0$, $H_\xi = 0$, $H_\eta \neq 0$ at $\upsilon = \eta = 0$) where

$$h^2E_\upsilon = h\frac{\partial}{\partial\upsilon}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2\Pi_{e\upsilon}\right)\right] + k^2h^2\Pi_{e\upsilon} \sim \left(\frac{\partial^2}{\partial\upsilon^2} + k^2h^2\right)\Pi_{e\upsilon} = \left(2\gamma\frac{\partial^2}{\partial\tau_x^2} + k^2h^2\right)\Pi_{e\upsilon}$$

$$h^2E_\eta = h\frac{\partial}{\partial\eta}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2\Pi_{e\upsilon}\right)\right] \sim \frac{\partial^2}{\partial\eta\partial\upsilon}\Pi_{e\upsilon} = 2\gamma\frac{\partial^2}{\partial\tau_y\partial\tau_x}\Pi_{e\upsilon}$$

$$h^2E_\xi = h\frac{\partial}{\partial\xi}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2\Pi_{e\upsilon}\right)\right] \sim \frac{\partial^2}{\partial\xi\partial\upsilon}\Pi_{e\upsilon} = \sqrt{2\gamma}\frac{\partial^2}{\partial\xi\partial\tau_x}\Pi_{e\upsilon}$$

$$h^2H_\xi = i\omega\varepsilon_0\frac{\partial}{\partial\eta}\left(h\Pi_{e\upsilon}\right) \sim i\omega\varepsilon_0 h\frac{\partial}{\partial\eta}\Pi_{e\upsilon} = i\omega\varepsilon_0\sqrt{2\gamma}h\frac{\partial}{\partial\tau_y}\Pi_{e\upsilon}$$

$$h^2H_\eta = -i\omega\varepsilon_0\frac{\partial}{\partial\xi}\left(h\Pi_{e\upsilon}\right) \sim -i\omega\varepsilon_0 h\frac{\partial}{\partial\xi}\Pi_{e\upsilon} = -i\omega\varepsilon_0 h\frac{\partial}{\partial\xi}\Pi_{e\upsilon}$$

then we have the resonance conditions

$$\mathrm{Re}\left[U_+'\left(s_a,0\right) + e^{i\Phi_{0a}}U_+^{*\prime}\left(s_a,0\right)\right] = 0$$

$$\mathrm{Re}\left[U_+'\left(s_b,0\right) + e^{i\Phi_{0b}}U_+^{*\prime}\left(s_b,0\right)\right] = 0$$

or

$$-\frac{U_+'\left(s_a,0\right)}{U_+^{*\prime}\left(s_a,0\right)} = e^{i\Phi_{0a}}$$

$$-\frac{U_+'\left(s_b,0\right)}{U_+^{*\prime}\left(s_b,0\right)} = e^{i\Phi_{0b}}$$

Using this with the Wronskian

$$U_+'U_+^* - U_+^{*\prime}U_+ = i$$

gives

$$\mathrm{Re}\left[U_+\left(s_a,0\right) + e^{i\Phi_{0a}}U_+^*\left(s_a,0\right)\right] = \frac{\mathrm{Im}\left[U_+'\left(s_a,0\right)\right]}{\left|U_+'\left(s_a,0\right)\right|^2}$$

$$\mathrm{Re}\left[U_+(s_b,0)+e^{i\Phi_{0b}}U_+^*(s_b,0)\right]=\frac{\mathrm{Im}\left[U_+'(s_b,0)\right]}{\left|U_+'(s_b,0)\right|^2}$$

On axis $\upsilon=\eta=0$ or $\tau_x=\tau_y=0$ and

$$\sigma=\operatorname{arcsinh}(\tan\xi)=\frac{1}{2}\ln\left(\frac{d+z}{d-z}\right)$$

$$\sin\xi=z/d$$

$$\cos\xi=\sqrt{1-z^2/d^2}$$

$$\Pi_{e\upsilon}=2\psi_a(s_a,0)\,\psi_b(s_b,0)\cos\left[kz-s\frac{1}{2}\ln\left(\frac{d+z}{d-z}\right)\right]/\sqrt{1-z^2/d^2}$$

$$=2c^2\frac{\mathrm{Im}\left[U_+'(s_a,0)\right]}{\left|U_+'(s_a,0)\right|^2}\frac{\mathrm{Im}\left[U_+'(s_b,0)\right]}{\left|U_+'(s_b,0)\right|^2}\cos\left[k_pz+p_0(z)\right]/\sqrt{1-z^2/d^2}$$

where

$$p_0(z)=\frac{1}{2}s\left\{\ln\left(\frac{d+\ell}{d-\ell}\right)(z/\ell)-\ln\left(\frac{d+z}{d-z}\right)\right\}$$

This raises the question: how are the two variations s_a and s_b (as well as the reflection coefficient phases Φ_{0a} and Φ_{0b}) connected to the azimuthal quantum number m? In the axisymmetric case we might argue that the two dimensional randomness is collapsed to one dimension, and therefore these are related. In fact our main interest is in the lowest transverse mode ($m=0$) with the vector nature put in through the selected component of the Hertz potential. Does this mean that in the axisymmetric geometry we can take $s_a=s_b$ (and $\Phi_{0a}=\Phi_{0b}$)?

Alternatively for the other polarization

$$\Pi_{e\eta}=\Phi=W(\upsilon,\eta,\xi)\,e^{i\gamma\sin\xi}+W(\upsilon,\eta,-\xi)\,e^{-i\gamma\sin\xi}$$

$$\frac{\partial^2}{\partial\xi^2}\Pi_{e\eta}+\frac{1}{h}\frac{\partial h}{\partial\xi}\frac{\partial}{\partial\xi}\Pi_{e\eta}+\Pi_{e\eta}\frac{\partial}{\partial\xi}\left\{\frac{1}{h}\frac{\partial h}{\partial\xi}\right\}+\frac{\partial}{\partial\upsilon}\left\{\frac{1}{h}\frac{\partial}{\partial\upsilon}(h\Pi_{e\eta})\right\}$$

$$+h\frac{\partial}{\partial\eta}\left[\frac{1}{h^3}\frac{\partial}{\partial\eta}\left(h^2\Pi_{e\eta}\right)\right]+k^2h^2\Pi_{e\eta}=0$$

or

$$\frac{1}{h}\frac{\partial}{\partial\xi}\left(h\frac{\partial W}{\partial\xi}\right)+i2\gamma\cos\xi\frac{\partial W}{\partial\xi}$$

$$+\frac{1}{h}\frac{\partial}{\partial \upsilon}\left(h\frac{\partial}{\partial \upsilon}W\right)+\frac{1}{h}\frac{\partial}{\partial \eta}\left(h\frac{\partial}{\partial \eta}W\right)$$

$$+\left[\gamma^2\zeta^2+\left(i\gamma\cos\xi+\frac{\partial}{\partial\xi}\right)\left(\frac{1}{h}\frac{\partial h}{\partial\xi}\right)+\frac{\partial}{\partial\upsilon}\left(\frac{1}{h}\frac{\partial h}{\partial\upsilon}\right)+h\frac{\partial}{\partial\eta}\left(\frac{2}{h^2}\frac{\partial h}{\partial\eta}\right)\right]W=0$$

and the procedure follows the same lines as in the preceding polarization state. We might expect that for arbitrary outer chaotic regions both polarization states would be generated and coexist.

In the next section we will find it more convenient to solve for the quasirectangular Hertz potential component in the prolate spheroidal system.

4.3.3 Asymptotic Solution Of Parabolic Equation In Prolate Spheroidal System

The parabolic equation in quasirectangular coordinates for the Hertz potential component $\Pi_{e\upsilon}$ or $\Pi_{e\eta}$ (or W) resulting from the vector wave equation is asymptotically the same (particularly for the $m=0$ mode) as that arising from the three dimensional scalar Helmholtz equation. Hence we can transform the parabolic equation for the Hertz potential from the quasirectangular system back to prolate spheroidal coordinates.

Even Case The potential is then

$$\Pi_{e\upsilon}=\frac{2}{\cos\xi}\psi_m(\tau,s)\cos(\gamma\sin\xi-s\sigma)\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\} \tag{313}$$

$$\psi_m=c_m\frac{\sqrt{2}}{\tau}\,\mathrm{Re}\left[W_+ + e^{i\Phi_0}W_+^*\right] \tag{314}$$

$$W_+(\tau,m,s)=W_{is/2,m/2}\left(-i\tau^2/2\right) \tag{315}$$

$$\tau=\sqrt{2\gamma}\zeta \tag{316}$$

$$k\ell-s_p\sigma_0=(p-1/2)\pi=k_p\ell\,,\ p=1,2,3,... \tag{317}$$

and the same holds for the $\Pi_{e\eta}$ polarization state.

Odd Case As in the scalar case for the mode that is odd along the orbit

$$\Pi_{e\upsilon}=\frac{2}{\cos\xi}\psi_m(\tau,s)\sin(\gamma\sin\xi-s\sigma)\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\} \tag{318}$$

where

$$k\ell-s_p\sigma_0=p\pi=k_p\ell\,,\ p=1,2,3,... \tag{319}$$

Behavior Of Zero Mode Near Orbit Since our main focus will be $m=0$ we now examine the behavior of the zero mode

$$\psi_0 = c_0 \frac{\sqrt{2}}{\tau} \operatorname{Re}\left[W_+ + e^{i\Phi_0} W_+^*\right]$$

near the orbit at $\tau \to 0$

$$W_+(\tau, 0, s) \sim -\left(-i\tau^2/2\right)^{1/2} \frac{1}{\Gamma(1/2 - is/2)} \left[\ln\left(-i\tau^2/2\right)\left(1 - s\tau^2/4\right)\right.$$

$$\left. + \psi(1/2 - is/2) - 2\psi(1) + \left(s\tau^2/4\right)\{-\psi(1/2 - is/2) + 2\psi(1) + 2\} + O\left(\tau^4 \ln \tau\right)\right]$$

$$W_+'(\tau, 0, s) \sim i\tau \frac{1}{2}\left(-i\tau^2/2\right)^{-1/2} \frac{1}{\Gamma(1/2 - is/2)}$$

$$\left[\ln\left(-i\tau^2/2\right) + \{\psi(1/2 - is/2) - 2\psi(1)\} + 2 + O\left(\tau^2 \ln \tau\right)\right] \ , \ \tau \to 0 \ , \ m = 0$$

Now taking

$$\tau \frac{\partial \psi_0}{\partial \tau} = c_0 \sqrt{2} \operatorname{Re}\left[W_+' + e^{i\Phi_0} W_+^{*\prime}\right] - \psi_0$$

If this is set to zero as $\tau \to 0$ to solve for the reflection phase

$$\operatorname{Re}\left[\left(W_+' - \frac{1}{\tau} W_+\right) + e^{i\Phi_0}\left(W_+' - \frac{1}{\tau} W_+\right)^*\right] \to 0 \, , \, \tau \to 0$$

we find the same result as in the scalar case

$$e^{i\Phi_0} = -\frac{\left(W_+' - \frac{1}{\tau} W_+\right)}{\left(W_+' - \frac{1}{\tau} W_+\right)^*} = i \frac{\Gamma(1/2 + is/2)}{\Gamma(1/2 - is/2)} + O\left(\tau^2\right)$$

Such a condition on the electromagnetic field implies that ($E_\xi = 0$ and $H_\xi = 0$ as $\zeta \to 0$) by means of

$$h^2 E_\xi \sim \frac{\partial^2}{\partial \xi \partial \upsilon} \Pi_{e\upsilon} = \frac{\partial \zeta}{\partial \upsilon} \sqrt{2\gamma} \frac{\partial^2}{\partial \xi \partial \tau} \Pi_{e\upsilon}$$

$$h^2 H_\xi \sim i\omega\varepsilon_0 h \frac{\partial}{\partial \eta} \Pi_{e\upsilon} = i\omega\varepsilon_0 h \frac{\partial \zeta}{\partial \eta} \sqrt{2\gamma} \frac{\partial}{\partial \tau} \Pi_{e\upsilon}$$

Thus near $\tau \to 0$ we again find

$$\psi_0(0, s_p) = c_0 \operatorname{Re}\left[e^{\pi s_p/2} e^{i\pi/4} \Gamma(1/2 + is_p/2)\right] \tag{320}$$

and in general

$$\psi_0(\tau, s) \sim c_0 \frac{\sqrt{2}}{\tau} \operatorname{Re}\left[-\left(-i\tau^2/2\right)^{1/2} \frac{1}{\Gamma(1/2 - is/2)}\right.$$

$$\left\{\ln\left(-i\tau^2/2\right)\left(1-s\tau^2/4\right)+\psi\left(1/2-is/2\right)-2\psi\left(1\right)+\left(s\tau^2/4\right)\left(-\psi\left(1/2-is/2\right)+2\psi\left(1\right)+2\right)+O\left(\tau^4\ln\tau\right)\right\}$$

$$-i\left(i\tau^2/2\right)^{1/2}\frac{1}{\Gamma\left(1/2-is/2\right)}$$

$$\left\{\ln\left(i\tau^2/2\right)\left(1-s\tau^2/4\right)+\psi\left(1/2+is/2\right)-2\psi\left(1\right)+\left(s\tau^2/4\right)\left(-\psi\left(1/2+is/2\right)+2\psi\left(1\right)+2\right)+O\left(\tau^4\ln\tau\right)\right\}\Big]$$

$$\sim c_0\,\mathrm{Re}\left[-e^{-i\pi/4}\frac{1}{\Gamma\left(1/2-is/2\right)}\right.$$

$$\left\{\ln\left(-i\tau^2/2\right)\left(1-s\tau^2/4\right)+\psi\left(1/2-is/2\right)-2\psi\left(1\right)+\left(s\tau^2/4\right)\left(-\psi\left(1/2-is/2\right)+2\psi\left(1\right)+2\right)+O\left(\tau^4\ln\tau\right)\right\}$$

$$+e^{-i\pi/4}\frac{1}{\Gamma\left(1/2-is/2\right)}$$

$$\left\{\ln\left(i\tau^2/2\right)\left(1-s\tau^2/4\right)+\psi\left(1/2+is/2\right)-2\psi\left(1\right)+\left(s\tau^2/4\right)\left(-\psi\left(1/2+is/2\right)+2\psi\left(1\right)+2\right)+O\left(\tau^4\ln\tau\right)\right\}\Big]$$

$$\sim c_0\,\mathrm{Re}\left[e^{-i\pi/4}\frac{1}{\Gamma\left(1/2-is/2\right)}\right.$$

$$\left\{i\pi\left(1-s\tau^2/4\right)+\left(\psi\left(1/2+is/2\right)-\psi\left(1/2-is/2\right)\right)+\left(s\tau^2/4\right)\left(\psi\left(1/2-is/2\right)-\psi\left(1/2+is/2\right)\right)+O\left(\tau^4\ln\tau\right)\right\}\Big]$$

$$\sim c_0\,\mathrm{Re}\left[\frac{e^{i\pi/4}}{\Gamma\left(1/2-is/2\right)}\right]\left[\pi-i\left\{\psi\left(1/2+is/2\right)-\psi\left(1/2-is/2\right)\right\}\right]\left(1-s\tau^2/4\right)+O\left(\tau^4\ln\tau\right)$$

Noting that

$$\psi\left(1/2+is/2\right)-\psi\left(1/2-is/2\right)=\pi\cot\pi\left(1/2-is/2\right)$$

$$=\pi\frac{\cos\pi\left(1/2-is/2\right)}{\sin\pi\left(1/2-is/2\right)}=\pi\frac{\sin\left(i\pi s/2\right)}{\cos\left(i\pi s/2\right)}=-\frac{\pi}{i}\frac{\sinh\left(\pi s/2\right)}{\cosh\left(\pi s/2\right)}=i\pi\tanh\left(\pi s/2\right)$$

Then

$$\psi_0\left(\tau,s\right)\sim c_0\,\mathrm{Re}\left[\frac{e^{i\pi/4}}{\Gamma\left(1/2-is/2\right)}\right]\pi\left[1+\tanh\left(\pi s/2\right)\right]\left(1-s\tau^2/4\right)+O\left(\tau^4\ln\tau\right)$$

$$\sim c_0\,\mathrm{Re}\left[e^{i\pi/4}\Gamma\left(1/2+is/2\right)\cosh\pi\left(s/2\right)\right]\left[1+\tanh\left(\pi s/2\right)\right]\left(1-s\tau^2/4\right)+O\left(\tau^4\ln\tau\right)$$

$$\sim c_0 \operatorname{Re}\left[e^{i\pi/4}\Gamma\left(1/2+is/2\right)\right]e^{\pi s/2}\left(1-s\tau^2/4\right)+O\left(\tau^4\ln\tau\right)$$

$$\tau\frac{\partial}{\partial\tau}\psi_0\left(\tau,s\right)\sim -c_0 \operatorname{Re}\left[e^{i\pi/4}\Gamma\left(1/2+is/2\right)\right]e^{\pi s/2}\left(s\tau^2/2\right)+O\left(\tau^4\ln\tau\right)\sim -\left(s\tau^2/2\right)\psi_0$$

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\psi_0\left(\tau,s\right)\sim -c_0 \operatorname{Re}\left[e^{i\pi/4}\Gamma\left(1/2+is/2\right)\right]e^{\pi s/2}\left(s/2\right)+O\left(\tau^4\ln\tau\right)\sim -\frac{s}{2}\psi_0$$

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial}{\partial\tau}\psi_0\right)\sim -c_0 \operatorname{Re}\left[e^{i\pi/4}\Gamma\left(1/2+is/2\right)\right]e^{\pi s/2}s+O\left(\tau^2\ln\tau\right)\sim -s\psi_0+O\left(\tau^2\ln\tau\right)$$

In addition

$$\tau\frac{\partial\psi_0}{\partial\tau}\left(\tau,s\right)=c_0\sqrt{2}\operatorname{Re}\left[\left(W_+'-\frac{1}{\tau}W_+\right)+e^{i\Phi_0}\left(W_+'-\frac{1}{\tau}W_+\right)^*\right]$$

$$\tau\frac{\partial^2\psi_0}{\partial\omega\partial\tau}\left(\tau,s\right)\approx c_0\sqrt{2}\operatorname{Re}\left[i\frac{\partial\Phi_0}{\partial\omega}e^{i\Phi_0}\left(W_+'-\frac{1}{\tau}W_+\right)^*\right]$$

$$\approx -c_0\sqrt{2}\operatorname{Re}\left[\frac{\partial\Phi_0}{\partial\omega}\frac{\Gamma\left(1/2+is/2\right)}{\Gamma\left(1/2-is/2\right)}\left(W_+'-\frac{1}{\tau}W_+\right)^*\right]$$

and

$$\tau\frac{\partial^2\psi_0}{\partial\omega\partial\tau}\left(0,s\right)\approx 2c_0\operatorname{Re}\left[\frac{e^{i\pi/4}}{\Gamma\left(1/2-is/2\right)}\right]\frac{\partial\Phi_0}{\partial\omega}$$

4.4 Vector Normalization Condition

The method used for normalization of the eigenfunction components by Antonsen [1] is now put into the framework of the electromagnetic energy theorem [13]

$$\nabla\cdot\left(\frac{\partial\underline{E}}{\partial\omega}\times\underline{H}^*+\underline{E}^*\times\frac{\partial\underline{H}}{\partial\omega}\right)=i\left[\frac{\partial\left(\omega\mu\right)}{\partial\omega}\underline{H}\cdot\underline{H}^*+\frac{\partial\left(\omega\varepsilon\right)}{\partial\omega}\underline{E}\cdot\underline{E}^*\right]-\frac{\partial\underline{E}}{\partial\omega}\cdot\underline{J}^*-\underline{E}^*\cdot\frac{\partial\underline{J}}{\partial\omega}\tag{321}$$

Integrating over the cavity volume and using the divergence theorem (and inserting the electrical properties of free space)

$$\oint_S\left(\frac{\partial\underline{E}}{\partial\omega}\times\underline{H}^*+\underline{E}^*\times\frac{\partial\underline{H}}{\partial\omega}\right)\cdot\underline{n}dS=i\int_V\left(\mu_0\underline{H}\cdot\underline{H}^*+\varepsilon_0\underline{E}\cdot\underline{E}^*\right)dV-\int_V\left(\frac{\partial\underline{E}}{\partial\omega}\cdot\underline{J}^*+\underline{E}^*\cdot\frac{\partial\underline{J}}{\partial\omega}\right)dV$$

where the unit vector $\underline{n}$ in the divergence theorem points out of the cavity region.

4.4.1 Source Free Form Of Theorem

The source free form is thus

$$\oint_S \left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega}\right) \cdot \underline{n} dS = i \int_V \left(\mu_0 \underline{H} \cdot \underline{H}^* + \varepsilon_0 \underline{E} \cdot \underline{E}^*\right) dV \tag{322}$$

Using $\underline{n} \times \underline{E} = 0$ on the cavity walls, the surface integral on the cavity boundary vanishes

$$\int_{S_{cavity}} \left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega}\right) \cdot \underline{n} dS = \int_{S_{cavity}} \left[\left(\underline{n} \times \frac{\partial \underline{E}}{\partial \omega}\right) \cdot \underline{H}^* + \left(\underline{n} \times \underline{E}^*\right) \cdot \frac{\partial \underline{H}}{\partial \omega}\right] dS = 0$$

However a part of the closed surface S_{scar} is taken to surround the scarred orbit $\zeta = \sqrt{\upsilon^2 + \eta^2} \to 0$

$$\int_{S_{scar}} \left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega}\right) \cdot \underline{n} dS = i \int_V \left(\mu_0 \underline{H} \cdot \underline{H}^* + \varepsilon_0 \underline{E} \cdot \underline{E}^*\right) dV \tag{323}$$

where the unit normal here $\underline{n}$ points into the scarred region. We take the fields to be [17], [18]

$$\underline{E} = \nabla \times \nabla \times \underline{\Pi}_e = \nabla \left(\nabla \cdot \underline{\Pi}_e\right) + k^2 \underline{\Pi}_e$$

$$\underline{H} = -i\omega\varepsilon \nabla \times \underline{\Pi}_e$$

where for the first polarization state [5]

$$\Pi_{e\upsilon} = \Phi$$

$$\Pi_{e\eta} = \Pi_{e\xi} = 0$$

Note that as $\zeta = \sqrt{\upsilon^2 + \eta^2} \to 0$ the fields become $E_\upsilon \neq 0$, $E_\eta = 0$, $E_\xi = 0$, $H_\xi = 0$, $H_\eta \neq 0$ at $\upsilon = \eta = 0$, where

$$h^2 E_\upsilon = h \frac{\partial}{\partial \upsilon}\left[\frac{1}{h^3} \frac{\partial}{\partial \upsilon}\left(h^2 \Pi_{e\upsilon}\right)\right] + k^2 h^2 \Pi_{e\upsilon} \sim \left(\frac{\partial^2}{\partial \upsilon^2} + k^2 h^2\right) \Pi_{e\upsilon} = \left(2\gamma \frac{\partial^2}{\partial \tau_x^2} + k^2 h^2\right) \Pi_{e\upsilon}$$

$$h^2 E_\eta = h \frac{\partial}{\partial \eta}\left[\frac{1}{h^3} \frac{\partial}{\partial \upsilon}\left(h^2 \Pi_{e\upsilon}\right)\right] \sim \frac{\partial^2}{\partial \eta \partial \upsilon} \Pi_{e\upsilon} = 2\gamma \frac{\partial^2}{\partial \tau_y \partial \tau_x} \Pi_{e\upsilon}$$

$$h^2 E_\xi = h \frac{\partial}{\partial \xi}\left[\frac{1}{h^3} \frac{\partial}{\partial \upsilon}\left(h^2 \Pi_{e\upsilon}\right)\right] \sim \frac{\partial^2}{\partial \xi \partial \upsilon} \Pi_{e\upsilon} = \sqrt{2\gamma} \frac{\partial^2}{\partial \xi \partial \tau_x} \Pi_{e\upsilon}$$

$$h^2 H_\xi = i\omega\varepsilon_0 \frac{\partial}{\partial \eta}\left(h \Pi_{e\upsilon}\right) \sim i\omega\varepsilon_0 h \frac{\partial}{\partial \eta} \Pi_{e\upsilon} = i\omega\varepsilon_0 \sqrt{2\gamma} h \frac{\partial}{\partial \tau_y} \Pi_{e\upsilon}$$

$$h^2 H_\eta = -i\omega\varepsilon_0 \frac{\partial}{\partial \xi}\left(h \Pi_{e\upsilon}\right) \sim -i\omega\varepsilon_0 h \frac{\partial}{\partial \xi} \Pi_{e\upsilon} = -i\omega\varepsilon_0 h \frac{\partial}{\partial \xi} \Pi_{e\upsilon}$$

Thus in the boundary integral with (note that the inward unit normal is oppositely directed from the prolate spheroidal unit vector) $\underline{n} = -\underline{e}_\zeta$ and

$$\underline{e}_\upsilon \cdot \underline{e}_\zeta = \frac{h_\zeta \zeta \upsilon}{\sqrt{h_\varphi^2 \eta^2 + h_\zeta^2 \upsilon^2 \zeta^2}} \sim \cos\varphi$$

$$\underline{e}_\eta \cdot \underline{e}_\zeta = \frac{h_\zeta \zeta \eta}{\sqrt{h_\varphi^2 \upsilon^2 + h_\zeta^2 \eta^2 \zeta^2}} \sim \sin\varphi$$

$$\underline{e}_\upsilon \times \underline{e}_\eta \sim \underline{e}_\xi$$

$$\underline{e}_\xi \times \underline{e}_\upsilon \sim \underline{e}_\eta$$

$$\underline{e}_\eta \times \underline{e}_\xi \sim \underline{e}_\upsilon$$

we note that $\underline{E} \times \underline{H}^*$ captures $E_\upsilon \underline{e}_\upsilon \times H_\xi^* \underline{e}_\xi = -E_\upsilon H_\xi^* \underline{e}_\eta$, $E_\upsilon \underline{e}_\upsilon \times H_\eta^* \underline{e}_\eta = E_\upsilon H_\eta^* \underline{e}_\xi$, $E_\eta \underline{e}_\eta \times H_\xi^* \underline{e}_\xi = E_\eta H_\xi^* \underline{e}_\upsilon$ and $E_\xi \underline{e}_\xi \times H_\eta^* \underline{e}_\eta = -E_\xi H_\eta^* \underline{e}_\upsilon$. But at the center $E_\eta = 0$, $E_\xi = 0$, $H_\xi = 0$ and only $E_\upsilon H_\eta^* \underline{e}_\xi$ survives, but is orthogonal to $\underline{n} = \underline{e}_\zeta$. Thus the ω derivative is responsible for a contribution

$$\left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^*\right) \cdot \underline{e}_\zeta = -\frac{\partial E_\xi}{\partial \omega} H_\eta^* \underline{e}_\upsilon \cdot \underline{e}_\zeta \sim -\frac{\partial E_\xi}{\partial \omega} H_\eta^* \cos\varphi = -\frac{\partial E_\xi}{\partial \omega} H_\eta^* \frac{\upsilon}{\sqrt{\upsilon^2 + \eta^2}}$$

$$\left(\underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega}\right) \cdot \underline{e}_\zeta = -E_\upsilon^* \frac{\partial H_\xi}{\partial \omega} \underline{e}_\eta \cdot \underline{e}_\zeta \sim -E_\upsilon^* \frac{\partial H_\xi}{\partial \omega} \sin\varphi = -E_\upsilon^* \frac{\partial H_\xi}{\partial \omega} \frac{\eta}{\sqrt{\upsilon^2 + \eta^2}}$$

Noting that E_ξ is odd in $\tau_x = \sqrt{2\gamma}\upsilon = \sqrt{2\gamma}\zeta\cos\varphi$ and that H_ξ is odd in $\tau_y = \sqrt{2\gamma}\eta = \sqrt{2\gamma}\zeta\sin\varphi$ and it appears that these will contribute. The ω derivatives may not have even or odd behavior anyway. Thus

$$\left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega}\right) \cdot \underline{n} \sim \left(\frac{\partial E_\xi}{\partial \omega} H_\eta^* \frac{\upsilon}{\sqrt{\upsilon^2 + \eta^2}} + E_\upsilon^* \frac{\partial H_\xi}{\partial \omega} \frac{\eta}{\sqrt{\upsilon^2 + \eta^2}}\right)$$

$$\sim i\omega\varepsilon_0 \frac{1}{h} \frac{\sqrt{2\gamma}}{\sqrt{\tau_x^2 + \tau_y^2}} \left[\frac{1}{h^2} \frac{\partial^3 \Pi_{e\upsilon}}{\partial\omega\partial\xi\partial\tau_x} \frac{\partial \Pi_{e\upsilon}^*}{\partial \xi} \tau_x + \frac{\partial^2 \Pi_{e\upsilon}}{\partial\omega\partial\tau_y} \tau_y \left(\frac{2\gamma}{h^2} \frac{\partial^2}{\partial \tau_x^2} + k^2\right) \Pi_{e\upsilon}^*\right]$$

Transforming back to prolate spheroidal coordinates

$$\left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega}\right) \cdot \underline{n}$$

$$\sim i\omega\varepsilon_0 \frac{1}{h} \left[\frac{1}{h^2} \frac{\partial^3 \Pi_{e\upsilon}}{\partial\omega\partial\xi\partial\upsilon} \frac{\partial \Pi_{e\upsilon}^*}{\partial \xi} \cos\varphi + \frac{\partial^2 \Pi_{e\upsilon}}{\partial\omega\partial\eta} \sin\varphi \left(\frac{1}{h^2} \frac{\partial^2}{\partial \upsilon^2} + k^2\right) \Pi_{e\upsilon}^*\right]$$

Noting the identities

$$\frac{\partial W}{\partial \upsilon} \sim \frac{\partial W}{\partial \zeta} \cos\varphi - \frac{1}{\zeta} \frac{\partial W}{\partial \varphi} \sin\varphi$$

$$\frac{\partial^2 W}{\partial \upsilon^2} \sim \frac{\partial^2 W}{\partial \zeta^2}\cos^2\varphi + \frac{1}{\zeta}\frac{\partial W}{\partial \zeta}\sin^2\varphi - \frac{1}{\zeta}\frac{\partial^2 W}{\partial \zeta \partial \varphi} 2\sin\varphi\cos\varphi + \frac{1}{\zeta^2}\frac{\partial W}{\partial \varphi} 2\sin\varphi\cos\varphi + \frac{1}{\zeta^2}\frac{\partial^2 W}{\partial \varphi^2}\sin^2\varphi$$

$$\frac{\partial W}{\partial \eta} \sim \frac{\partial W}{\partial \zeta}\sin\varphi + \frac{1}{\zeta}\frac{\partial W}{\partial \varphi}\cos\varphi$$

and for the axisymmetric case $m = 0$

$$\frac{\partial W}{\partial \upsilon} \sim \frac{\partial W}{\partial \zeta}\cos\varphi$$

$$\frac{\partial^2 W}{\partial \upsilon^2} \sim \frac{\partial^2 W}{\partial \zeta^2}\cos^2\varphi + \frac{1}{\zeta}\frac{\partial W}{\partial \zeta}\sin^2\varphi$$

$$\frac{\partial W}{\partial \eta} \sim \frac{\partial W}{\partial \zeta}\sin\varphi$$

and

$$\int_0^{2\pi} \frac{\partial W}{\partial \upsilon}\cos\varphi d\varphi \sim \frac{\partial W}{\partial \zeta}\int_0^{2\pi}\cos^2\varphi d\varphi = \pi\frac{\partial W}{\partial \zeta}$$

$$\int_0^{2\pi} \frac{\partial W}{\partial \eta}\frac{\partial^2 W}{\partial \upsilon^2}\sin\varphi d\varphi \sim \frac{\partial W}{\partial \zeta}\frac{\partial^2 W}{\partial \zeta^2}\int_0^{2\pi}\cos^2\varphi\sin^2\varphi d\varphi + \frac{1}{\zeta}\left(\frac{\partial W}{\partial \zeta}\right)^2\int_0^{2\pi}\sin^4\varphi d\varphi$$

$$\sim \frac{1}{4}\frac{\partial W}{\partial \zeta}\frac{\partial^2 W}{\partial \zeta^2}\int_0^{2\pi}(1+\cos 2\varphi)(1-\cos 2\varphi)\, d\varphi + \frac{1}{4}\frac{1}{\zeta}\left(\frac{\partial W}{\partial \zeta}\right)^2\int_0^{2\pi}(1-\cos 2\varphi)^2 d\varphi$$

$$\sim \frac{1}{4}\frac{\partial W}{\partial \zeta}\frac{\partial^2 W}{\partial \zeta^2}\int_0^{2\pi}\left(1-\cos^2 2\varphi\right) d\varphi + \frac{1}{4}\frac{1}{\zeta}\left(\frac{\partial W}{\partial \zeta}\right)^2\int_0^{2\pi}\left(1-2\cos 2\varphi+\cos^2 2\varphi\right) d\varphi$$

$$\sim \frac{\pi}{4}\frac{\partial W}{\partial \zeta}\left(\frac{\partial^2 W}{\partial \zeta^2} + 3\frac{1}{\zeta}\frac{\partial W}{\partial \zeta}\right) = \frac{\pi}{4}\frac{\partial W}{\partial \zeta}\frac{1}{\zeta}\frac{\partial}{\partial \zeta}\left(\zeta\frac{\partial W}{\partial \zeta} + 2W\right)$$

$$\int_0^{2\pi} \frac{\partial W}{\partial \eta}\sin\varphi d\varphi \sim \frac{\partial W}{\partial \zeta}\int_0^{2\pi}\sin^2\varphi d\varphi = \pi\frac{\partial W}{\partial \zeta}$$

or

$$\int_0^{2\pi}\left(\frac{\partial \underline{E}}{\partial \omega}\times\underline{H}^* + \underline{E}^*\times\frac{\partial \underline{H}}{\partial \omega}\right)\cdot\underline{n} d\varphi$$

$$\sim i\omega\varepsilon_0\frac{\pi}{h}\left[\frac{1}{h^2}\frac{\partial^3 \Pi_{ev}}{\partial\omega\partial\xi\partial\zeta}\frac{\partial \Pi_{ev}^*}{\partial \xi} + \frac{\partial^2 \Pi_{ev}}{\partial\omega\partial\zeta}\left(\frac{1}{4}\frac{1}{h^2}\frac{\partial^2}{\partial \zeta^2} + \frac{3}{4}\frac{1}{h^2}\frac{1}{\zeta}\frac{\partial}{\partial\zeta} + k^2\right)\Pi_{ev}^*\right]$$

Therefore we find (noting that $h_\xi \sim h$ and $h_\varphi \sim h\zeta$)

$$i\int_V \left(\mu_0 \underline{H}\cdot\underline{H}^* + \varepsilon_0 \underline{E}\cdot\underline{E}^*\right) dV = \int_{S_{scar}} \left(\frac{\partial \underline{E}}{\partial \omega}\times \underline{H}^* + \underline{E}^*\times\frac{\partial \underline{H}}{\partial \omega}\right)\cdot \underline{n} dS$$

$$= \int_{-\xi_0}^{\xi_0} \left(\frac{\partial \underline{E}}{\partial \omega}\times \underline{H}^* + \underline{E}^*\times\frac{\partial \underline{H}}{\partial \omega}\right)\cdot \underline{n} h_\xi d\xi h_\varphi d\varphi$$

$$\sim i\omega\varepsilon_0\pi \int_{-\xi_0}^{\xi_0} \zeta \left[\frac{1}{h^2}\frac{\partial^3 \Pi_{ev}}{\partial\omega\partial\xi\partial\zeta}\frac{\partial \Pi_{ev}^*}{\partial \xi} + \frac{\partial^2 \Pi_{ev}}{\partial\omega\partial\zeta}\left(\frac{1}{4}\frac{1}{h^2}\frac{\partial^2}{\partial\zeta^2} + \frac{3}{4}\frac{1}{h^2}\frac{1}{\zeta}\frac{\partial}{\partial\zeta} + k^2\right)\Pi_{ev}^*\right] h d\xi$$

$$\sim i\omega\varepsilon_0\pi \int_{-\xi_0}^{\xi_0} \left[\zeta\frac{\partial^3 \Pi_{ev}}{\partial\omega\partial\xi\partial\zeta}\frac{\partial \Pi_{ev}^*}{\partial \xi} + \zeta\frac{\partial^2 \Pi_{ev}}{\partial\omega\partial\zeta}\left\{\frac{1}{4}\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial \Pi_{ev}^*}{\partial\zeta} + 2\Pi_{ev}^*\right) + k^2h^2\Pi_{ev}^*\right\}\right] d\xi/h$$

$$\sim i\omega\varepsilon_0\frac{\pi}{d} \int_{-\xi_0}^{\xi_0} \left[\tau\frac{\partial^3 \Pi_{ev}}{\partial\omega\partial\xi\partial\tau}\frac{\partial \Pi_{ev}^*}{\partial \xi} + \tau\frac{\partial^2 \Pi_{ev}}{\partial\omega\partial\tau}\left\{2\gamma\frac{1}{4}\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial \Pi_{ev}^*}{\partial\tau} + 2\Pi_{ev}^*\right) + \gamma^2\cos^2\xi\Pi_{ev}^*\right\}\right] d\xi/\cos\xi$$

$$\sim i\omega\varepsilon_0\frac{\pi}{d} \int_{-\xi_0}^{\xi_0} \left[\tau\frac{\partial^3 \Pi_{ev}}{\partial\omega\partial\xi\partial\tau}\frac{\partial \Pi_{ev}^*}{\partial \xi} + \gamma\tau\frac{\partial^2 \Pi_{ev}}{\partial\omega\partial\tau}\Pi_{ev}^*\left(\gamma\cos^2\xi - s\right)\right] d\xi/\cos\xi$$

where we used

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\psi_0(\tau) \sim -c_0 \operatorname{Re}\left[e^{i\pi/4}\Gamma(1/2+is/2)\right]e^{\pi s/2}(s/2) + O\left(\tau^4\ln\tau\right) \sim -\frac{s}{2}\psi_0$$

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial}{\partial\tau}\psi_0\right) \sim -c_0 \operatorname{Re}\left[e^{i\pi/4}\Gamma(1/2+is/2)\right]e^{\pi s/2}s + O\left(\tau^2\ln\tau\right) \sim -s\psi_0 + O\left(\tau^2\ln\tau\right)$$

to evaluate the last term in brackets. Now from the function

$$\Pi_{ev} = \frac{2}{\cos\xi}\psi_0(\tau,s)\cos(\gamma\sin\xi - s\sigma)$$

$$= 2\psi_0(\tau,s)\cosh\sigma\cos(\gamma\tanh\sigma - s\sigma)$$

$$\psi_0(0,s) = c_0 \operatorname{Re}\left[e^{i\pi/4}\Gamma(1/2+is/2)\right]e^{\pi s/2}$$

$$\tau\frac{\partial^2\psi_0}{\partial\omega\partial\tau}(0,s) \approx 2c_0 \operatorname{Re}\left\{\frac{e^{i\pi/4}}{\Gamma(1/2-is/2)}\right\}\frac{\partial\Phi_0}{\partial\omega}$$

$$\frac{\partial\Pi_{ev}}{\partial\xi} = 2\psi_0(\tau,s)\frac{\partial}{\partial\xi}\left[\frac{\cos(\gamma\sin\xi - s\sigma)}{\cos\xi}\right] = 2\psi_0(\tau,s)\frac{\partial\sigma}{\partial\xi}\frac{\partial}{\partial\sigma}\left[\frac{\cos(\gamma\sin\xi - s\sigma)}{\cos\xi}\right]$$

$$= 2\psi_0(\tau,s)\cosh\sigma\frac{\partial}{\partial\sigma}\left\{\cosh\sigma\cos(\gamma\tanh\sigma - s\sigma)\right\}$$

$$= 2\psi_0(\tau,s)\left\{\sinh\sigma\cosh\sigma\cos(\gamma\tanh\sigma - s\sigma) - \left(\gamma - s\cosh^2\sigma\right)\sin(\gamma\tanh\sigma - s\sigma)\right\}$$

where we have used

$$\sinh\sigma = \tan\xi$$

$$\cosh\sigma = \sec\xi$$

$$\tanh\sigma = \sin\xi$$

$$d\sigma = \sec\xi d\xi$$

we find

$$i\int_V \left(\mu_0 \underline{H}\cdot\underline{H}^* + \varepsilon_0 \underline{E}\cdot\underline{E}^*\right) dV = \int_{S_{scar}} \left(\frac{\partial \underline{E}}{\partial\omega}\times\underline{H}^* + \underline{E}^*\times\frac{\partial\underline{H}}{\partial\omega}\right)\cdot\underline{n}dS$$

$$\sim i\omega\varepsilon_0\frac{\pi}{d}\int_{-\sigma_0}^{\sigma_0}\left[\tau\frac{\partial^3\Pi_{ev}}{\partial\omega\partial\xi\partial\tau}\frac{\partial\Pi_{ev}^*}{\partial\xi} + \gamma\tau\frac{\partial^2\Pi_{ev}}{\partial\omega\partial\tau}\Pi_{ev}^*\left(\gamma\mathrm{sech}^2\sigma - s\right)\right]d\sigma$$

$$\sim i\omega\varepsilon_0\frac{\pi}{d}4\left\{\tau\frac{\partial^2\psi_0}{\partial\omega\partial\tau}\left(0,s\right)\right\}\psi_0\left(0,s\right)\int_{-\sigma_0}^{\sigma_0}$$

$$\left[\left\{\sinh\sigma\cosh\sigma\cos\left(\gamma\tanh\sigma - s\sigma\right) - \left(\gamma - s\cosh^2\sigma\right)\sin\left(\gamma\tanh\sigma - s\sigma\right)\right\}^2\right.$$

$$\left.+\gamma\cos^2\left(\gamma\tanh\sigma - s\sigma\right)\left(\gamma - s\cosh^2\sigma\right)\right]d\sigma$$

Averaging over the sinusoids for large γ gives

$$i\int_V \left(\mu_0 \underline{H}\cdot\underline{H}^* + \varepsilon_0 \underline{E}\cdot\underline{E}^*\right) dV = \int_{S_{scar}} \left(\frac{\partial \underline{E}}{\partial\omega}\times\underline{H}^* + \underline{E}^*\times\frac{\partial\underline{H}}{\partial\omega}\right)\cdot\underline{n}dS$$

$$\sim i\omega\varepsilon_0\frac{\pi}{d}2\left\{\tau\frac{\partial^2\psi_0}{\partial\omega\partial\tau}\left(0,s\right)\right\}\psi_0\left(0,s\right)$$

$$\int_{-\sigma_0}^{\sigma_0}\left[\left\{\sinh^2\sigma\cosh^2\sigma + \left(\gamma - s\cosh^2\sigma\right)^2\right\}\right.$$

$$\left.+\gamma\left(\gamma - s\cosh^2\sigma\right)\right]d\sigma$$

For the time being we retain only the leading order γ^2 terms

$$\int_V \left(\mu_0 \underline{H}\cdot\underline{H}^* + \varepsilon_0 \underline{E}\cdot\underline{E}^*\right) dV$$

$$\sim 8\omega\varepsilon_0\frac{\pi}{d}\left\{\tau\frac{\partial^2\psi_0}{\partial\omega\partial\tau}\left(0,s\right)\right\}\psi_0\left(0,s\right)\gamma^2\sigma_0$$

$$\sim \omega\varepsilon_0 k^2 d\pi 16 c_0^2 \,\mathrm{Re}\left\{\frac{e^{i\pi/4}}{\Gamma(1/2 - is/2)}\right\}\frac{\partial\Phi_0}{\partial\omega}\,\mathrm{Re}\left[e^{i\pi/4}\Gamma(1/2+is/2)\right]e^{\pi s/2}\sigma_0$$

$$\sim \varepsilon_0 k^4 d\pi 8 c_0^2 \frac{d\Phi_0}{dk^2}\frac{\mathrm{Re}^2\left[e^{i\pi/4}\Gamma(1/2+is/2)\right]}{\left|\Gamma(1/2-is/2)\right|^2}e^{\pi s/2}\ln(\Lambda_+)$$

The phase Φ_0 indicates the reflection phase of the pth component. Again following Antonsen [1] the average derivative is set by taking $\Delta\Phi_0 = 2\pi$ and the spacing between eigenvalues to be given by the Weyl asymptotic result for the vector case $\Delta k \sim \pi^2/\left(k^2 V\right)$, or $\Delta k^2 = 2k\Delta k \sim 2\pi^2/(kV)$ [16]. Note in the acoustic case $\Delta k \sim 2\pi^2/\left(k^2 V\right)$ [15] and $\Delta k^2 = 2k\Delta k \sim 4\pi^2/(Vk)$. In our case we are interested in the scar amplitude on axis (like the even-even eigenvalues in the 2D problem). Setting the total cross sectional area of the axisymmetric cavity to $A = 2A_0$

$$\frac{1}{2}A = A_0 = \int_{-\infty}^{\infty}\rho(z)\,dz$$

The modal spacing in the scalar case is (this corresponds to a single azimuthal parity since the two parities are degenerate)

$$\frac{dk^2}{dN} = 8\pi/A \tag{324}$$

Note that we will take half this spacing for the electromagnetic vector modes (again this corresponds to a single azimuthal parity)

$$\frac{dk^2}{dN} = 4\pi/A \tag{325}$$

Note that by choosing the eigenvalue spacing to be one half that of the scalar case, results in a one half being inserted into the theoretical strength of the square amplitude. However, we are picking only the even modes with respect to z so that (this symmetry doubles the square amplitude)

$$\frac{dk^2}{dN} = 8\pi/A$$

Thus we take [1]

$$\left(\frac{d\Phi_0}{dk^2}\right)^{-1} = \frac{4}{A}v^2$$

where v is the Gaussian random variable with unit variance discussed previously. If we had used the eigenvalue spacing for the 3D electromagnetic cavity $\Delta k^2 = 2k\Delta k \sim 2\pi^2/(Vk)$, but even along the z direction, we would have obtained

$$\left(\frac{d\Phi_0}{dk^2}\right)^{-1} = \frac{2\pi}{Vk}v^2$$

We recall that in the 2D case we introduced a factor of two (we also used a factor of two for evenness along the orbit) to account for evenness about the normal to the orbit, but here we are assuming that the axisymmetric case (say $\cos\varphi$ parity) is handled by starting with the axisymmetric 3D scalar problem.

Thus the energy theorem along with the outer phase derivative connects the normalization constant c_0

with the integration of the field energy throughout the volume

$$\int_V \left(\mu_0 \underline{H} \cdot \underline{H}^* + \varepsilon_0 \underline{E} \cdot \underline{E}^*\right) dV$$

$$\sim \varepsilon_0 8 d\pi \left(\frac{d\Phi_0}{dk^2}\right) \frac{\mathrm{Re}^2 \left[e^{i\pi/4} \Gamma\left(1/2 + is/2\right)\right]}{\left|\Gamma\left(1/2 - is/2\right)\right|^2} e^{\pi s/2} \ln\left(\Lambda_+\right) \left(k^2 c_0\right)^2 \tag{326}$$

We suspect there is an equal contribution to the energy from the corresponding other component of electric Hertz potential (the magnetic Hertz potential is constructed later), but this would correspond to the other degenerate parity of the field in azimuth. Note that the scalar normalization was

$$\int_V |u|^2 \, dV \sim 4\pi d \frac{\mathrm{Re}^2 \left[e^{i\pi/4} \Gamma\left(1/2 + is_p/2\right)\right]}{\left|\Gamma\left(1/2 - is_p/2\right)\right|^2} \left(\frac{d\Phi_0}{dk^2}\right) e^{\pi s_p/2} \ln\left(\Lambda_+\right) c_0^2$$

Thus the factor of two apparent here in the vector case results from including equal contributions from the electric and magnetic energies. In the scalar case we took

$$\int_V |u|^2 \, dV = 1$$

In the vector case we will take

$$\int_V \left(\mu_0 \underline{H} \cdot \underline{H}^* + \varepsilon_0 \underline{E} \cdot \underline{E}^*\right) dV = 2\varepsilon_0 \tag{327}$$

Because the modes in the vector case are twice as dense as in the scalar case, we actually end up with one half the squared amplitude of the scalar case.

4.4.2 Summary of Results

A summary of the results for the axisymmetric mode $m = 0$ are (note that this mode is a vector mode, with electric field on axis $\zeta = \sqrt{\upsilon^2 + \eta^2} \to 0$ polarized in the $\underline{e}_\upsilon$ direction).

$$\Pi_{e\upsilon} = \frac{2}{\cos\xi} \psi_0\left(\tau, s\right) \cos\left(\gamma \sin\xi - s\sigma\right) \tag{328}$$

$$\psi_0 = c_0 \frac{\sqrt{2}}{\tau} \mathrm{Re}\left[W_+ + e^{i\Phi_0} W_+^*\right] \tag{329}$$

$$W_+\left(\tau, 0, s\right) = W_{is/2,0}\left(-i\tau^2/2\right) \tag{330}$$

$$\tau = \sqrt{2\gamma}\zeta \tag{331}$$

$$k\ell - s_p \sigma_0 = \left(p - 1/2\right)\pi = k_p \ell \,,\; p = 1, 2, 3, ... \tag{332}$$

Note that

$$\psi_0(0, s_p) = c_0 \operatorname{Re}\left[e^{\pi s_p/2} e^{i\pi/4} \Gamma(1/2 + i s_p/2)\right] \ , \ \tau \to 0 \tag{333}$$

and near the orbit

$$\psi_0(\tau, s) \sim c_0 \operatorname{Re}\left[e^{i\pi/4} \Gamma(1/2 + is/2)\right] e^{\pi s/2} \left(1 - s\tau^2/4\right) + O\left(\tau^4 \ln \tau\right)$$

$$\tau \frac{\partial^2 \psi_0}{\partial \omega \partial \tau}(0, s) \approx 2c_0 \operatorname{Re}\left[\frac{e^{i\pi/4}}{\Gamma(1/2 - is/2)}\right] \frac{\partial \Phi_0}{\partial \omega}$$

The outer region reflection phase we take [1] to be

$$\left(\frac{d\Phi_0}{dk^2}\right)^{-1} = \frac{4}{A} v^2 \tag{334}$$

where A is the cross sectional area of the axisymmetric cavity and where v is the Gaussian random variable with unit variance and density

$$f(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2} \tag{335}$$

The normalization constant c_0 is connected to the volume energy by means of

$$\int_V \left(\mu_0 \underline{H} \cdot \underline{H}^* + \varepsilon_0 \underline{E} \cdot \underline{E}^*\right) dV$$

$$\sim 8d\pi \left(\frac{d\Phi_0}{dk^2}\right) \frac{\operatorname{Re}^2\left[e^{i\pi/4}\Gamma(1/2 + is/2)\right]}{\left|\Gamma(1/2 - is/2)\right|^2} e^{\pi s/2} \ln(\Lambda_+) \varepsilon_0 \left(k^2 c_0\right)^2 \tag{336}$$

Note that the scalar form of the normalization was similar

$$\int_V |u|^2 dV \sim 4d\pi \left(\frac{d\Phi_0}{dk^2}\right) \frac{\operatorname{Re}^2\left[e^{i\pi/4}\Gamma(1/2 + is_p/2)\right]}{\left|\Gamma(1/2 - is_p/2)\right|^2} e^{\pi s_p/2} \ln(\Lambda_+) c_0^2$$

The fields near the axis are

$$E_v \sim \left(\frac{1}{h^2}\frac{\partial^2}{\partial v^2} + k^2\right) \Pi_{ev} = \left(\frac{2\gamma}{h^2}\frac{\partial^2}{\partial \tau_x^2} + k^2\right) \Pi_{ev}$$

$$\sim \left(\frac{2k}{d\cos^2\xi} \cos^2\varphi \frac{\partial^2}{\partial \tau^2} + k^2\right) \Pi_{ev} \sim k^2 \Pi_{ev} \tag{337}$$

$$H_\eta \sim \frac{-i\omega\varepsilon_0}{h} \frac{\partial}{\partial \xi} \Pi_{ev} \sim \frac{-i\omega\varepsilon_0}{d\cos\xi} \frac{\partial}{\partial \xi} \Pi_{ev}$$

where

$$h = d\sqrt{\cos^2\xi + \zeta^2} = d\sqrt{\cos^2\xi + v^2 + \eta^2} \sim d\cos\xi$$

$$\tau_x = \sqrt{2\gamma}\upsilon = \sqrt{2\gamma}\zeta\cos\varphi$$

$$\tau_y = \sqrt{2\gamma}\eta = \sqrt{2\gamma}\zeta\sin\varphi$$

$$\tau = \sqrt{\tau_x^2 + \tau_y^2} = \sqrt{2\gamma}\zeta$$

In the case where the eigenfunction is odd with respect to the orbit center

$$k\ell - s_p\sigma_0 = p\pi = k_p\ell\,,\; p = 1, 2, 3, ...$$

$$\Pi_{ev}\left(\tau, \xi\right) = \frac{2}{\cos\xi}\psi_0\left(\tau, s\right)\sin\left(\gamma\sin\xi - s\sigma\right)$$

Cylindrical Form If we transform back to Cylindrical coordinates on axis, then we rewrite the potential

$$\Pi_{ev}\left(0, \xi\right) = \frac{2}{\cos\xi}\psi_0\left(0, s\right)\cos\left(\gamma\sin\xi - s\sigma\right)$$

$$\tanh\sigma = \sin\xi$$

$$z \to d\sin\xi$$

$$\sigma = \mathrm{Arctanh}\left(z/d\right) = \frac{1}{2}\ln\left(\frac{d+z}{d-z}\right)$$

as

$$\Pi_{ev}\left(0, z\right) = \frac{2\psi_0\left(0, s_p\right)}{\sqrt{1 - z^2/d^2}}\cos\left[k_p z + p_0\left(z\right)\right] \tag{338}$$

$$p_0\left(z\right) = \left(k - k_p\right)z - \frac{1}{2}s_p\ln\left(\frac{d+z}{d-z}\right)$$

$$= \frac{1}{2}s_p\left\{\left(z/\ell\right)\ln\left(\frac{d+\ell}{d-\ell}\right) - \ln\left(\frac{d+z}{d-z}\right)\right\} \tag{339}$$

where

$$\sigma_0 = \frac{1}{2}\ln\left(\frac{d+\ell}{d-\ell}\right) = \frac{1}{4}\ln\left(\Lambda_+\right)$$

$$\Lambda_{\pm} = \left(\frac{d+\ell}{d-\ell}\right)^{\pm 2} \tag{340}$$

and the separation constant s is

$$s_p = (k - k_p)\,\ell/\sigma_0 = 2\,(k - k_p)\,L/\ln(\Lambda_+) \tag{341}$$

$$k_p \ell = (p - 1/2)\,\pi$$

In the case where the solution is odd along the orbit

$$\Pi_{ev}(0, z) = \frac{2\psi_0(0, s_p)}{\sqrt{1 - z^2/d^2}} \sin[k_p z + p_0(z)]$$

$$k\ell - s_p \sigma_0 = p\pi = k_p \ell\,,\ p = 1, 2, 3, ...$$

4.5 Vector Scar Projection

We now discuss the projections of the vector scar solution along the orbit.

4.5.1 Trigonometric (Fourier) Projection

The trigonometric projection is taken as

$$V_p = 2\int_0^{\ell} \cos(k_p z)\,E_v(0, z)\,dz$$

$$\sim 2\int_0^{\ell} \cos(k_p z)\,k^2 \Pi_{ev}(0, z)\,dz \tag{342}$$

Using

$$\Pi_{ev}(0, z) = \frac{2\psi_0(0, s_p)}{\sqrt{1 - z^2/d^2}} \cos[k_p z + p_0(z)]$$

gives

$$V_p \sim 4k^2 \psi_0(0, s_p) \int_0^{\ell} \cos(k_p z) \cos[k_p z + p_0(z)] \frac{dz}{\sqrt{1 - z^2/d^2}} \tag{343}$$

$$\sim 2k^2 \psi_0(0, s_p) \int_0^{\ell} \{\cos(p_0(z)) + \cos[2k_p z + p_0(z)]\} \frac{dz}{\sqrt{1 - z^2/d^2}}$$

Averaging over the trigonometric functions gives

$$V_p \sim 2k^2 c_0 \operatorname{Re}\left[e^{\pi s_p/2} e^{i\pi/4} \Gamma\left(1/2 + is_p/2\right)\right] \int_0^\ell \cos\left(p_0\left(z\right)\right) \frac{dz}{\sqrt{1 - z^2/d^2}}$$

where

$$p_0\left(z\right) = \frac{1}{2} s_p \left\{\left(z/\ell\right) \ln\left(\frac{d+\ell}{d-\ell}\right) - \ln\left(\frac{d+z}{d-z}\right)\right\}$$

$$\psi_0\left(0, s_p\right) = c_0 \operatorname{Re}\left[e^{\pi s_p/2} e^{i\pi/4} \Gamma\left(1/2 + is_p/2\right)\right] \tag{344}$$

In the limit where $d >> \ell$ we find that $p_0\left(z\right) \to 0$ (the same as $s_p \to 0$) and the amplitude divergence factor $\sqrt{1 - z^2/d^2} \to 1$ so that

$$V_p \sim 2k^2 c_0 \operatorname{Re}\left[e^{\pi s_p/2} e^{i\pi/4} \Gamma\left(1/2 + is_p/2\right)\right] \ell$$

Then using the normalization condition (we have changed this to twice ε_0, since the electric and magnetic energies are assumed to be equal, in order to be more consistent with the scalar normalization)

$$2\varepsilon_0 = \int_V \left(\mu_0 \underline{H} \cdot \underline{H}^* + \varepsilon_0 \underline{E} \cdot \underline{E}^*\right) dV$$

$$\sim 8d\pi \left(\frac{d\Phi_0}{dk^2}\right) \frac{\operatorname{Re}^2\left[e^{i\pi/4} \Gamma\left(1/2 + is/2\right)\right]}{\left|\Gamma\left(1/2 - is/2\right)\right|^2} e^{\pi s/2} \ln\left(\Lambda_+\right) \varepsilon_0 \left(k^2 c_0\right)^2$$

$$\left(\frac{d\Phi_0}{dk^2}\right)^{-1} = \frac{4}{A} v^2$$

$$k^2 c_0 = \frac{v \left|\Gamma\left(1/2 - is/2\right)\right|}{\sqrt{\pi A d \ln\left(\Lambda_+\right)} \operatorname{Re}\left[e^{i\pi/4} \Gamma\left(1/2 + is/2\right)\right] e^{\pi s/4}}$$

we find

$$V_p \sim \frac{2v \left|\Gamma\left(1/2 - is/2\right)\right| e^{\pi s/4} \ell}{\sqrt{\pi A d \ln\left(\Lambda_+\right)}}$$

Now taking

$$\left\langle L V_p^2 \right\rangle = L^2 G_1\left(s\right)/A \tag{345}$$

we find

$$G_1\left(s\right) = \frac{\left|\Gamma\left(1/2 - is/2\right)\right|^2 e^{\pi s/2} \left(2\ell/d\right)}{\pi \ln\left(\Lambda_+\right)}$$

$$= \frac{e^{\pi s/2} \left(2\ell/d\right)}{\cosh\left(\pi s/2\right) \ln\left(\Lambda_+\right)} = \frac{2L/d}{\left(1 + e^{-\pi s}\right) \ln\left(\Lambda_+\right)} \tag{346}$$

where we have used

$$|\Gamma(1/2 + is/2)|^2 = \frac{\pi}{\cosh(\pi s/2)}$$

Now let us use

$$\ln(\Lambda_+) = 2\ln\left(\frac{d+\ell}{d-\ell}\right) \sim 2L/d$$

to yield

$$G_1(s) \sim \frac{1}{(1+e^{-\pi s})}$$

4.5.2 Elliptic System Projection

If we take the projection to be defined by a surface integral about the scar (the quasirectangular unit vector $\underline{e}_v$ is part of this projection operator) we will obtain (note that this limit definition of the integration around the scar produces a definition without the extra amplitude divergence factors in the kernel of the projection operator)

$$\begin{aligned} V_p &= \lim_{\zeta\to 0}\frac{1}{\zeta d}\int_{-\xi_0}^{\xi_0}\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos(\gamma\sin\xi - s\sigma)}{\cos\xi}E_v(\zeta,\varphi,\xi)\,h_\varphi d\varphi h_\xi d\xi \\ &= \lim_{\zeta\to 0}\frac{1}{\zeta d}\int_{-\xi_0}^{\xi_0}\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos(\gamma\sin\xi - s\sigma)}{\cos\xi}k^2\Pi_{ev}(\zeta,\xi)\,h_\varphi d\varphi h_\xi d\xi \end{aligned} \tag{347}$$

where

$$E_v \sim \left(\frac{1}{h^2}\frac{\partial^2}{\partial v^2} + k^2\right)\Pi_{ev} \sim k^2\Pi_{ev}$$

and the Hertz potential on axis is

$$\Pi_{ev}(0,\xi) = \frac{2}{\cos\xi}\psi_0(0,s)\cos(\gamma\sin\xi - s\sigma)$$

$$\tanh\sigma = \sin\xi$$

$$(k-k_p)\,\ell = s\sigma_0 = s\ln(\tan\xi_0 + \sec\xi_0) = s\ln\sqrt{\frac{d+\ell}{d-\ell}}$$

$$= s\frac{1}{4}\ln(\Lambda_+)$$

and the metric coefficients near the axis are

$$h_\zeta = h_\xi = d\sqrt{\sinh^2\zeta + \cos^2\xi} \sim d\sqrt{\cos^2\xi + \zeta^2} \sim d\cos\xi$$

$$h_\varphi = d \sinh \zeta \cos \xi \sim \zeta d \cos \xi$$

Thus we find

$$V_p \sim 2k^2 d\psi_0 (0, s) \int_{-\xi_0}^{\xi_0} \cos^2 (\gamma \sin \xi - s\sigma) \, d\xi$$

Now for large γ we average over the sinusoid and approximate the integration as ξ_0

$$V_p \sim 2k^2 d\xi_0 \psi_0 (0, s) \tag{348}$$

where

$$\psi_0 (0, s_p) = c_0 \, \mathrm{Re} \left[e^{\pi s_p/2} e^{i\pi/4} \Gamma (1/2 + is_p/2) \right]$$

Inserting the amplitude coefficient from the energy theorem

$$2\varepsilon_0 = \int_V (\mu_0 \underline{H} \cdot \underline{H}^* + \varepsilon_0 \underline{E} \cdot \underline{E}^*) \, dV$$

$$\sim 8d\pi \left(\frac{d\Phi_0}{dk^2} \right) \frac{\mathrm{Re}^2 \left[e^{i\pi/4} \Gamma (1/2 + is/2) \right]}{\left| \Gamma (1/2 - is/2) \right|^2} e^{\pi s/2} \ln (\Lambda_+) \, \varepsilon_0 \left(k^2 c_0 \right)^2$$

$$\left(\frac{d\Phi_0}{dk^2} \right)^{-1} = \frac{4}{A} v^2$$

or with the volume integral taken as $2\varepsilon_0$

$$k^2 c_0 = v \frac{e^{-\pi s_p/4} \left| \Gamma (1/2 + is_p/2) \right|}{\sqrt{A\pi d \ln (\Lambda_+)} \, \mathrm{Re} \left[e^{i\pi/4} \Gamma (1/2 + is_p/2) \right]}$$

gives

$$\psi_0 (0, s_p) = v \frac{e^{\pi s_p/4} \left| \Gamma (1/2 + is_p/2) \right|}{k^2 \sqrt{A\pi d \ln (\Lambda_+)}}$$

Thus the projection is

$$V_p \sim 2dv \frac{e^{\pi s_p/4} \left| \Gamma (1/2 + is_p/2) \right|}{\sqrt{A\pi d \ln (\Lambda_+)}} \xi_0$$

Now using

$$\xi_0 = \arcsin (\ell/d)$$

and

$$\left| \Gamma (1/2 - is_p/2) \right|^2 = \frac{\pi}{\cosh (\pi s_p/2)}$$

we find

$$\left\langle LV_p^2\right\rangle = 4Ld^2 \frac{e^{\pi s_p/2}\left|\Gamma\left(1/2+is_p/2\right)\right|^2}{A\pi d\ln\left(\Lambda_+\right)}\xi_0^2$$

$$= 4Ld\frac{e^{\pi s_p/2}}{A\ln\left(\Lambda_+\right)\cosh\left(\pi s_p/2\right)}\arcsin^2\left(\ell/d\right)$$

$$= \frac{Ld}{A\ln\left(\Lambda_+\right)\left(1+e^{-\pi s_p}\right)}8\arcsin^2\left(\ell/d\right)$$

Then with

$$\left\langle LV_p^2\right\rangle = L^2G_1/A$$

we find

$$G_1\left(s_p\right) = \frac{(d/L)}{\ln\left(\Lambda_+\right)\left(1+e^{-\pi s_p}\right)}8\arcsin^2\left(\ell/d\right) \tag{349}$$

The limit $d >> \ell$ gives

$$\ln\left(\Lambda_+\right) = 2\ln\left(\frac{d+\ell}{d-\ell}\right) \sim 2L/d$$

and

$$G_1\left(s_p\right) \rightarrow \frac{2L/d}{\ln\left(\Lambda_+\right)\left(1+e^{-\pi s_p}\right)} \sim \frac{1}{1+e^{-\pi s_p}}$$

4.5.3 Scar (Galerkin) Projection

From an application point of view we may want a the field projection taken as a simple line integral along the orbit

$$V_p = \int_{-\xi_0}^{\xi_0}\frac{\cos\left(\gamma\sin\xi - s\sigma\right)}{\cos\xi}E_v\left(0,\xi\right)h_\xi d\xi$$

$$= \int_{-\xi_0}^{\xi_0}\frac{\cos\left(\gamma\sin\xi - s\sigma\right)}{\cos\xi}k^2\Pi_{ev}\left(0,\xi\right)h_\xi d\xi \tag{350}$$

where the Hertz potential on axis is

$$\Pi_{ev}\left(0,\xi\right) = \frac{2}{\cos\xi}\psi_0\left(0,s\right)\cos\left(\gamma\sin\xi - s\sigma\right)$$

$$\tanh\sigma = \sin\xi$$

$$d\sigma = \sec\xi d\xi$$

$$\sigma_0 = \frac{1}{2}\ln\left(\frac{d+\ell}{d-\ell}\right) = \frac{1}{4}\ln\left(\Lambda_+\right)$$

$$\left(k - k_p\right)\ell = s\sigma_0$$

and the metric coefficient near the axis is

$$h_\xi = d\sqrt{\sinh^2\zeta + \cos^2\xi} \sim d\sqrt{\cos^2\xi + \zeta^2} \sim d\cos\xi$$

Thus we find

$$V_p = 2k^2 d\psi_0\left(0, s\right)\int_{-\xi_0}^{\xi_0}\cos^2\left(\gamma\sin\xi - s\sigma\right)\frac{d\xi}{\cos\xi}$$

$$= 2k^2 d\psi_0\left(0, s\right)\int_{-\sigma_0}^{\sigma_0}\cos^2\left(\gamma\tanh\sigma - s\sigma\right)d\sigma$$

Now for large γ we average over the sinusoid and approximate the integration as σ_0

$$V_p \sim 2k^2 d\sigma_0\psi_0\left(0, s\right)$$

Plugging in

$$\psi_0\left(0, s_p\right) = v\frac{e^{\pi s_p/4}\left|\Gamma\left(1/2 + is_p/2\right)\right|}{k^2\sqrt{A\pi d\ln\left(\Lambda_+\right)}}$$

gives

$$V_p \sim \frac{1}{2}ve^{\pi s_p/4}\left|\Gamma\left(1/2 + is_p/2\right)\right|\sqrt{\frac{d\ln\left(\Lambda_+\right)}{\pi A}}$$

Now using

$$\left|\Gamma\left(1/2 - is_p/2\right)\right|^2 = \frac{\pi}{\cosh\left(\pi s_p/2\right)}$$

we find

$$\left\langle LV_p^2\right\rangle = \frac{e^{\pi s_p/2}}{\cosh\left(\pi s_p/2\right)}\frac{Ld\ln\left(\Lambda_+\right)}{4A} = \frac{1}{1 + e^{-\pi s_p}}\frac{Ld\ln\left(\Lambda_+\right)}{2A}$$

Then taking

$$\left\langle LV_p^2\right\rangle = L^2 G_1/A$$

gives

$$G_1 = \frac{\left(d/L\right)\ln\left(\Lambda_+\right)/2}{1 + e^{-\pi s_p}}$$

$$\ln\left(\Lambda_+\right) = 2\ln\left(\frac{d+\ell}{d-\ell}\right)$$

Note that if we take the limit $\ell << d$ then $\ln\left(\Lambda_+\right) \sim 2L/d$ and

$$G_1 \sim \frac{1}{1+e^{-\pi s_p}}$$

4.5.4 Scar (Galerkin) Projection Using Cartesian Form

This can also be carried out using the Cartesian form of the asymptotic solution

$$V_p = 2\int_0^{\ell} \left(1-z^2/d^2\right)^{-1/2} \cos\left[k_p z + p_0(z)\right] E_v(0,z)\, dz$$

$$\sim 2\int_0^{\ell} \left(1-z^2/d^2\right)^{-1/2} \cos\left[k_p z + p_0(z)\right] k^2 \Pi_{ev}(0,z)\, dz \tag{351}$$

where

$$p_0(z) = \frac{1}{2} s_p \left\{ \ln\left(\frac{d+\ell}{d-\ell}\right)(z/\ell) - \ln\left(\frac{d+z}{d-z}\right) \right\}$$

$$\Pi_{ev}(0,z) = \frac{2\psi_0(0,s_p)}{\sqrt{1-z^2/d^2}} \cos\left[k_p z + p_0(z)\right]$$

In the case where the parity along the orbit is odd the projection is taken as

$$V_p = 2\int_0^{\ell} \left(1-z^2/d^2\right)^{-1/2} \sin\left[k_p z + p_0(z)\right] E_v(0,z)\, dz$$

$$\sim 2\int_0^{\ell} \left(1-z^2/d^2\right)^{-1/2} \sin\left[k_p z + p_0(z)\right] k^2 \Pi_{ev}(0,z)\, dz \tag{352}$$

Notice that the only difference between the 3D scalar projections and these 3D vector projections is that u is replaced by $k^2\Pi_{ev}$. Because the vector normalization condition is the same for $k^2 c_0$ as the scalar condition for c_0, the statistics of the projection will be similar. The actual potential is made up of a sum of different values of p, but because the pth components are asymptotically orthogonal at high frequencies, if the eigenfunction is made up of a sum

$$\Pi_{ev}(0,z) \sim \sum_p \Pi_{evp}(0,z) \tag{353}$$

this projection will pick out the p term of the sum

$$V_p \sim 4k^2\psi_0(0,s_p) \int_0^{\ell} \left(1-z^2/d^2\right)^{-1} \cos^2\left[k_p z + p_0(z)\right] dz \tag{354}$$

Now averaging over the rapidly varying $k_p z$ we find

$$V_p \sim 2k^2 d\psi_0(0,s_p) \int_0^{\ell/d} \left(1-z^2\right)^{-1} dz = k^2 d\psi_0(0,s_p) \int_0^{\ell/d} \left(\frac{1}{1-z} + \frac{1}{1+z}\right) dz$$

$$\sim c_0 \operatorname{Re}\left[e^{\pi s_p/2} e^{i\pi/4} \Gamma\left(1/2 + is_p/2\right)\right] k^2 d \ln \left|\frac{1+\ell/d}{1-\ell/d}\right|$$

$$\sim v \frac{e^{\pi s_p/4}\left|\Gamma\left(1/2 + is_p/2\right)\right|}{\sqrt{A\pi d \ln \Lambda_+}} d \ln \left|\frac{d+\ell}{d-\ell}\right|$$

$$\sim \frac{1}{2} v e^{\pi s_p/4}\left|\Gamma\left(1/2 + is_p/2\right)\right| \sqrt{\frac{d \ln\left(\Lambda_+\right)}{\pi A}}$$

where in the preceding result we have chosen the normalization

$$\int_V \left(\mu_0 \underline{H} \cdot \underline{H}^* + \varepsilon_0 \underline{E} \cdot \underline{E}^*\right) dV = 2\varepsilon_0$$

to find the amplitude as given by

$$8\pi d \left(\frac{4}{A} v^2\right)^{-1} \frac{\operatorname{Re}^2\left[e^{i\pi/4} \Gamma\left(1/2 + is_p/2\right)\right]}{\left|\Gamma\left(1/2 - is_p/2\right)\right|^2} e^{\pi s_p/2} \ln\left(\Lambda_+\right) \left(k^2 c_0\right)^2 \sim 2$$

Recall that in the 3D scalar case we had set

$$\int_V |u|^2 \, dV = 1$$

In this 3D vector case, taking the average of the square, yields

$$\left\langle L V_p^2 \right\rangle \sim e^{\pi s_p/2} \left|\Gamma\left(1/2 + is_p/2\right)\right|^2 \frac{L d \ln\left(\Lambda_+\right)}{4\pi A} \sim \frac{e^{\pi s_p/2}}{\cosh\left(\pi s_p/2\right)} \frac{L d \ln\left(\Lambda_+\right)}{4A}$$

where we used

$$\left|\Gamma\left(1/2 - is_p/2\right)\right|^2 = \frac{\pi}{\cosh\left(\pi s_p/2\right)}$$

Now writing

$$\left\langle L V_p^2 \right\rangle = L^2 G_1\left(s_p\right)/A$$

gives

$$G_1\left(s_p\right) = \frac{1}{1 + e^{-\pi s_p}} \left(d/L\right) \ln\left(\Lambda_+\right)/2$$

the same result.

Trigonometric Projection Limit If we expand for $\ell << d$

$$\frac{1}{2} \ln\left(\Lambda_+\right) = \ln\left(\frac{d+\ell}{d-\ell}\right) \sim 2\ell/d = L/d$$

we obtain the 3D trigonometric projection

$$\left\langle LV_p^2\right\rangle = L^2 G_1\left(s_p\right)/A \tag{355}$$

where

$$G_1\left(s_p\right) \sim \frac{1}{1+e^{-\pi s_p}}$$

Recall that in 2D we had taken

$$\left\langle \sqrt{kL}V_p^2\right\rangle = L^2 G_1\left(s_p\right)/A$$

where

$$G_1\left(s_p\right) = \frac{2\sqrt{L/d}\exp\left(\pi s_p/2\right)\left|\Gamma\left(1/4 - is_p/2\right)\right|^2}{\left(\pi/2\right)\ln\left(\Lambda_+\right)}$$

and in the $\ell << d$ limit

$$G_1\left(s_p\right) \sim \frac{4}{\left(1+e^{-\pi s_p}\right)\sqrt{L/d}}$$

where we had set

$$\int_A |u|^2\, dS = 1$$

4.6 Vector Random Plane Wave

The random plane wave construction is now discussed in the 3D vector formulation. Initially this is not constructed in axisymmetric geometry but in full 3D. It is compared with the axisymmetric $m = 0$ solution only because this is the solution that exists as the scar orbit is approached in the electromagnetic case. We need to include the random vector direction in this calculation, which is done later in the section.

4.6.1 Vector 3D Random Plane Wave Construction

The vector field random plane wave representation can be written as

$$\underline{E}_r = \lim_{N\to\infty}\sqrt{2/\left(VN\right)}\,\mathrm{Re}\left[\sum_{j=1}^{N} a_j\left(\cos\varphi_{pj}\underline{e}_j + \sin\varphi_{pj}\underline{e}'_j\right)e^{i\alpha_j + i\underline{k}_j\cdot\underline{r}}\right] \tag{356}$$

where the polarization angle φ_{pj} is uniformly distributed on 0 to 2π, $\underline{e}_j$ is perpendicular to $\underline{k}_j$ and $\underline{e}'_j = \left(\underline{k}_j \times \underline{e}_j\right)/k$. Now if we take

$$\underline{E}_r\cdot\underline{E}_r = \lim_{N\to\infty}\sqrt{2/\left(VN\right)}\,\mathrm{Re}\left[\sum_{j=1}^{N} a_j\left(\cos\varphi_{pj}\underline{e}_j + \sin\varphi_{pj}\underline{e}'_j\right)e^{i\alpha_j + i\underline{k}_j\cdot\underline{r}}\right]$$

$$\sqrt{2/(VN)}\,\mathrm{Re}\left[\sum_{j'=1}^{N} a_{j'}\left(\cos\varphi_{pj'}\underline{e}_{j'}+\sin\varphi_{pj'}\underline{e}'_{j'}\right)e^{i\alpha_{j'}+i\underline{k}_{j'}\cdot\underline{r}}\right]$$

and

$$\langle\underline{E}_r\cdot\underline{E}_r\rangle_{a_j}=\lim_{N\to\infty}\sqrt{2/(VN)}\sum_{j=1}^{N}\lim_{N'\to\infty}\sqrt{2/(VN')}\sum_{j'=1}^{N'}\langle a_j a_{j'}\rangle_{a_j}\cos\left(\alpha_j+\underline{k}_j\cdot\underline{r}\right)\cos\left(\alpha_{j'}+\underline{k}_{j'}\cdot\underline{r}\right)$$

$$\left(\cos\varphi_{pj}\underline{e}_j+\sin\varphi_{pj}\underline{e}'_j\right)\cdot\left(\cos\varphi_{pj'}\underline{e}_{j'}+\sin\varphi_{pj'}\underline{e}'_{j'}\right)$$

where the amplitudes a_j for different j values are regarded as independent and the cross terms vanish

$$\langle\underline{E}_r\cdot\underline{E}_r\rangle_{a_j}=\lim_{N\to\infty}\frac{2}{VN}\sum_{j=1}^{N}\cos^2\left(\alpha_j+\underline{k}_j\cdot\underline{r}\right)$$

and averaging over the phases

$$\left\langle\cos^2\left(\alpha_j+\underline{k}_j\cdot\underline{r}\right)\right\rangle_{\alpha_j}=\frac{1}{2\pi}\frac{1}{2}\int_0^{2\pi}\left[1+\cos 2\left(\alpha_j+\underline{k}_j\cdot\underline{r}\right)\right]d\alpha_j=\frac{1}{2}$$

Thus

$$\langle\underline{E}_r\cdot\underline{E}_r\rangle_{a_j,\alpha_j}=\lim_{N\to\infty}\frac{2}{VN}\sum_{j=1}^{N}\frac{1}{2}=\frac{1}{V}$$

4.6.2 Vector 3D Trigonometric (Fourier) Projection

The trigonometric projection (d large) on the vector random plane wave field is

$$V_{pr}=\int_{-\ell}^{\ell}\cos(k_p z)\,E_v(0,z)\,dz=\int_{-\ell}^{\ell}\cos(k_p z)\,\underline{e}_v\cdot\underline{E}_r(0,z)\,dz \tag{357}$$

or

$$V_{pr}=\lim_{N\to\infty}\sqrt{2/(VN)}\sum_{j=1}^{N}a_j\underline{e}_v\cdot\left(\cos\varphi_{pj}\underline{e}_j+\sin\varphi_{pj}\underline{e}'_j\right)\int_{-\ell}^{\ell}\cos(k_p z)\cos\left(\alpha_j+kz\cos\theta_j\right)dz$$

$$=\lim_{N\to\infty}\sqrt{2/(VN)}\sum_{j=1}^{N}a_j\underline{e}_x\cdot\left(\cos\varphi_{pj}\underline{e}_j+\sin\varphi_{pj}\underline{e}'_j\right)\int_{-\ell}^{\ell}\cos(k_p z)\cos\left(\alpha_j+kz\cos\theta_j\right)dz$$

$$\underline{k}_j=k\left(\underline{e}_x\sin\theta_j\cos\varphi_j+\underline{e}_y\sin\theta_j\sin\varphi_j+\underline{e}_z\cos\theta_j\right)$$

$$\underline{e}_j\cdot\underline{k}_j=0$$

Let us take

$$\underline{e}_j = \underline{e}_x \sin\theta_{0j}\cos\varphi_{0j} + \underline{e}_y \sin\theta_{0j}\sin\varphi_{pj} + \underline{e}_z \cos\theta_{0j}$$

with

$$\underline{e}_j \cdot \underline{k}_j = \sin\theta_j\cos\varphi_j\sin\theta_{0j}\cos\varphi_{0j} + \sin\theta_j\sin\varphi_j\sin\theta_{0j}\sin\varphi_{0j} + \cos\theta_j\cos\theta_{0j} = 0$$

$$= \sin\theta_j\sin\theta_{0j}\cos\left(\varphi_j - \varphi_{0j}\right) + \cos\theta_j\cos\theta_{0j}$$

We can define the unit vector $\underline{e}_j$ by taking $\theta_{0j} = \pi/2$ and $\varphi_{0j} = \varphi_j - \pi/2$ or

$$\underline{e}_j = \underline{e}_x \sin\varphi_j - \underline{e}_y \cos\varphi_j$$

Then we take

$$\underline{e}'_j = \left(\underline{e}_x \sin\theta_j\cos\varphi_j + \underline{e}_y \sin\theta_j\sin\varphi_j + \underline{e}_z\cos\theta_j\right) \times \left(\underline{e}_x\sin\varphi_j - \underline{e}_y\cos\varphi_j\right)$$

$$= \underline{e}_x\cos\theta_j\cos\varphi_j + \underline{e}_y\cos\theta_j\sin\varphi_j - \underline{e}_z\sin\theta_j$$

Then the projection becomes

$$V_{pr} = \lim_{N\to\infty}\sqrt{2/\left(VN\right)}\sum_{j=1}^{N} a_j\left(\cos\varphi_{pj}\sin\varphi_j + \sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right)\int_{-\ell}^{\ell}\cos\left(k_p z\right)\cos\left(\alpha_j + kz\cos\theta_j\right)dz$$

Now form the square

$$V_{pr}^2 = \lim_{N\to\infty}\sqrt{2/\left(VN\right)}\sum_{j=1}^{N}$$

$$\lim_{N'\to\infty}\sqrt{2/\left(VN'\right)}\sum_{j'=1}^{N'} a_j a_{j'}\left(\cos\varphi_{pj}\sin\varphi_j + \sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right)\left(\cos\varphi_{pj'}\sin\varphi_{j'} + \sin\varphi_{pj'}\cos\theta_{j'}\cos\varphi_{j'}\right)$$

$$\int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\cos\left(k_p z\right)\cos\left(k_p z'\right)\cos\left(\alpha_j + kz\cos\theta_j\right)\cos\left(\alpha_{j'} + kz'\cos\theta_{j'}\right)dzdz'$$

If we average over the amplitudes a_j and regard different j values as independent the cross terms vanish

$$\left\langle V_{pr}^2\right\rangle_{a_j,\alpha_j} = \lim_{N\to\infty}\frac{2}{VN}\sum_{j=1}^{N}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\cos\left(k_p z\right)\cos\left(k_p z'\right)\left\langle\cos\left(\alpha_j + kz\cos\theta_j\right)\cos\left(\alpha_j + kz'\cos\theta_j\right)\right\rangle_{\alpha_j}dzdz'$$

$$\left\langle\left(\cos\varphi_{pj}\sin\varphi_j + \sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right)^2\right\rangle_{\varphi_j,\varphi_{pj}}$$

The final additional factor arising from the vector nature of the field averages to

$$\left\langle \left(\cos\varphi_{pj}\sin\varphi_j + \sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right)^2 \right\rangle_{\varphi_j,\varphi_{pj}} = \frac{1}{4}\left(1+\cos^2\theta_j\right)$$

and before we had the average over the phases as

$$\left\langle \cos\left(\alpha_j + kz\cos\theta_j\right)\cos\left(\alpha_j + kz'\cos\theta_j\right)\right\rangle_{\alpha_j} = \frac{1}{2\pi}\frac{1}{2}\int_0^{2\pi}\left[\cos\left(k\left(z-z'\right)\cos\theta_j\right) + \cos\left(2\alpha_j + k\left(z+z'\right)\cos\theta_j\right)\right]d\alpha_j$$

$$= \frac{1}{2}\cos\left(k\left(z-z'\right)\cos\theta_j\right) = \frac{1}{2}\cos\left(kz\cos\theta_j\right)\cos\left(kz'\cos\theta_j\right) + \frac{1}{2}\sin\left(kz\cos\theta_j\right)\sin\left(kz'\cos\theta_j\right)$$

This means that

$$\left\langle V_{pr}^2\right\rangle_{a_j,\alpha_j,\varphi_j,\varphi_{pj}} = \lim_{N\to\infty}\frac{1}{4VN}\sum_{j=1}^{N}\left(1+\cos^2\theta_j\right)\int_{-\ell}^{\ell}\cos\left(k_pz\right)\cos\left(kz\cos\theta_j\right)dz\int_{-\ell}^{\ell}\cos\left(k_pz'\right)\cos\left(kz'\cos\theta_j\right)dz'$$

$$= \lim_{N\to\infty}\frac{1/4}{4VN}\sum_{j=1}^{N}\left(1+\cos^2\theta_j\right)\int_{-\ell}^{\ell}\left[\cos\left(k_pz - kz\cos\theta_j\right) + \cos\left(k_pz + kz\cos\theta_j\right)\right]dz$$

$$\int_{-\ell}^{\ell}\left[\cos\left(k_pz' - kz'\cos\theta_j\right) + \cos\left(k_pz' + kz'\cos\theta_j\right)\right]dz'$$

or

$$\left\langle V_{pr}^2\right\rangle = \lim_{N\to\infty}\frac{1/4}{VN}\sum_{j=1}^{N}\left(1+\cos^2\theta_j\right)\left[\frac{\sin\left(k_p - k\cos\theta_j\right)\ell}{k_p - k\cos\theta_j} + \frac{\sin\left(k_p + k\cos\theta_j\right)\ell}{k_p + k\cos\theta_j}\right]^2$$

Now replace the summation with the averaging over the sphere of 4π solid angle

$$\left\langle V_{pr}^2\right\rangle = \frac{1/4}{4\pi V}\int_0^{2\pi}d\varphi\int_0^{\pi}\left[\frac{\sin\left(k_p - k\cos\theta\right)\ell}{k_p - k\cos\theta} + \frac{\sin\left(k_p + k\cos\theta\right)\ell}{k_p + k\cos\theta}\right]^2\left(1+\cos^2\theta\right)\sin\theta d\theta$$

$$= \frac{1/4}{2V}\int_0^{\pi}\left[\frac{\sin\left(k_p - k\cos\theta\right)\ell}{k_p - k\cos\theta} + \frac{\sin\left(k_p + k\cos\theta\right)\ell}{k_p + k\cos\theta}\right]^2\left(1+\cos^2\theta\right)\sin\theta d\theta$$

Changing variables to $x = \cos\theta$ gives

$$\left\langle V_{pr}^2\right\rangle = \frac{1}{2V}\int_{-1}^{1}\left[\frac{\sin\left(k_p - kx\right)\ell}{k_p - kx} + \frac{\sin\left(k_p + kx\right)\ell}{k_p + kx}\right]^2\left(1+x^2\right)dx$$

$$= \frac{1}{2V}\int_{-1}^{1}\left[\frac{\sin^2\left(k_p - kx\right)\ell}{\left(k_p - kx\right)^2} + 2\frac{\sin\left(k_p - kx\right)\ell}{\left(k_p - kx\right)}\frac{\sin\left(k_p + kx\right)\ell}{\left(k_p + kx\right)} + \frac{\sin^2\left(k_p + kx\right)\ell}{\left(k_p + kx\right)^2}\right]\left(1+x^2\right)dx$$

Instead we note that when $k \to k_p$ the first term peaks for $\theta \to 0$ and the second term peaks for $\theta \to \pi$. Thus letting $\theta = \zeta/\sqrt{k\ell/2}$ gives (note that $1+\cos^2\theta \sim 2$)

$$\left\langle V_{pr}^2\right\rangle = \frac{1/4}{V}\int_0^\pi \left[\frac{\sin\left(k_p - k + k\theta^2/2\right)\ell}{k_p - k + k\theta^2/2}\right]^2 \theta d\theta + \int_0^\pi \left[\frac{\sin\left(k_p + k - k\left(\theta-\pi\right)^2/2\right)\ell}{k_p + k - k\left(\theta-\pi\right)^2/2}\right]^2 \left(\pi-\theta\right)d\theta$$

$$= \frac{\ell^2}{Vk\ell}\int_0^\infty \left[\frac{\sin\left(\left(k_p-k\right)\ell+\zeta^2\right)}{\left(k_p-k\right)\ell+\zeta^2}\right]^2 \zeta d\zeta$$

Now taking

$$\lambda = 2\left(k-k_p\right)L$$

gives

$$\left\langle V_{pr}^2\right\rangle = \frac{L^2}{V2kL}\int_0^\infty \left[\frac{\sin\left(-\lambda/4+\zeta^2\right)}{-\lambda/4+\zeta^2}\right]^2 \zeta d\zeta$$

Letting $\zeta^2 = \xi$ gives

$$\left\langle V_{pr}^2\right\rangle = \frac{L^2}{V4kL}\int_0^\infty \left[\frac{\sin\left(-\lambda/4+\xi\right)}{-\lambda/4+\xi}\right]^2 d\xi$$

$$= \frac{L^2}{V4kL}\int_{-\lambda/4}^\infty \sin^2\xi\frac{d\xi}{\xi^2}$$

$$= \frac{L^2}{V4kL}\left[-\frac{\sin^2\left(\lambda/4\right)}{\lambda/4} + \int_{-\lambda/4}^\infty 2\sin\xi\cos\xi\frac{d\xi}{\xi}\right]$$

$$= \frac{L^2}{V4kL}\left[-\frac{\sin^2\left(\lambda/4\right)}{\lambda/4} + \int_{-\lambda/4}^\infty \sin\left(2\xi\right)\frac{d\xi}{\xi}\right]$$

$$= \frac{L^2}{V4kL}\left[-\frac{\sin^2\left(\lambda/4\right)}{\lambda/4} + \int_{-\lambda/2}^\infty \sin\xi\frac{d\xi}{\xi}\right]$$

$$= \frac{L^2}{V4kL}\left[-\frac{\sin^2\left(\lambda/4\right)}{\lambda/4} + \frac{\pi}{2} + \mathrm{Si}\left(\lambda/2\right)\right]$$

Now letting

$$\left\langle kLV_p^2\right\rangle_r = L^2G\left(\lambda\right)/V \tag{358}$$

gives

$$G\left(\lambda\right) = \frac{1}{4}\left[-\frac{\sin^2\left(\lambda/4\right)}{\lambda/4} + \frac{\pi}{2} + \mathrm{Si}\left(\lambda/2\right)\right] \tag{359}$$

Thus the 3D vector case gives exactly half the result of the 3D scalar case.

4.6.3 Extraction Of m=1 Mode From Vector Plane Wave Representation

The vector field random plane wave representation can be written as

$$\underline{E}_r = \lim_{N\to\infty} \sqrt{2/(VN)}\,\mathrm{Re}\left[\sum_{j=1}^{N} a_j \left(\cos\varphi_{pj}\underline{e}_j + \sin\varphi_{pj}\underline{e}_j'\right) e^{i\alpha_j + i\underline{k}_j\cdot\underline{r}}\right] \tag{360}$$

where a_j are real random numbers with $\langle a_j^2 \rangle = 1$, $\left|\underline{k}_j\right| = k$ are random vectors uniformly distributed in angles, and the random phases α_j are uniformly distributed on a 2π interval. We now wish to extract the part that has $\cos\varphi$ dependence. If we transform the unit vectors to cylindrical coordinates

$$\underline{E}_r = \lim_{N\to\infty} \sqrt{2/(VN)} \sum_{j=1}^{N} a_j \cos\left(\underline{k}_j \cdot \underline{r} + \alpha_j\right)$$

$$\left[\underline{e}_\rho \left\{\cos\varphi_{pj}\sin\left(\varphi_j-\varphi\right) + \sin\varphi_{pj}\cos\theta_j\cos\left(\varphi_j-\varphi\right)\right\} + \underline{e}_\varphi\left\{\sin\varphi_{pj}\cos\theta_j\sin\left(\varphi_j-\varphi\right) - \cos\varphi_{pj}\cos\left(\varphi_j-\varphi\right)\right\}\right.$$

$$\left. -\underline{e}_z \sin\varphi_{pj}\sin\theta_j\right]$$

$$= \lim_{N\to\infty} \sqrt{2/(VN)} \sum_{j=1}^{N} a_j \cos\left(k\rho\sin\theta_j\cos\left(\varphi_j-\varphi\right) + kz\cos\theta_j + \alpha_j\right)$$

$$\left[\underline{e}_\rho \left\{\cos\varphi_{pj}\sin\left(\varphi_j-\varphi\right) + \sin\varphi_{pj}\cos\theta_j\cos\left(\varphi_j-\varphi\right)\right\} + \underline{e}_\varphi\left\{\sin\varphi_{pj}\cos\theta_j\sin\left(\varphi_j-\varphi\right) - \cos\varphi_{pj}\cos\left(\varphi_j-\varphi\right)\right\}\right.$$

$$\left. -\underline{e}_z \sin\varphi_{pj}\sin\theta_j\right]$$

Now let us multiply by $\cos\varphi$ and integrate over the interval from $-\pi$ to π

$$\int_0^{2\pi} \underline{E}_r \cos\varphi d\varphi = \int_{-\pi+\varphi_j}^{\pi+\varphi_j} \underline{E}_r \cos\varphi d\varphi$$

$$= \lim_{N\to\infty} \sqrt{2/(VN)} \sum_{j=1}^{N} a_j \int_{-\pi}^{\pi} \cos\left(k\rho\sin\theta_j\cos u + kz\cos\theta_j + \alpha_j\right)\left(\cos u\cos\varphi_j - \sin u\sin\varphi_j\right)$$

$$\left[\underline{e}_\rho\left\{-\cos\varphi_{pj}\sin u + \sin\varphi_{pj}\cos\theta_j\cos u\right\} + \underline{e}_\varphi\left\{-\sin\varphi_{pj}\cos\theta_j\sin u - \cos\varphi_{pj}\cos u\right\} - \underline{e}_z\sin\varphi_{pj}\sin\theta_j\right] du$$

The $\cos u \sin u$ and $\sin u$ terms vanish (since the $\cos u$ in the phase is even in u) and

$$\int_0^{2\pi} \underline{E}_r \cos\varphi d\varphi = \lim_{N\to\infty} \sqrt{2/(VN)} \sum_{j=1}^{N} 2a_j \int_0^{\pi} \cos\left(k\rho\sin\theta_j\cos u + kz\cos\theta_j + \alpha_j\right)$$

$$\left[\underline{e}_\rho \left\{\cos\varphi_{pj}\sin^2 u\sin\varphi_j + \sin\varphi_{pj}\cos\theta_j\cos^2 u\cos\varphi_j\right\} + \underline{e}_\varphi \left\{\sin\varphi_{pj}\cos\theta_j\sin^2 u\sin\varphi_j - \cos\varphi_{pj}\cos^2 u\cos\varphi_j\right\}\right.$$

$$\left. -\underline{e}_z \sin\varphi_{pj}\sin\theta_j\cos u\cos\varphi_j\right] du$$

$$= \lim_{N\to\infty}\sqrt{2/(VN)}\sum_{j=1}^{N} a_j \int_0^\pi \cos\left(k\rho\sin\theta_j\cos u + kz\cos\theta_j + \alpha_j\right)$$

$$\left[\underline{e}_\rho \left\{\cos\varphi_{pj}\left(1-\cos 2u\right)\sin\varphi_j + \sin\varphi_{pj}\cos\theta_j\left(1+\cos 2u\right)\cos\varphi_j\right\}\right.$$

$$+\underline{e}_\varphi \left\{\sin\varphi_{pj}\cos\theta_j\left(1-\cos 2u\right)\sin\varphi_j - \cos\varphi_{pj}\left(1+\cos 2u\right)\cos\varphi_j\right\}$$

$$\left. -\underline{e}_z 2\sin\varphi_{pj}\sin\theta_j\cos u\cos\varphi_j\right] du$$

Now using the identities [20]

$$\int_0^\pi \cos\left(z\cos u\right)\cos\left(nu\right)du = \pi\cos\left(n\pi/2\right)J_n\left(z\right)$$

$$\int_0^\pi \sin\left(z\cos u\right)\cos\left(nu\right)du = \pi\sin\left(n\pi/2\right)J_n\left(z\right)$$

yields

$$\int_0^{2\pi} \underline{E}_r\cos\varphi d\varphi = \pi\lim_{N\to\infty}\sqrt{2/(VN)}\sum_{j=1}^{N} a_j \left[\underline{e}_z 2\sin\left(kz\cos\theta_j + \alpha_j\right)\sin\varphi_{pj}\sin\theta_j J_1\left(k\rho\sin\theta_j\right)\cos\varphi_j\right.$$

$$+\underline{e}_\rho \left\{\cos\varphi_{pj}\left(J_0\left(k\rho\sin\theta_j\right) + J_2\left(k\rho\sin\theta_j\right)\right)\sin\varphi_j + \sin\varphi_{pj}\cos\theta_j\left(J_0\left(k\rho\sin\theta_j\right) - J_2\left(k\rho\sin\theta_j\right)\right)\cos\varphi_j\right\}$$

$$\cos\left(kz\cos\theta_j + \alpha_j\right)$$

$$+\underline{e}_\varphi \left\{\sin\varphi_{pj}\cos\theta_j\left(J_0\left(k\rho\sin\theta_j\right) + J_2\left(k\rho\sin\theta_j\right)\right)\sin\varphi_j - \cos\varphi_{pj}\left(J_0\left(k\rho\sin\theta_j\right) - J_2\left(k\rho\sin\theta_j\right)\right)\cos\varphi_j\right\}$$

$$\left.\cos\left(kz\cos\theta_j + \alpha_j\right)\right]$$

Thus the $m = 1$ random plane wave (even in φ) is (we are removing the integration of $\cos^2\varphi$ from 0 to 2π by means of the $1/\pi$)

$$\underline{E}_r^{(m=1)} = \frac{1}{\pi}\cos\varphi\int_0^{2\pi}\underline{E}_r\cos\varphi d\varphi =$$

$$\lim_{N\to\infty}\sqrt{2/(VN)}\cos\varphi\sum_{j=1}^{N}a_j\left[\underline{e}_z 2\sin\left(kz\cos\theta_j+\alpha_j\right)\sin\varphi_{pj}\sin\theta_j J_1\left(k\rho\sin\theta_j\right)\cos\varphi_j\right.$$

$$+\underline{e}_\rho\left\{\cos\varphi_{pj}\left(J_0\left(k\rho\sin\theta_j\right)+J_2\left(k\rho\sin\theta_j\right)\right)\sin\varphi_j+\sin\varphi_{pj}\cos\theta_j\left(J_0\left(k\rho\sin\theta_j\right)-J_2\left(k\rho\sin\theta_j\right)\right)\cos\varphi_j\right\}$$

$$\cos\left(kz\cos\theta_j+\alpha_j\right)$$

$$+\underline{e}_\varphi\left\{\sin\varphi_{pj}\cos\theta_j\left(J_0\left(k\rho\sin\theta_j\right)+J_2\left(k\rho\sin\theta_j\right)\right)\sin\varphi_j-\cos\varphi_{pj}\left(J_0\left(k\rho\sin\theta_j\right)-J_2\left(k\rho\sin\theta_j\right)\right)\cos\varphi_j\right\}$$

$$\left.\cos\left(kz\cos\theta_j+\alpha_j\right)\right] \tag{361}$$

Note that

$$J_0\left(k\rho\sin\theta_j\right)+J_2\left(k\rho\sin\theta_j\right) = 2\frac{J_1\left(k\rho\sin\theta_j\right)}{k\rho\sin\theta_j}$$

$$J_0\left(k\rho\sin\theta_j\right)-J_2\left(k\rho\sin\theta_j\right) = 2J_1'\left(k\rho\sin\theta_j\right)$$

For the projection we will examine $\underline{e}_x\cdot\underline{E}_r = \underline{e}_\rho\cdot\underline{E}_r$ at $\varphi = 0$. First let us check the normalization over the cavity by taking the dot product of this field with itself

$$\left\langle\underline{E}_r^{(m=1)}\cdot\underline{E}_r^{(m=1)}\right\rangle_{a_j} = \lim_{N\to\infty}\left\{2/(VN)\right\}\cos^2\varphi\sum_{j=1}^{N}\left[4\sin^2\left(kz\cos\theta_j+\alpha_j\right)\sin^2\varphi_{pj}\sin^2\theta_j J_1^2\left(k\rho\sin\theta_j\right)\cos^2\varphi_j\right.$$

$$+\left\{\cos\varphi_{pj}\left(J_0\left(k\rho\sin\theta_j\right)+J_2\left(k\rho\sin\theta_j\right)\right)\sin\varphi_j+\sin\varphi_{pj}\cos\theta_j\left(J_0\left(k\rho\sin\theta_j\right)-J_2\left(k\rho\sin\theta_j\right)\right)\cos\varphi_j\right\}^2$$

$$\cos^2\left(kz\cos\theta_j+\alpha_j\right)$$

$$+\left\{\sin\varphi_{pj}\cos\theta_j\left(J_0\left(k\rho\sin\theta_j\right)+J_2\left(k\rho\sin\theta_j\right)\right)\sin\varphi_j-\cos\varphi_{pj}\left(J_0\left(k\rho\sin\theta_j\right)-J_2\left(k\rho\sin\theta_j\right)\right)\cos\varphi_j\right\}^2$$

$$\left.\cos^2\left(kz\cos\theta_j+\alpha_j\right)\right]$$

where we have averaged over the amplitudes a_j and used their independence. Next we average over the phase α_j

$$\left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)} \right\rangle_{a_j,\alpha_j} = \frac{1}{2} \lim_{N\to\infty} \{2/(VN)\} \cos^2\varphi \sum_{j=1}^{N} \left[4\sin^2\varphi_{pj} \sin^2\theta_j J_1^2 (k\rho\sin\theta_j) \cos^2\varphi_j \right.$$

$$+ \left\{ \cos\varphi_{pj} \left(J_0(k\rho\sin\theta_j) + J_2(k\rho\sin\theta_j)\right) \sin\varphi_j + \sin\varphi_{pj} \cos\theta_j \left(J_0(k\rho\sin\theta_j) - J_2(k\rho\sin\theta_j)\right) \cos\varphi_j \right\}^2$$

$$\left. + \left\{ \sin\varphi_{pj} \cos\theta_j \left(J_0(k\rho\sin\theta_j) + J_2(k\rho\sin\theta_j)\right) \sin\varphi_j - \cos\varphi_{pj} \left(J_0(k\rho\sin\theta_j) - J_2(k\rho\sin\theta_j)\right) \cos\varphi_j \right\}^2 \right]$$

Then averaging over the polarization angles φ_{pj} gives

$$\left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)} \right\rangle_{a_j,\alpha_j,\varphi_{pj}} = \frac{1}{4} \lim_{N\to\infty} \{2/(VN)\} \cos^2\varphi \sum_{j=1}^{N} \left[4\sin^2\theta_j J_1^2 (k\rho\sin\theta_j) \cos^2\varphi_j \right.$$

$$\left. + \left(1+\cos^2\theta_j\right) \left\{ \left(J_0(k\rho\sin\theta_j) - J_2(k\rho\sin\theta_j)\right)^2 \cos^2\varphi_j + \left(J_0(k\rho\sin\theta_j) + J_2(k\rho\sin\theta_j)\right)^2 \sin^2\varphi_j \right\} \right]$$

Next averaging over φ_j gives

$$\left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)} \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j} = \frac{1}{8} \lim_{N\to\infty} \{2/(VN)\} \cos^2\varphi \sum_{j=1}^{N} \left[4\sin^2\theta_j J_1^2 (k\rho\sin\theta_j) \right.$$

$$\left. + \left(1+\cos^2\theta_j\right) \left\{ \left(J_0(k\rho\sin\theta_j) - J_2(k\rho\sin\theta_j)\right)^2 + \left(J_0(k\rho\sin\theta_j) + J_2(k\rho\sin\theta_j)\right)^2 \right\} \right]$$

or

$$\left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)} \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j} = \lim_{N\to\infty} \{1/(VN)\} \cos^2\varphi \sum_{j=1}^{N} \left[\sin^2\theta_j J_1^2 (k\rho\sin\theta_j) \right.$$

$$\left. + \left(1+\cos^2\theta_j\right) \left\{ \left(J_1'(k\rho\sin\theta_j)\right)^2 + \left(\frac{J_1(k\rho\sin\theta_j)}{k\rho\sin\theta_j}\right)^2 \right\} \right]$$

Averaging over θ_j gives

$$\left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)} \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j} = (1/V) \cos^2\varphi \frac{1}{2} \int_0^{\pi} \sin\theta_j \left[\sin^2\theta_j J_1^2 (k\rho\sin\theta_j) \right.$$

$$\left. + \left(1+\cos^2\theta_j\right) \left\{ \left(J_1'(k\rho\sin\theta_j)\right)^2 + \left(\frac{J_1(k\rho\sin\theta_j)}{k\rho\sin\theta_j}\right)^2 \right\} \right] d\theta_j$$

Changing variables to $u_j = \cos\theta_j$ gives

$$\left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)} \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j,\theta_j} = (1/V) \cos^2\varphi \int_0^1 \left[\left(1-u_j^2\right) J_1^2 \left(k\rho\sqrt{1-u_j^2}\right) \right.$$

$$+\left(1+u_j^2\right)\left\{\left(J_1'\left(k\rho\sqrt{1-u_j^2}\right)\right)^2+\left(\frac{J_1\left(k\rho\sqrt{1-u_j^2}\right)}{k\rho\sqrt{1-u_j^2}}\right)^2\right\}\Bigg] du_j$$

Now changing to $w_j=\sqrt{1-u_j^2}$

$$\left\langle \underline{E}_r^{(m=1)}\cdot\underline{E}_r^{(m=1)}\right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j,\theta_j}=(1/V)\cos^2\varphi\int_0^1\left[w_j^2J_1^2\left(k\rho w_j\right)\right.$$

$$\left.+\left(2-w_j^2\right)\left\{\left(J_1'\left(k\rho w_j\right)\right)^2+\left(\frac{J_1\left(k\rho w_j\right)}{k\rho w_j}\right)^2\right\}\right]\frac{w_j dw_j}{\sqrt{1-w_j^2}} \tag{362}$$

If we integrate over the volume before taking any limiting cases of $k\rho$ (such as $k\rho\to\infty$), we find

$$\int_V\left\langle \underline{E}_r^{(m=1)}\cdot\underline{E}_r^{(m=1)}\right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j} dV=\frac{2\pi}{V}\int_0^1\int_0^{\rho_{\max}}\ell\left(\rho\right)\left[w_j^2J_1^2\left(k\rho w_j\right)\right.$$

$$\left.+\left(2-w_j^2\right)\left\{\left(J_1'\left(k\rho w_j\right)\right)^2+\left(\frac{J_1\left(k\rho w_j\right)}{k\rho w_j}\right)^2\right\}\right]\rho d\rho\frac{w_j dw_j}{\sqrt{1-w_j^2}}$$

For $k\rho=0$

$$\left\langle \underline{E}_r^{(m=1)}\cdot\underline{E}_r^{(m=1)}\left(\rho=0\right)\right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j,\theta_j}=\left(\frac{1}{2V}\right)\cos^2\varphi\int_0^1\left(\frac{1}{\sqrt{1-w_j^2}}+\sqrt{1-w_j^2}\right)w_j dw_j$$

$$=\frac{1}{V}\frac{5}{6}\cos^2\varphi$$

For $k\rho\to\infty$ we might consider expanding the Bessel Functions in their asymptotic forms

$$J_1\left(z\right)\sim\sqrt{2/\left(\pi z\right)}\cos\left(z-3\pi/4\right)$$

Taking the integration to average over the squares of the sinusoids we would arrive at

$$\left\langle \underline{E}_r^{(m=1)}\cdot\underline{E}_r^{(m=1)}\right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j}\sim\frac{2}{\pi k\rho}(1/V)\cos^2\varphi\int_0^1\frac{dw_j}{\sqrt{1-w_j^2}}=\frac{1}{k\rho}(1/V)\cos^2\varphi\ ,\ \ k\rho\to\infty$$

If we integrate over the cavity volume using this asymptotic form for large $k\rho$ we find

$$\int_V\left\langle \underline{E}_r^{(m=1)}\cdot\underline{E}_r^{(m=1)}\right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j} dV=\int_0^{\rho_{\max}}\int_0^{2\pi}d\varphi\int_{-\ell(\rho)}^{\ell(\rho)}\left\langle \underline{E}_r^{(m=1)}\cdot\underline{E}_r^{(m=1)}\right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j} dz\rho d\rho$$

$$= \int_0^{\rho_{\max}} \left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)} \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j,\varphi=0} 2\pi\ell(\rho)\,\rho d\rho$$

If we use the asymptotic form we find

$$\int_V \left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)} \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j} dV \sim \frac{2\pi}{kV} \int_0^{\rho_{\max}} \ell(\rho)\,d\rho = \frac{\pi A}{2kV} \tag{363}$$

where A is the total cross sectional area of the cavity. Note that if we had included the $\sin\varphi$ terms from the 3D random plane wave representation we would expect this result to be doubled.

It is instructive to examine the $\sin\varphi$ terms. So let us multiply by $\sin\varphi$ and integrate over the interval from $-\pi$ to π

$$\int_0^{2\pi} \underline{E}_r \sin\varphi d\varphi = \int_{-\pi+\varphi_j}^{\pi+\varphi_j} \underline{E}_r \sin\varphi d\varphi$$

$$= \lim_{N\to\infty} \sqrt{2/(VN)} \sum_{j=1}^{N} a_j \int_{-\pi}^{\pi} \cos(k\rho\sin\theta_j\cos u + kz\cos\theta_j + \alpha_j) \left(\sin u\cos\varphi_j + \cos u\sin\varphi_j\right)$$

$$\left[\underline{e}_\rho \left\{-\cos\varphi_{pj}\sin u + \sin\varphi_{pj}\cos\theta_j\cos u\right\} + \underline{e}_\varphi \left\{-\sin\varphi_{pj}\cos\theta_j\sin u - \cos\varphi_{pj}\cos u\right\}\right.$$

$$\left. -\underline{e}_z \sin\varphi_{pj}\sin\theta_j\right] du$$

The $\cos u\sin u$ and $\sin u$ terms vanish (since the $\cos u$ in the phase is even in u) and

$$\int_0^{2\pi} \underline{E}_r \sin\varphi d\varphi = \lim_{N\to\infty} \sqrt{2/(VN)} \sum_{j=1}^{N} 2a_j \int_0^{\pi} \cos(k\rho\sin\theta_j\cos u + kz\cos\theta_j + \alpha_j)$$

$$\left[\underline{e}_\rho \left\{-\cos\varphi_{pj}\cos\varphi_j\sin^2 u + \sin\varphi_{pj}\sin\varphi_j\cos\theta_j\cos^2 u\right\}\right.$$

$$+\underline{e}_\varphi \left\{-\sin\varphi_{pj}\cos\varphi_j\cos\theta_j\sin^2 u - \cos\varphi_{pj}\sin\varphi_j\cos^2 u\right\}$$

$$\left. -\underline{e}_z \sin\varphi_{pj}\sin\theta_j\cos u\sin\varphi_j\right] du$$

$$= \lim_{N\to\infty} \sqrt{2/(VN)} \sum_{j=1}^{N} a_j \int_0^{\pi} \cos(k\rho\sin\theta_j\cos u + kz\cos\theta_j + \alpha_j)$$

$$\left[\underline{e}_\rho \left\{-\cos\varphi_{pj}(1-\cos 2u)\cos\varphi_j + \sin\varphi_{pj}\cos\theta_j(1+\cos 2u)\sin\varphi_j\right\}\right.$$

$$+\underline{e}_\varphi \left\{-\sin\varphi_{pj}\cos\theta_j(1-\cos 2u)\cos\varphi_j - \cos\varphi_{pj}(1+\cos 2u)\sin\varphi_j\right\}$$

$$-\underline{e}_z 2 \sin\varphi_{pj} \sin\theta_j \cos u \sin\varphi_j \big] \, du$$

Now using the identities [20]

$$\int_0^\pi \cos(z\cos u)\cos(nu)\,du = \pi\cos(n\pi/2)\,J_n(z)$$

$$\int_0^\pi \sin(z\cos u)\cos(nu)\,du = \pi\sin(n\pi/2)\,J_n(z)$$

yields

$$\int_0^{2\pi} \underline{E}_r \sin\varphi d\varphi = \pi \lim_{N\to\infty} \sqrt{2/(VN)} \sum_{j=1}^{N} a_j \big[\underline{e}_z 2\sin(kz\cos\theta_j + \alpha_j)\sin\varphi_{pj}\sin\theta_j J_1(k\rho\sin\theta_j)\sin\varphi_j$$

$$+\underline{e}_\rho \left\{-\cos\varphi_{pj}\left(J_0(k\rho\sin\theta_j) + J_2(k\rho\sin\theta_j)\right)\cos\varphi_j + \sin\varphi_{pj}\cos\theta_j\left(J_0(k\rho\sin\theta_j) - J_2(k\rho\sin\theta_j)\right)\sin\varphi_j\right\}$$

$$\cos(kz\cos\theta_j + \alpha_j)$$

$$+\underline{e}_\varphi \left\{-\sin\varphi_{pj}\cos\theta_j\left(J_0(k\rho\sin\theta_j) + J_2(k\rho\sin\theta_j)\right)\cos\varphi_j - \cos\varphi_{pj}\left(J_0(k\rho\sin\theta_j) - J_2(k\rho\sin\theta_j)\right)\sin\varphi_j\right\}$$

$$\cos(kz\cos\theta_j + \alpha_j)\big]$$

Thus the $m = 1$ random plane wave (odd in φ) is (we are removing the integration of $\sin^2\varphi$ from 0 to 2π by means of the $1/\pi$)

$$\underline{E}_r^{(m=1)} = \frac{1}{\pi}\sin\varphi \int_0^{2\pi} \underline{E}_r \sin\varphi d\varphi =$$

$$\lim_{N\to\infty} \sqrt{2/(VN)} \sin\varphi \sum_{j=1}^{N} a_j \big[\underline{e}_z 2\sin(kz\cos\theta_j + \alpha_j)\sin\varphi_{pj}\sin\theta_j J_1(k\rho\sin\theta_j)\sin\varphi_j$$

$$+\underline{e}_\rho \left\{-\cos\varphi_{pj}\left(J_0(k\rho\sin\theta_j) + J_2(k\rho\sin\theta_j)\right)\cos\varphi_j + \sin\varphi_{pj}\cos\theta_j\left(J_0(k\rho\sin\theta_j) - J_2(k\rho\sin\theta_j)\right)\sin\varphi_j\right\}$$

$$\cos(kz\cos\theta_j + \alpha_j)$$

$$+\underline{e}_\varphi \left\{-\sin\varphi_{pj}\cos\theta_j\left(J_0(k\rho\sin\theta_j) + J_2(k\rho\sin\theta_j)\right)\cos\varphi_j - \cos\varphi_{pj}\left(J_0(k\rho\sin\theta_j) - J_2(k\rho\sin\theta_j)\right)\sin\varphi_j\right\}$$

$$\cos\left(kz\cos\theta_j+\alpha_j\right)]$$

Again let us check the normalization over the cavity by taking the dot product of this field with itself

$$\left\langle \underline{E}_r^{(m=1)}\cdot\underline{E}_r^{(m=1)}\right\rangle_{a_j} = \lim_{N\to\infty}\left\{2/\left(VN\right)\right\}\sin^2\varphi\sum_{j=1}^{N}\left[4\sin^2\left(kz\cos\theta_j+\alpha_j\right)\sin^2\varphi_{pj}\sin^2\theta_j J_1^2\left(k\rho\sin\theta_j\right)\sin^2\varphi_j\right.$$

$$+\left\{-\cos\varphi_{pj}\left(J_0\left(k\rho\sin\theta_j\right)+J_2\left(k\rho\sin\theta_j\right)\right)\cos\varphi_j+\sin\varphi_{pj}\cos\theta_j\left(J_0\left(k\rho\sin\theta_j\right)-J_2\left(k\rho\sin\theta_j\right)\right)\sin\varphi_j\right\}^2$$

$$\cos^2\left(kz\cos\theta_j+\alpha_j\right)$$

$$+\left\{\sin\varphi_{pj}\cos\theta_j\left(J_0\left(k\rho\sin\theta_j\right)+J_2\left(k\rho\sin\theta_j\right)\right)\cos\varphi_j+\cos\varphi_{pj}\left(J_0\left(k\rho\sin\theta_j\right)-J_2\left(k\rho\sin\theta_j\right)\right)\sin\varphi_j\right\}^2$$

$$\left.\cos^2\left(kz\cos\theta_j+\alpha_j\right)\right]$$

where we have averaged over the amplitudes a_j and used their independence. Next we average over the phase α_j

$$\left\langle \underline{E}_r^{(m=1)}\cdot\underline{E}_r^{(m=1)}\right\rangle_{a_j,\alpha_j} = \frac{1}{2}\lim_{N\to\infty}\left\{2/\left(VN\right)\right\}\sin^2\varphi\sum_{j=1}^{N}\left[4\sin^2\varphi_{pj}\sin^2\theta_j J_1^2\left(k\rho\sin\theta_j\right)\sin^2\varphi_j\right.$$

$$+\left\{-\cos\varphi_{pj}\left(J_0\left(k\rho\sin\theta_j\right)+J_2\left(k\rho\sin\theta_j\right)\right)\cos\varphi_j+\sin\varphi_{pj}\cos\theta_j\left(J_0\left(k\rho\sin\theta_j\right)-J_2\left(k\rho\sin\theta_j\right)\right)\sin\varphi_j\right\}^2$$

$$\left.+\left\{\sin\varphi_{pj}\cos\theta_j\left(J_0\left(k\rho\sin\theta_j\right)+J_2\left(k\rho\sin\theta_j\right)\right)\cos\varphi_j+\cos\varphi_{pj}\left(J_0\left(k\rho\sin\theta_j\right)-J_2\left(k\rho\sin\theta_j\right)\right)\sin\varphi_j\right\}^2\right]$$

Then averaging over the polarization angles φ_{pj} gives

$$\left\langle \underline{E}_r^{(m=1)}\cdot\underline{E}_r^{(m=1)}\right\rangle_{a_j,\alpha_j,\varphi_{pj}} = \frac{1}{4}\lim_{N\to\infty}\left\{2/\left(VN\right)\right\}\sin^2\varphi\sum_{j=1}^{N}\left[4\sin^2\theta_j J_1^2\left(k\rho\sin\theta_j\right)\sin^2\varphi_j\right.$$

$$\left.+\left(1+\cos^2\theta_j\right)\left\{\left(J_0\left(k\rho\sin\theta_j\right)-J_2\left(k\rho\sin\theta_j\right)\right)^2\sin^2\varphi_j+\left(J_0\left(k\rho\sin\theta_j\right)+J_2\left(k\rho\sin\theta_j\right)\right)^2\cos^2\varphi_j\right\}\right]$$

Next averaging over φ_j gives

$$\left\langle \underline{E}_r^{(m=1)}\cdot\underline{E}_r^{(m=1)}\right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j} = \frac{1}{8}\lim_{N\to\infty}\left\{2/\left(VN\right)\right\}\sin^2\varphi\sum_{j=1}^{N}\left[4\sin^2\theta_j J_1^2\left(k\rho\sin\theta_j\right)\right.$$

$$\left.+\left(1+\cos^2\theta_j\right)\left\{\left(J_0\left(k\rho\sin\theta_j\right)-J_2\left(k\rho\sin\theta_j\right)\right)^2+\left(J_0\left(k\rho\sin\theta_j\right)+J_2\left(k\rho\sin\theta_j\right)\right)^2\right\}\right]$$

or

$$\left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)} \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j} = \lim_{N\to\infty} \{1/(VN)\} \sin^2\varphi \sum_{j=1}^{N} \left[\sin^2\theta_j J_1^2(k\rho\sin\theta_j)\right.$$

$$\left. + \left(1+\cos^2\theta_j\right)\left\{\left(J_1'(k\rho\sin\theta_j)\right)^2 + \left(\frac{J_1(k\rho\sin\theta_j)}{k\rho\sin\theta_j}\right)^2\right\}\right]$$

Averaging over θ_j gives

$$\left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)} \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j} = (1/V)\sin^2\varphi \frac{1}{2}\int_0^{\pi} \sin\theta_j \left[\sin^2\theta_j J_1^2(k\rho\sin\theta_j)\right.$$

$$\left. + \left(1+\cos^2\theta_j\right)\left\{\left(J_1'(k\rho\sin\theta_j)\right)^2 + \left(\frac{J_1(k\rho\sin\theta_j)}{k\rho\sin\theta_j}\right)^2\right\}\right] d\theta_j$$

Changing variables to $u_j = \cos\theta_j$ gives

$$\left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)} \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j} = (1/V)\sin^2\varphi \int_0^{1} \left[\left(1-u_j^2\right) J_1^2\left(k\rho\sqrt{1-u_j^2}\right)\right.$$

$$\left. + \left(1+u_j^2\right)\left\{\left(J_1'\left(k\rho\sqrt{1-u_j^2}\right)\right)^2 + \left(\frac{J_1\left(k\rho\sqrt{1-u_j^2}\right)}{k\rho\sqrt{1-u_j^2}}\right)^2\right\}\right] du_j$$

Now changing to $w_j = \sqrt{1-u_j^2}$

$$\left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)} \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j} = (1/V)\sin^2\varphi \int_0^{1} \left[w_j^2 J_1^2(k\rho w_j)\right.$$

$$\left. + \left(2-w_j^2\right)\left\{\left(J_1'(k\rho w_j)\right)^2 + \left(\frac{J_1(k\rho w_j)}{k\rho w_j}\right)^2\right\}\right] \frac{w_j dw_j}{\sqrt{1-w_j^2}}$$

For $k\rho \to \infty$ we might consider expanding the Bessel Functions in their asymptotic forms

$$J_1(z) \sim \sqrt{2/(\pi z)}\cos(z - 3\pi/4)$$

Taking the integration to average over the squares of the sinusoids we would arrive at

$$\left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)} \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j} \sim \frac{2}{\pi k\rho}(1/V)\sin^2\varphi \int_0^1 \frac{dw_j}{\sqrt{1-w_j^2}} = \frac{1}{k\rho}(1/V)\sin^2\varphi\ ,\ k\rho\to\infty$$

We next integrate over the cavity volume

$$\int_V \left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)} \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j} dV = \int_0^{\rho_{\max}} \int_0^{2\pi} d\varphi \int_{-\ell(\rho)}^{\ell(\rho)} \left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)} \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j} dz\rho d\rho$$

$$= \int_0^{\rho_{\max}} \left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)} \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_{j,}\varphi=\pi/2} 2\pi\ell(\rho)\,\rho d\rho$$

If we use the asymptotic form we find the same result as previously for the $\cos\varphi$ contribution

$$\int_V \left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)} \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j} dV \sim \frac{2\pi}{kV} \int_0^{\rho_{\max}} \ell(\rho)\, d\rho = \frac{\pi A}{2kV}$$

Note that by not adding this into the normalization (which would reduce the final result by a factor of two) we are being consistent with the theory, only using the potential component Π_{ev} and not adding in the contribution of $\Pi_{e\eta}$, which is also consistent with the simulation. In other words only a single parity was included in the simulation, the random plane wave normalization, and the theory.

The desired projection had we retained the $\sin\varphi$ terms is

$$\underline{E}_r^{(m=1)}(\rho=0) \cdot \underline{e}_x = \underline{E}_r^{(m=1)}(\rho=0) \cdot \underline{e}_\rho \cos\varphi - \underline{E}_r^{(m=1)}(\rho=0) \cdot \underline{e}_\varphi \sin\varphi$$

with

$$\underline{E}_r^{(m=1)} =$$

$$\lim_{N\to\infty} \sqrt{2/(VN)} \sum_{j=1}^{N} a_j \left[\underline{e}_z 2\sin(kz\cos\theta_j + \alpha_j)\sin\varphi_{pj}\sin\theta_j J_1(k\rho\sin\theta_j)\cos(\varphi_j - \varphi)\right.$$

$$+\underline{e}_\rho \left\{\cos\varphi_{pj}\left(J_0(k\rho\sin\theta_j) + J_2(k\rho\sin\theta_j)\right)\sin(\varphi_j-\varphi) + \sin\varphi_{pj}\cos\theta_j\left(J_0(k\rho\sin\theta_j) - J_2(k\rho\sin\theta_j)\right)\cos(\varphi_j-\varphi)\right\}$$

$$\cos(kz\cos\theta_j + \alpha_j)$$

$$+\underline{e}_\varphi \left\{\sin\varphi_{pj}\cos\theta_j\left(J_0(k\rho\sin\theta_j) + J_2(k\rho\sin\theta_j)\right)\sin(\varphi_j-\varphi) - \cos\varphi_{pj}\left(J_0(k\rho\sin\theta_j) - J_2(k\rho\sin\theta_j)\right)\cos(\varphi_j-\varphi)\right\}$$

$$\left.\cos(kz\cos\theta_j + \alpha_j)\right]$$

Thus

$$\underline{E}_r^{(m=1)}(\rho=0) \cdot \underline{e}_x =$$

$$\lim_{N\to\infty} \sqrt{2/(VN)} \sum_{j=1}^{N} a_j \left[\cos\varphi \left\{\cos\varphi_{pj}\sin(\varphi_j-\varphi) + \sin\varphi_{pj}\cos\theta_j\cos(\varphi_j-\varphi)\right\}\right.$$

$$-\sin\varphi\left\{\sin\varphi_{pj}\cos\theta_j\sin\left(\varphi_j-\varphi\right)-\cos\varphi_{pj}\cos\left(\varphi_j-\varphi\right)\right\}\Big]\cos\left(kz\cos\theta_j+\alpha_j\right)$$

$$=\lim_{N\to\infty}\sqrt{2/\left(VN\right)}\sum_{j=1}^{N}a_j\left\{\cos\varphi_{pj}\sin\varphi_j+\sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right\}\cos\left(kz\cos\theta_j+\alpha_j\right) \tag{364}$$

which is the same as the following expression based on the even solution when we take $\varphi = 0$.

4.6.4 Trigonometric (Fourier) Projection of Random Plane Wave m=1 Mode

In the even case, taking the limit $\varphi, \rho \to 0$ the projection is given by (if we had retained the $\sin\varphi$ terms from the 3D random plane wave form, we could alternatively have taken the result $-\underline{E}_r^{(m=1)}\left(\rho=0,\varphi=\pi/2\right)\cdot\underline{e}_\varphi$)

$$\underline{E}_r^{(m=1)}\left(\rho=0,\varphi=0\right)\cdot\underline{e}_\rho=$$

$$\lim_{N\to\infty}\sqrt{2/\left(VN\right)}\sum_{j=1}^{N}a_j\left\{\cos\varphi_{pj}\sin\varphi_j+\sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right\}\cos\left(kz\cos\theta_j+\alpha_j\right)$$

and the projection is taken as

$$V_{pr}=\int_{-\ell}^{\ell}\underline{E}_r^{(m=1)}\cdot\underline{e}_\rho\cos\left(k_pz\right)dz=$$

$$\lim_{N\to\infty}\sqrt{2/\left(VN\right)}\sum_{j=1}^{N}a_j\left\{\cos\varphi_{pj}\sin\varphi_j+\sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right\}$$

$$\int_{-\ell}^{\ell}\cos\left(kz\cos\theta_j+\alpha_j\right)\cos\left(k_pz\right)dz$$

$$=\frac{1}{2}\lim_{N\to\infty}\sqrt{2/\left(VN\right)}\sum_{j=1}^{N}a_j\left\{\cos\varphi_{pj}\sin\varphi_j+\sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right\}$$

$$\int_{-\ell}^{\ell}\left[\cos\left(kz\cos\theta_j+\alpha_j-k_pz\right)+\cos\left(kz\cos\theta_j+\alpha_j+k_pz\right)\right]dz$$

$$=\frac{1}{2}\lim_{N\to\infty}\sqrt{2/\left(VN\right)}\sum_{j=1}^{N}a_j\left\{\cos\varphi_{pj}\sin\varphi_j+\sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right\}$$

$$\left[\frac{\sin\left(k\ell\cos\theta_j+\alpha_j-k_p\ell\right)+\sin\left(k\ell\cos\theta_j-\alpha_j-k_p\ell\right)}{k\cos\theta_j-k_p}+\frac{\sin\left(k\ell\cos\theta_j+\alpha_j+k_p\ell\right)+\sin\left(k\ell\cos\theta_j-\alpha_j+k_p\ell\right)}{k\cos\theta_j+k_p}\right]$$

$$=\lim_{N\to\infty}\sqrt{2/\left(VN\right)}\sum_{j=1}^{N}a_j\left\{\cos\varphi_{pj}\sin\varphi_j+\sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right\}\cos\alpha_j$$

$$\left[\frac{\sin\left(k\ell\cos\theta_j-k_p\ell\right)}{k\cos\theta_j-k_p}+\frac{\sin\left(k\ell\cos\theta_j+k_p\ell\right)}{k\cos\theta_j+k_p}\right]$$

Then we take

$$\left\langle V_{pr}^2\right\rangle_{a_j}=\lim_{N\to\infty}\left\{2/\left(VN\right)\right\}\sum_{j=1}^{N}\left\{\cos^2\varphi_{pj}\sin^2\varphi_j+\sin^2\varphi_{pj}\cos^2\theta_j\cos^2\varphi_j\right\}\cos^2\alpha_j$$

$$\left[\frac{\sin\left(k\ell\cos\theta_j-k_p\ell\right)}{k\cos\theta_j-k_p}+\frac{\sin\left(k\ell\cos\theta_j+k_p\ell\right)}{k\cos\theta_j+k_p}\right]^2$$

and

$$\left\langle V_{pr}^2\right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j,\theta_j}=\frac{1}{8V}\int_0^{\pi}\left(1+\cos^2\theta_j\right)\left[\frac{\sin\left(k\ell\cos\theta_j-k_p\ell\right)}{k\cos\theta_j-k_p}+\frac{\sin\left(k\ell\cos\theta_j+k_p\ell\right)}{k\cos\theta_j+k_p}\right]^2\sin\theta_j d\theta_j$$

$$=\frac{\ell^2}{8V}\int_{-1}^{1}\left(1+u^2\right)\left[\frac{\sin\left(ku-k_p\right)\ell}{\left(ku-k_p\right)\ell}+\frac{\sin\left(ku+k_p\right)\ell}{\left(ku+k_p\right)\ell}\right]^2du$$

$$=\frac{\ell^2}{4V}\int_{0}^{1}\left(1+u_j^2\right)\left[\frac{\sin\left(ku_j-k_p\right)\ell}{\left(ku_j-k_p\right)\ell}+\frac{\sin\left(ku_j+k_p\right)\ell}{\left(ku_j+k_p\right)\ell}\right]^2du_j$$

Instead we note that when $k\to k_p$ the first term peaks for $\theta_j\to 0$ and the second term peaks for $\theta_j\to\pi$. Thus letting $\theta_j=\zeta/\sqrt{k\ell/2}$ gives (note that $1+\cos^2\theta_j\sim 2$)

$$\left\langle V_{pr}^2\right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j,\theta_j}=\frac{1}{8V}\int_0^{\pi}\left(1+\cos^2\theta_j\right)\left[\frac{\sin\left(k\ell\cos\theta_j-k_p\ell\right)}{k\cos\theta_j-k_p}+\frac{\sin\left(k\ell\cos\theta_j+k_p\ell\right)}{k\cos\theta_j+k_p}\right]^2\sin\theta_j d\theta_j$$

$$\sim\frac{1/4}{V}\int_0^{\pi}\left[\frac{\sin\left(k_p-k+k\theta_j^2/2\right)\ell}{k_p-k+k\theta_j^2/2}\right]^2\theta_j d\theta_j+\int_0^{\pi}\left[\frac{\sin\left(k_p+k-k\left(\theta_j-\pi\right)^2/2\right)\ell}{k_p+k-k\left(\theta_j-\pi\right)^2/2}\right]^2\left(\pi-\theta_j\right)d\theta_j$$

$$=\frac{\ell^2}{Vk\ell}\int_0^{\infty}\left[\frac{\sin\left(\left(k_p-k\right)\ell+\zeta^2\right)}{\left(k_p-k\right)\ell+\zeta^2}\right]^2\zeta d\zeta$$

Now taking

$$\lambda=2\left(k-k_p\right)L$$

gives

$$\left\langle V_{pr}^2\right\rangle=\frac{L^2}{V2kL}\int_0^{\infty}\left[\frac{\sin\left(-\lambda/4+\zeta^2\right)}{-\lambda/4+\zeta^2}\right]^2\zeta d\zeta$$

Letting $\zeta^2=\xi$ gives

$$\langle V_{pr}^2 \rangle = \frac{L^2}{V4kL} \int_0^\infty \left[\frac{\sin(-\lambda/4+\xi)}{-\lambda/4+\xi} \right]^2 d\xi$$

$$= \frac{L^2}{V4kL} \int_{-\lambda/4}^\infty \sin^2 \xi \frac{d\xi}{\xi^2}$$

$$= \frac{L^2}{V4kL} \left[-\frac{\sin^2(\lambda/4)}{\lambda/4} + \int_{-\lambda/4}^\infty 2\sin\xi \cos\xi \frac{d\xi}{\xi} \right]$$

$$= \frac{L^2}{V4kL} \left[-\frac{\sin^2(\lambda/4)}{\lambda/4} + \int_{-\lambda/4}^\infty \sin(2\xi) \frac{d\xi}{\xi} \right]$$

$$= \frac{L^2}{V4kL} \left[-\frac{\sin^2(\lambda/4)}{\lambda/4} + \int_{-\lambda/2}^\infty \sin\xi \frac{d\xi}{\xi} \right]$$

$$= \frac{L^2}{V4kL} \left[-\frac{\sin^2(\lambda/4)}{\lambda/4} + \frac{\pi}{2} + \mathrm{Si}(\lambda/2) \right]$$

Now letting

$$\left\langle kLV_p^2 \right\rangle_r = L^2 G_0(\lambda)/V \tag{365}$$

gives

$$G_0(\lambda) = \frac{1}{4} \left[-\frac{\sin^2(\lambda/4)}{\lambda/4} + \frac{\pi}{2} + \mathrm{Si}(\lambda/2) \right] \tag{366}$$

If we renormalize by the asymptotic result

$$\int_V \left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)} \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j} dV \sim \frac{\pi A}{2kV} \tag{367}$$

we find

$$\left\langle LV_p^2 \right\rangle_r = L^2 G(\lambda)/A \tag{368}$$

$$G(\lambda) = \frac{2}{\pi} G_0(\lambda) = \frac{1}{4} \left[-\frac{2}{\pi} \frac{\sin^2(\lambda/4)}{\lambda/4} + 1 + \frac{2}{\pi} \mathrm{Si}(\lambda/2) \right] \tag{369}$$

Introducing the symmetry about $z = 0$, which doubles this result, gives

$$G_s(\lambda) = 2G(\lambda) = \frac{1}{2} \left[-\frac{2}{\pi} \frac{\sin^2(\lambda/4)}{\lambda/4} + 1 + \frac{2}{\pi} \mathrm{Si}(\lambda/2) \right] \tag{370}$$

This has now rigorously established the vector random plane wave result, accounting for all parts of the $\cos\varphi$ (and $\sin\varphi$) fields in the cylindrical system, to be consistent with the axisymmetric numerical simulation. The asymptotic level (unity) for $\lambda >> 1$ is compared in the figure below with the histogram from the numerical simulation, and appears to do a reasonable job of representing the histogram value. In addition this provides a check on the asymptotic level of the vector scar theory (in particular the asymptotic eigenvalue density being twice that of the scalar $m = 1$ axisymmetric case).

Note that

$$G_s(\lambda) \to 0 \ , \ \lambda \to -\infty \tag{371}$$

$$G_s(0) = 1/2 \tag{372}$$

$$G_s(\infty) = 1 \tag{373}$$

The $m = 1$ vector case gives exactly half the result of the axisymmetric scalar case (as would be anticipated from the energy theorem with the modal spacing being taken as one half the scalar case)! Note that the factor $2kV/(\pi A)$ relating the axisymmetric projection to the non-axisymmetric projection can be thought of as a symmetry enhancement in axisymmetric 3D, analogous to the factors of two in the Cartesian 2D geometry. At high frequencies this value can be large $2kV/(\pi A) >> 1$; it thus represents the major enhancement of the squared field along the scar in the convex wall geometry, and is shared with the random plane wave (with axisymmetric symmetry enforced) behavior.

Simple Vector Random Plane Wave m=1 Mode Trigonometric Projection In the vector case taking the trigonometric projection

$$V_{pr} = \int_{-\ell}^{\ell} \cos(k_p z) E_v(0,z)\, dz = \int_{-\ell}^{\ell} \cos(k_p z)\, \underline{e}_v \cdot \underline{E}_r(0,z)\, dz \tag{374}$$

$$= \int_{-\ell}^{\ell} \cos(k_p z)\, \underline{e}_x \cdot \underline{E}_r(0,z)\, dz$$

$$V_{pr} = \lim_{N\to\infty} \sqrt{4k/(\pi A N)} \sum_{j=1}^{N} a_j \underline{e}_x \cdot \left(\cos\varphi_{pj} \underline{e}_j + \sin\varphi_{pj} \underline{e}'_j\right) \int_{-\ell}^{\ell} \cos(k_p z) \cos(\alpha_j + kz\cos\theta_j)\, dz \tag{375}$$

$$= \lim_{N\to\infty} \sqrt{4k/(\pi A N)} \sum_{j=1}^{N} a_j \left(\cos\varphi_{pj} \sin\varphi_j + \sin\varphi_{pj} \cos\theta_j \cos\varphi_j\right) \int_{-\ell}^{\ell} \cos(k_p z) \cos(\alpha_j + kz\cos\theta_j)\, dz$$

But this is the same as the preceding non-axisymmetric expression except that it is multiplied by $\sqrt{2kV/(\pi A)}$ yielding

$$\left\langle L V_p^2 \right\rangle_r = L^2 G(\lambda)/A$$

with

$$G(\lambda) = \frac{1}{4}\left[-\frac{\sin^2(\lambda/4)}{\lambda/4} + \frac{\pi}{2} + \mathrm{Si}(\lambda/2)\right]\frac{2}{\pi}$$

For the even case along the orbit we expect to multiply by two

$$G_s(\lambda) = 2G(\lambda) = \frac{1}{2}\left[-\frac{4}{\pi}\frac{\sin^2(\lambda/4)}{\lambda/2} + 1 + \frac{2}{\pi}\,\mathrm{Si}(\lambda/2)\right]$$

$$s_p = \lambda/\ln(\Lambda_+) \tag{376}$$

4.6.5 Statistics Of Random Plane Wave Field Component Off Axis

The random plane wave field for a single Cartesian component is

$$E_{rx} = \underline{e}_x \cdot \underline{E}_r = \lim_{N\to\infty}\sqrt{2/(VN)}\sum_{j=1}^{N} a_j \cos\left(k\rho\sin\theta_j\cos\left(\varphi_j-\varphi\right) + kz\cos\theta_j + \alpha_j\right)$$

$$\left[\cos\varphi\left\{\cos\varphi_{pj}\sin\left(\varphi_j-\varphi\right) + \sin\varphi_{pj}\cos\theta_j\cos\left(\varphi_j-\varphi\right)\right\} - \sin\varphi\left\{\sin\varphi_{pj}\cos\theta_j\sin\left(\varphi_j-\varphi\right) - \cos\varphi_{pj}\cos\left(\varphi_j-\varphi\right)\right\}\right]$$

$$= \lim_{N\to\infty}\sqrt{2/(VN)}\sum_{j=1}^{N} a_j \cos\left(k\rho\sin\theta_j\cos\left(\varphi_j-\varphi\right) + kz\cos\theta_j + \alpha_j\right)$$

$$\left(\cos\varphi_{pj}\sin\varphi_j + \sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right) \tag{377}$$

Then squaring and averaging over the amplitudes a_j gives

$$\left\langle E_{rx}^2\right\rangle_{a_j} = \lim_{N\to\infty}\frac{2}{VN}\sum_{j=1}^{N}\cos^2\left(k\rho\sin\theta_j\cos\left(\varphi_j-\varphi\right) + kz\cos\theta_j + \alpha_j\right)\left(\cos\varphi_{pj}\sin\varphi_j + \sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right)^2$$

Now averaging over the phase α_j

$$\left\langle E_{rx}^2\right\rangle_{a_j,\alpha_j} = \lim_{N\to\infty}\frac{1}{VN}\sum_{j=1}^{N}\left(\cos\varphi_{pj}\sin\varphi_j + \sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right)^2$$

Averaging over φ_{pj} gives

$$\left\langle E_{rx}^2\right\rangle_{a_j,\alpha_j,\varphi_{pj}} = \lim_{N\to\infty}\frac{1}{2VN}\sum_{j=1}^{N}\left(\sin^2\varphi_j + \cos^2\theta_j\cos^2\varphi_j\right)$$

Averaging over φ_j gives

$$\left\langle E_{rx}^2\right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j} = \lim_{N\to\infty}\frac{1}{4VN}\sum_{j=1}^{N}\left(1+\cos^2\theta_j\right)$$

Finally averaging over θ_j yields

$$\left\langle E_{rx}^2\right\rangle = \frac{1}{4V}\frac{1}{2}\int_0^{\pi}\left(1+\cos^2\theta_j\right)\sin\theta_j d\theta_j = \frac{1}{8V}\int_{-1}^{1}\left(1+u^2\right)du = \frac{1}{4V}\left(1+\frac{1}{3}\right) = \frac{1}{3V}$$

The vector field we constructed this from

$$\underline{E}_r = \lim_{N\to\infty} \sqrt{2/(VN)}\,\mathrm{Re}\left[\sum_{j=1}^{N} a_j \left(\cos\varphi_{pj}\underline{e}_j + \sin\varphi_{pj}\underline{e}'_j\right) e^{i\alpha_j + i\underline{k}_j\cdot\underline{r}}\right]$$

where the polarization angle φ_{pj} is uniformly distributed on 0 to 2π, $\underline{e}_j$ is perpendicular to $\underline{k}_j$ and $\underline{e}'_j = \left(\underline{k}_j \times \underline{e}_j\right)/k$, has the desired normalization

$$\langle \underline{E}_r \cdot \underline{E}_r \rangle_{a_j,\alpha_j} = \lim_{N\to\infty} \frac{2}{VN} \sum_{j=1}^{N} \frac{1}{2} = \frac{1}{V} \tag{378}$$

Hence the single Cartesian field component has one third the contribution to the energy density of the vector total

$$\left\langle V E_{rx}^2 \right\rangle = \frac{1}{3} \tag{379}$$

We really want the x component of the $m = 1$ mode in φ. The $m = 1$ random plane wave (even in φ) is (we are removing the integration of $\cos^2\varphi$ from 0 to 2π by means of the $1/\pi$)

$$\underline{E}_r^{(m=1)} = \frac{1}{\pi}\cos\varphi \int_0^{2\pi} \underline{E}_r \cos\varphi d\varphi =$$

$$\lim_{N\to\infty} \sqrt{2/(VN)} \cos\varphi \sum_{j=1}^{N} a_j \left[\underline{e}_z 2\sin\left(kz\cos\theta_j + \alpha_j\right)\sin\varphi_{pj}\sin\theta_j J_1\left(k\rho\sin\theta_j\right)\cos\varphi_j\right.$$

$$+\underline{e}_\rho \left\{\cos\varphi_{pj}\left(J_0\left(k\rho\sin\theta_j\right) + J_2\left(k\rho\sin\theta_j\right)\right)\sin\varphi_j + \sin\varphi_{pj}\cos\theta_j\left(J_0\left(k\rho\sin\theta_j\right) - J_2\left(k\rho\sin\theta_j\right)\right)\cos\varphi_j\right\}$$

$$\cos\left(kz\cos\theta_j + \alpha_j\right)$$

$$+\underline{e}_\varphi \left\{\sin\varphi_{pj}\cos\theta_j\left(J_0\left(k\rho\sin\theta_j\right) + J_2\left(k\rho\sin\theta_j\right)\right)\sin\varphi_j - \cos\varphi_{pj}\left(J_0\left(k\rho\sin\theta_j\right) - J_2\left(k\rho\sin\theta_j\right)\right)\cos\varphi_j\right\}$$

$$\left.\cos\left(kz\cos\theta_j + \alpha_j\right)\right]$$

Note that

$$J_0\left(k\rho\sin\theta_j\right) + J_2\left(k\rho\sin\theta_j\right) = 2\frac{J_1\left(k\rho\sin\theta_j\right)}{k\rho\sin\theta_j}$$

$$J_0\left(k\rho\sin\theta_j\right) - J_2\left(k\rho\sin\theta_j\right) = 2J'_1\left(k\rho\sin\theta_j\right)$$

Thus

$$E_{rx}^{(m=1)} = \underline{e}_x \cdot \underline{E}_r^{(m=1)} = \lim_{N\to\infty} \sqrt{2/(VN)} \cos\varphi \sum_{j=1}^{N} a_j \, [$$

$$\cos\varphi \left\{ \cos\varphi_{pj} \left(J_0\left(k\rho\sin\theta_j\right) + J_2\left(k\rho\sin\theta_j\right)\right) \sin\varphi_j + \sin\varphi_{pj} \cos\theta_j \left(J_0\left(k\rho\sin\theta_j\right) - J_2\left(k\rho\sin\theta_j\right)\right) \cos\varphi_j \right\}$$

$$\cos\left(kz\cos\theta_j + \alpha_j\right)$$

$$-\sin\varphi \left\{ \sin\varphi_{pj} \cos\theta_j \left(J_0\left(k\rho\sin\theta_j\right) + J_2\left(k\rho\sin\theta_j\right)\right) \sin\varphi_j - \cos\varphi_{pj} \left(J_0\left(k\rho\sin\theta_j\right) - J_2\left(k\rho\sin\theta_j\right)\right) \cos\varphi_j \right\}$$

$$\cos\left(kz\cos\theta_j + \alpha_j\right)]$$

Then if we square this result and average over a_j

$$\left\langle E_{rx}^{(m=1)2} \right\rangle_{a_j} = \lim_{N\to\infty} \frac{2}{VN} \cos^2\varphi \sum_{j=1}^{N} [$$

$$\cos\varphi \left\{ \cos\varphi_{pj} \left(J_0\left(k\rho\sin\theta_j\right) + J_2\left(k\rho\sin\theta_j\right)\right) \sin\varphi_j + \sin\varphi_{pj} \cos\theta_j \left(J_0\left(k\rho\sin\theta_j\right) - J_2\left(k\rho\sin\theta_j\right)\right) \cos\varphi_j \right\}$$

$$\cos\left(kz\cos\theta_j + \alpha_j\right)$$

$$-\sin\varphi \left\{ \sin\varphi_{pj} \cos\theta_j \left(J_0\left(k\rho\sin\theta_j\right) + J_2\left(k\rho\sin\theta_j\right)\right) \sin\varphi_j - \cos\varphi_{pj} \left(J_0\left(k\rho\sin\theta_j\right) - J_2\left(k\rho\sin\theta_j\right)\right) \cos\varphi_j \right\}$$

$$\cos\left(kz\cos\theta_j + \alpha_j\right)]^2$$

Next averaging over α_j gives

$$\left\langle E_{rx}^{(m=1)2} \right\rangle_{a_j,\alpha_j} = \lim_{N\to\infty} \frac{1}{VN} \cos^2\varphi \sum_{j=1}^{N} [$$

$$\cos^2\varphi \left\{ \cos\varphi_{pj} \left(J_0\left(k\rho\sin\theta_j\right) + J_2\left(k\rho\sin\theta_j\right)\right) \sin\varphi_j + \sin\varphi_{pj} \cos\theta_j \left(J_0\left(k\rho\sin\theta_j\right) - J_2\left(k\rho\sin\theta_j\right)\right) \cos\varphi_j \right\}^2$$

$$-2\sin\varphi \left\{ \sin\varphi_{pj} \cos\theta_j \left(J_0\left(k\rho\sin\theta_j\right) + J_2\left(k\rho\sin\theta_j\right)\right) \sin\varphi_j - \cos\varphi_{pj} \left(J_0\left(k\rho\sin\theta_j\right) - J_2\left(k\rho\sin\theta_j\right)\right) \cos\varphi_j \right\}$$

$$\cos\varphi \left\{ \cos\varphi_{pj} \left(J_0\left(k\rho\sin\theta_j\right) + J_2\left(k\rho\sin\theta_j\right)\right) \sin\varphi_j + \sin\varphi_{pj} \cos\theta_j \left(J_0\left(k\rho\sin\theta_j\right) - J_2\left(k\rho\sin\theta_j\right)\right) \cos\varphi_j \right\}$$

$$+\sin^2\varphi\left\{\sin\varphi_{pj}\cos\theta_j\left(J_0\left(k\rho\sin\theta_j\right)+J_2\left(k\rho\sin\theta_j\right)\right)\sin\varphi_j-\cos\varphi_{pj}\left(J_0\left(k\rho\sin\theta_j\right)-J_2\left(k\rho\sin\theta_j\right)\right)\cos\varphi_j\right\}^2$$

]

Averaging over φ_{pj} gives

$$\left\langle E_{rx}^{(m=1)2}\right\rangle_{a_j,\alpha_j,\varphi_{pj}}=\lim_{N\to\infty}\frac{1}{2VN}\cos^2\varphi\sum_{j=1}^{N}[$$

$$\cos^2\varphi\left\{\left(J_0\left(k\rho\sin\theta_j\right)+J_2\left(k\rho\sin\theta_j\right)\right)^2\sin^2\varphi_j+\cos^2\theta_j\left(J_0\left(k\rho\sin\theta_j\right)-J_2\left(k\rho\sin\theta_j\right)\right)^2\cos^2\varphi_j\right\}$$

$$-2\sin\varphi\cos\varphi\left(\cos^2\theta_j-1\right)\sin\varphi_j\cos\varphi_j\left(J_0\left(k\rho\sin\theta_j\right)-J_2\left(k\rho\sin\theta_j\right)\right)\left(J_0\left(k\rho\sin\theta_j\right)+J_2\left(k\rho\sin\theta_j\right)\right)$$

$$+\sin^2\varphi\left\{\cos^2\theta_j\left(J_0\left(k\rho\sin\theta_j\right)+J_2\left(k\rho\sin\theta_j\right)\right)^2\sin^2\varphi_j+\left(J_0\left(k\rho\sin\theta_j\right)-J_2\left(k\rho\sin\theta_j\right)\right)^2\cos^2\varphi_j\right\}$$

]

Next averaging over φ_j gives

$$\left\langle E_{rx}^{(m=1)2}\right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j}=\lim_{N\to\infty}\frac{1}{4VN}\cos^2\varphi\sum_{j=1}^{N}[$$

$$\cos^2\varphi\left\{\left(J_0\left(k\rho\sin\theta_j\right)+J_2\left(k\rho\sin\theta_j\right)\right)^2+\cos^2\theta_j\left(J_0\left(k\rho\sin\theta_j\right)-J_2\left(k\rho\sin\theta_j\right)\right)^2\right\}$$

$$+\sin^2\varphi\left\{\cos^2\theta_j\left(J_0\left(k\rho\sin\theta_j\right)+J_2\left(k\rho\sin\theta_j\right)\right)^2+\left(J_0\left(k\rho\sin\theta_j\right)-J_2\left(k\rho\sin\theta_j\right)\right)^2\right\}$$

]

Finally averaging over θ_j gives

$$\left\langle E_{rx}^{(m=1)2}\right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j,\theta_j}=\frac{1}{8V}\cos^2\varphi\,[$$

$$\cos^2\varphi\int_0^{\pi}\sin\theta_j\left\{\left(J_0\left(k\rho\sin\theta_j\right)+J_2\left(k\rho\sin\theta_j\right)\right)^2+\cos^2\theta_j\left(J_0\left(k\rho\sin\theta_j\right)-J_2\left(k\rho\sin\theta_j\right)\right)^2\right\}d\theta_j$$

$$+\sin^2\varphi\int_0^{\pi}\sin\theta_j\left\{\cos^2\theta_j\left(J_0\left(k\rho\sin\theta_j\right)+J_2\left(k\rho\sin\theta_j\right)\right)^2+\left(J_0\left(k\rho\sin\theta_j\right)-J_2\left(k\rho\sin\theta_j\right)\right)^2\right\}d\theta_j$$

]

Letting $u_j = \cos\theta_j$

$$\left\langle E_{rx}^{(m=1)2} \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j,\theta_j} = \frac{1}{4V}\cos^2\varphi\,[$$

$$\cos^2\varphi \int_0^1 \left\{ \left(J_0\left(k\rho\sqrt{1-u_j^2}\right) + J_2\left(k\rho\sqrt{1-u_j^2}\right)\right)^2 + u_j^2\left(J_0\left(k\rho\sqrt{1-u_j^2}\right) - J_2\left(k\rho\sqrt{1-u_j^2}\right)\right)^2 \right\} du_j$$

$$+\sin^2\varphi \int_0^1 \left\{ u_j^2\left(J_0\left(k\rho\sqrt{1-u_j^2}\right) + J_2\left(k\rho\sqrt{1-u_j^2}\right)\right)^2 + \left(J_0\left(k\rho\sqrt{1-u_j^2}\right) - J_2\left(k\rho\sqrt{1-u_j^2}\right)\right)^2 \right\} du_j$$

At $\varphi = 0$

$$\left\langle E_{rx}^{(m=1)2} \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j,\theta_j} = \frac{1}{4V}$$

$$\int_0^1 \left\{ \left(J_0\left(k\rho\sqrt{1-u_j^2}\right) + J_2\left(k\rho\sqrt{1-u_j^2}\right)\right)^2 + u_j^2\left(J_0\left(k\rho\sqrt{1-u_j^2}\right) - J_2\left(k\rho\sqrt{1-u_j^2}\right)\right)^2 \right\} du_j$$

or

$$\left\langle E_{rx}^{(m=1)2} \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j,\theta_j} = \frac{1}{V}$$

$$\int_0^1 \left\{ \left(\frac{J_1\left(k\rho\sqrt{1-u_j^2}\right)}{k\rho\sqrt{1-u_j^2}} \right)^2 + u_j^2\left(J_1'\left(k\rho\sqrt{1-u_j^2}\right)\right)^2 \right\} du_j$$

If we were to let $w_j = \sqrt{1-u_j^2}$

$$\left\langle E_{rx}^{(m=1)2} \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j,\theta_j} = \frac{1}{V}$$

$$\int_0^1 \left\{ \left(\frac{J_1\left(k\rho w_j\right)}{k\rho w_j} \right)^2 + \left(1-w_j^2\right)\left(J_1'\left(k\rho w_j\right)\right)^2 \right\} \frac{w_j dw_j}{\sqrt{1-w_j^2}}$$

On axis $\rho = 0$ we find

$$\left\langle E_{rx}^{(m=1)2}\left(\rho=0\right) \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j,\theta_j} = \frac{1}{4V}\int_0^1 \left(1+u_j^2\right) du_j = \frac{1}{3V} \tag{380}$$

Note that, the $m = 1$ mode was originally extracted from the random plane wave representation, which was itself normalized using the full set of azimuthal modes

$$\left\langle \underline{E}_r \cdot \underline{E}_r \right\rangle = \frac{1}{V} \tag{381}$$

Hence, in terms of a normalization involving the random plane wave representation (including all the azimuthal modes), there is no enhancement along the axis (since $1/(3V)$ is what one would expect of a single component). However, because the resonant frequencies of these different m modes will not, in general, be degenerate, this indicates that the spatial normalization of the $m = 1$ modes alone will be required at the resonant frequencies of these modes. Thus, the enhancement on axis $\rho = 0$ will be real for these modes. This fact therefore represents a statement about the enhancement of the tail statistics resulting from symmetry conditions (in this case affecting the $m = 1$ mode).

Thus, in the $m = 1$ case, we note that

$$\int_V \left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)} \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j} dV \sim \frac{2\pi}{kV} \int_0^{\rho_{\max}} \ell(\rho)\, d\rho = \frac{\pi A}{2kV}$$

where A is the total cross sectional area of the axisymmetric cavity. Hence, if we renormalize (multiplying the field square by the ratio of the 3D normalization (unity) to the axisymmetric normalization ($\frac{\pi A}{2kV}$), we would find that the mean square field at $\varphi = 0$ (we expect this to double the mean over the azimuth for the $\cos\varphi$ parity) is

$$\left\langle V E_{rx}^{(m=1)2} \right\rangle = \frac{2kV}{\pi A} \int_0^1 \left\{ \left(\frac{J_1(k\rho w_j)}{k\rho w_j} \right)^2 + \left(1 - w_j^2\right) \left(J_1'(k\rho w_j)\right)^2 \right\} \frac{w_j dw_j}{\sqrt{1 - w_j^2}} \tag{382}$$

On axis this mean square is

$$\left\langle V E_{rx}^{(m=1)2}(\rho = 0) \right\rangle = \frac{2kV}{3\pi A} \tag{383}$$

Using the asymptotic form for the Bessel functions for large $k\rho$

$$J_1(z) \sim \sqrt{2/(\pi z)} \cos(z - 3\pi/4) \ , \ z >> 1$$

gives the mean square

$$\left\langle V E_{rx}^{(m=1)2} \right\rangle \sim \frac{2V}{\pi^2 A \rho} \int_0^1 \sqrt{1 - w_j^2} dw_j = \frac{V}{2\pi\rho A} \ , \ k\rho >> 1 \tag{384}$$

4.7 Comparisons Between Simulation And Electromagnetic Scar Theory

In this section the projection definition will be taken as

$$V_p = 2 \int_0^\ell \left(1 - z^2/d^2\right)^{-1/2} \cos\left[k_p z + p_0(z)\right] E_v(0, z)\, dz \tag{385}$$

in the even case or

$$V_p = 2 \int_0^\ell \left(1 - z^2/d^2\right)^{-1/2} \sin\left[k_p z + p_0(z)\right] E_v(0, z)\, dz \tag{386}$$

in the odd case. The amplitude divergence factor $\left(1 - z^2/d^2\right)^{-1/2}$ (included in these) can lead to convergence problems in the stadium sections below, where the definitions involving a surface integration around the scar are preferred, but in this case the foci are far outside of the cavity region and the results with or without these factors are expected to be nearly the same. The normalization in the numerical simulation is taken as (note that the radial coordinate weight ρ must be present here to get the volume weighting correct)

$$\int_{A/2} \rho \underline{E} \cdot \underline{E}^* dS = 1$$

(at the peak of the azimuthal variation, since the variation $e^{j\varphi}$ is assumed in the numerical calculation). This means that

$$\int_V \underline{E} \cdot \underline{E}^* dV = \pi \int_{A/2} \underline{E} \cdot \underline{E}^* \rho dS = \pi$$

where a factor of one half has been taken into account because of the azimuthal variation associated with the $\cos\varphi$ or $\sin\varphi$ mode (if we allowed both degenerate azimuthal modes to be present then the result is 2π). Thus we divided the amplitude of the square of the simulation results by π to achieve the proper mean square volume integration. Furthermore in the simulation we did not restrict ourselves to the even modes in z. This means that we needed to multiply by a factor of 2 (just like with $G_s(\lambda) = 2G(\lambda)$). Although the odd definition was also used in the simulation for half the values of k_p, we also divided by the total number of s values that fell within the bin in the histogram (hence, the addition of the odd definition simply improved the resolution but did not double the value, and we must double the value after the fact). We believe this is true because there are different k_p values for the odd modes, and hence their addition does not double the histogram results.

In the theoretical model we enforced

$$\int_V \left(\frac{\mu_0}{\varepsilon_0} \underline{H} \cdot \underline{H}^* + \underline{E} \cdot \underline{E}^* \right) dV = 2 \tag{387}$$

or equating the electric and magnetic energies (where we included the azimuthal variation in the volume integration, resulting in a one half factor)

$$\int_V \underline{E} \cdot \underline{E}^* dV = \pi \int_{A/2} \underline{E} \cdot \underline{E}^* \rho dS = 1 \tag{388}$$

The other polarization state could be included by considering either Π_{mv} or $\Pi_{e\eta}$. Since the simulation did not include the second state of azimuthal parity, we leave it out in the theoretical model as well. This raises questions about consistency between the simulation with cylindrical coordinates versus the high frequency theoretical model using the quasirectangular Hertz potentials? Do the modes require the pair Π_{ev}, Π_{mv}, or for the ninety degree parity case $\Pi_{e\eta}, \Pi_{m\eta}$, and what is the level of the accompanying magnetic potential (either Π_{mv} or $\Pi_{m\eta}$); does it contribute to the normalization? But if this were so, why would we get the appropriate level versus the random plane wave for large s? Consistency with the random plane wave limit is thought to exist because we did not include the other parity in the random plane wave normalization as discussed in the preceding sections. We will demonstrate below that these other choices of potential are needed to include other polarizations with respect to the azimuth, and at high frequencies to leading order only one seems to be needed.

Figure 10 shows a simulation of an axisymmetric scar in a rotationally symmetric bowtie cavity [19]. Figure 11 shows a comparison of the preceding theory with a histogram from the numerical simulation

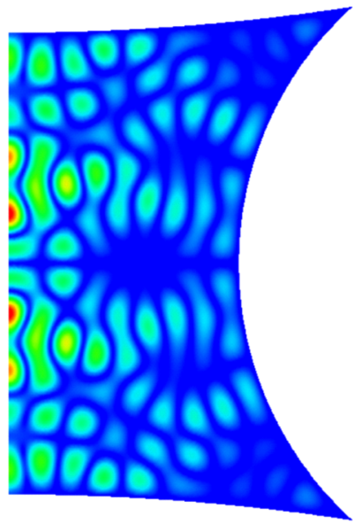

Figure 10. Simulation of axisymmetric scar in rotationally symmetric bowtie cavity.

of the axisymmetric scar [19]. At high frequencies we note that the factor $2kV/(\pi A) >> 1$ relating the axisymmetric projection to the non-axisymmetric projection can be thought of as a symmetry enhancement in axisymmetric 3D, analogous to the factors of two in the Cartesian 2D geometry; it thus represents the major enhancement of the squared field along the scar in the convex wall geometry, and is shared with the random plane wave (with axisymmetric symmetry enforced) behavior. As we found in the 2D case, we expect this high frequency enhancement to persist even if the outer boundary breaks the axisymmetry. Figures 12 and 13 show a numerical simulation with finer gridding and the combined two results.

The modal density for modes that are even along the orbit (the density of modes that are odd are expected to be the same) can be used to find the number of expected modes in the frequency range between 1 GHz and 3 GHz

$$\frac{dN}{dk^2} \sim A/(8\pi)$$

$$N \sim (k_2 - k_1)(k_2 + k_1)\, A/(8\pi)$$

$$k_1 \approx 20.98\ \mathrm{m}^{-1},\ k_2 \approx 62.79\ \mathrm{m}^{-1}$$

$$A \approx 4.677086\ \mathrm{m}^2$$

$$N \approx 654$$

In the simulation for the quarter bowtie which should capture the odd modes along the orbit (since we used a perfect electric conductor at the center) we observed 626 modes between 1 GHz and 3 GHz, which is close to the expected number.

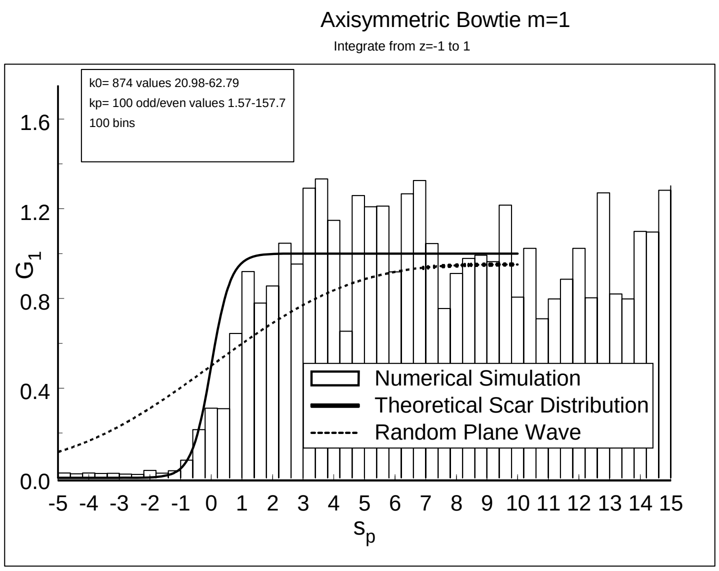

Figure 11. Comparison of theoretical vector scar prediction with histogram from numerical simulation. The random plane wave results are also shown.

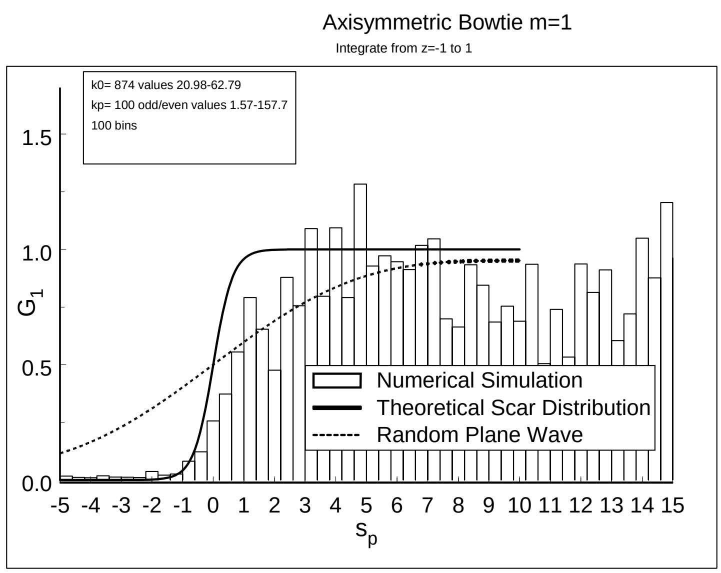

Figure 12. Electric field statistics from simulation (histogram) and from 3D scar theory (solid curve) for a rotationally symmetric bowtie cavity (dotted curve is non-scarred, random plane wave result). The abscissa is proportional to the frequency separation between the modal resonance and the orbital scar frequency. A finer grid was used in this simulation.

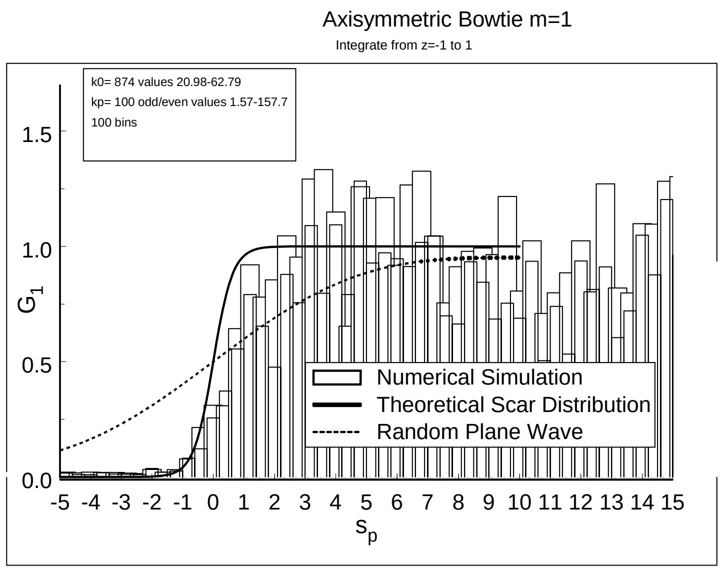

Figure 13. Electric field statistics from simulation (histogram) and from 3D axisymmetric scar theory (solid curve) for a rotationally symmetric bowtie cavity (dotted curve is non-scarred, random plane wave result). The abscissa is proportional to the frequency separation between the modal resonance and the orbital scar frequency. The narrow bins are from a additional simulation with much finer gridding of the cavity walls.

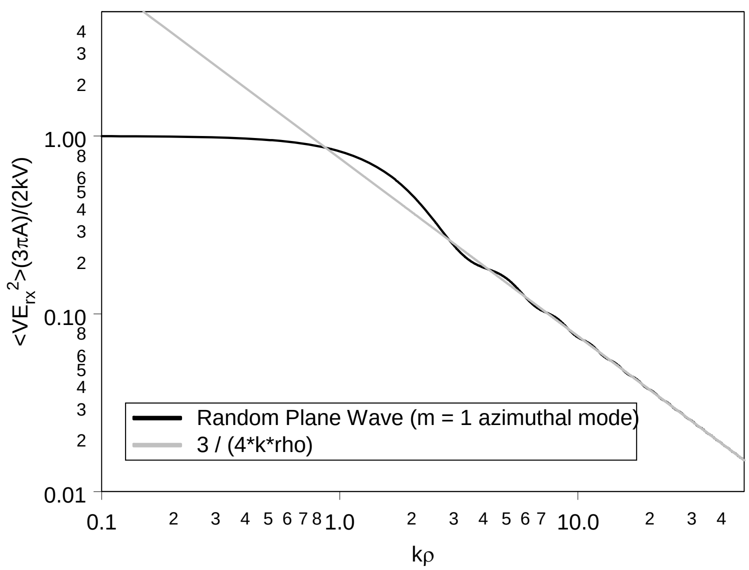

Figure 14. The normalized field squared behavior at at various distances from the symmetric axis of the axisymmetric cavity compared to the simple asymptotic result far from the axis.

4.7.1 Point Statistics

We now compare field statistics at points in the cavity from the numerical simulation and the random plane wave results (in the convex walled cavity there is not much difference between these as we observed in two-dimensions [2]). The point statistic on axis for the random plane wave (from the $\cos\varphi$ parity with $\varphi = 0$) is

$$\left\langle V E_{rx}^{(m=1)2} \left(\rho = 0\right) \right\rangle = \frac{2kV}{3\pi A} \tag{389}$$

and in general (for the $\cos\varphi$ parity at $\varphi = 0$)

$$\left\langle V E_{rx}^{(m=1)2} \right\rangle = \frac{2kV}{\pi A} \int_0^1 \left\{ \left(\frac{J_1\left(k\rho w_j\right)}{k\rho w_j} \right)^2 + \left(1 - w_j^2\right) \left(J_1'\left(k\rho w_j\right)\right)^2 \right\} \frac{w_j dw_j}{\sqrt{1 - w_j^2}} \tag{390}$$

$$\left\langle V E_{rx}^{(m=1)2} \right\rangle \sim \frac{2V}{\pi^2 A\rho} \int_0^1 \sqrt{1 - w_j^2} dw_j = \frac{V}{2\pi\rho A} = \frac{3}{4k\rho} \frac{2kV}{3\pi A} \ , \ k\rho >> 1 \tag{391}$$

with normalization

$$\left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)} \right\rangle = \frac{1}{V} \tag{392}$$

The random plane wave in full 3D gives

$$\left\langle VE_{rx}^2 \right\rangle = \frac{1}{3}$$

with normalization

$$\langle \underline{E}_r \cdot \underline{E}_r \rangle = \frac{1}{V}$$

Thus the enhancement factor in the point statistics due to the symmetry is again $2kV/(\pi A)$. For the range $k = 20.98 - 62.79$ m^{-1} we expect that $2kV/(\pi A) = 23.73 - 71.02$. Thus we expect that $\left\langle VE_{rx}^2 (\rho = 0) \right\rangle = 7.91 - 23.67$. To average over frequency we use the modal spacing associated with the eigenvalues along the orbit in the $m = 1$ vector case

$$\frac{dk^2}{dN} = 2k\frac{dk}{dN} = 4\pi/A$$

Thus

$$\left\langle VE_{rx}^{(m=1)2} (\rho = 0) \right\rangle_k = \frac{1}{N}\sum_n \frac{2kV}{3\pi A} = \frac{1}{N(k_2) - N(k_1)}\int_{N(k_1)}^{N(k_2)} \frac{2kV}{3\pi A} dN$$

$$= \int_{k_1}^{k_2} \frac{2kV}{3\pi A}\frac{dN}{dk}dk / \int_{k_1}^{k_2} \frac{dN}{dk}dk = \int_{k_1}^{k_2} \frac{2kV}{3\pi A}kdk / \int_{k_1}^{k_2} kdk$$

$$= \frac{2V}{3\pi A}\frac{2}{3}\frac{k_2^3 - k_1^3}{k_2^2 - k_1^2} = \frac{2V}{3\pi A}\langle k \rangle \approx 17.75 \approx 2(8.88)$$

where for the numerical simulation comparisons we used

$$\langle k \rangle = \frac{2}{3}\frac{k_2^2 + k_2 k_1 + k_1^2}{k_2 + k_1} \approx 45.36 \text{ m}^{-1}$$

The factor of two associated with the $\varphi = 0$ evaluation of the $\cos\varphi$ parity was separated out here. This normalized field square function is shown in Figure 14. Comparisons of the level

Far away from the axis the value is nearly independent of k

$$\left\langle VE_{rx}^{(m=1)2} \right\rangle_k \sim \frac{V}{2\pi\rho A}\ ,\ k\rho >> 1$$

Note that we can apply this asymptotic formula even to the $\rho = 0.1$ m case since $k\rho > 2.1$ for all these off-axis cases (see the preceding graph showing a comparison of the asymptotic formula to the numerical evaluation of the random plane wave point statistics for $k\rho$).

$$\left\langle VE_{rx}^{(m=1)2} (\rho = 0.1 \text{ m}) \right\rangle \sim 2(1.4676)$$

$$\left\langle VE_{rx}^{(m=1)2} (\rho = 0.2 \text{ m}) \right\rangle \sim 2(0.7338)$$

$$\left\langle VE_{rx}^{(m=1)2} (\rho = 0.4 \text{ m}) \right\rangle \sim 2(0.3669)$$

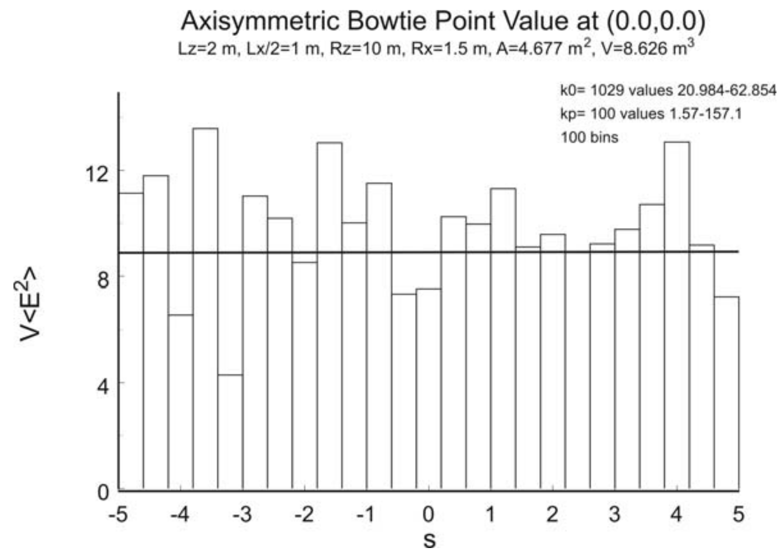

Figure 15. Field point statistic at the center of the orbit as a function of the scar frequency separation s. The horizontal line is the random plane wave level.

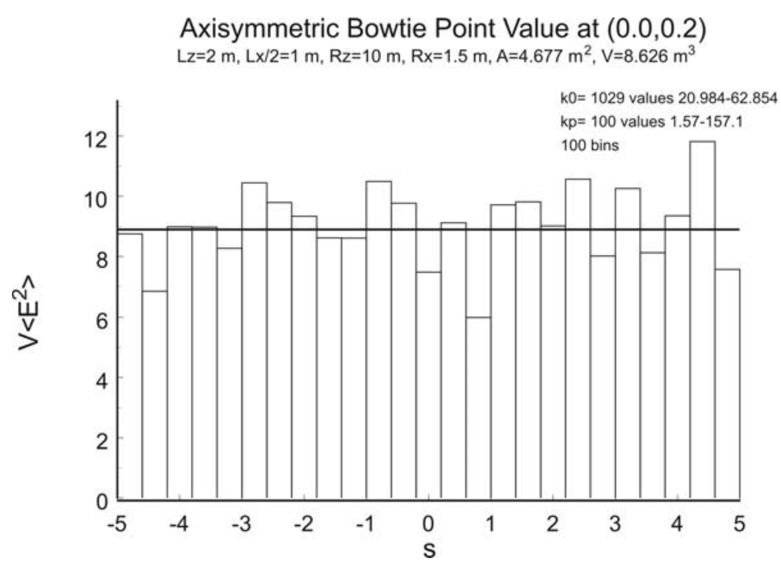

Figure 16. Field point statistic at 0.2 m from the center of, but along the orbit, as a function of the scar frequency separation s. The horizontal line is the random plane wave level.

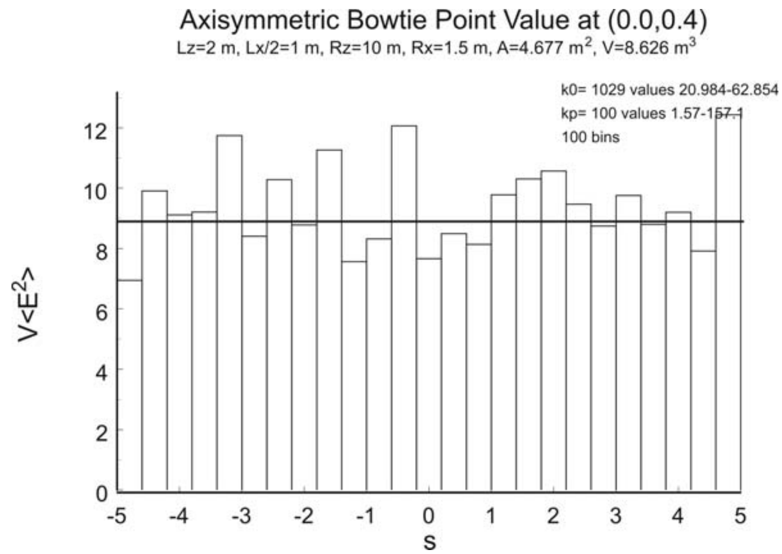

Figure 17. Field point statistic at 0.4 m from the center of, but along the orbit, as a function of the scar frequency separation s. The horizontal line is the random plane wave level.

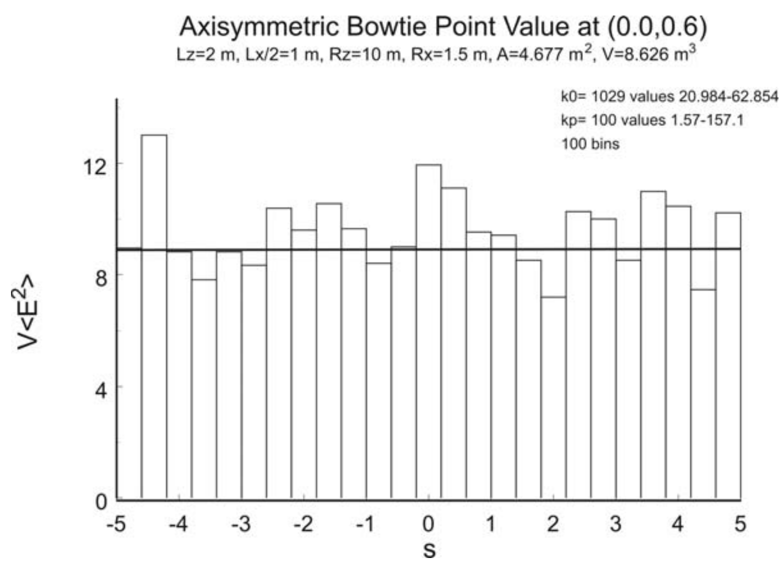

Figure 18. Field point statistic at 0.6 m from the center of, but along the orbit, as a function of the scar frequency separation s. The horizontal line is the random plane wave level.

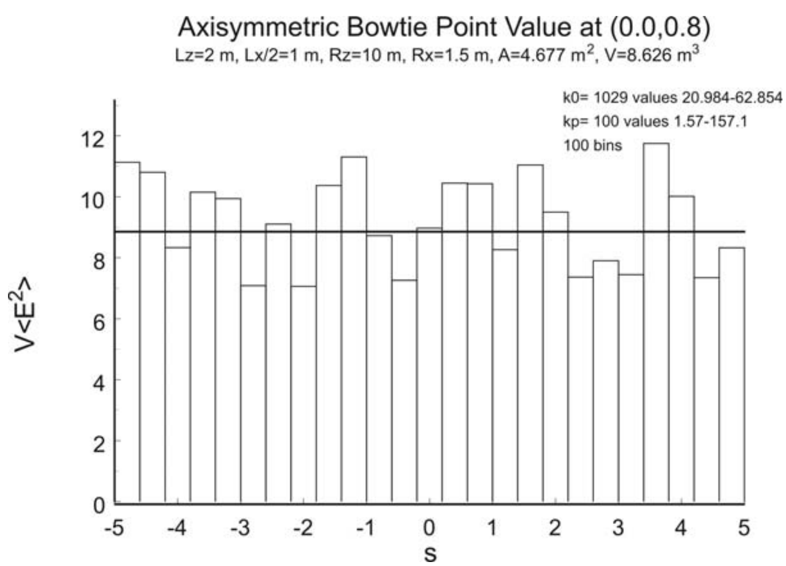

Figure 19. Field point statistic at 0.8 m from the center of, but along the orbit, as a function of the scar frequency separation s. The horizontal line is the random plane wave level.

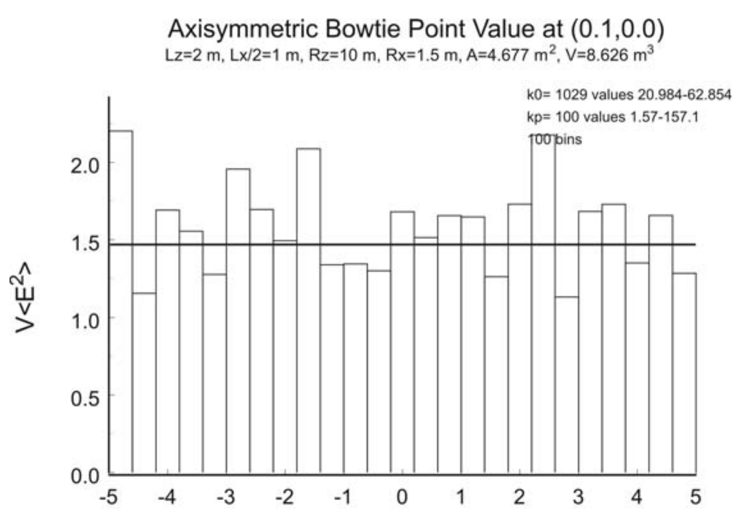

Figure 20. Field point statistic at a radial distance of 0.1 m from the orbit at its axial center, as a function of the scar frequency separation s. The horizontal line is the random plane wave level.

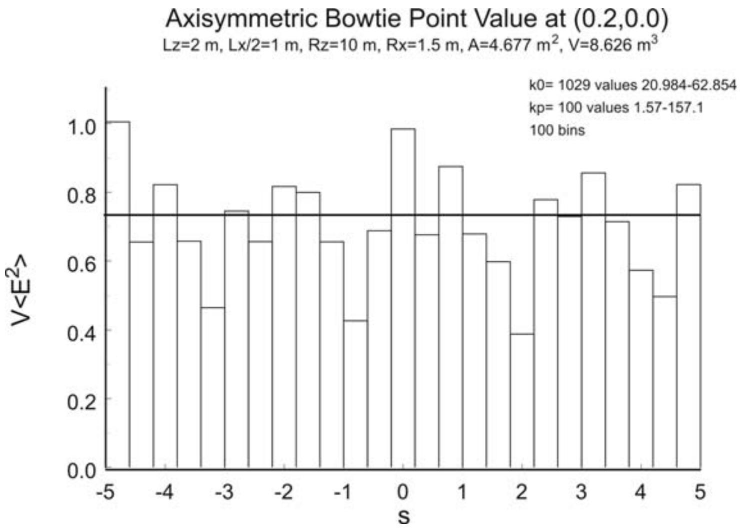

Figure 21. Field point statistic at a radial distance of 0.2 m from the orbit at its axial center, as a function of the scar frequency separation s. The horizontal line is the random plane wave level.

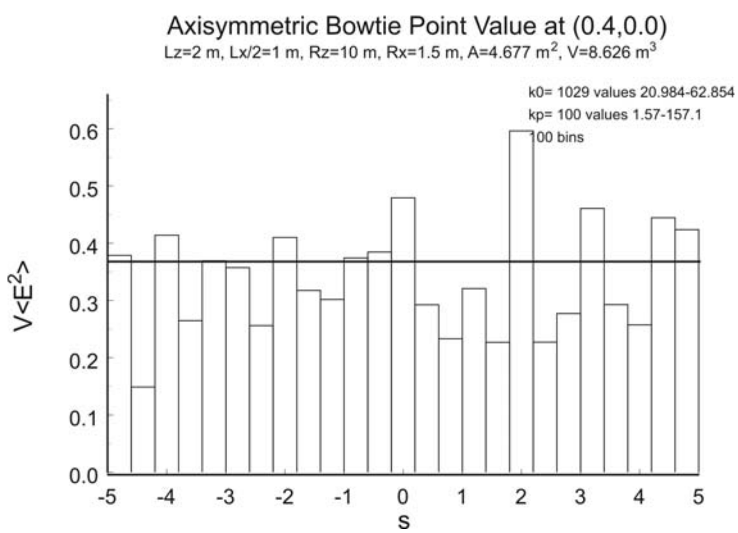

Figure 22. Field point statistic at a radial distance of 0.4 m from the orbit at its axial center, as a function of the scar frequency separation s. The horizontal line is the random plane wave level.

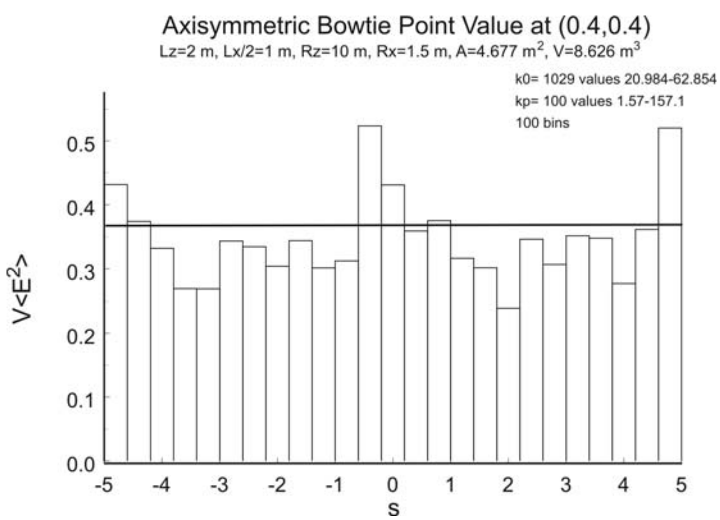

Figure 23. Field point statistic at a radial distance of 0.4 m from the orbit, and displaced by 0.4 m from its axial center, as a function of the scar frequency separation s. The horizontal line is the random plane wave level.

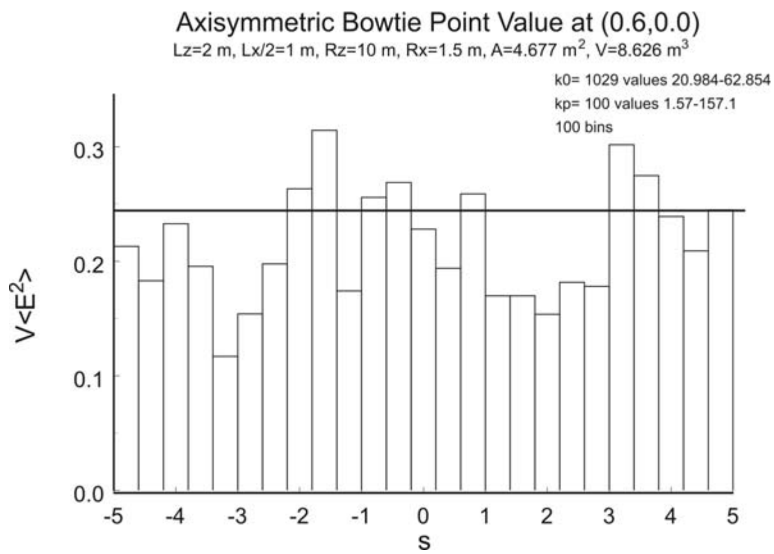

Figure 24. Field point statistic at a radial distance of 0.6 m from the orbit at its axial center, as a function of the scar frequency separation s. The horizontal line is the random plane wave level.

$$\left\langle VE_{rx}^{(m=1)2}\left(\rho=0.6\text{ m}\right)\right\rangle \sim 2\left(0.2446\right)$$

$$\left\langle VE_{rx}^{(m=1)2}\left(\rho=0.8\text{ m}\right)\right\rangle \sim 2\left(0.1835\right)$$

$$\left\langle VE_{rx}^{(m=1)2}\left(\rho=1\text{ m}\right)\right\rangle \sim 2\left(0.1468\right)$$

This last figure may not represent an accurate comparison since the cavity half dimensions are each 1 m. Thus we are approaching the outer boundary in the simulation for this case.

4.8 Magnetic Hertz Potential

We now repeat the set up of the vector problem using the magnetic Hertz potential to examine the differences in the application of the boundary conditions and in the field polarization at the scar orbit. The electromagnetic field in a source free region satisfies

$$\nabla\times\underline{H}=-i\omega\varepsilon\underline{E}$$

$$\nabla\times\underline{E}=i\omega\mu\underline{H}$$

$$\nabla\cdot\underline{H}=0$$

$$\nabla\cdot\underline{E}=0$$

and the boundary conditions on the walls (in our case these hold on the end mirrors $\xi=\pm\xi_0$)

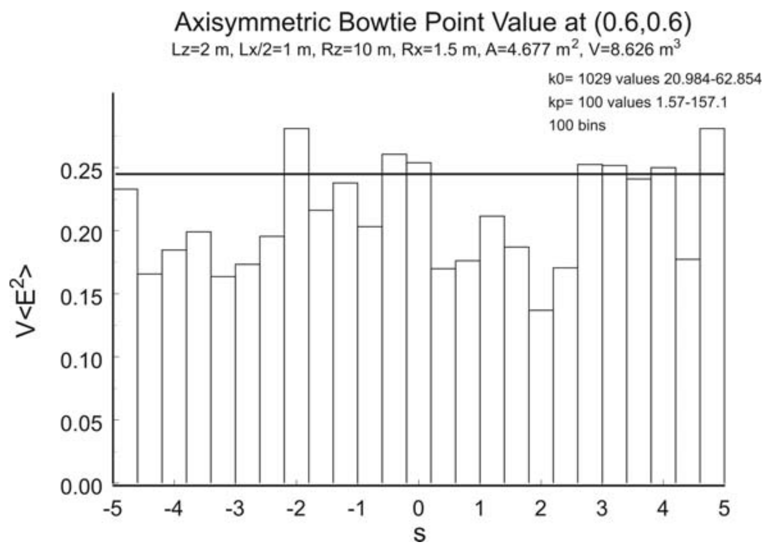

Figure 25. Field point statistic at a radial distance of 0.6 m from the orbit, and displaced by 0.6 m from its axial center, as a function of the scar frequency separation s. The horizontal line is the random plane wave level.

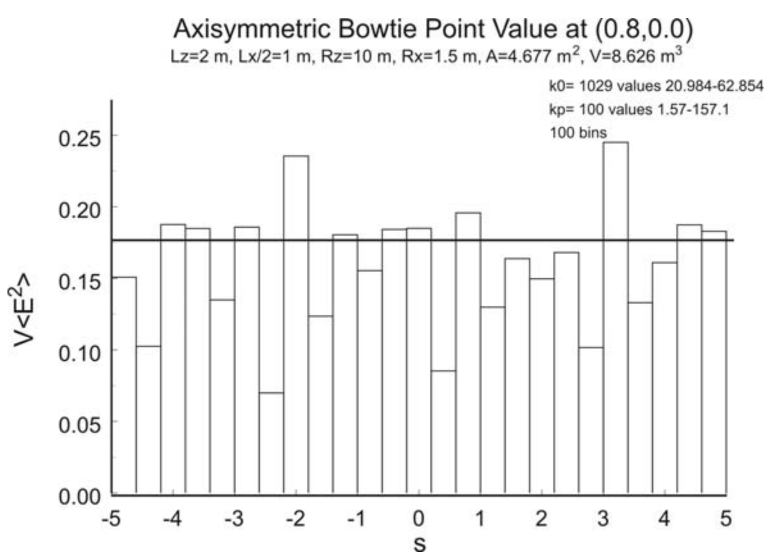

Figure 26. Field point statistic at a radial distance of 0.8 m from the orbit at its axial center, as a function of the scar frequency separation s. The horizontal line is the random plane wave level.

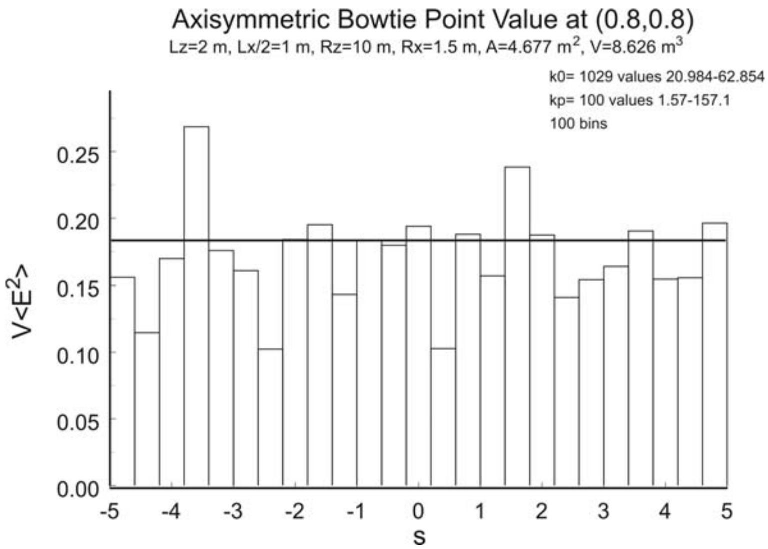

Figure 27. Field point statistic at a radial distance of 0.8 m from the orbit, and displaced by 0.8 m from its axial center, as a function of the scar frequency separation s. The horizontal line is the random plane wave level.

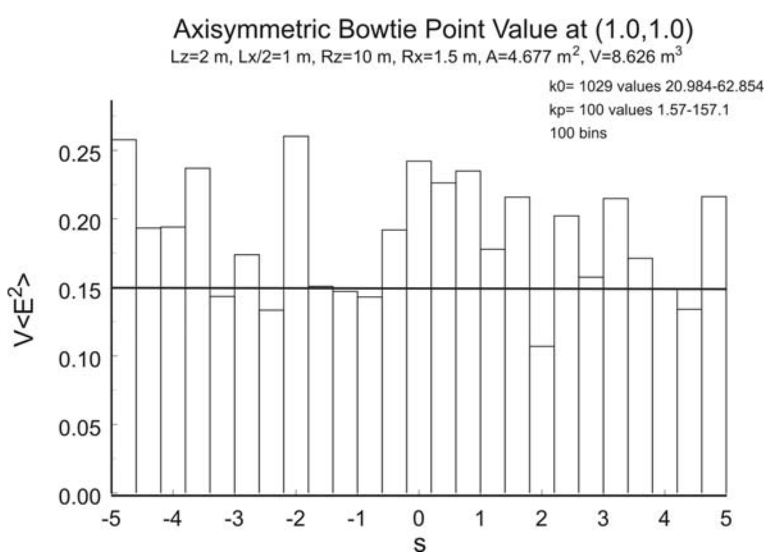

Figure 28. Field point statistic at a radial distance of 1 m from the orbit, and displaced by 1 m from its axial center, as a function of the scar frequency separation s. The horizontal line is the random plane wave level.

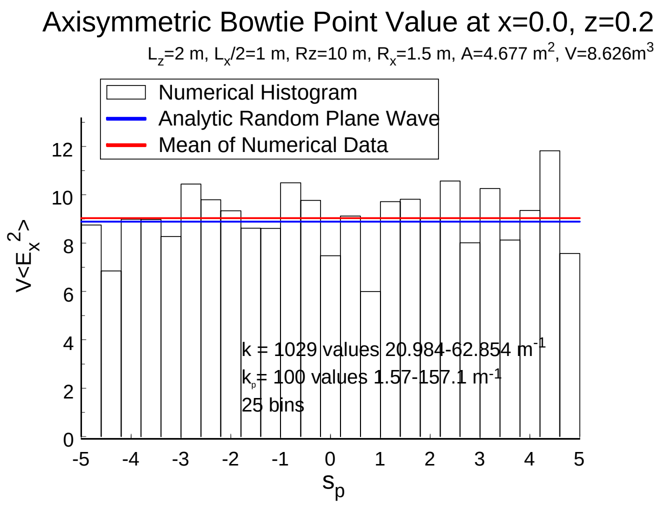

Figure 29. Field point statistic at 0.2 m from the center of, but along the orbit, as a function of the scar frequency separation s. The numerical mean value is also shown as the horizontal red line. The blue line is the random plane wave level.

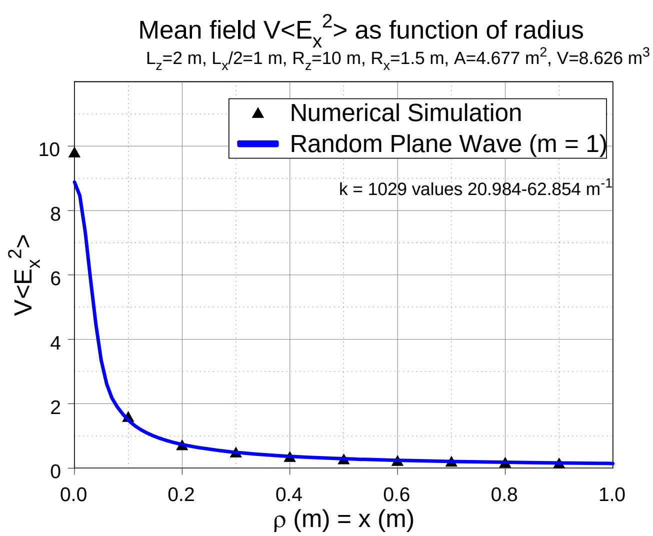

Figure 30. Variation of field point statistic as a function of radial distance from the orbit. The black diamonds are from the numerical simulation and the blue curve is from the random plane wave theory.

$$E_\zeta = E_\varphi = H_\xi = 0$$

or in the quasi-rectangular system

$$E_\upsilon = E_\eta = H_\xi = 0$$

We now use the magnetic Hertz potential $\underline{\Pi}_m$ with the fields given by

$$\underline{H} = \nabla \times \nabla \times \underline{\Pi}_m \tag{393}$$

$$\underline{E} = i\omega\mu\nabla \times \underline{\Pi}_m$$

where the Hertz vector satisfies the vector wave equation

$$-\nabla \times \nabla \times \underline{\Pi}_m + \nabla\left(\nabla \cdot \underline{\Pi}_m\right) + k^2\underline{\Pi}_m = \left(\nabla^2 + k^2\right)\underline{\Pi}_m = 0 \tag{394}$$

$$k^2 = \omega^2\mu\varepsilon$$

and the magnetic field is thus given by

$$\underline{H} = \nabla\left(\nabla \cdot \underline{\Pi}_m\right) + k^2\underline{\Pi}_m \tag{395}$$

At high frequencies we can make one of the following sets of approximations [5]

$$\Pi_{m\upsilon} = \Phi$$

$$\Pi_{m\eta} = 0 = \Pi_{m\xi}$$

or

$$\Pi_{m\eta} = \Phi$$

$$\Pi_{m\upsilon} = 0 = \Pi_{m\xi}$$

We cannot satisfy (394) exactly with this approximate choice, so instead we require that the potential $\Pi_{m\upsilon}$ (or in the second case $\Pi_{m\eta}$) satisfies the equation resulting from equating H_υ (or in the second case H_η) from (393) and (395). Thus the equation for the potential is [5]

$$\left[-\nabla \times \nabla \times \left(\underline{\Pi}_{m\upsilon}\underline{e}_\upsilon\right) + \nabla\left\{\nabla \cdot \left(\underline{\Pi}_{m\upsilon}\underline{e}_\upsilon\right)\right\} + k^2\left(\Pi_{m\upsilon}\underline{e}_\upsilon\right)\right] \cdot \underline{e}_\upsilon = 0$$

or

$$\left[-\nabla \times \nabla \times \left(\Pi_{m\eta}\underline{e}_\eta\right) + \nabla\left\{\nabla \cdot \left(\Pi_{m\eta}\underline{e}_\eta\right)\right\} + k^2\left(\Pi_{m\eta}\underline{e}_\eta\right)\right] \cdot \underline{e}_\eta = 0$$

4.8.1 Approximate Orthogonality And Fields

When we assume approximate orthogonality the equations simplify (as given in Vaynshteyn [5]). We can derive the following equations by using the orthogonal curvilinear coordinate results for gradient, divergence and curl [10], using the same metric coefficient h for all three coordinates

$$h = d\sqrt{\cos^2\xi + \zeta^2} = d\sqrt{\cos^2\xi + \upsilon^2 + \eta^2} \sim d\cos\xi$$

The potential $\Pi_{e\upsilon} = \Phi$ then satisfies [5]

$$\frac{\partial}{\partial\xi}\left\{\frac{1}{h}\frac{\partial}{\partial\xi}\left(h\Pi_{m\upsilon}\right)\right\} + \frac{\partial}{\partial\eta}\left\{\frac{1}{h}\frac{\partial}{\partial\eta}\left(h\Pi_{m\upsilon}\right)\right\}$$

$$+h\frac{\partial}{\partial\upsilon}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2\Pi_{m\upsilon}\right)\right] + k^2h^2\Pi_{m\upsilon} = 0$$

Note that the first and second terms are from $(-\nabla\times\nabla\times\underline{\Pi}_m)\cdot\underline{e}_\upsilon$ and the third term results from $(\nabla\nabla\cdot\underline{\Pi}_m)\cdot\underline{e}_\upsilon$. The fields are

$$h^2H_\upsilon = h\frac{\partial}{\partial\upsilon}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2\Pi_{m\upsilon}\right)\right] + k^2h^2\Pi_{m\upsilon}$$

$$h^2H_\eta = h\frac{\partial}{\partial\eta}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2\Pi_{m\upsilon}\right)\right]$$

$$h^2H_\xi = h\frac{\partial}{\partial\xi}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2\Pi_{m\upsilon}\right)\right]$$

and

$$h^2E_\xi = -i\omega\mu_0\frac{\partial}{\partial\eta}\left(h\Pi_{m\upsilon}\right)$$

$$h^2E_\eta = i\omega\mu_0\frac{\partial}{\partial\xi}\left(h\Pi_{m\upsilon}\right)$$

Alternatively the potential $\Pi_{m\eta} = \Phi$ satisfies

$$\frac{\partial}{\partial\xi}\left\{\frac{1}{h}\frac{\partial}{\partial\xi}\left(h\Pi_{m\eta}\right)\right\} + \frac{\partial}{\partial\upsilon}\left\{\frac{1}{h}\frac{\partial}{\partial\upsilon}\left(h\Pi_{m\eta}\right)\right\}$$

$$+h\frac{\partial}{\partial\eta}\left[\frac{1}{h^3}\frac{\partial}{\partial\eta}\left(h^2\Pi_{m\eta}\right)\right] + k^2h^2\Pi_{m\eta} = 0$$

The fields are

$$h^2H_\upsilon = h\frac{\partial}{\partial\upsilon}\left[\frac{1}{h^3}\frac{\partial}{\partial\eta}\left(h^2\Pi_{m\eta}\right)\right]$$

$$h^2 H_\eta = h\frac{\partial}{\partial\eta}\left[\frac{1}{h^3}\frac{\partial}{\partial\eta}\left(h^2\Pi_{m\eta}\right)\right] + k^2h^2\Pi_{m\eta}$$

$$h^2 H_\xi = h\frac{\partial}{\partial\xi}\left[\frac{1}{h^3}\frac{\partial}{\partial\eta}\left(h^2\Pi_{m\eta}\right)\right]$$

and

$$h^2 E_\xi = i\omega\mu_0\frac{\partial}{\partial\upsilon}\left(h\Pi_{m\eta}\right)$$

$$h^2 E_\upsilon = -i\omega\mu_0\frac{\partial}{\partial\xi}\left(h\Pi_{m\eta}\right)$$

4.8.2 Asymptotic Solution Of Quasirectangular Equations

Let us take

$$\Pi_{m\upsilon} = \Phi = W\left(\upsilon,\eta,\xi\right)e^{i\gamma\sin\xi} + W\left(\upsilon,\eta,-\xi\right)e^{-i\gamma\sin\xi}$$

and insert this into

$$\frac{\partial^2}{\partial\xi^2}\Pi_{m\upsilon} + \left(\frac{1}{h}\frac{\partial h}{\partial\xi}\right)\frac{\partial}{\partial\xi}\Pi_{m\upsilon} + \Pi_{m\upsilon}\frac{\partial}{\partial\xi}\left(\frac{1}{h}\frac{\partial h}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left\{\frac{1}{h}\frac{\partial}{\partial\eta}\left(h\Pi_{m\upsilon}\right)\right\}$$

$$+h\frac{\partial}{\partial\upsilon}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2\Pi_{m\upsilon}\right)\right] + k^2h^2\Pi_{m\upsilon} = 0$$

then W satisfies

$$\frac{\partial^2 W}{\partial\xi^2} + \left(i2\gamma\cos\xi + \frac{1}{h}\frac{\partial h}{\partial\xi}\right)\frac{\partial W}{\partial\xi}$$

$$+h\frac{\partial}{\partial\upsilon}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2W\right)\right] + \frac{\partial}{\partial\eta}\left\{\frac{1}{h}\frac{\partial}{\partial\eta}\left(hW\right)\right\}$$

$$+\left[k^2h^2 - \gamma^2\cos^2\xi - i\gamma\sin\xi + \left(i\gamma\cos\xi + \frac{\partial}{\partial\xi}\right)\left(\frac{1}{h}\frac{\partial h}{\partial\xi}\right)\right]W = 0$$

or

$$\frac{1}{h}\frac{\partial}{\partial\xi}\left(h\frac{\partial W}{\partial\xi}\right) + i2\gamma\cos\xi\frac{\partial W}{\partial\xi}$$

$$+\frac{1}{h}\frac{\partial}{\partial\upsilon}\left(h\frac{\partial}{\partial\upsilon}W\right) + \frac{1}{h}\frac{\partial}{\partial\eta}\left(h\frac{\partial}{\partial\eta}W\right)$$

$$+\left[\gamma^2\zeta^2 - i\gamma\sin\xi + \left(i\gamma\cos\xi + \frac{\partial}{\partial\xi}\right)\left(\frac{1}{h}\frac{\partial h}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{1}{h}\frac{\partial h}{\partial\eta}\right) + h\frac{\partial}{\partial\upsilon}\left(\frac{2}{h^2}\frac{\partial h}{\partial\upsilon}\right)\right]W = 0$$

We now neglect the first term $\partial^2 W/\partial\xi^2$ and let

$$h \sim d\cos\xi$$

giving

$$(i2\gamma\cos\xi - \tan\xi)\frac{\partial W}{\partial\xi} + \frac{\partial^2 W}{\partial\upsilon^2} + \frac{\partial^2 W}{\partial\eta^2}$$

$$+\left[\gamma^2\left(\upsilon^2+\eta^2\right) - i2\gamma\sin\xi - \sec^2\xi\right]W = 0$$

Now take

$$W = f(\xi)\,\Psi(\upsilon,\eta,\sigma)$$

where we want

$$\frac{f'}{f} = \frac{i2\gamma\sin\xi + \sec^2\xi}{i2\gamma\cos\xi - \tan\xi}$$

or

$$1/f = \cos\xi + \frac{i}{2\gamma}\tan\xi$$

Using this along with the transformation

$$\sigma = \int_0^\xi \frac{d\xi}{\cos\xi} = \operatorname{arcsinh}(\tan\xi) = \ln(\tan\xi + \sec\xi)$$

gives

$$i2\gamma\frac{\partial\Psi}{\partial\sigma} + \frac{\partial^2\Psi}{\partial\upsilon^2} + \frac{\partial^2\Psi}{\partial\eta^2}$$

$$+\gamma^2\left(\upsilon^2+\eta^2\right)\Psi = 0$$

We could also have dropped the terms $\tan\xi$ and $\sec^2\xi$ terms compared to the $\gamma\cos\xi$ and the $\gamma\sin\xi$ term (these terms could be accounted for by higher order terms in the asymptotic series just like the $\partial^2 W/\partial\xi^2$ term and the higher order terms $\partial h/\partial\upsilon$ and $\partial h/\partial\eta$) to obtain directly

$$i2\gamma\cos\xi\frac{\partial W}{\partial\xi} + \frac{\partial^2 W}{\partial\upsilon^2} + \frac{\partial^2 W}{\partial\eta^2}$$

$$+\left[\gamma^2\left(\upsilon^2+\eta^2\right) - i2\gamma\sin\xi\right]W = 0$$

which is the same as the preceding scalar case. For this leading term approach we would take

$$W = \frac{1}{\cos\xi}\Psi(\upsilon,\eta,\sigma)$$

and obtain the same equation for Ψ. Scaling the independent variables by letting

$$\tau_x = \sqrt{2\gamma}\upsilon$$

$$\tau_y = \sqrt{2\gamma}\eta$$

gives

$$i\frac{\partial \Psi}{\partial \sigma} + \frac{\partial^2 \Psi}{\partial \tau_x^2} + \frac{\partial^2 \Psi}{\partial \tau_y^2} + \frac{1}{4}\left(\tau_x^2 + \tau_y^2\right)\Psi = 0$$

Separating the variables

$$\Psi\left(\tau_x, \tau_y, \sigma\right) = \psi_a\left(\tau_x\right)\psi_b\left(\tau_y\right)\psi_c\left(\sigma\right)$$

gives

$$\left(\frac{i}{\psi_c}\frac{\partial \psi_c}{\partial \sigma}\right) + \left(\frac{1}{\psi_a}\frac{\partial^2 \psi_a}{\partial \tau_x^2} + \tau_x^2/4\right) + \left(\frac{1}{\psi_b}\frac{\partial^2 \psi_b}{\partial \tau_y^2} + \tau_y^2/4\right) = 0$$

Each parenthetical term is independent of the variables associated with the others and therefore each equals a constant. We take the first term as s

$$\frac{\partial \psi_c}{\partial \sigma} = -is\psi_c$$

or

$$\psi_c = e^{-is\sigma}$$

Then we can write

$$\frac{\partial^2 \psi_a}{\partial \tau_x^2} + \left(s_a + \tau_x^2/4\right)\psi_a = 0$$

$$\frac{\partial^2 \psi_b}{\partial \tau_y^2} + \left(s_b + \tau_y^2/4\right)\psi_b = 0$$

$$s_a + s_b = s$$

$$\Psi\left(\tau_x, \tau_y, \sigma\right) = e^{-is_a\sigma}\psi_a\left(s_a, \tau_x\right)e^{-is_b\sigma}\psi_b\left(s_b, \tau_y\right) = e^{-is\sigma}\psi_a\left(s_a, \tau_x\right)\psi_b\left(s_b, \tau_y\right)$$

This is a form of the equation of the parabolic cylinder functions

$$\frac{\partial^2 \psi}{\partial \tau^2} + \left(\frac{\tau^2}{4} + s\right)\psi = 0 \tag{396}$$

The solution that is outgoing in τ is [1]

$$U_+\left(s, \tau\right) = e^{-\pi(s+i/2)/4}U\left(-is, \tau e^{-i\pi/4}\right) \tag{397}$$

where $U\left(a, z\right)$ is the standard solution [11]. Following [1] the total transverse solution is taken as the incident plus reflected form

$$\psi(s,\tau) = c\,\mathrm{Re}\left[U_+(s,\tau) + e^{i\Phi_0} U_+^*(s,\tau)\right] \tag{398}$$

where the constant c is used for normalization. The transverse boundary condition in τ is a reflection with a random phase $\Phi_0\left(k^2\right)$ which was introduced by Antonsen to match to the chaotic region of the cavity; it describes the phase relation between a wave leaving the vicinity of the unstable periodic orbit and one returning [1] with the variation of the pth component along the orbit. Figure 9 schematically illustrates a wave bouncing back and forth between mirrors in the region of the scarred orbit; it leaves the vicinity of the orbit and eventually returns from the outer chaotic region with transverse reflection phase $\Phi_0\left(k^2\right)$. (For purposes of simplification this figure does not include the vertical evenness of the cavity, which confines the wave leaving and the wave reflected to either the upper half or lower half of the cavity.) Thus $\psi_a = \psi_a(s_a, \tau_x)$ and $\psi_b = \psi_b(s_b, \tau_y)$ are elliptic cylinder functions like in the two-dimensional case [6], [1].

The boundary conditions imply

$$E_\upsilon = E_\eta = H_\xi = 0\ ,\ \xi = \pm\xi_0$$

or using the preceding relations for the field in terms of the Hertz potential

$$h^2 H_\upsilon = h\frac{\partial}{\partial\upsilon}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2\Pi_{m\upsilon}\right)\right] + k^2h^2\Pi_{m\upsilon}$$

$$h^2 H_\eta = h\frac{\partial}{\partial\eta}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2\Pi_{m\upsilon}\right)\right]$$

$$h^2 H_\xi = h\frac{\partial}{\partial\xi}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2\Pi_{m\upsilon}\right)\right]$$

and

$$h^2 E_\xi = -i\omega\mu_0\frac{\partial}{\partial\eta}\left(h\Pi_{m\upsilon}\right)$$

$$h^2 E_\eta = i\omega\mu_0\frac{\partial}{\partial\xi}\left(h\Pi_{m\upsilon}\right)$$

$$h = d\sqrt{\cos^2\xi + \zeta^2} = d\sqrt{\cos^2\xi + \upsilon^2 + \eta^2} \sim d\cos\xi$$

or

$$h^2 H_\xi = h\frac{\partial}{\partial\xi}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2\Pi_{m\upsilon}\right)\right]$$

$$h^2 E_\eta = i\omega\mu_0\frac{\partial}{\partial\xi}\left(h\Pi_{m\upsilon}\right)$$

we see that we must take the potential to satisfy on the mirrors

$$h\frac{\partial\Pi_{m\upsilon}}{\partial\xi}+\Pi_{m\upsilon}\frac{\partial h}{\partial\xi}\sim\frac{\partial}{\partial\xi}\left(h\Pi_{m\upsilon}\right)=0\sim\frac{\partial}{\partial\xi}\left(\frac{1}{h}\Pi_{m\upsilon}\right)=\frac{1}{h^2}\left(h\frac{\partial\Pi_{m\upsilon}}{\partial\xi}-\Pi_{m\upsilon}\frac{\partial h}{\partial\xi}\right)\ ,\ \xi=\pm\xi_0$$

or

$$\cos\xi\frac{\partial\Pi_{m\upsilon}}{\partial\xi}-\Pi_{m\upsilon}\sin\xi\sim 0\sim\cos\xi\frac{\partial\Pi_{m\upsilon}}{\partial\xi}+\Pi_{m\upsilon}\sin\xi\ ,\ \xi=\pm\xi_0$$

Then in the present case

$$\Pi_{m\upsilon}=\Phi=W\left(\upsilon,\eta,\xi\right)e^{i\gamma\sin\xi}+W\left(\upsilon,\eta,-\xi\right)e^{-i\gamma\sin\xi}$$

$$W=\frac{1}{\cos\xi}\Psi\left(\upsilon,\eta,\sigma\right)$$

$$\tau_x=\sqrt{2\gamma}\upsilon$$

$$\tau_y=\sqrt{2\gamma}\eta$$

$$\Psi\left(\tau_x,\tau_y,\sigma\right)=\psi_a\left(\tau_x\right)\psi_b\left(\tau_y\right)\psi_c\left(\sigma\right)$$

$$\psi_c=e^{-is\sigma}$$

$$\psi\left(s,\tau\right)=c\,\mathrm{Re}\left[U_+\left(s,\tau\right)+e^{i\Phi_0}U_+^*\left(s,\tau\right)\right]$$

$$i\gamma\cos\xi_0\left[\Psi\left(\tau_x,\tau_y,\pm\sigma_0\right)e^{\pm i\gamma\sin\xi_0}-\Psi\left(\tau_x,\tau_y,\mp\sigma_0\right)e^{\mp i\gamma\sin\xi_0}\right]$$

$$\sim\pm\frac{\sin\xi_0}{\cos\xi_0}\left[\Psi\left(\tau_x,\tau_y,\pm\sigma_0\right)e^{\pm i\gamma\sin\xi_0}+\Psi\left(\tau_x,\tau_y,\mp\sigma_0\right)e^{\mp i\gamma\sin\xi_0}\right]$$

$$\Psi\left(\tau_x,\tau_y,\pm\sigma_0\right)e^{\pm i\gamma\sin\xi_0}\sim\Psi\left(\tau_x,\tau_y,\mp\sigma_0\right)e^{\mp i\gamma\sin\xi_0}$$

or

$$\Psi\left(\tau_x,\tau_y,\sigma_0\right)e^{i\gamma\sin\xi_0}\sim\Psi\left(\tau_x,\tau_y,-\sigma_0\right)e^{-i\gamma\sin\xi_0}$$

or

$$\psi_a\left(\tau_x\right)\psi_b\left(\tau_y\right)e^{i\gamma\sin\xi_0-is\sigma_0}\sim\psi_a\left(\tau_x\right)\psi_b\left(\tau_y\right)e^{-i\gamma\sin\xi_0+is\sigma_0}$$

or

$$e^{i2\gamma\sin\xi_0-i2s\sigma_0}=e^{i2\pi p}\ ,\ p=0,1,2,...$$

or

$$\gamma \sin \xi_0 - s\sigma_0 = \pi p$$

or

$$(k - k_p)\,\ell = s\sigma_0 = s \ln (\tan \xi_0 + \sec \xi_0) = s \ln \sqrt{\frac{d+\ell}{d-\ell}}$$

$$= s\frac{1}{4} \ln (\Lambda_+)$$

where

$$\Lambda_\pm = \left(\frac{d+\ell}{d-\ell}\right)^{\pm 2} = \left[1 + L/R \pm \sqrt{(1 + L/R)^2 - 1}\right]^2$$

are the stability exponents of the orbit and

$$k_p \ell = p\pi \ , \ p = 1, 2, ...$$

where p is a large integer representing the number of half wave variations along the z axis.

Alternatively in the case where the potential is odd along the orbit we take

$$\Pi_{mv} = \Phi = W (\upsilon, \eta, \xi)\, e^{i\gamma \sin \xi} - W (\upsilon, \eta, -\xi)\, e^{-i\gamma \sin \xi}$$

and boundary condition

$$\begin{aligned} &i\gamma \cos \xi_0 \left[\Psi (\tau_x, \tau_y, \pm\sigma_0)\, e^{\pm i\gamma \sin \xi_0} + \Psi (\tau_x, \tau_y, \mp\sigma_0)\, e^{\mp i\gamma \sin \xi_0}\right] \\ &\sim \pm\frac{\sin \xi_0}{\cos \xi_0} \left[\Psi (\tau_x, \tau_y, \pm\sigma_0)\, e^{\pm i\gamma \sin \xi_0} - \Psi (\tau_x, \tau_y, \mp\sigma_0)\, e^{\mp i\gamma \sin \xi_0}\right] \end{aligned}$$

$$\Psi (\tau_x, \tau_y, \pm\sigma_0)\, e^{\pm i\gamma \sin \xi_0} \sim -\Psi (\tau_x, \tau_y, \mp\sigma_0)\, e^{\mp i\gamma \sin \xi_0}$$

or

$$\Psi (\tau_x, \tau_y, \sigma_0)\, e^{i\gamma \sin \xi_0} \sim -\Psi (\tau_x, \tau_y, -\sigma_0)\, e^{-i\gamma \sin \xi_0}$$

or

$$\psi_a (\tau_x)\, \psi_b (\tau_y)\, e^{i\gamma \sin \xi_0 - is\sigma_0} \sim -\psi_a (\tau_x)\, \psi_b (\tau_y)\, e^{-i\gamma \sin \xi_0 + is\sigma_0}$$

or

$$e^{i2\gamma \sin \xi_0 - i2s\sigma_0} = e^{i2\pi(p-1/2)} \ , \ p = 1, 2, ...$$

or

$$\gamma \sin \xi_0 - s\sigma_0 = \pi (p - 1/2)$$

and

$$(k-k_p)\,\ell = s\sigma_0 = s\ln(\tan\xi_0+\sec\xi_0) = s\ln\sqrt{\frac{d+\ell}{d-\ell}}$$

$$= s\frac{1}{4}\ln(\Lambda_+)$$

$$k_p\ell = (p-1/2)\,\pi\ ,\ \ p=1,2,...$$

The quantities s_a and s_b are the transverse numbers of half wave variations (not necessarily integers in this unstable case, and usually random functions versus eigenvalue), where $s=s_a+s_b$. The preceding result thus connects the eigenvalue k minus the wavenumber in z, or k_p, to the sum of the transverse variations.

$$\Pi_{m\upsilon} = 2\psi_a(s_a,\tau_x)\,\psi_b(s_b,\tau_y)\cos(\gamma\sin\xi - s\sigma)\,/\cos\xi$$

$$\psi_a(s_a,\tau_x) = c_a\,\mathrm{Re}\left[U_+(s_a,\tau_x)+e^{i\Phi_{0a}}U_+^*(s_a,\tau_x)\right]$$

$$\psi_b(s_b,\tau_y) = c_b\,\mathrm{Re}\left[U_+(s_b,\tau_y)+e^{i\Phi_{0b}}U_+^*(s_b,\tau_y)\right]$$

Note that if the eigenfunction $\Pi_{m\upsilon}$ is chosen to be even with respect to the normals of the orbit η and υ (say, PEC at $\eta=0$ and PMC at $\upsilon=0$, resulting in $H_\upsilon\neq 0$, $H_\eta=0$, $H_\xi=0$, $E_\xi=0$, $E_\eta\neq 0$ at $\upsilon=\eta=0$) where

$$h^2H_\upsilon = h\frac{\partial}{\partial\upsilon}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2\Pi_{m\upsilon}\right)\right]+k^2h^2\Pi_{m\upsilon}\sim\left(\frac{\partial^2}{\partial\upsilon^2}+k^2h^2\right)\Pi_{m\upsilon} = \left(2\gamma\frac{\partial^2}{\partial\tau_x^2}+k^2h^2\right)\Pi_{m\upsilon}$$

$$h^2H_\eta = h\frac{\partial}{\partial\eta}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2\Pi_{m\upsilon}\right)\right]\sim\frac{\partial^2}{\partial\eta\partial\upsilon}\Pi_{m\upsilon} = 2\gamma\frac{\partial^2}{\partial\tau_y\partial\tau_x}\Pi_{m\upsilon}$$

$$h^2H_\xi = h\frac{\partial}{\partial\xi}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2\Pi_{m\upsilon}\right)\right]\sim\frac{\partial^2}{\partial\xi\partial\upsilon}\Pi_{m\upsilon} = \sqrt{2\gamma}\frac{\partial^2}{\partial\xi\partial\tau_x}\Pi_{m\upsilon}$$

and

$$h^2E_\xi = -i\omega\mu_0\frac{\partial}{\partial\eta}\left(h\Pi_{m\upsilon}\right)\sim -i\omega\mu_0 h\frac{\partial}{\partial\eta}\Pi_{m\upsilon} = -i\omega\mu_0\sqrt{2\gamma}h\frac{\partial}{\partial\tau_y}\Pi_{m\upsilon}$$

$$h^2E_\eta = i\omega\mu_0\frac{\partial}{\partial\xi}\left(h\Pi_{m\upsilon}\right)\sim i\omega\mu_0 h\frac{\partial}{\partial\xi}\Pi_{m\upsilon} = i\omega\mu_0 h\frac{\partial}{\partial\xi}\Pi_{m\upsilon}$$

then we have the resonance conditions

$$\mathrm{Re}\left[U_+'(s_a,0)+e^{i\Phi_{0a}}U_+^{*\prime}(s_a,0)\right]=0$$

$$\mathrm{Re}\left[U_+'(s_b,0)+e^{i\Phi_{0b}}U_+^{*\prime}(s_b,0)\right]=0$$

or

$$-\frac{U_+'(s_a,0)}{U_+^{*\prime}(s_a,0)} = e^{i\Phi_{0a}}$$

$$-\frac{U_+'(s_b,0)}{U_+^{*\prime}(s_b,0)} = e^{i\Phi_{0b}}$$

Using this with the Wronskian

$$U_+' U_+^* - U_+^{*\prime} U_+ = i$$

gives

$$\mathrm{Re}\left[U_+(s_a,0) + e^{i\Phi_{0a}} U_+^*(s_a,0)\right] = \frac{\mathrm{Im}\left[U_+'(s_a,0)\right]}{\left|U_+'(s_a,0)\right|^2}$$

$$\mathrm{Re}\left[U_+(s_b,0) + e^{i\Phi_{0b}} U_+^*(s_b,0)\right] = \frac{\mathrm{Im}\left[U_+'(s_b,0)\right]}{\left|U_+'(s_b,0)\right|^2}$$

On axis $\upsilon = \eta = 0$ or $\tau_x = \tau_y = 0$ and

$$\sigma = \mathrm{arcsinh}\,(\tan\xi) = \frac{1}{2}\ln\left(\frac{d+z}{d-z}\right)$$

$$\sin\xi = z/d$$

$$\cos\xi = \sqrt{1 - z^2/d^2}$$

$$\Pi_{m\upsilon} = 2\psi_a(s_a,0)\,\psi_b(s_b,0)\cos\left[kz - s\frac{1}{2}\ln\left(\frac{d+z}{d-z}\right)\right]/\sqrt{1-z^2/d^2}$$

$$= 2c^2\frac{\mathrm{Im}\left[U_+'(s_a,0)\right]}{\left|U_+'(s_a,0)\right|^2}\frac{\mathrm{Im}\left[U_+'(s_b,0)\right]}{\left|U_+'(s_b,0)\right|^2}\cos\left[k_p z + p_0(z)\right]/\sqrt{1-z^2/d^2}$$

where

$$p_0(z) = \frac{1}{2}s\left\{\ln\left(\frac{d+\ell}{d-\ell}\right)(z/\ell) - \ln\left(\frac{d+z}{d-z}\right)\right\}$$

Our main interest is again the lowest transverse mode ($m = 0$) with the vector nature put in through the selected component of the Hertz potential.

In the odd case we have

$$\Pi_{m\upsilon} = \Phi = 2i\psi_a(\tau_x)\,\psi_b(\tau_y)\sin(\gamma\sin\xi - s\sigma)/\cos\xi$$

or

$$\Pi_{m\upsilon} = 2i\psi_a\left(s_a,0\right)\psi_b\left(s_b,0\right)\sin\left[kz - s\frac{1}{2}\ln\left(\frac{d+z}{d-z}\right)\right]/\sqrt{1-z^2/d^2}$$

$$= 2ic^2\frac{\operatorname{Im}\left[U_+'\left(s_a,0\right)\right]}{\left|U_+'\left(s_a,0\right)\right|^2}\frac{\operatorname{Im}\left[U_+'\left(s_b,0\right)\right]}{\left|U_+'\left(s_b,0\right)\right|^2}\sin\left[k_p z + p_0\left(z\right)\right]/\sqrt{1-z^2/d^2}$$

Alternatively for the other polarization

$$\Pi_{m\eta} = \Phi = W\left(\upsilon,\eta,\xi\right)e^{i\gamma\sin\xi} + W\left(\upsilon,\eta,-\xi\right)e^{-i\gamma\sin\xi}$$

$$\frac{\partial^2}{\partial\xi^2}\Pi_{m\eta} + \frac{1}{h}\frac{\partial h}{\partial\xi}\frac{\partial}{\partial\xi}\Pi_{m\eta} + \Pi_{m\eta}\frac{\partial}{\partial\xi}\left\{\frac{1}{h}\frac{\partial h}{\partial\xi}\right\} + \frac{\partial}{\partial\upsilon}\left\{\frac{1}{h}\frac{\partial}{\partial\upsilon}\left(h\Pi_{m\eta}\right)\right\}$$

$$+h\frac{\partial}{\partial\eta}\left[\frac{1}{h^3}\frac{\partial}{\partial\eta}\left(h^2\Pi_{m\eta}\right)\right] + k^2h^2\Pi_{m\eta} = 0$$

or

$$\frac{1}{h}\frac{\partial}{\partial\xi}\left(h\frac{\partial W}{\partial\xi}\right) + i2\gamma\cos\xi\frac{\partial W}{\partial\xi}$$

$$+\frac{1}{h}\frac{\partial}{\partial\upsilon}\left(h\frac{\partial}{\partial\upsilon}W\right) + \frac{1}{h}\frac{\partial}{\partial\eta}\left(h\frac{\partial}{\partial\eta}W\right)$$

$$+\left[\gamma^2\zeta^2 + \left(i\gamma\cos\xi + \frac{\partial}{\partial\xi}\right)\left(\frac{1}{h}\frac{\partial h}{\partial\xi}\right) + \frac{\partial}{\partial\upsilon}\left(\frac{1}{h}\frac{\partial h}{\partial\upsilon}\right) + h\frac{\partial}{\partial\eta}\left(\frac{2}{h^2}\frac{\partial h}{\partial\eta}\right)\right]W = 0$$

and the procedure follows the same lines as in the preceding polarization state.

We would expect that for arbitrary outer chaotic regions both polarization states would be generated and coexist.

4.8.3 Asymptotic Solution Of Parabolic Equation In Prolate Spheroidal System

The parabolic equation in quasirectangular coordinates for the Hertz potential component $\Pi_{m\upsilon}$ or $\Pi_{m\eta}$ (or W) resulting from the vector wave equation is the same as that arising from the three dimensional scalar Helmholtz equation. Hence we can transform the parabolic equation for the Hertz potential from the quasirectangular system back to prolate spheroidal coordinates. The potential is then

$$\Pi_{m\upsilon} = \frac{2}{\cos\xi}\psi_m\left(\tau,s\right)\cos\left(\gamma\sin\xi - s\sigma\right)\left\{\begin{array}{c}\cos\left(m\varphi\right)\\ \sin\left(m\varphi\right)\end{array}\right\}$$

$$\psi_m = c_m\frac{\sqrt{2}}{\tau}\operatorname{Re}\left[W_+ + e^{i\Phi_0}W_+^*\right]$$

$$W_+\left(\tau,m,s\right) = W_{is/2,m/2}\left(-i\tau^2/2\right)$$

$$\tau = \sqrt{2\gamma}\zeta$$

$$k\ell - s_p\sigma_0 = (p - 1/2)\,\pi = k_p\ell\,,\; p = 1, 2, 3, ...$$

In the odd case

$$\Pi_{mv} = \frac{2i}{\cos\xi}\psi_m\,(\tau, s)\sin\left(\gamma\sin\xi - s\sigma\right)\left\{\begin{array}{c}\cos\left(m\varphi\right)\\ \sin\left(m\varphi\right)\end{array}\right\}$$

The same holds for the $\Pi_{m\eta}$ polarization state.

Behavior Of Zero Mode Near Orbit The behavior of the zero mode $m = 0$

$$\psi_0 = c_0\frac{\sqrt{2}}{\tau}\,\mathrm{Re}\left[W_+ + e^{i\Phi_0}W_+^*\right]$$

is now discussed near $\tau \to 0$. Now taking

$$\tau\frac{\partial\psi_0}{\partial\tau} = c_0\sqrt{2}\,\mathrm{Re}\left[W_+' + e^{i\Phi_0}W_+^{*\prime}\right] - \psi_0 \to 0$$

gives

$$e^{i\Phi_0} = i\frac{\Gamma\left(1/2 + is/2\right)}{\Gamma\left(1/2 - is/2\right)} + O\left(\tau^2\right)$$

and

$$\psi_0\,(0, s_p) = c_0\,\mathrm{Re}\left[e^{\pi s_p/2}e^{i\pi/4}\Gamma\left(1/2 + is_p/2\right)\right]$$

$$\psi_0\,(\tau, s) \sim c_0\,\mathrm{Re}\left[e^{i\pi/4}\Gamma\left(1/2 + is/2\right)\right]e^{\pi s/2}\left(1 - s\tau^2/4\right) + O\left(\tau^4\ln\tau\right)$$

$$\tau\frac{\partial}{\partial\tau}\psi_0\,(\tau) \sim -c_0\,\mathrm{Re}\left[e^{i\pi/4}\Gamma\left(1/2 + is/2\right)\right]e^{\pi s/2}\left(s\tau^2/2\right) + O\left(\tau^4\ln\tau\right) \sim -\left(s\tau^2/2\right)\psi_0$$

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\psi_0\,(\tau) \sim -c_0\,\mathrm{Re}\left[e^{i\pi/4}\Gamma\left(1/2 + is/2\right)\right]e^{\pi s/2}\left(s/2\right) + O\left(\tau^4\ln\tau\right) \sim -\frac{s}{2}\psi_0$$

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial}{\partial\tau}\psi_0\right) \sim -c_0\,\mathrm{Re}\left[e^{i\pi/4}\Gamma\left(1/2 + is/2\right)\right]e^{\pi s/2}s + O\left(\tau^2\ln\tau\right) \sim -s\psi_0 + O\left(\tau^2\ln\tau\right)$$

$$\tau\frac{\partial^2\psi_0}{\partial\omega\partial\tau}\,(0, s) \approx 2c_0\,\mathrm{Re}\left[\frac{e^{i\pi/4}}{\Gamma\left(1/2 - is/2\right)}\right]\frac{\partial\Phi_0}{\partial\omega}$$

4.8.4 Vector Normalization Condition

The method used for normalization of the eigenfunction components by Antonsen [1] is now put into the framework of the electromagnetic energy theorem [13]

$$\nabla \cdot \left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega} \right) = i \left[\frac{\partial (\omega \mu)}{\partial \omega} \underline{H} \cdot \underline{H}^* + \frac{\partial (\omega \varepsilon)}{\partial \omega} \underline{E} \cdot \underline{E}^* \right] - \frac{\partial \underline{E}}{\partial \omega} \cdot \underline{J}^* - \underline{E}^* \cdot \frac{\partial \underline{J}}{\partial \omega}$$

Integrating over the cavity volume and using the divergence theorem (and inserting the electrical properties of free space)

$$\oint_S \left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega} \right) \cdot \underline{n} dS = i \int_V (\mu_0 \underline{H} \cdot \underline{H}^* + \varepsilon_0 \underline{E} \cdot \underline{E}^*) \, dV - \int_V \left(\frac{\partial \underline{E}}{\partial \omega} \cdot \underline{J}^* + \underline{E}^* \cdot \frac{\partial \underline{J}}{\partial \omega} \right) dV$$

where the unit vector $\underline{n}$ in the divergence theorem points out of the cavity region.

4.8.5 Source Free Form Of Theorem

The source free form is thus

$$\oint_S \left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega} \right) \cdot \underline{n} dS = i \int_V (\mu_0 \underline{H} \cdot \underline{H}^* + \varepsilon_0 \underline{E} \cdot \underline{E}^*) \, dV$$

Using $\underline{n} \times \underline{E} = 0$ on the cavity walls, the surface integral on the cavity boundary vanishes

$$\int_{S_{cavity}} \left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega} \right) \cdot \underline{n} dS = \int_{S_{cavity}} \left[\left(\underline{n} \times \frac{\partial \underline{E}}{\partial \omega} \right) \cdot \underline{H}^* + (\underline{n} \times \underline{E}^*) \cdot \frac{\partial \underline{H}}{\partial \omega} \right] dS = 0$$

However a part of the closed surface S_{scar} is taken to surround the scarred orbit $\zeta = \sqrt{\upsilon^2 + \eta^2} \to 0$

$$\int_{S_{scar}} \left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega} \right) \cdot \underline{n} dS = i \int_V (\mu_0 \underline{H} \cdot \underline{H}^* + \varepsilon_0 \underline{E} \cdot \underline{E}^*) \, dV$$

where the unit normal here $\underline{n}$ points into the scarred region. We take the fields to be [17], [18]

$$\underline{H} = \nabla \times \nabla \times \underline{\Pi}_m = \nabla (\nabla \cdot \underline{\Pi}_m) + k^2 \underline{\Pi}_m$$

$$\underline{E} = i\omega\mu \nabla \times \underline{\Pi}_m$$

where for the first polarization state [5]

$$\Pi_{m\upsilon} = \Phi$$

$$\Pi_{m\eta} = \Pi_{m\xi} = 0$$

Note that as $\zeta = \sqrt{\upsilon^2 + \eta^2} \to 0$ the fields become $H_\upsilon \neq 0$, $H_\eta = 0$, $H_\xi = 0$, $E_\xi = 0$, $E_\eta \neq 0$ at $\upsilon = \eta = 0$, where

$$h^2 H_\upsilon = h \frac{\partial}{\partial \upsilon} \left[\frac{1}{h^3} \frac{\partial}{\partial \upsilon} \left(h^2 \Pi_{m\upsilon} \right) \right] + k^2 h^2 \Pi_{m\upsilon} \sim \left(\frac{\partial^2}{\partial \upsilon^2} + k^2 h^2 \right) \Pi_{m\upsilon} = \left(2\gamma \frac{\partial^2}{\partial \tau_x^2} + k^2 h^2 \right) \Pi_{m\upsilon}$$

$$h^2 H_\eta = h\frac{\partial}{\partial\eta}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2\Pi_{m\upsilon}\right)\right] \sim \frac{\partial^2}{\partial\eta\partial\upsilon}\Pi_{m\upsilon} = 2\gamma\frac{\partial^2}{\partial\tau_y\partial\tau_x}\Pi_{m\upsilon}$$

$$h^2 H_\xi = h\frac{\partial}{\partial\xi}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2\Pi_{m\upsilon}\right)\right] \sim \frac{\partial^2}{\partial\xi\partial\upsilon}\Pi_{m\upsilon} = \sqrt{2\gamma}\frac{\partial^2}{\partial\xi\partial\tau_x}\Pi_{m\upsilon}$$

$$h^2 E_\xi = -i\omega\mu_0\frac{\partial}{\partial\eta}\left(h\Pi_{m\upsilon}\right) \sim -i\omega\mu_0 h\frac{\partial}{\partial\eta}\Pi_{m\upsilon} = -i\omega\mu_0\sqrt{2\gamma}h\frac{\partial}{\partial\tau_y}\Pi_{m\upsilon}$$

$$h^2 E_\eta = i\omega\mu_0\frac{\partial}{\partial\xi}\left(h\Pi_{m\upsilon}\right) \sim i\omega\mu_0 h\frac{\partial}{\partial\xi}\Pi_{m\upsilon} = i\omega\mu_0 h\frac{\partial}{\partial\xi}\Pi_{m\upsilon}$$

Thus in the boundary integral with (note that the inward unit normal is oppositely directed from the prolate spheroidal unit vector) $\underline{n} = -\underline{e}_\zeta$ and

$$\underline{e}_\upsilon \cdot \underline{e}_\zeta = \frac{h_\zeta\zeta\upsilon}{\sqrt{h_\varphi^2\eta^2 + h_\zeta^2\upsilon^2\zeta^2}} \sim \cos\varphi$$

$$\underline{e}_\eta \cdot \underline{e}_\zeta = \frac{h_\zeta\zeta\eta}{\sqrt{h_\varphi^2\upsilon^2 + h_\zeta^2\eta^2\zeta^2}} \sim \sin\varphi$$

$$\underline{e}_\upsilon \times \underline{e}_\eta \sim \underline{e}_\xi$$

$$\underline{e}_\xi \times \underline{e}_\upsilon \sim \underline{e}_\eta$$

$$\underline{e}_\eta \times \underline{e}_\xi \sim \underline{e}_\upsilon$$

we note that $\underline{E} \times \underline{H}^*$ captures $E_\eta\underline{e}_\eta \times H_\upsilon^*\underline{e}_\upsilon = -E_\eta H_\upsilon^*\underline{e}_\xi$, $E_\eta\underline{e}_\eta \times H_\xi^*\underline{e}_\xi = E_\eta H_\xi^*\underline{e}_\upsilon$, $E_\xi\underline{e}_\xi \times H_\upsilon^*\underline{e}_\upsilon = E_\xi H_\upsilon^*\underline{e}_\eta$, and $E_\xi\underline{e}_\xi \times H_\eta^*\underline{e}_\eta = -E_\xi H_\eta^*\underline{e}_\upsilon$. But at the center $H_\eta = 0$, $H_\xi = 0$, $E_\xi = 0$ and only $-E_\eta H_\upsilon^*\underline{e}_\xi$ survives, but is orthogonal to $\underline{n} = \underline{e}_\zeta$. Thus the ω derivative is responsible for a contribution

$$\left(\frac{\partial\underline{E}}{\partial\omega} \times \underline{H}^*\right)\cdot\underline{e}_\zeta = \frac{\partial E_\xi}{\partial\omega}H_\upsilon^*\underline{e}_\eta\cdot\underline{e}_\zeta \sim \frac{\partial E_\xi}{\partial\omega}H_\upsilon^*\sin\varphi = \frac{\partial E_\xi}{\partial\omega}H_\upsilon^*\frac{\eta}{\sqrt{\upsilon^2+\eta^2}}$$

$$\left(\underline{E}^* \times \frac{\partial\underline{H}}{\partial\omega}\right)\cdot\underline{e}_\zeta = E_\eta^*\frac{\partial H_\xi}{\partial\omega}\underline{e}_\upsilon\cdot\underline{e}_\zeta \sim E_\eta^*\frac{\partial H_\xi}{\partial\omega}\cos\varphi = E_\eta^*\frac{\partial H_\xi}{\partial\omega}\frac{\upsilon}{\sqrt{\upsilon^2+\eta^2}}$$

Noting that H_ξ is odd in $\tau_x = \sqrt{2\gamma}\upsilon = \sqrt{2\gamma}\zeta\cos\varphi$ and that E_ξ is odd in $\tau_y = \sqrt{2\gamma}\eta = \sqrt{2\gamma}\zeta\sin\varphi$ it appears that these will contribute. The ω derivatives may not have even or odd behavior anyway. Thus

$$\left(\frac{\partial\underline{E}}{\partial\omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial\underline{H}}{\partial\omega}\right)\cdot\underline{n} \sim \left(\frac{\partial E_\xi}{\partial\omega}H_\upsilon^*\frac{\eta}{\sqrt{\upsilon^2+\eta^2}} + E_\eta^*\frac{\partial H_\xi}{\partial\omega}\frac{\upsilon}{\sqrt{\upsilon^2+\eta^2}}\right)$$

$$\sim -i\omega\mu_0\frac{1}{h}\frac{\sqrt{2\gamma}}{\sqrt{\tau_x^2+\tau_y^2}}\left[\frac{\partial^2\Pi_{m\upsilon}}{\partial\omega\partial\tau_y}\tau_y\left(\frac{2\gamma}{h^2}\frac{\partial^2}{\partial\tau_x^2}+k^2\right)\Pi_{m\upsilon}^* + \frac{1}{h^2}\frac{\partial\Pi_{m\upsilon}^*}{\partial\xi}\frac{\partial^3\Pi_{m\upsilon}}{\partial\omega\partial\xi\partial\tau_x}\tau_x\right]$$

Transforming back to prolate spheroidal coordinates

$$\left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega}\right) \cdot \underline{n}$$

$$\sim -i\omega\mu_0 \frac{1}{h}\left[\frac{\partial^2 \Pi_{mv}}{\partial \omega \partial \eta} \sin\varphi \left(\frac{1}{h^2}\frac{\partial^2}{\partial v^2} + k^2\right)\Pi^*_{mv} + \frac{1}{h^2}\frac{\partial^3 \Pi_{mv}}{\partial \omega \partial \xi \partial v}\frac{\partial \Pi^*_{mv}}{\partial \xi}\cos\varphi\right]$$

Noting the identities

$$\frac{\partial W}{\partial v} \sim \frac{\partial W}{\partial \zeta}\cos\varphi - \frac{1}{\zeta}\frac{\partial W}{\partial \varphi}\sin\varphi$$

$$\frac{\partial^2 W}{\partial v^2} \sim \frac{\partial^2 W}{\partial \zeta^2}\cos^2\varphi + \frac{1}{\zeta}\frac{\partial W}{\partial \zeta}\sin^2\varphi - \frac{1}{\zeta}\frac{\partial^2 W}{\partial \zeta \partial \varphi} 2\sin\varphi\cos\varphi + \frac{1}{\zeta^2}\frac{\partial W}{\partial \varphi} 2\sin\varphi\cos\varphi + \frac{1}{\zeta^2}\frac{\partial^2 W}{\partial \varphi^2}\sin^2\varphi$$

$$\frac{\partial W}{\partial \eta} \sim \frac{\partial W}{\partial \zeta}\sin\varphi + \frac{1}{\zeta}\frac{\partial W}{\partial \varphi}\cos\varphi$$

and for the axisymmetric case $m = 0$

$$\frac{\partial W}{\partial v} \sim \frac{\partial W}{\partial \zeta}\cos\varphi$$

$$\frac{\partial^2 W}{\partial v^2} \sim \frac{\partial^2 W}{\partial \zeta^2}\cos^2\varphi + \frac{1}{\zeta}\frac{\partial W}{\partial \zeta}\sin^2\varphi$$

$$\frac{\partial W}{\partial \eta} \sim \frac{\partial W}{\partial \zeta}\sin\varphi$$

and

$$\int_0^{2\pi} \frac{\partial W}{\partial v}\cos\varphi d\varphi \sim \frac{\partial W}{\partial \zeta}\int_0^{2\pi}\cos^2\varphi d\varphi = \pi\frac{\partial W}{\partial \zeta}$$

$$\int_0^{2\pi} \frac{\partial W}{\partial \eta}\frac{\partial^2 W}{\partial v^2}\sin\varphi d\varphi \sim \frac{\partial W}{\partial \zeta}\frac{\partial^2 W}{\partial \zeta^2}\int_0^{2\pi}\cos^2\varphi\sin^2\varphi d\varphi + \frac{1}{\zeta}\left(\frac{\partial W}{\partial \zeta}\right)^2\int_0^{2\pi}\sin^4\varphi d\varphi$$

$$\sim \frac{1}{4}\frac{\partial W}{\partial \zeta}\frac{\partial^2 W}{\partial \zeta^2}\int_0^{2\pi}(1+\cos 2\varphi)(1-\cos 2\varphi)\,d\varphi + \frac{1}{4}\frac{1}{\zeta}\left(\frac{\partial W}{\partial \zeta}\right)^2\int_0^{2\pi}(1-\cos 2\varphi)^2\,d\varphi$$

$$\sim \frac{1}{4}\frac{\partial W}{\partial \zeta}\frac{\partial^2 W}{\partial \zeta^2}\int_0^{2\pi}\left(1-\cos^2 2\varphi\right)d\varphi + \frac{1}{4}\frac{1}{\zeta}\left(\frac{\partial W}{\partial \zeta}\right)^2\int_0^{2\pi}\left(1-2\cos 2\varphi+\cos^2 2\varphi\right)d\varphi$$

$$\sim \frac{\pi}{4}\frac{\partial W}{\partial \zeta}\left(\frac{\partial^2 W}{\partial \zeta^2} + 3\frac{1}{\zeta}\frac{\partial W}{\partial \zeta}\right) = \frac{\pi}{4}\frac{\partial W}{\partial \zeta}\frac{1}{\zeta}\frac{\partial}{\partial \zeta}\left(\zeta\frac{\partial W}{\partial \zeta} + 2W\right)$$

$$\int_0^{2\pi} \frac{\partial W}{\partial \eta} \sin\varphi d\varphi \sim \frac{\partial W}{\partial \zeta} \int_0^{2\pi} \sin^2 \varphi d\varphi = \pi \frac{\partial W}{\partial \zeta}$$

or

$$\int_0^{2\pi} \left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega} \right) \cdot \underline{n} d\varphi$$

$$\sim -i\omega\mu_0 \frac{\pi}{h} \left[\frac{\partial^2 \Pi_{mv}}{\partial \omega \partial \zeta} \left(\frac{1}{4} \frac{1}{h^2} \frac{\partial^2}{\partial \zeta^2} + \frac{3}{4} \frac{1}{h^2} \frac{1}{\zeta} \frac{\partial}{\partial \zeta} + k^2 \right) \Pi_{mv}^* + \frac{1}{h^2} \frac{\partial^3 \Pi_{mv}}{\partial \omega \partial \xi \partial \zeta} \frac{\partial \Pi_{mv}^*}{\partial \xi} \right]$$

Therefore we find (noting that $h_\xi \sim h$ and $h_\varphi \sim h\zeta$)

$$i \int_V \left(\mu_0 \underline{H} \cdot \underline{H}^* + \varepsilon_0 \underline{E} \cdot \underline{E}^* \right) dV = \int_{S_{scar}} \left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega} \right) \cdot \underline{n} dS$$

$$= \int_{-\xi_0}^{\xi_0} \left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega} \right) \cdot \underline{n} h_\xi d\xi h_\varphi d\varphi$$

$$\sim -i\omega\mu_0 \pi \int_{-\xi_0}^{\xi_0} \zeta \left[\frac{\partial^2 \Pi_{mv}}{\partial \omega \partial \zeta} \left(\frac{1}{4} \frac{1}{h^2} \frac{\partial^2}{\partial \zeta^2} + \frac{3}{4} \frac{1}{h^2} \frac{1}{\zeta} \frac{\partial}{\partial \zeta} + k^2 \right) \Pi_{mv}^* + \frac{1}{h^2} \frac{\partial^3 \Pi_{mv}}{\partial \omega \partial \xi \partial \zeta} \frac{\partial \Pi_{mv}^*}{\partial \xi} \right] h d\xi$$

$$\sim -i\omega\mu_0 \pi \int_{-\xi_0}^{\xi_0} \left[\zeta \frac{\partial^3 \Pi_{mv}}{\partial \omega \partial \xi \partial \zeta} \frac{\partial \Pi_{mv}^*}{\partial \xi} + \zeta \frac{\partial^2 \Pi_{mv}}{\partial \omega \partial \zeta} \left\{ \frac{1}{4} \frac{1}{\zeta} \frac{\partial}{\partial \zeta} \left(\zeta \frac{\partial \Pi_{mv}^*}{\partial \zeta} + 2\Pi_{mv}^* \right) + k^2 h^2 \Pi_{mv}^* \right\} \right] d\xi / h$$

$$\sim -i\omega\mu_0 \frac{\pi}{d} \int_{-\xi_0}^{\xi_0} \left[\tau \frac{\partial^3 \Pi_{mv}}{\partial \omega \partial \xi \partial \tau} \frac{\partial \Pi_{mv}^*}{\partial \xi} + \tau \frac{\partial^2 \Pi_{mv}}{\partial \omega \partial \tau} \left\{ 2\gamma \frac{1}{4} \frac{1}{\tau} \frac{\partial}{\partial \tau} \left(\tau \frac{\partial \Pi_{mv}^*}{\partial \tau} + 2\Pi_{mv}^* \right) + \gamma^2 \cos^2 \xi \Pi_{mv}^* \right\} \right] d\xi / \cos\xi$$

$$\sim -i\omega\mu_0 \frac{\pi}{d} \int_{-\xi_0}^{\xi_0} \left[\tau \frac{\partial^3 \Pi_{mv}}{\partial \omega \partial \xi \partial \tau} \frac{\partial \Pi_{mv}^*}{\partial \xi} + \gamma\tau \frac{\partial^2 \Pi_{mv}}{\partial \omega \partial \tau} \Pi_{mv}^* \left(\gamma \cos^2 \xi - s \right) \right] d\xi / \cos\xi$$

where we again used

$$\frac{1}{\tau} \frac{\partial}{\partial \tau} \psi_0(\tau) \sim -c_0 \operatorname{Re}\left[e^{i\pi/4} \Gamma(1/2 + is/2) \right] e^{\pi s/2} (s/2) + O\left(\tau^4 \ln \tau\right) \sim -\frac{s}{2} \psi_0$$

$$\frac{1}{\tau} \frac{\partial}{\partial \tau} \left(\tau \frac{\partial}{\partial \tau} \psi_0 \right) \sim -c_0 \operatorname{Re}\left[e^{i\pi/4} \Gamma(1/2 + is/2) \right] e^{\pi s/2} s + O\left(\tau^2 \ln \tau\right) \sim -s\psi_0 + O\left(\tau^2 \ln \tau\right)$$

to evaluate the limit of the final terms in the brackets. Now from the function

$$\Pi_{mv} = \frac{2}{\cos\xi} \psi_0(\tau, s) \cos(\gamma \sin\xi - s\sigma)$$

$$= 2\psi_0(\tau, s) \cosh\sigma \cos(\gamma \tanh\sigma - s\sigma)$$

$$\psi_0(0,s) = c_0 \operatorname{Re}\left[e^{i\pi/4}\Gamma(1/2+is/2)\right] e^{\pi s/2}$$

$$\tau \frac{\partial^2 \psi_0}{\partial\omega\partial\tau}(0,s) \approx 2c_0 \operatorname{Re}\left\{\frac{e^{i\pi/4}}{\Gamma(1/2-is/2)}\right\} \frac{\partial\Phi_0}{\partial\omega}$$

$$\frac{\partial\Pi_{mv}}{\partial\xi} = 2\psi_0(\tau,s)\frac{\partial}{\partial\xi}\left[\frac{\cos(\gamma\sin\xi - s\sigma)}{\cos\xi}\right] = 2\psi_0(\tau,s)\frac{\partial\sigma}{\partial\xi}\frac{\partial}{\partial\sigma}\left[\frac{\cos(\gamma\sin\xi - s\sigma)}{\cos\xi}\right]$$

$$= 2\psi_0(\tau,s)\cosh\sigma\frac{\partial}{\partial\sigma}\left\{\cosh\sigma\cos(\gamma\tanh\sigma - s\sigma)\right\}$$

where we have used

$$\sinh\sigma = \tan\xi$$

$$\cosh\sigma = \sec\xi$$

$$\tanh\sigma = \sin\xi$$

$$d\sigma = \sec\xi d\xi$$

we find

$$i\int_V \left(\mu_0 \underline{H}\cdot\underline{H}^* + \varepsilon_0 \underline{E}\cdot\underline{E}^*\right) dV = \int_{S_{scar}} \left(\frac{\partial \underline{E}}{\partial\omega}\times\underline{H}^* + \underline{E}^*\times\frac{\partial \underline{H}}{\partial\omega}\right)\cdot\underline{n}dS$$

$$\sim -i\omega\mu_0 4\frac{\pi}{d}\tau\frac{\partial^2\psi_0}{\partial\omega\partial\tau}(0,s)\psi_0(0,s)$$

$$\int_{-\sigma_0}^{\sigma_0}\left[\left\{\sinh\sigma\cosh\sigma\cos(\gamma\tanh\sigma - s\sigma) - \sin(\gamma\tanh\sigma - s\sigma)\left(\gamma - s\cosh^2\sigma\right)\right\}^2\right.$$

$$\left.+\gamma\cos^2(\gamma\tanh\sigma - s\sigma)\left(\gamma - s\cosh^2\sigma\right)\right] d\sigma$$

Averaging over the rapidly varying sinusoids for γ large gives

$$i\int_V \left(\mu_0 \underline{H}\cdot\underline{H}^* + \varepsilon_0 \underline{E}\cdot\underline{E}^*\right) dV = \int_{S_{scar}} \left(\frac{\partial \underline{E}}{\partial\omega}\times\underline{H}^* + \underline{E}^*\times\frac{\partial \underline{H}}{\partial\omega}\right)\cdot\underline{n}dS$$

$$\sim -i\omega\mu_0 2\frac{\pi}{d}\tau\frac{\partial^2\psi_0}{\partial\omega\partial\tau}(0,s)\psi_0(0,s)$$

$$\int_{-\sigma_0}^{\sigma_0}\left[\left\{\sinh^2\sigma\cosh^2\sigma + \left(\gamma - s\cosh^2\sigma\right)^2\right\}\right.$$

$$+\gamma\left(\gamma - s\cosh^2\sigma\right)\Big] d\sigma$$

For the time being we retain only the leading order γ^2 terms

$$\int_V \left(\mu_0 \underline{H}\cdot\underline{H}^* + \varepsilon_0 \underline{E}\cdot\underline{E}^*\right) dV$$

$$\sim -8\omega\mu_0 \frac{\pi}{d}\tau \frac{\partial^2\psi_0}{\partial\omega\partial\tau}(0,s)\,\psi_0(0,s)\,\gamma^2\sigma_0$$

$$\sim -\omega\mu_0 k^2 d\pi 16 c_0^2 \operatorname{Re}\left\{\frac{e^{i\pi/4}}{\Gamma(1/2 - is/2)}\right\}\frac{\partial\Phi_0}{\partial\omega}\operatorname{Re}\left[e^{i\pi/4}\Gamma(1/2+is/2)\right] e^{\pi s/2}\sigma_0$$

$$\sim -\mu_0 k^4 d\pi 8 c_0^2 \frac{d\Phi_0}{dk^2}\frac{\operatorname{Re}^2\left[e^{i\pi/4}\Gamma(1/2+is/2)\right]}{\left|\Gamma(1/2-is/2)\right|^2} e^{\pi s/2}\ln(\Lambda_+)$$

The phase Φ_0 indicates the reflection phase of the pth component. Following Antonsen [1] the average derivative is set by taking $\Delta\Phi_0 = 2\pi$ and the spacing between eigenvalues to be given by the Weyl asymptotic result for the vector case $\Delta k \sim \pi^2/\left(k^2V\right)$, or $\Delta k^2 = 2k\Delta k \sim 2\pi^2/(kV)$ [16]. Note in the acoustic case $\Delta k \sim 2\pi^2/\left(k^2V\right)$ [15] and $\Delta k^2 = 2k\Delta k \sim 4\pi^2/(Vk)$. In our case we are interested in the scar amplitude on axis (like the even-even eigenvalues in the 2D problem). Setting the total cross sectional area of the axisymmetric cavity to $A = 2A_0$

$$\frac{1}{2}A = A_0 = \int_{-\infty}^{\infty}\rho(z)\,dz$$

The modal spacing in the scalar case is (this is for a single azimuthal parity, since the two parities are degenerate)

$$\frac{dk^2}{dN} = 8\pi/A \tag{399}$$

Note that we will take half this spacing for the electromagnetic vector modes (again this is for a single azimuthal parity, since the two parities are degenerate)

$$\frac{dk^2}{dN} = 4\pi/A \tag{400}$$

Note that by choosing the average eigenvalue spacing to be one half that of the scalar case, results in a one half being inserted into the theoretical strength of the square amplitude. However, we are picking only the even modes with respect to z so that (this symmetry doubles the square amplitude)

$$\frac{dk^2}{dN} = 8\pi/A$$

Thus we take [1]

$$\left(\frac{d\Phi_0}{dk^2}\right)^{-1} = \frac{4}{A}v^2$$

where v is the Gaussian random variable with unit variance discussed previously. If we had used the eigenvalue spacing for the 3D electromagnetic cavity $\Delta k^2 = 2k\Delta k \sim 2\pi^2/(Vk)$, but even along the z direction, we would have obtained

$$\left(\frac{d\Phi_0}{dk^2}\right)^{-1} = \frac{2\pi}{Vk}v^2$$

We recall that in the 2D case we introduced a factor of two (we also used a factor of two for evenness along the orbit) to account for evenness about the normal to the orbit, but here we are assuming that the axisymmetric case (say $\cos\varphi$ parity) is handled by starting with the axisymmetric 3D scalar problem.

Thus the energy theorem along with the outer phase derivative connects the normalization constant c_0 with the integration of the field energy throughout the volume

$$\int_V (\mu_0 \underline{H} \cdot \underline{H}^* + \varepsilon_0 \underline{E} \cdot \underline{E}^*)\, dV$$

$$\sim -\mu_0 8d\pi \left(\frac{d\Phi_0}{dk^2}\right) \frac{\mathrm{Re}^2\left[e^{i\pi/4}\Gamma\left(1/2+is/2\right)\right]}{\left|\Gamma\left(1/2-is/2\right)\right|^2} e^{\pi s/2} \ln\left(\Lambda_+\right)\left(k^2 c_0\right)^2$$

$$\sim \varepsilon_0 8d\pi \left(\frac{d\Phi_0}{dk^2}\right) \frac{\mathrm{Re}^2\left[e^{i\pi/4}\Gamma\left(1/2+is/2\right)\right]}{\left|\Gamma\left(1/2-is/2\right)\right|^2} e^{\pi s/2} \ln\left(\Lambda_+\right)\left(-ik^2 \eta_0 c_0\right)^2$$

Note that the scalar normalization was

$$\int_V |u|^2\, dV \sim 4d\pi \left(\frac{d\Phi_0}{dk^2}\right) \frac{\mathrm{Re}^2\left[e^{i\pi/4}\Gamma\left(1/2+is_p/2\right)\right]}{\left|\Gamma\left(1/2-is_p/2\right)\right|^2} e^{\pi s_p/2} \ln\left(\Lambda_+\right) c_0^2$$

and the previous vector normalization was

$$\int_V (\mu_0 \underline{H} \cdot \underline{H}^* + \varepsilon_0 \underline{E} \cdot \underline{E}^*)\, dV$$

$$\sim \varepsilon_0 8d\pi \left(\frac{d\Phi_0}{dk^2}\right) \frac{\mathrm{Re}^2\left[e^{i\pi/4}\Gamma\left(1/2+is/2\right)\right]}{\left|\Gamma\left(1/2-is/2\right)\right|^2} e^{\pi s/2} \ln\left(\Lambda_+\right)\left(k^2 c_0\right)^2$$

In the odd case

$$\Pi_{mv} = \frac{2i}{\cos\xi}\psi_0\left(\tau, s\right) \sin\left(\gamma \sin\xi - s\sigma\right)$$

$$= 2i\psi_0\left(\tau, s\right) \cosh\sigma \sin\left(\gamma \tanh\sigma - s\sigma\right)$$

$$\psi_0\left(0, s\right) = c_0 \,\mathrm{Re}\left[e^{i\pi/4}\Gamma\left(1/2+is/2\right)\right] e^{\pi s/2}$$

$$\tau\frac{\partial^2\psi_0}{\partial\omega\partial\tau}\left(0, s\right) \approx 2c_0 \,\mathrm{Re}\left\{\frac{e^{i\pi/4}}{\Gamma\left(1/2-is/2\right)}\right\}\frac{\partial\Phi_0}{\partial\omega}$$

$$\frac{\partial \Pi_{mv}}{\partial \xi} = 2i\psi_0(\tau,s)\frac{\partial}{\partial \xi}\left[\frac{\sin(\gamma\sin\xi - s\sigma)}{\cos\xi}\right] = 2i\psi_0(\tau,s)\frac{\partial\sigma}{\partial\xi}\frac{\partial}{\partial\sigma}\left[\frac{\sin(\gamma\sin\xi - s\sigma)}{\cos\xi}\right]$$

$$= 2i\psi_0(\tau,s)\cosh\sigma\frac{\partial}{\partial\sigma}\{\cosh\sigma\sin(\gamma\tanh\sigma - s\sigma)\}$$

where we have used

$$\sinh\sigma = \tan\xi$$

$$\cosh\sigma = \sec\xi$$

$$\tanh\sigma = \sin\xi$$

$$d\sigma = \sec\xi d\xi$$

Therefore we find (noting that $h_\xi \sim h$ and $h_\varphi \sim h\zeta$)

$$i\int_V \left(\mu_0\underline{H}\cdot\underline{H}^* + \varepsilon_0\underline{E}\cdot\underline{E}^*\right)dV = \int_{S_{scar}}\left(\frac{\partial \underline{E}}{\partial\omega}\times\underline{H}^* + \underline{E}^*\times\frac{\partial\underline{H}}{\partial\omega}\right)\cdot\underline{n}dS$$

$$\sim -i\omega\mu_0\frac{\pi}{d}\int_{-\xi_0}^{\xi_0}\left[\tau\frac{\partial^3\Pi_{mv}}{\partial\omega\partial\xi\partial\tau}\frac{\partial\Pi_{mv}^*}{\partial\xi} + \gamma\tau\frac{\partial^2\Pi_{mv}}{\partial\omega\partial\tau}\Pi_{mv}^*\left(\gamma\cos^2\xi - s\right)\right]d\xi/\cos\xi$$

$$\sim -i\omega\mu_0 4\frac{\pi}{d}\tau\frac{\partial^2\psi_0}{\partial\omega\partial\tau}(0,s)\,\psi_0(0,s)$$

$$\int_{-\sigma_0}^{\sigma_0}\left[\left\{\sinh\sigma\cosh\sigma\sin(\gamma\tanh\sigma - s\sigma) + \cos(\gamma\tanh\sigma - s\sigma)\left(\gamma - s\cosh^2\sigma\right)\right\}^2\right.$$

$$\left.+\gamma\sin^2(\gamma\tanh\sigma - s\sigma)\left(\gamma - s\cosh^2\sigma\right)\right]d\sigma$$

Averaging over the rapidly varying sinusoids for γ large gives

$$i\int_V \left(\mu_0\underline{H}\cdot\underline{H}^* + \varepsilon_0\underline{E}\cdot\underline{E}^*\right)dV = \int_{S_{scar}}\left(\frac{\partial \underline{E}}{\partial\omega}\times\underline{H}^* + \underline{E}^*\times\frac{\partial\underline{H}}{\partial\omega}\right)\cdot\underline{n}dS$$

$$\sim -i\omega\mu_0 2\frac{\pi}{d}\tau\frac{\partial^2\psi_0}{\partial\omega\partial\tau}(0,s)\,\psi_0(0,s)$$

$$\int_{-\sigma_0}^{\sigma_0}\left[\left\{\sinh^2\sigma\cosh^2\sigma + \left(\gamma - s\cosh^2\sigma\right)^2\right\}\right.$$

$$\left.+\gamma\left(\gamma - s\cosh^2\sigma\right)\right]d\sigma$$

For the time being we retain only the leading order γ^2 terms

$$\int_V \left(\mu_0 \underline{H} \cdot \underline{H}^* + \varepsilon_0 \underline{E} \cdot \underline{E}^*\right) dV$$

$$\sim -8\omega\mu_0 \frac{\pi}{d} \tau \frac{\partial^2 \psi_0}{\partial\omega\partial\tau} (0, s)\, \psi_0 (0, s)\, \gamma^2 \sigma_0$$

$$\sim -\omega\mu_0 k^2 d\pi 16 c_0^2 \,\mathrm{Re} \left\{ \frac{e^{i\pi/4}}{\Gamma (1/2 - is/2)} \right\} \frac{\partial \Phi_0}{\partial \omega} \,\mathrm{Re} \left[e^{i\pi/4} \Gamma (1/2 + is/2) \right] e^{\pi s/2} \sigma_0$$

The phase Φ_0 indicates the reflection phase of the pth component. Thus we take [1]

$$\left(\frac{d\Phi_0}{dk^2} \right)^{-1} = \frac{4}{A} v^2$$

where v is the Gaussian random variable with unit variance discussed previously. Thus the energy theorem along with the outer phase derivative connects the normalization constant c_0 with the integration of the field energy throughout the volume

$$\int_V \left(\mu_0 \underline{H} \cdot \underline{H}^* + \varepsilon_0 \underline{E} \cdot \underline{E}^*\right) dV$$

$$\sim -\mu_0 k^4 8 d\pi \left(\frac{d\Phi_0}{dk^2} \right) \frac{\mathrm{Re}^2 \left[e^{i\pi/4} \Gamma (1/2 + is/2) \right]}{\left| \Gamma (1/2 - is/2) \right|^2} e^{\pi s/2} \ln (\Lambda_+)\, c_0^2$$

$$\sim \varepsilon_0 8 d\pi \left(\frac{d\Phi_0}{dk^2} \right) \frac{\mathrm{Re}^2 \left[e^{i\pi/4} \Gamma (1/2 + is/2) \right]}{\left| \Gamma (1/2 - is/2) \right|^2} e^{\pi s/2} \ln (\Lambda_+) \left(-ik^2 \eta_0 c_0 \right)^2$$

4.8.6 Summary of Results

A summary of the results for the axisymmetric mode $m = 0$ are (note that this mode is a vector mode, with magnetic field on axis $\zeta = \sqrt{\upsilon^2 + \eta^2} \to 0$ polarized in the $\underline{e}_\upsilon$ direction).

Even Case In the even case

$$\Pi_{m\upsilon} = \frac{2}{\cos\xi} \psi_0 (\tau, s) \cos (\gamma \sin\xi - s\sigma)$$

$$\psi_0 = c_0 \frac{\sqrt{2}}{\tau} \,\mathrm{Re} \left[W_+ + e^{i\Phi_0} W_+^* \right]$$

$$W_+ (\tau, 0, s) = W_{is/2,0} \left(-i\tau^2/2 \right)$$

$$\tau = \sqrt{2\gamma} \zeta$$

$$k\ell - s_p \sigma_0 = p\pi = k_p \ell\,,\; p = 1, 2, 3, ...$$

Note that

$$\psi_0(0, s_p) = c_0 \operatorname{Re}\left[e^{\pi s_p/2} e^{i\pi/4} \Gamma(1/2 + is_p/2)\right], \; \tau \to 0$$

and near the orbit

$$\psi_0(\tau, s) \sim c_0 \operatorname{Re}\left[e^{i\pi/4} \Gamma(1/2 + is/2)\right] e^{\pi s/2} \left(1 - s\tau^2/4\right) + O\left(\tau^4 \ln \tau\right)$$

$$\tau \frac{\partial^2 \psi_0}{\partial \omega \partial \tau}(0, s) \approx 2c_0 \operatorname{Re}\left[\frac{e^{i\pi/4}}{\Gamma(1/2 - is/2)}\right] \frac{\partial \Phi_0}{\partial \omega}$$

The outer region reflection phase we take [1] to be

$$\left(\frac{d\Phi_0}{dk^2}\right)^{-1} = \frac{4}{A} v^2$$

where A is the cross sectional area of the axisymmetric cavity and where v is the Gaussian random variable with unit variance and density

$$f(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$$

The normalization constant c_0 is connected to the volume energy by means of

$$\int_V \left(\mu_0 \underline{H} \cdot \underline{H}^* + \varepsilon_0 \underline{E} \cdot \underline{E}^*\right) dV$$

$$\sim 8d\pi \left(\frac{d\Phi_0}{dk^2}\right) \frac{\operatorname{Re}^2\left[e^{i\pi/4} \Gamma(1/2 + is/2)\right]}{\left|\Gamma(1/2 - is/2)\right|^2} e^{\pi s/2} \ln(\Lambda_+) \varepsilon_0 \left(-ik^2 \eta_0 c_0\right)^2$$

Note that the scalar form of the normalization was similar

$$\int_V |u|^2 \, dV \sim 4d\pi \left(\frac{d\Phi_0}{dk^2}\right) \frac{\operatorname{Re}^2\left[e^{i\pi/4} \Gamma(1/2 + is_p/2)\right]}{\left|\Gamma(1/2 - is_p/2)\right|^2} e^{\pi s_p/2} \ln(\Lambda_+) c_0^2$$

The fields near the axis are

$$H_v \sim \left(\frac{1}{h^2} \frac{\partial^2}{\partial v^2} + k^2\right) \Pi_{mv} = \left(\frac{2\gamma}{h^2} \frac{\partial^2}{\partial \tau_x^2} + k^2\right) \Pi_{mv}$$

$$\sim \left(\frac{2k}{d \cos^2 \xi} \cos^2 \varphi \frac{\partial^2}{\partial \tau^2} + k^2\right) \Pi_{mv} \sim k^2 \Pi_{mv}$$

$$E_\eta \sim \frac{i\omega\mu_0}{h} \frac{\partial}{\partial \xi} \Pi_{mv} \sim \frac{i\omega\mu_0}{d \cos \xi} \frac{\partial}{\partial \xi} \Pi_{mv}$$

where

$$h = d\sqrt{\cos^2\xi + \zeta^2} = d\sqrt{\cos^2\xi + \upsilon^2 + \eta^2} \sim d\cos\xi$$

$$\tau_x = \sqrt{2\gamma}\upsilon = \sqrt{2\gamma}\zeta\cos\varphi$$

$$\tau_y = \sqrt{2\gamma}\eta = \sqrt{2\gamma}\zeta\sin\varphi$$

$$\tau = \sqrt{\tau_x^2 + \tau_y^2} = \sqrt{2\gamma}\zeta$$

Odd Case In the odd case we have

$$k\ell - s_p\sigma_0 = (p - 1/2)\,\pi = k_p\ell\,,\ p = 1, 2, 3, ...$$

$$\Pi_{m\upsilon}(\tau, \xi) = \frac{2i}{\cos\xi}\psi_0(\tau, s)\sin(\gamma\sin\xi - s\sigma)$$

Remark On Normalization Our approach for checking the normalization is now the following. We take the two potentials $\Pi_{e\upsilon}$ and $\Pi_{m\upsilon}$ each normalized to have the same volume integrals, but opposite parities so that the resonant frequencies are the same. We then look at the projection of the field E_η (which is made up of contributions from both potentials) to see if it has the same statistics as the projection of the field E_υ. If these two field component projections have the same statistics this indicates that the two potentials have been normalized properly with respect to each other?

Cylindrical Form If we transform back to Cylindrical coordinates on axis, then we rewrite the potential

$$\Pi_{m\upsilon}(0, \xi) = \frac{2}{\cos\xi}\psi_0(0, s)\cos(\gamma\sin\xi - s\sigma)$$

$$\tanh\sigma = \sin\xi$$

$$z \to d\sin\xi$$

$$\sigma = \text{Arctanh}\,(z/d) = \frac{1}{2}\ln\left(\frac{d+z}{d-z}\right)$$

as

$$\Pi_{m\upsilon}(0, z) = \frac{2\psi_0(0, s_p)}{\sqrt{1 - z^2/d^2}}\cos\left[k_p z + p_0(z)\right]$$

$$p_0(z) = (k - k_p) z - \frac{1}{2} s_p \ln\left(\frac{d+z}{d-z}\right)$$

where

$$\sigma_0 = \frac{1}{2} \ln\left(\frac{d+\ell}{d-\ell}\right) = \frac{1}{4} \ln \Lambda_+$$

the two exact stability exponents [12] can be written as

$$\Lambda_\pm = \left(\frac{d+\ell}{d-\ell}\right)^{\pm 2}$$

The separation constant s is

$$s_p = (k - k_p) \ell / \sigma_0 = 2 (k - k_p) L / \ln(\Lambda_+)$$

In the odd case (dropping the i factor by simply redefining the coefficient)

$$\Pi_{mv}(0, z) = \frac{2\psi_0(0, s_p)}{\sqrt{1 - z^2/d^2}} \sin[k_p z + p_0(z)]$$

$$k\ell - s_p \sigma_0 = (p - 1/2)\pi = k_p \ell\,,\ p = 1, 2, 3, \ldots$$

4.8.7 Vector Scar Projection

We now discuss the projections of the scar solution along the orbit with the magnetic Hertz potential.

Elliptic System Projection If we take the projection to be defined by a surface integral about the scar (note that the quasirectangular unit vector $\underline{e}_\eta$ is part of this projection operator) this limit definition of the integration around the scar produces an integral without the extra amplitude divergence factors in the kernel of the projection operator (noting that $h_\xi \sim h$, $h_\varphi \sim h\zeta$, and $h \sim d \cos \xi$)

$$V_p = \lim_{\zeta \to 0} \frac{1}{\zeta d} \int_{-\xi_0}^{\xi_0} \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin(\gamma \sin \xi - s\sigma)}{\cos \xi} E_\eta(\zeta, \varphi, \xi) h_\varphi d\varphi h_\xi d\xi$$

$$= \lim_{\zeta \to 0} \frac{1}{\zeta d} \int_{-\xi_0}^{\xi_0} \frac{1}{2\pi} \frac{\sin(\gamma \sin \xi - s\sigma)}{\cos \xi} i\omega\mu_0 \frac{1}{h} \frac{\partial}{\partial \xi} \Pi_{mv}(\zeta, \xi) h_\varphi d\varphi h_\xi d\xi$$

$$= \lim_{\zeta \to 0} \int_{-\xi_0}^{\xi_0} \frac{1}{2\pi} \int_0^{2\pi} \sin(\gamma \sin \xi - s\sigma) i\omega\mu_0 \frac{\partial}{\partial \xi} \Pi_{mv}(\zeta, \xi) d\varphi d\xi$$

$$\sim -i\omega\mu_0 2\psi_0(0, s) \gamma \int_{-\xi_0}^{\xi_0} \sin^2(\gamma \sin \xi - s\sigma) d\xi$$

$$\sim -i\omega\mu_0 2\gamma\psi_0(0, s) \xi_0$$

where we have used

$$h^2 E_\eta = i\omega\mu_0 \frac{\partial}{\partial\xi}\left(h\Pi_{mv}\right) \sim i\omega\mu_0 h \frac{\partial}{\partial\xi}\Pi_{mv}$$

Thus

$$V_p \sim -i\eta_0 2k^2 d\xi_0 \psi_0\left(0,s\right)$$

which is $-i\eta_0$ times the V_p of the preceding case using Π_{ev}, but this is accounted for by the different result from the normalization (the electromagnetic energy theorem). Hence this case at high frequencies seems to be a rotation of polarization to E_η from E_v in the previous Π_{ev} case.

Scar (Galerkin) Projection Alternatively we may not want a surface integration definition of the field projection, but instead a simple line integral along the orbit. Because the field from the magnetic Hertz potential will be added to the field from the electric Hertz potential the projection operator must be defined consistently. Thus we take (note that the derivative involves $1/h_\xi$)

$$V_p = \int_{-\xi_0}^{\xi_0} \frac{\sin\left(\gamma\sin\xi - s\sigma\right)}{\cos\xi} E_\eta\left(0,\xi\right) h_\xi d\xi$$

$$\sim i\omega\mu_0 \int_{-\xi_0}^{\xi_0} \frac{\sin\left(\gamma\sin\xi - s\sigma\right)}{\cos\xi} \frac{\partial}{\partial\xi}\Pi_{mv} d\xi$$

$$\sim -i\omega\mu_0 2\gamma\psi_0\left(0,s\right) \int_{-\xi_0}^{\xi_0} \frac{\sin^2\left(\gamma\sin\xi - s\sigma\right)}{\cos\xi} d\xi$$

$$\sim -i\omega\mu_0 2\gamma\psi_0\left(0,s\right) \int_{-\sigma_0}^{\sigma_0} \sin^2\left(\gamma\tanh\sigma - s\sigma\right) d\sigma$$

$$\sim -i\omega\mu_0 2\gamma\psi_0\left(0,s\right)\sigma_0 \sim -i\eta_0 2k^2 d\sigma_0 \psi_0\left(0,s\right)$$

where

$$\sigma_0 = \frac{1}{2}\ln\left(\frac{d+\ell}{d-\ell}\right) = \frac{1}{4}\ln\left(\Lambda_+\right)$$

which again is $-i\eta_0$ times the V_p of the preceding case using Π_{ev}, but this is accounted for by the different result from the normalization (the electromagnetic energy theorem).

For the odd case the projection of E_η derived from Π_{mv} is

$$V_p = \int_{-\xi_0}^{\xi_0} \frac{\cos\left(\gamma\sin\xi - s\sigma\right)}{\cos\xi} E_\eta\left(0,\xi\right) h_\xi d\xi$$

$$\sim i\omega\mu_0 \int_{-\xi_0}^{\xi_0} \frac{\cos\left(\gamma\sin\xi - s\sigma\right)}{\cos\xi} \frac{\partial}{\partial\xi}\Pi_{mv} d\xi$$

again with

$$h^2 E_\eta = i\omega\mu_0 \frac{\partial}{\partial\xi}\left(h\Pi_{mv}\right) \sim i\omega\mu_0 h \frac{\partial}{\partial\xi}\Pi_{mv}$$

and where the Hertz potential on axis in this odd case is (dropping the i factor by redefining the coefficient)

$$\Pi_{mv}\left(0,\xi\right) = \frac{2}{\cos\xi}\psi_0\left(0,s\right)\sin\left(\gamma\sin\xi - s\sigma\right)$$

Thus we find

$$V_p \sim i\omega\mu_0\gamma 2\psi_0\left(0,s\right)\int_{-\xi_0}^{\xi_0}\frac{\cos^2\left(\gamma\sin\xi - s\sigma\right)}{\cos\xi}d\xi$$

$$\sim i\omega\mu_0\gamma 2\psi_0\left(0,s\right)\int_{-\sigma_0}^{\sigma_0}\cos^2\left(\gamma\tanh\sigma - s\sigma\right)d\sigma$$

$$\sim i\omega\mu_0\gamma 2\psi_0\left(0,s\right)\sigma_0 \sim i\eta_0 2k^2 d\sigma_0\psi_0\left(0,s\right)$$

4.8.8 Remarks On Polarization And Hertz Potential Contributions

These projection results for E_η derived from Π_{mv} are therefore identical to the projection with E_v derived from Π_{ev}. If we can show that the contribution to the projection of E_η from Π_{ev} is of lower order, then we will have succeeded in showing that the projection of E_η is the same as that of E_v using the corresponding Hertz potentials Π_{mv} or Π_{ev}, with each carrying the same contributions to the energy normalization condition. In summary, if we show that the Hertz potential Π_{ev} does not contribute significantly to the projection of E_η, it also would follow that with the use of the two Hertz potentials Π_{ev} and $\Pi_{e\eta}$ that only Π_{ev} contributes significantly to the projection of E_v and only $\Pi_{e\eta}$ contributes significantly to the projection of E_η, even though each contributes equally to the normalization conditions.

The contribution from the other Hertz potential Π_{ev} is

$$h^2 E_\eta = h\frac{\partial}{\partial\eta}\left[\frac{1}{h^3}\frac{\partial}{\partial v}\left(h^2\Pi_{ev}\right)\right] \sim \frac{\partial^2}{\partial\eta\partial v}\Pi_{ev} = 2\gamma\frac{\partial^2}{\partial\tau_y\partial\tau_x}\Pi_{ev}$$

where

$$\Pi_{ev} = \frac{2}{\cos\xi}\psi_0\left(\tau,s\right)\cos\left(\gamma\sin\xi - s\sigma\right)$$

$$= 2\psi_0\left(\tau,s\right)\cosh\sigma\cos\left(\gamma\tanh\sigma - s\sigma\right)$$

and

$$v = \zeta\cos\varphi$$

$$\eta = \zeta\sin\varphi$$

Stepping back to the original form of the electric field in the direction of the electric Hertz potential

$$E_\upsilon \sim \left(\frac{1}{h^2}\frac{\partial^2}{\partial \upsilon^2}+k^2\right)\Pi_{e\upsilon} = \left(\frac{2\gamma}{h^2}\frac{\partial^2}{\partial \tau_x^2}+k^2\right)\Pi_{e\upsilon}$$

$$\sim \left(\frac{2k}{d\cos^2\xi}\cos^2\varphi\frac{\partial^2}{\partial \tau^2}+k^2\right)\Pi_{e\upsilon} \sim k^2\Pi_{e\upsilon}$$

we see that the $O\left(k^2d^2\right) = O\left(\gamma^2\right)$ term was dominant. Hence the other component E_η, involving twin transverse derivatives

$$h^2E_\eta = h\frac{\partial}{\partial \eta}\left[\frac{1}{h^3}\frac{\partial}{\partial \upsilon}\left(h^2\Pi_{e\upsilon}\right)\right] \sim \frac{\partial^2}{\partial\eta\partial\upsilon}\Pi_{e\upsilon} = 2\gamma\frac{\partial^2}{\partial\tau_y\partial\tau_x}\Pi_{e\upsilon}$$

is of $O\left(1/\gamma\right) = O\left(1/\left(kd\right)\right)$ of the component in the direction of the potential, and hence can be ignored. Note that the electric field contribution E_η arising from $\Pi_{m\upsilon}$ was taken as the same order as E_υ arising from $\Pi_{e\upsilon}$ due to the normalization condition in the energy theorem.

5 CONCAVE MIRRORS AND ROTATIONAL STADIUM CAVITY

The rotated stadium-like cavity has three regions to consider. The inner region 1 has a field description similar to the bowtie cavity. The outer region 2 is a complementary region outside the foci and the focal region 3 involves a product form of special functions near the foci.

5.1 Curved Trajectory Analysis

Vaynshteyn has treatments for stable modes between concave mirrors. Here we wish to consider the generalization to unstable modes between concave mirrors. Following Vaynshteyn [5] we have Figure 31 where

$$r = d\sinh\zeta\cos\xi$$

$$z = d\cosh\zeta\sin\xi$$

$$0 < \zeta < \infty$$

$$-\pi/2 < \xi < \pi/2\ ,\ 0 < \varphi < 2\pi$$

We take

$$L = 2\ell$$

where

$$\ell = d\cosh\zeta_0 \tag{401}$$

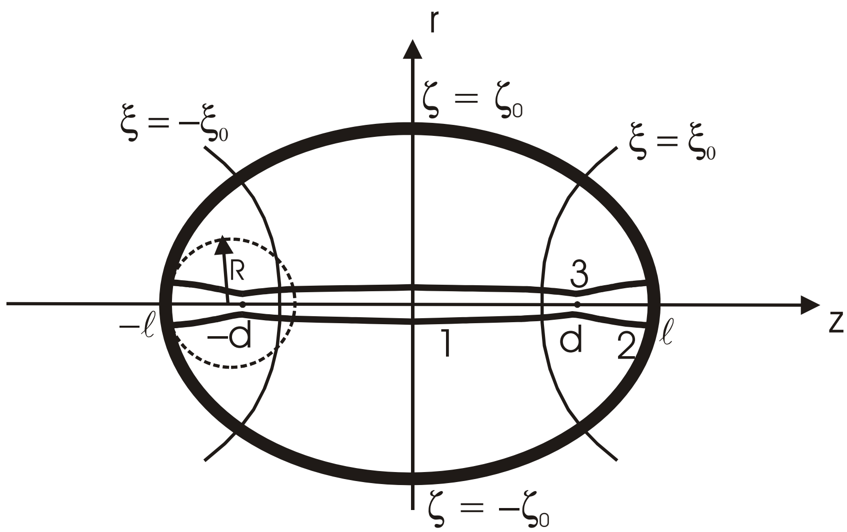

Figure 31. Fitting of prolate spheroidal coordinate system to radius of curvature of rotationally symmetric stadium cavity.

On the mirror we have

$$r = d \sinh \zeta_0 \cos \xi$$

$$z = d \cosh \zeta_0 \sin \xi$$

$$\sqrt{1 - [r/(d \sinh \zeta_0)]^2} = \sin \xi$$

As $r \to 0$ we have

$$\sin \xi \sim 1 - \frac{1}{2} [r/(d \sinh \zeta_0)]^2$$

and

$$z \sim d \cosh \zeta_0 - \frac{r^2}{2d} \frac{\cosh \zeta_0}{\sinh^2 \zeta_0}$$

On a circle of radius R, centered at $(0, z_0)$ where $z_0 = d \cosh \zeta_0 - R$, we can write

$$(z - z_0)^2 + r^2 = R^2$$

$$z = d \cosh \zeta_0 - R + \sqrt{R^2 - r^2} \sim d \cosh \zeta_0 - \frac{r^2}{2R}$$

Thus

$$R = d \sinh^2 \zeta_0 / \cosh \zeta_0$$

Also

$$R = d (\cosh \zeta_0 - 1/\cosh \zeta_0)$$

$$\ell = d \cosh \zeta_0$$

and

$$R = \ell - d^2/\ell$$

$$d = \sqrt{\ell (\ell - R)} = \ell \sqrt{1 - R/\ell} \tag{402}$$

In examples below we will take $\ell \approx 0.11176$ m and $R \approx 0.10160$ m, and $d = 0.0336969$ m.

5.2 Scalar Field High Frequency Approximation

Figure 31 shows the regions near the scarred orbit on the major axis. The modes of the Helmholtz equation

$$\nabla^2 u + k^2 u = 0$$
are now investigated. This can be written in these three-dimensions as [5]

$$\frac{1}{\cos\xi}\frac{\partial}{\partial\xi}\left(\cos\xi\frac{\partial u}{\partial\xi}\right)+\left(\frac{1}{\cos^2\xi}+\frac{1}{\sinh^2\zeta}\right)\frac{\partial^2 u}{\partial\varphi^2}$$

$$+\frac{1}{\sinh\zeta}\frac{\partial}{\partial\zeta}\left(\sinh\zeta\frac{\partial u}{\partial\zeta}\right)+\gamma^2\left(\cosh^2\zeta-\sin^2\xi\right)u=0$$
where

$$\gamma = kd = k\ell\sqrt{1-R/\ell} \tag{403}$$
We assume $\gamma >> 1$. On the mirror we want

$$u=0\,,\,\zeta=\zeta_0\,,\,0<|\xi|<\pi/2 \tag{404}$$

5.2.1 Modal Description In Region One: Between Foci

We will solve the problem separately in the several regions of the stadium cavity. The high frequency analysis in the first region is the same as for the bowtie cavity, which we repeat here for convenience. We assume in the first region that we are inside the foci with $-\xi_0 < \xi < \xi_0$ and that near the orbit we have $\sinh^2\zeta << 1$. We assume $\gamma >> 1$ and that $\sinh^2\zeta << 1$. We take the function u to be even about the z axis. We seek a solution of the form [5]

$$u = W\left(\zeta,\varphi,\xi\right)e^{i\gamma\sin\xi}+W\left(\zeta,\varphi,-\xi\right)e^{-i\gamma\sin\xi}$$
Substituting into the Helmholtz equation gives

$$\frac{1}{\cos\xi}\frac{\partial}{\partial\xi}\left(\cos\xi\frac{\partial W}{\partial\xi}\right)+\left(i2\gamma\cos\xi\frac{\partial W}{\partial\xi}\right)+\left(\frac{1}{\cos^2\xi}+\frac{1}{\sinh^2\zeta}\right)\frac{\partial^2 W}{\partial\varphi^2}$$

$$+\frac{1}{\sinh\zeta}\frac{\partial}{\partial\zeta}\left(\sinh\zeta\frac{\partial W}{\partial\zeta}\right)+\gamma\left(\gamma\sinh^2\zeta-i2\sin\xi\right)W=0$$
Now for high frequencies $\gamma >> 1$ we ignore the term

$$\frac{1}{\cos\xi}\frac{\partial}{\partial\xi}\left(\cos\xi\frac{\partial W}{\partial\xi}\right)$$
and taking $\sinh^2\zeta << 1$ we can replace $\sinh\zeta$ by ζ and neglect the term

$$\frac{1}{\cos^2\xi}\frac{\partial^2 W}{\partial\varphi^2}$$
Then

$$\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial W}{\partial\zeta}\right)+i2\gamma\cos\xi\frac{\partial W}{\partial\xi}$$

$$+\frac{1}{\zeta^2}\frac{\partial^2 W}{\partial\varphi^2}+\left(\gamma^2\zeta^2-i2\gamma\sin\xi\right)W=0$$

Next taking

$$W=\frac{1}{\cos\xi}\Psi$$

$$\tau=\sqrt{2\gamma}\zeta$$

$$\sigma=\int_0^{\xi}\frac{d\xi}{\cos\xi}=\operatorname{arcsinh}\left(\tan\xi\right)$$

$$\sigma_0=\operatorname{arcsinh}\left(\tan\xi_0\right)=\ln\left(\tan\xi_0+\sec\xi_0\right)$$

These give

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial W}{\partial\tau}\right)+i\cos\xi\frac{\partial\sigma}{\partial\xi}\frac{\partial W}{\partial\sigma}$$

$$+\frac{1}{\tau^2}\frac{\partial^2 W}{\partial\varphi^2}+\left(\tau^2/4-i\sin\xi\right)W=0$$

or

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial\Psi}{\partial\tau}\right)+\frac{1}{\tau^2}\frac{\partial^2\Psi}{\partial\varphi^2}+i\frac{\partial\Psi}{\partial\sigma}+\frac{\tau^2}{4}\Psi=0$$

Taking

$$\Psi=\Psi_m\left(\tau,\sigma\right)\cos\left(m\varphi\right)$$

or

$$\Psi=\Psi_m\left(\tau,\sigma\right)\sin\left(m\varphi\right)$$

gives

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial\Psi_m}{\partial\tau}\right)+i\frac{\partial\Psi_m}{\partial\sigma}+\left(\frac{\tau^2}{4}-\frac{m^2}{\tau^2}\right)\Psi_m=0$$

Letting

$$\Psi_m\left(\tau,\sigma\right)=e^{-is\sigma}\psi_m\left(\tau,s\right)$$

gives

$$\frac{1}{\tau}\frac{d}{d\tau}\left(\tau\frac{d\psi_m}{d\tau}\right)+\left(\frac{\tau^2}{4}+s-\frac{m^2}{\tau^2}\right)\psi_m=0$$

If we let $\tau^2=2v$

$$\frac{1}{\tau}\frac{d}{d\tau}\left(\tau\frac{d\psi_m}{d\tau}\right)=\frac{1}{\tau}\frac{dv}{d\tau}\frac{d}{dv}\left(\tau\frac{dv}{d\tau}\frac{d\psi_m}{dv}\right)=2\frac{d}{dv}\left(v\frac{d\psi_m}{dv}\right)$$

$$\frac{d}{dv}\left(v\frac{d\psi_m}{dv}\right)+\left(\frac{v}{4}+\frac{s}{2}-\frac{m^2}{4v}\right)\psi_m=0$$

Now taking

$$\psi_m=\frac{1}{\sqrt{v}}w_m$$

gives

$$v\frac{d\psi_m}{dv}=\sqrt{v}\left(\frac{dw_m}{dv}-\frac{w_m}{2v}\right)$$

$$\frac{d}{dv}\left(v\frac{d\psi_m}{dv}\right)=\sqrt{v}\frac{d^2w_m}{dv^2}+\sqrt{v}\left(\frac{w_m}{4v^2}\right)$$

$$\frac{d^2w_m}{dv^2}+\left(\frac{1}{4}+\frac{s/2}{v}+\frac{1-m^2}{4v^2}\right)w_m=0$$

This is a form of Whittaker's equation

$$\frac{d^2w}{dz^2}+\left(-\frac{1}{4}+\frac{\kappa}{z}+\frac{1-4\mu^2}{4z^2}\right)w=0$$

$$w_m=M_{\kappa,\mu}(z),W_{\kappa,\mu}(z)$$

Therefore in our equation with $z=-iv$

$$\frac{d^2w_m}{dz^2}+\left(-\frac{1}{4}+\frac{is/2}{z}+\frac{1-m^2}{4z^2}\right)w_m=0$$

$$\kappa=is/2$$

$$\pm m=2\mu$$

$$w_m=M_{\kappa,\mu}(-iv),W_{\kappa,\mu}(-iv)$$

Noting that

$$W_{\kappa,\mu}(z)\sim z^{\kappa}e^{-z/2}\ ,\ \ |z|\to\infty$$

Let us take

$$W_+(v,m,s)=W_{\kappa,\mu}(-iv)$$

and then

$$w_m = c_m \,\mathrm{Re}\left[W_+ + e^{i\Phi_0} W_+^*\right]$$

The potential is then

$$u = \frac{2}{\cos\xi}\psi_m\left(\tau, s\right)\cos\left(\gamma\sin\xi - s\sigma\right)\left\{\begin{array}{c}\cos\left(m\varphi\right)\\ \sin\left(m\varphi\right)\end{array}\right\}$$

$$\psi_m = c_m \frac{\sqrt{2}}{\tau}\,\mathrm{Re}\left[W_+ + e^{i\Phi_0} W_+^*\right]$$

$$W_+\left(\tau, m, s\right) = W_{is/2, m/2}\left(-i\tau^2/2\right)$$

5.2.2 Modal Description In Region Two: Outside Foci

In the second region outside the foci we assume that $0 < \zeta < \zeta_0$ and that $\cos^2\xi << 1$. We seek a second solution of the form

$$u = W\left(\zeta, \varphi, \xi\right) e^{i\gamma\cosh\zeta} + W\left(\zeta, \varphi, -\xi\right) e^{-i\gamma\cosh\zeta}\ ,\ \ |\xi| > \xi_0 \tag{405}$$

$$r = d\sinh\zeta\cos\xi$$

$$z = d\cosh\zeta\sin\xi$$

Note that the sign change in the exponential goes with the sign change of ξ before the limit $\xi \to \pi/2$ is applied. The parity in ξ is actually required since the Region 1 matching (which is even in ξ) will make the disjoint region two's have even parity also. The fact that this introduces the standing wave in Region 2 is comforting.

Substituting into the Helmholtz equation

$$\frac{1}{\cos\xi}\frac{\partial}{\partial\xi}\left(\cos\xi\frac{\partial u}{\partial\xi}\right) + \left(\frac{1}{\cos^2\xi} + \frac{1}{\sinh^2\zeta}\right)\frac{\partial^2 u}{\partial\varphi^2}$$

$$+\frac{1}{\sinh\zeta}\frac{\partial}{\partial\zeta}\left(\sinh\zeta\frac{\partial u}{\partial\zeta}\right) + \gamma^2\left(\cosh^2\zeta - \sin^2\xi\right)u = 0$$

and using

$$\frac{1}{\sinh\zeta}\frac{\partial}{\partial\zeta}\left(\sinh\zeta\frac{\partial u}{\partial\zeta}\right) = \frac{1}{\sinh\zeta}\frac{\partial}{\partial\zeta}\left(\sinh\zeta\frac{\partial}{\partial\zeta}We^{i\gamma\cosh\zeta}\right)$$

$$= \frac{1}{\sinh\zeta}\frac{\partial}{\partial\zeta}\left(\sinh\zeta\frac{\partial W}{\partial\zeta}e^{i\gamma\cosh\zeta}\right) + i\gamma\frac{1}{\sinh\zeta}\frac{\partial}{\partial\zeta}\left(\sinh^2\zeta We^{i\gamma\cosh\zeta}\right)$$

$$= e^{i\gamma\cosh\zeta}\left[\frac{1}{\sinh\zeta}\frac{\partial}{\partial\zeta}\left(\sinh\zeta\frac{\partial W}{\partial\zeta}\right) + i2\gamma\sinh\zeta\frac{\partial W}{\partial\zeta} + i2\gamma\cosh\zeta W - \gamma^2\sinh^2\zeta W\right]$$

gives

$$\frac{1}{\cos\xi}\frac{\partial}{\partial\xi}\left(\cos\xi\frac{\partial W}{\partial\xi}\right) + \left(\frac{1}{\cos^2\xi} + \frac{1}{\sinh^2\zeta}\right)\frac{\partial^2 W}{\partial\varphi^2}$$

$$+\frac{1}{\sinh\zeta}\frac{\partial}{\partial\zeta}\left(\sinh\zeta\frac{\partial W}{\partial\zeta}\right) + i2\gamma\sinh\zeta\frac{\partial W}{\partial\zeta} + \left(i2\gamma\cosh\zeta + \gamma^2\cos^2\xi\right)W = 0$$

Let us now substitute the second term $W(\zeta,\varphi,-\xi)\,e^{-i\gamma\cosh\zeta}$ into the Helmholtz equation

$$\frac{1}{\cos\xi}\frac{\partial}{\partial\xi}\left(\cos\xi\frac{\partial W}{\partial\xi}\right) + \left(\frac{1}{\cos^2\xi} + \frac{1}{\sinh^2\zeta}\right)\frac{\partial^2 W}{\partial\varphi^2}$$

$$+\frac{1}{\sinh\zeta}\frac{\partial}{\partial\zeta}\left(\sinh\zeta\frac{\partial W}{\partial\zeta}\right) - i2\gamma\sinh\zeta\frac{\partial W}{\partial\zeta} + \left(-i2\gamma\cosh\zeta + \gamma^2\cos^2\xi\right)W = 0$$

The original equation is not recovered by choosing a change in sign of ξ. It can be recovered by a $\pm i\pi$ shift in ζ where $\cosh(\zeta \pm i\pi) = -\cosh\zeta$ and $\sinh(\zeta \pm i\pi) = -\sinh\zeta$. Let us take

$$u = W(\zeta,\varphi,\xi)\,e^{i\gamma\cosh\zeta} + W(\zeta - i\pi,\varphi,\xi)\,e^{-i\gamma\cosh\zeta}\ ,\ \ |\xi| > \xi_0 \tag{406}$$

Now for high frequencies $\gamma >> 1$ we neglect the term

$$\frac{1}{\sinh\zeta}\frac{\partial}{\partial\zeta}\left(\sinh\zeta\frac{\partial W}{\partial\zeta}\right)$$

and for $\cos^2\xi << 1$ the term

$$\frac{1}{\sinh^2\zeta}\frac{\partial^2 W}{\partial\varphi^2}$$

to give

$$\frac{1}{\cos\xi}\frac{\partial}{\partial\xi}\left(\cos\xi\frac{\partial W}{\partial\xi}\right) + \frac{1}{\cos^2\xi}\frac{\partial^2 W}{\partial\varphi^2}$$

$$+i2\gamma\sinh\zeta\frac{\partial W}{\partial\zeta} + \left(i2\gamma\cosh\zeta + \gamma^2\cos^2\xi\right)W = 0$$

Using $\cos^2\xi = \cos^2(\pi/2 - |\xi| - \pi/2) = \sin^2(\pi/2 - |\xi|) \approx (\pi/2 - |\xi|)^2 = \xi'^2$, we have

$$\frac{1}{\xi'}\frac{\partial}{\partial\xi'}\left(\xi'\frac{\partial W}{\partial\xi'}\right) + \frac{1}{\xi'^2}\frac{\partial^2 W}{\partial\varphi^2}$$

$$+i2\gamma\sinh\zeta\frac{\partial W}{\partial\zeta} + \left(i2\gamma\cosh\zeta + \gamma^2\xi'^2\right)W = 0 \tag{407}$$

We will focus on the $\xi > 0$ side so we can define $\xi' = \pi/2 - \xi$, $\left|\xi'\right| < \pi/2 - \xi_0$. To generalize to both sides of Region 2 we can take $\xi' = \pm\pi/2 - \xi$. Now letting

$$W = \frac{1}{\sinh\zeta}\Psi \tag{408}$$

and

$$\tau' = \sqrt{2\gamma}\xi' \tag{409}$$

$$\sigma' = \int_{\infty}^{\zeta} \frac{d\zeta}{\sinh\zeta} = \ln\left[\tanh\left(\zeta/2\right)\right] \tag{410}$$

gives

$$\sinh\zeta\frac{\partial}{\partial\zeta} = \sinh\zeta\frac{\partial\sigma'}{\partial\zeta}\frac{\partial}{\partial\sigma'} = \frac{\partial}{\partial\sigma'}$$

$$\frac{1}{\tau'}\frac{\partial}{\partial\tau'}\left(\tau'\frac{\partial\Psi}{\partial\tau'}\right) + \frac{1}{\tau'^2}\frac{\partial^2\Psi}{\partial\varphi^2}$$

$$+i\frac{\partial\Psi}{\partial\sigma'} + \frac{\tau'^2}{4}\Psi = 0$$

Note that

$$\sigma' = \int_{\infty}^{\zeta - i\pi} \frac{d\zeta}{\sinh\zeta} = -\int_{\infty}^{\zeta} \frac{d\zeta}{\sinh\zeta}$$

$$\sigma_0' = \ln\left[\tanh\left(\zeta_0/2\right)\right]$$

$$\sinh\left(\zeta - i\pi\right) \to e^{-i\pi}\sinh\zeta \tag{411}$$

Because of the mirror boundary condition this equation must be integrated under the restriction that

$$\Psi\left(\tau', \varphi, \sigma_0'\right) = \Psi\left(\tau', \varphi, -\sigma_0'\right)e^{i\chi}$$

$$\chi = -2\gamma\cosh\zeta_0 + \pi\left(2n-1\right) + \pi = -kL + \pi\left(2n-1\right) + \pi$$

Letting

$$\Psi = \Psi_m\left(\tau, \sigma\right)\cos\left(m\varphi\right)$$

or

$$\Psi = \Psi_m\left(\tau, \sigma\right)\sin\left(m\varphi\right)$$

gives

$$\frac{1}{\tau'}\frac{\partial}{\partial\tau'}\left(\tau'\frac{\partial\Psi}{\partial\tau'}\right)+i\frac{\partial\Psi}{\partial\sigma'}+\left(\frac{\tau'^2}{4}-\frac{m^2}{\tau'^2}\right)\Psi=0$$

and taking

$$\Psi_m\left(\tau',\sigma'\right)=e^{-is'\sigma'}\psi_m\left(\tau',s'\right) \tag{412}$$

gives

$$\frac{1}{\tau'}\frac{d}{d\tau'}\left(\tau'\frac{d\psi_m}{d\tau'}\right)+\left(\frac{\tau'^2}{4}+s'-\frac{m^2}{\tau'^2}\right)\psi_m=0 \tag{413}$$

The mirror condition gives

$$1=e^{i2s'\sigma_0'+i\chi}$$

$$2s'\sigma_0'+\chi=2n'\pi$$

$$2s'\sigma_0'=2n'\pi+kL-\pi\left(2n-1\right)-\pi=2\left(n'-n\right)\pi+kL$$

$$=-2p\pi+kL=\left(k-k_p\right)L \tag{414}$$

where

$$k_pL=2k_p\ell=\pi 2p \tag{415}$$

$$p=1,2,... \tag{416}$$

We take the real part at the end of the Region 2 construction.

5.2.3 More Symmetrical Version Of Region Two Solution

It turns out to be convenient to take the solution in Region 2 as

$$u=e^{-i\pi/2}\left[W\left(\zeta,\varphi,\xi\right)e^{i\gamma\cosh\zeta}+W\left(\zeta-i\pi,\varphi,\xi\right)e^{-i\gamma\cosh\zeta}\right]\ ,\ \ |\xi|>\xi_0 \tag{417}$$

This choice eliminates a factor $e^{i\pi/2}$ that would appear in subsequent sections.

5.2.4 More General Version Of Region Two Solution

To make sure we have not missed any choices in the second solution, suppose we take the solution in Region 2 to be more generally

$$u=e^{-i\pi/2-i\Phi_1/2}\left[W\left(\zeta,\varphi,\xi\right)e^{i\gamma\cosh\zeta}+e^{i\Phi_1}W\left(\zeta-i\pi,\varphi,\xi\right)e^{-i\gamma\cosh\zeta}\right]\ ,\ \ |\xi|>\xi_0$$

$$= \frac{1}{\sinh\zeta}\left[\Psi_m(\tau',\sigma')\,e^{i\gamma\cosh\zeta-i\pi/2-i\Phi_1/2}+\Psi_m(\tau',-\sigma')\,e^{-i\gamma\cosh\zeta+i\pi/2+i\Phi_1/2}\right]\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}$$

$$= \frac{2}{\sinh\zeta}\psi_m(\tau',s')\cos(\gamma\cosh\zeta-s'\sigma'-\pi/2-\Phi_1/2)\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\} \tag{418}$$

where

$$\psi_m(\tau',s') = c_m\frac{\sqrt{2}}{\tau'}\,\mathrm{Re}\left[W_+(\tau',m,s')+e^{i\Phi_0'}W_+^*(\tau',m,s')\right] \tag{419}$$

$$W_+(\tau',m,s') = W_{is'/2,m/2}\left(-i\tau'^2/2\right)$$

$$\tau' = \sqrt{2\gamma}\xi' \tag{420}$$

$$\xi' = \pm\pi/2-\xi \tag{421}$$

$$2s'\sigma_0' = 2s'\ln[\tanh(\zeta_0/2)] = (k-k_p)\,L \tag{422}$$

$$\sigma' = \int_\infty^\zeta \frac{d\zeta}{\sinh\zeta} = \ln[\tanh(\zeta/2)]$$

The mirror condition gives

$$\cos(\gamma\cosh\zeta_0-s'\sigma_0'-\pi/2-\Phi_1/2) = \cos(k_p\ell-\pi/2-\Phi_1/2) = 0$$

or

$$k_p\ell-\pi/2-\Phi_1/2 = \pi(p-1/2) \tag{423}$$

At this point we can think of k_p and hence s' as still arbitrary (k_p and s' are directly related). The introduced phase Φ_1 is selected to match the mirror boundary condition.

5.2.5 Modified Region One Solution

Because the solution must be even in ξ we can write in Region 1

$$u = C\left[W(\zeta,\varphi,\xi)\,e^{i\gamma\sin\xi}+W(\zeta,\varphi,-\xi)\,e^{-i\gamma\sin\xi}\right]\ ,\ \ |\xi|<\xi_0$$

$$= C\frac{1}{\cos\xi}\left[\Psi(\tau,\varphi,\sigma)\,e^{i\gamma\sin\xi}+\Psi(\tau,\varphi,-\sigma)\,e^{-i\gamma\sin\xi}\right]$$

$$= 2C\frac{\psi_m(\tau,s)}{\cos\xi}\cos(\gamma\sin\xi-s\sigma)\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}$$

$$= 2C\frac{\psi_m(\tau,s)}{\sin\xi'}\cos\left(\gamma\cos\xi' - s\sigma\right)\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\},\ \xi' = \pi/2 - \xi \tag{424}$$

For matching purposes (with Region 2) it is convenient to insert the imaginary unit into the definition

$$\psi_m(\tau,s) = c_m\frac{\sqrt{2}}{\tau}\operatorname{Re}\left[e^{-i\pi/2}W_+(\tau,m,s) + e^{i\pi/2}e^{i\Phi_0}W_+^*(\tau,m,s)\right] \tag{425}$$

$$\sigma = \int_0^\xi \frac{d\xi}{\cos\xi} = \operatorname{arcsinh}(\tan\xi)$$

$$= \ln(\tan\xi + \sec\xi) = \ln\left\{\tan\frac{1}{2}(\xi + \pi/2)\right\} = -\ln\left\{\tan\left(\xi'/2\right)\right\} \tag{426}$$

$$\tau = \sqrt{2\gamma}\zeta$$

$$W_+(\tau,m,s) = W_{is/2,m/2}\left(-i\tau^2/2\right)$$

5.2.6 Behavior Of Zero Mode Near Orbit For Stadium Solutions

The Region 1 behavior of the zero mode $m = 0$

$$\psi_0(\tau,s) = c_0\frac{\sqrt{2}}{\tau}\operatorname{Re}\left[e^{-i\pi/2}W_+(\tau,0,s) + e^{i\pi/2}e^{i\Phi_0}W_+^*(\tau,0,s)\right]$$

is now discussed near $\tau \to 0$. Noting the behavior of the Whittaker function

$$W_+(\tau,0,s) \sim -\left(-i\tau^2/2\right)^{1/2}\frac{1}{\Gamma(1/2 - is/2)}\left[\ln\left(-i\tau^2/2\right)\left(1 - s\tau^2/4\right)\right.$$

$$\left. +\psi(1/2 - is/2) - 2\psi(1) + \left(s\tau^2/4\right)\{-\psi(1/2 - is/2) + 2\psi(1) + 2\} + O\left(\tau^4\ln\tau\right)\right]$$

$$W_+'(\tau,0,s) \sim i\tau\frac{1}{2}\left(-i\tau^2/2\right)^{-1/2}\frac{1}{\Gamma(1/2 - is/2)}$$

$$\left[\ln\left(-i\tau^2/2\right) + \{\psi(1/2 - is/2) - 2\psi(1)\} + 2 + O\left(\tau^2\ln\tau\right)\right],\ \tau \to 0,\ m = 0$$

and taking

$$\tau\frac{\partial\psi_0}{\partial\tau} = c_0\sqrt{2}\operatorname{Re}\left[e^{-i\pi/2}W_+' + e^{i\pi/2}e^{i\Phi_0}W_+^{*\prime}\right] - \psi_0 \tag{427}$$

to vanish as $\tau \to 0$ gives the reflection phase

$$e^{i\Phi_0} = -\frac{e^{-i\pi/2}\left(W_+' - \frac{1}{\tau}W_+\right)}{e^{i\pi/2}\left(W_+' - \frac{1}{\tau}W_+\right)^*} = \frac{\left(W_+' - \frac{1}{\tau}W_+\right)}{\left(W_+' - \frac{1}{\tau}W_+\right)^*}$$

$$= -i\frac{\Gamma\left(1/2+is/2\right)}{\Gamma\left(1/2-is/2\right)}+O\left(\tau^2\right) \tag{428}$$

Thus near $\tau \to 0$ we find

$$\psi_0 = c_0 \,\mathrm{Re}\left[\frac{e^{i\pi/4}}{\Gamma\left(1/2-is/2\right)}\left\{\ln\left(-i\tau^2/2\right)-\ln\left(i\tau^2/2\right)+\psi\left(1/2-is/2\right)-\psi\left(1/2+is/2\right)\right\}+O\left(\tau^2\ln\tau\right)\right]$$

Noting that

$$\psi\left(1/2+is/2\right)-\psi\left(1/2-is/2\right)=\pi\cot\pi\left(1/2-is/2\right)$$

$$=\pi\frac{\cos\pi\left(1/2-is/2\right)}{\sin\pi\left(1/2-is/2\right)}=\pi\frac{\sin\left(i\pi s/2\right)}{\cos\left(i\pi s/2\right)}=-\frac{\pi}{i}\frac{\sinh\left(\pi s/2\right)}{\cosh\left(\pi s/2\right)}=i\pi\tanh\left(\pi s/2\right)$$

gives

$$\psi_0\left(0,s\right)=c_0\,\mathrm{Re}\left[-e^{i\pi/4}\frac{1}{\Gamma\left(1/2-is/2\right)}\left\{i\pi+\psi\left(1/2+is/2\right)-\psi\left(1/2-is/2\right)\right\}\right]$$

$$=c_0\,\mathrm{Re}\left[-e^{i\pi/4}\frac{1}{\Gamma\left(1/2-is/2\right)}\left\{i\pi+\pi\cot\pi\left(1/2-is/2\right)\right\}\right]$$

$$=c_0\,\mathrm{Re}\left[-e^{i\pi/4}\frac{1}{\Gamma\left(1/2-is/2\right)}\left\{i\pi+\pi\tan\left(i\pi s/2\right)\right\}\right]$$

$$=c_0\,\mathrm{Re}\left[\pi e^{-i\pi/4}\frac{1}{\Gamma\left(1/2-is/2\right)}\left\{1+\tanh\left(\pi s/2\right)\right\}\right]$$

$$=c_0\,\mathrm{Re}\left[\pi e^{-i\pi/4}\frac{1}{\Gamma\left(1/2-is/2\right)}\frac{e^{\pi s/2}}{\cosh\left(\pi s/2\right)}\right]$$

$$=c_0\,\mathrm{Re}\left[e^{-i\pi/4}e^{\pi s/2}\Gamma\left(1/2+is/2\right)\right]$$

or

$$\psi_0\left(0,s\right)=c_0\,\mathrm{Re}\left[e^{\pi s/2}e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]\ ,\ \tau\to 0 \tag{429}$$

which is $-i$ (inside the real part) times the preceding bowtie result. In general

$$\psi_0\left(\tau,s\right)\sim c_0\frac{\sqrt{2}}{\tau}\,\mathrm{Re}\left[-\left(-i\tau^2/2\right)^{1/2}\frac{-i}{\Gamma\left(1/2-is/2\right)}\right.$$

$$\left.\left\{\ln\left(-i\tau^2/2\right)\left(1-s\tau^2/4\right)+\psi\left(1/2-is/2\right)-2\psi\left(1\right)+\left(s\tau^2/4\right)\left(-\psi\left(1/2-is/2\right)+2\psi\left(1\right)+2\right)+O\left(\tau^4\ln\tau\right)\right\}\right.$$

$$+i\left(i\tau^2/2\right)^{1/2}\frac{i}{\Gamma\left(1/2-is/2\right)}$$

$$\left\{\ln\left(i\tau^2/2\right)\left(1-s\tau^2/4\right)+\psi\left(1/2+is/2\right)-2\psi\left(1\right)+\left(s\tau^2/4\right)\left(-\psi\left(1/2+is/2\right)+2\psi\left(1\right)+2\right)+O\left(\tau^4\ln\tau\right)\right\}\right]$$

$$\sim c_0\operatorname{Re}\left[-e^{-i\pi/4}\frac{-i}{\Gamma\left(1/2-is/2\right)}\right.$$

$$\left\{\ln\left(-i\tau^2/2\right)\left(1-s\tau^2/4\right)+\psi\left(1/2-is/2\right)-2\psi\left(1\right)+\left(s\tau^2/4\right)\left(-\psi\left(1/2-is/2\right)+2\psi\left(1\right)+2\right)+O\left(\tau^4\ln\tau\right)\right\}$$

$$-e^{-i\pi/4}\frac{i}{\Gamma\left(1/2-is/2\right)}$$

$$\left\{\ln\left(i\tau^2/2\right)\left(1-s\tau^2/4\right)+\psi\left(1/2+is/2\right)-2\psi\left(1\right)+\left(s\tau^2/4\right)\left(-\psi\left(1/2+is/2\right)+2\psi\left(1\right)+2\right)+O\left(\tau^4\ln\tau\right)\right\}\right]$$

$$\sim c_0\operatorname{Re}\left[e^{-i\pi/4}\frac{-i}{\Gamma\left(1/2-is/2\right)}\right.$$

$$\left\{i\pi\left(1-s\tau^2/4\right)+\left(\psi\left(1/2+is/2\right)-\psi\left(1/2-is/2\right)\right)+\left(s\tau^2/4\right)\left(\psi\left(1/2-is/2\right)-\psi\left(1/2+is/2\right)\right)+O\left(\tau^4\ln\tau\right)\right\}\right]$$

$$\sim c_0\operatorname{Re}\left[\frac{e^{-i\pi/4}}{\Gamma\left(1/2-is/2\right)}\right]\left[\pi-i\left\{\psi\left(1/2+is/2\right)-\psi\left(1/2-is/2\right)\right\}\right]\left(1-s\tau^2/4\right)+O\left(\tau^4\ln\tau\right)$$

Then

$$\psi_0\left(\tau,s\right)\sim c_0\operatorname{Re}\left[\frac{e^{-i\pi/4}}{\Gamma\left(1/2-is/2\right)}\right]\pi\left[1+\tanh\left(\pi s/2\right)\right]\left(1-s\tau^2/4\right)+O\left(\tau^4\ln\tau\right)$$

$$\sim c_0\operatorname{Re}\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\cosh\pi\left(s/2\right)\right]\left[1+\tanh\left(\pi s/2\right)\right]\left(1-s\tau^2/4\right)+O\left(\tau^4\ln\tau\right)$$

$$\sim c_0\operatorname{Re}\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]e^{\pi s/2}\left(1-s\tau^2/4\right)+O\left(\tau^4\ln\tau\right)$$

$$\tau\frac{\partial}{\partial\tau}\psi_0\left(\tau\right)\sim -c_0\operatorname{Re}\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]e^{\pi s/2}\left(s\tau^2/2\right)+O\left(\tau^4\ln\tau\right)\sim-\left(s\tau^2/2\right)\psi_0$$

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\psi_0\left(\tau\right)\sim -c_0\operatorname{Re}\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]e^{\pi s/2}\left(s/2\right)+O\left(\tau^4\ln\tau\right)\sim-\frac{s}{2}\psi_0$$

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial}{\partial\tau}\psi_0\right)\sim -c_0\operatorname{Re}\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]e^{\pi s/2}s+O\left(\tau^2\ln\tau\right)\sim-s\psi_0+O\left(\tau^2\ln\tau\right)$$

In addition

$$\tau\frac{\partial\psi_0}{\partial\tau}(\tau,s)=c_0\sqrt{2}\,\mathrm{Re}\left[e^{-i\pi/2}\left(W_+'-\frac{1}{\tau}W_+\right)+e^{i\Phi_0}e^{i\pi/2}\left(W_+'-\frac{1}{\tau}W_+\right)^*\right]$$

$$\tau\frac{\partial^2\psi_0}{\partial\omega\partial\tau}(\tau,s)\approx c_0\sqrt{2}\,\mathrm{Re}\left[i\frac{\partial\Phi_0}{\partial\omega}e^{i\Phi_0}e^{i\pi/2}\left(W_+'-\frac{1}{\tau}W_+\right)^*\right]$$

$$\approx -c_0\sqrt{2}\,\mathrm{Re}\left[i\frac{\partial\Phi_0}{\partial\omega}\frac{\Gamma(1/2+is/2)}{\Gamma(1/2-is/2)}\left(W_+'-\frac{1}{\tau}W_+\right)^*\right]$$

and

$$\tau\frac{\partial^2\psi_0}{\partial\omega\partial\tau}(0,s)\approx 2c_0\,\mathrm{Re}\left[\frac{e^{-i\pi/4}}{\Gamma(1/2-is/2)}\right]\frac{\partial\Phi_0}{\partial\omega}$$

The Region 2 behavior of the zero mode

$$\psi_0(\tau',s')=c_0\frac{\sqrt{2}}{\tau'}\,\mathrm{Re}\left[W_+(\tau',0,s')+e^{i\Phi_0'}W_+^*(\tau',0,s')\right]$$

is the same as the previous bowtie sections but with primed arguments, where

$$\tau'=\sqrt{2\gamma}\xi'$$

$$\xi'=\pm\pi/2-\xi$$

Again taking

$$\tau'\frac{\partial\psi_0}{\partial\tau'}=c_0\sqrt{2}\,\mathrm{Re}\left[W_+'+e^{i\Phi_0'}W_+^{*\prime}\right]-\psi_0$$

to vanish as $\tau'\to 0$ gives

$$e^{i\Phi_0'}=-\frac{\left(W_+'-\frac{1}{\tau'}W_+\right)}{\left(W_+'-\frac{1}{\tau'}W_+\right)^*}=i\frac{\Gamma(1/2+is'/2)}{\Gamma(1/2-is'/2)}+O\left(\tau'^2\right)$$

Thus near $\tau'\to 0$ we find

$$\psi_0(0,s')=c_0\,\mathrm{Re}\left[e^{\pi s'/2}e^{i\pi/4}\Gamma(1/2+is'/2)\right] \tag{430}$$

$$\psi_0(\tau',s')\sim c_0\,\mathrm{Re}\left[e^{i\pi/4}\Gamma(1/2+is'/2)\right]e^{\pi s'/2}\left(1-s'\tau'^2/4\right)+O\left(\tau'^4\ln\tau'\right)$$

$$\tau'\frac{\partial}{\partial\tau'}\psi_0(\tau',s')\sim -c_0\,\mathrm{Re}\left[e^{i\pi/4}\Gamma(1/2+is'/2)\right]e^{\pi s'/2}\left(s'\tau'^2/2\right)+O\left(\tau'^4\ln\tau'\right)\sim-\left(s'\tau'^2/2\right)\psi_0$$

$$\frac{1}{\tau'}\frac{\partial}{\partial\tau'}\psi_0\left(\tau',s'\right)\sim -c_0\,\mathrm{Re}\left[e^{i\pi/4}\Gamma\left(1/2+is'/2\right)\right]e^{\pi s'/2}\left(s'/2\right)+O\left(\tau'^4\ln\tau'\right)\sim -\frac{s'}{2}\psi_0$$

$$\frac{1}{\tau'}\frac{\partial}{\partial\tau'}\left(\tau'\frac{\partial}{\partial\tau'}\psi_0\right)\sim -c_0\,\mathrm{Re}\left[e^{i\pi/4}\Gamma\left(1/2+is'/2\right)\right]e^{\pi s'/2}s'+O\left(\tau'^2\ln\tau'\right)\sim -s'\psi_0+O\left(\tau'^2\ln\tau'\right)$$

In addition

$$\tau'\frac{\partial\psi_0}{\partial\tau'}\left(\tau',s'\right)=c_0\sqrt{2}\,\mathrm{Re}\left[\left(W_+'-\frac{1}{\tau}W_+\right)+e^{i\Phi_0'}\left(W_+'-\frac{1}{\tau}W_+\right)^*\right]$$

$$\tau'\frac{\partial^2\psi_0}{\partial\omega\partial\tau'}\left(\tau',s'\right)\approx c_0\sqrt{2}\,\mathrm{Re}\left[i\frac{\partial\Phi_0'}{\partial\omega}e^{i\Phi_0'}\left(W_+'-\frac{1}{\tau'}W_+\right)^*\right]$$

$$\approx -c_0\sqrt{2}\,\mathrm{Re}\left[\frac{\partial\Phi_0'}{\partial\omega}\frac{\Gamma\left(1/2+is'/2\right)}{\Gamma\left(1/2-is'/2\right)}\left(W_+'-\frac{1}{\tau'}W_+\right)^*\right]$$

and

$$\tau'\frac{\partial^2\psi_0}{\partial\omega\partial\tau'}\left(0,s'\right)\approx 2c_0\,\mathrm{Re}\left[\frac{e^{i\pi/4}}{\Gamma\left(1/2-is'/2\right)}\right]\frac{\partial\Phi_0'}{\partial\omega}$$

5.2.7 Approach Of Focal Point

We first take the limits of the outer two regions as the focal region is approached. However to allow flexibility in phase matching at the focus, we will allow the focal point to shift by a small amount in the Region 1 and Region 2 solutions by adding a small z shift relative to the Region 3 solution. This can also be viewed as distorting the the coordinate with $d\to d+\delta$, which corresponds to a slight reduction of the prolate spheroidal radius of curvature on axis, since $d=\ell\sqrt{1-R/\ell}$; because the prolate spheroid has increasing radii of curvature off axis, such a reduction on axis might actually represent the stadium mirrors better at a finite wavelength (this also explains why this shift is taken to be positive since R shrinks on axis). Therefore we let

$$d\to d+\delta \tag{431}$$

$$\gamma\to\gamma+k\delta \tag{432}$$

where $\delta << d$ is a geometrically small shift. Thus in Region 1 we find

$$u\sim 2Cc_m\frac{1}{\sqrt{\gamma}\zeta}\,\mathrm{Re}\left[e^{-i\pi/2}W_+\left(\sqrt{2\gamma}\zeta,m,s\right)+e^{i\pi/2}e^{i\Phi_0}W_+^*\left(\sqrt{2\gamma}\zeta,m,s\right)\right]$$

$$\xi'^{-1}\cos\left[\gamma\left(1-\xi'^2/2\right)+k\delta+s\ln\left(\xi'/2\right)\right]\left\{\begin{array}{c}\cos\left(m\varphi\right)\\ \sin\left(m\varphi\right)\end{array}\right\}\;,\;\xi'\to 0$$

In Region 2 we find

$$u \sim 2c_m \frac{1}{\sqrt{\gamma}\xi'} \operatorname{Re}\left[W_+\left(\sqrt{2\gamma}\xi', m, s'\right) + e^{i\Phi_0'} W_+^*\left(\sqrt{2\gamma}\xi', m, s'\right)\right]$$

$$\zeta^{-1} \cos\left[\gamma\left(1+\zeta^2/2\right) + k\delta - s'\ln(\zeta/2) - \pi/4 - \Phi_1/2\right] \left\{ \begin{array}{c} \cos(m\varphi) \\ \sin(m\varphi) \end{array} \right\}, \ \zeta \to 0$$

where

$$s' \ln\left[\tanh\left(\zeta_0/2\right)\right] = \left(k - k_p\right)\ell$$

$$k_p\ell - \pi/2 - \Phi_1/2 = \pi\left(p - 1/2\right)$$

Small Shift In Focal Position We can also view this shift as a change in the original coordinate system (r, z) or (ζ, ξ) or $\left(\zeta, \xi'\right)$

$$r = d\sinh\zeta\cos\xi = d\sinh\zeta\sin\xi' \sim d\zeta\xi'$$

$$z = d\cosh\zeta\sin\xi = d\cosh\zeta\cos\xi' \sim d\left(1 + \zeta^2/2 - \xi'^2/2\right)$$

to the focal coordinate system (r', z') or $\left(\widehat{\zeta}, \widehat{\xi}'\right)$ with

$$z - \delta = z' = d\cosh\widehat{\zeta}\cos\widehat{\xi}' \sim d\left(1 + \widehat{\zeta}^2/2 - \widehat{\xi}'^2/2\right)$$

$$r = r' = d\sinh\widehat{\zeta}\sin\widehat{\xi}' \sim d\widehat{\zeta}\widehat{\xi}'$$

Thus the small shift δ enters as an additive correction as in the preceding section.

5.2.8 Focal Region Three

Near the focus $\xi = \pi/2$ and $\zeta = 0$ we approximate the Helmholtz equation

$$\frac{1}{\cos\xi}\frac{\partial}{\partial\xi}\left(\cos\xi\frac{\partial u}{\partial\xi}\right) + \left(\frac{1}{\cos^2\xi} + \frac{1}{\sinh^2\zeta}\right)\frac{\partial^2 u}{\partial\varphi^2}$$

$$+\frac{1}{\sinh\zeta}\frac{\partial}{\partial\zeta}\left(\sinh\zeta\frac{\partial u}{\partial\zeta}\right) + \gamma^2\left(\cosh^2\zeta - \sin^2\xi\right)u = 0$$

or

$$\frac{1}{\cos\xi}\frac{\partial}{\partial\xi}\left(\cos\xi\frac{\partial u}{\partial\xi}\right) + \left(\frac{1}{\cos^2\xi} + \frac{1}{\sinh^2\zeta}\right)\frac{\partial^2 u}{\partial\varphi^2}$$

$$+\frac{1}{\sinh\zeta}\frac{\partial}{\partial\zeta}\left(\sinh\zeta\frac{\partial u}{\partial\zeta}\right) + \gamma^2\left(\sinh^2\zeta + \cos^2\xi\right)u = 0$$

or

$$\frac{1}{\sin\xi'}\frac{\partial}{\partial\xi'}\left(\sin\xi'\frac{\partial u}{\partial\xi'}\right)+\left(\frac{1}{\sin^2\xi'}+\frac{1}{\sinh^2\zeta}\right)\frac{\partial^2 u}{\partial\varphi^2}$$

$$+\frac{1}{\sinh\zeta}\frac{\partial}{\partial\zeta}\left(\sinh\zeta\frac{\partial u}{\partial\zeta}\right)+\gamma^2\left(\sinh^2\zeta+\sin^2\xi'\right)u=0$$

as $\xi'=\pi/2-\xi$ (we are assuming that $\zeta^2<<1$ and that $\xi'^2<<1$)

$$\frac{1}{\xi'}\frac{\partial}{\partial\xi'}\left(\xi'\frac{\partial u}{\partial\xi'}\right)+\left(\frac{1}{\xi'^2}+\frac{1}{\zeta^2}\right)\frac{\partial^2 u}{\partial\varphi^2}$$

$$+\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial u}{\partial\zeta}\right)+\gamma^2\left(\zeta^2+\xi'^2\right)u=0 \tag{433}$$

or taking

$$u=u_m\left(\zeta,\xi'\right)\left\{\begin{array}{c}\cos\left(m\varphi\right)\\ \sin\left(m\varphi\right)\end{array}\right\} \tag{434}$$

$$\frac{1}{\xi'}\frac{\partial}{\partial\xi'}\left(\xi'\frac{\partial u_m}{\partial\xi'}\right)+\left(\gamma^2\xi'^2-m^2/\xi'^2\right)u_m$$

$$+\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial u_m}{\partial\zeta}\right)+\left(\gamma^2\zeta^2-m^2/\zeta^2\right)u_m=0$$

Separating variables

$$u_m=R_m\left(\zeta\right)Z_m\left(\xi'\right) \tag{435}$$

$$\left[\frac{1}{Z_m}\frac{1}{\xi'}\frac{\partial}{\partial\xi'}\left(\xi'\frac{\partial Z_m}{\partial\xi'}\right)+\gamma^2\xi'^2-m^2/\xi'^2\right]$$

$$+\left[\frac{1}{R_m}\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial R_m}{\partial\zeta}\right)+\gamma^2\zeta^2-m^2/\zeta^2\right]=0$$

Taking the separation constant to be $2\gamma g$ we find

$$\frac{1}{\xi'}\frac{\partial}{\partial\xi'}\left(\xi'\frac{\partial Z_m}{\partial\xi'}\right)+\left(-2\gamma g+\gamma^2\xi'^2-m^2/\xi'^2\right)Z_m=0$$

$$\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial R_m}{\partial\zeta}\right)+\left(2\gamma g+\gamma^2\zeta^2-m^2/\zeta^2\right)R_m=0$$

Thus we find Whittaker equations in both directions. Letting

$$\xi'\sqrt{2\gamma}=\tau'$$

in the first and

$$\zeta\sqrt{2\gamma} = \tau$$

in the second, gives

$$\frac{1}{\tau'}\frac{\partial}{\partial\tau'}\left(\tau'\frac{\partial Z_m}{\partial\tau'}\right) + \left(-g + \frac{1}{4}\tau'^2 - \frac{m^2}{\tau'^2}\right) Z_m = 0$$

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial R_m}{\partial\tau}\right) + \left(g + \frac{1}{4}\tau^2 - \frac{m^2}{\tau^2}\right) R_m = 0$$

Therefore we take as the solution

$$u_m = X_m Z_m = C_0 c_m \frac{\sqrt{2}}{\tau'} \operatorname{Re}\left[W_+(\tau', m, -g) + B' W_+^*(\tau', m, -g)\right]$$

$$\frac{\sqrt{2}}{\tau} \operatorname{Re}\left[W_+(\tau, m, g) + B W_+^*(\tau, m, g)\right]$$

$$= C_0 c_m \frac{1}{\xi'\sqrt{\gamma}} \operatorname{Re}\left[W_+\left(\xi'\sqrt{2\gamma}, m, -g\right) + B' W_+^*\left(\xi'\sqrt{2\gamma}, m, -g\right)\right]$$

$$\frac{1}{\zeta\sqrt{\gamma}} \operatorname{Re}\left[e^{-i\pi/2} W_+\left(\zeta\sqrt{2\gamma}, m, g\right) + e^{i\pi/2} B W_+^*\left(\zeta\sqrt{2\gamma}, m, g\right)\right]$$

For purposes of matching with the other two regions we take $g = s = -s'$ and $B = e^{i\Phi_0}$ and $B' = e^{i\Phi_0'}$

$$u = C_0 c_m \frac{1}{\xi'\sqrt{\gamma}} \operatorname{Re}\left[W_+\left(\xi'\sqrt{2\gamma}, m, s'\right) + e^{i\Phi_0'} W_+^*\left(\xi'\sqrt{2\gamma}, m, s'\right)\right]$$

$$\frac{1}{\zeta\sqrt{\gamma}} \operatorname{Re}\left[e^{-i\pi/2} W_+\left(\zeta\sqrt{2\gamma}, m, -s'\right) + e^{i\pi/2} e^{i\Phi_0} W_+^*\left(\zeta\sqrt{2\gamma}, m, -s'\right)\right] \left\{\begin{array}{c} \cos(m\varphi) \\ \sin(m\varphi) \end{array}\right\} \tag{436}$$

Expanding as we leave the focal region using the asymptotic form

$$W_+(\tau, m, s) \sim e^{i\tau^2/4}\left(-i\tau^2/2\right)^{is/2} = e^{i\tau^2/4 + s\pi/4 + is\ln\left(\tau/\sqrt{2}\right)}\ ,\ \tau \to \infty$$

gives

$$u \sim 2C_0 e^{s'\pi/4} \cos(\Phi_0'/2)\, c_m \frac{1}{\zeta\sqrt{\gamma}} \operatorname{Re}\left[e^{-i\pi/2} W_+\left(\zeta\sqrt{2\gamma}, m, -s'\right) + e^{i\pi/2} e^{i\Phi_0} W_+^*\left(\zeta\sqrt{2\gamma}, m, -s'\right)\right]$$

$$\frac{1}{\xi'\sqrt{\gamma}} \cos\left[\xi'^2\gamma/2 + s'\ln\left(\xi'\sqrt{\gamma}\right) - \Phi_0'/2\right] \left\{\begin{array}{c} \cos(m\varphi) \\ \sin(m\varphi) \end{array}\right\},\ \text{Region } 3 \to 1$$

$$u \sim 2C_0 e^{-s'\pi/4} \cos(\Phi_0/2)\, c_m \frac{1}{\xi'\sqrt{\gamma}} \operatorname{Re}\left[W_+\left(\xi'\sqrt{2\gamma}, m, s'\right) + e^{i\Phi_0'} W_+^*\left(\xi'\sqrt{2\gamma}, m, s'\right)\right]$$

$$\frac{1}{\zeta\sqrt{\gamma}} \cos\left[\zeta^2\gamma/2 - s'\ln(\zeta\sqrt{\gamma}) - \pi/2 - \Phi_0/2\right] \left\{\begin{array}{c} \cos(m\varphi) \\ \sin(m\varphi) \end{array}\right\},\ \text{Region } 3 \to 2$$

These must match to the limiting forms of the outer solutions from the preceding sections

$$u = 2C\frac{\psi_m(\tau,s)}{\sin\xi'}\cos\left[\gamma\cos\xi' + s\ln\left\{\tan\left(\xi'/2\right)\right\}\right]\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\},\ \xi' = \pi/2-\xi$$

$$\psi_m(\tau,s) = c_m\frac{\sqrt{2}}{\tau}\,\mathrm{Re}\left[e^{-i\pi/2}W_+(\tau,m,s) + e^{i\pi/2}e^{i\Phi_0}W_+^*(\tau,m,s)\right]$$

or

$$u \sim 2C\frac{1}{\xi'}c_m\frac{1}{\zeta\sqrt{\gamma}}\,\mathrm{Re}\left[e^{-i\pi/2}W_+\left(\zeta\sqrt{2\gamma},m,s\right) + e^{i\pi/2}e^{i\Phi_0}W_+^*\left(\zeta\sqrt{2\gamma},m,s\right)\right]$$

$$\cos\left[\gamma\left(1-\xi'^2/2\right)+k\delta+s\ln\left(\xi'/2\right)\right]\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\},\ \xi'\to 0$$

in Region 1, and to

$$u = \frac{2}{\sinh\zeta}\psi_m(\tau',s')\cos\left[\gamma\cosh\zeta - s'\ln\{\tanh(\zeta/2)\} - \pi/2 - \Phi_1/2\right]\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}$$

$$\psi_m(\tau',s') = c_m\frac{\sqrt{2}}{\tau'}\,\mathrm{Re}\left[W_+(\tau',m,s') + e^{i\Phi_0'}W_+^*(\tau',m,s')\right]$$

or

$$u \sim \frac{2}{\zeta}c_m\frac{1}{\xi'\sqrt{\gamma}}\,\mathrm{Re}\left[W_+\left(\xi'\sqrt{2\gamma},m,s'\right) + e^{i\Phi_0'}W_+^*\left(\xi'\sqrt{2\gamma},m,s'\right)\right]$$

$$\cos\left[\gamma\left(1+\zeta^2/2\right)+k\delta-s'\ln(\zeta/2)-\pi/2-\Phi_1/2\right]\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\},\ \zeta\to 0$$

in Region 2, where

$$s'\ln\left[\tanh\left(\zeta_0/2\right)\right] = (k-k_p)\,\ell$$

$$k_p\ell - \pi/2 - \Phi_1/2 = \pi\,(p-1/2)$$

The functional behaviors are identical, but the phases only match if

$$s'\ln(2\sqrt{\gamma}) - \Phi_0'/2 = -\gamma - k\delta + n\pi \tag{437}$$

$$-s'\ln(2\sqrt{\gamma}) - \pi/2 - \Phi_0/2 = \gamma + k\delta - \pi/2 - \Phi_1/2 + n'\pi \tag{438}$$

and the amplitudes match if

$$C_0 e^{s'\pi/4}\cos\left(\Phi_0'/2\right)\frac{1}{\sqrt{\gamma}} = C\,(-1)^n \tag{439}$$

$$C_0 e^{-s'\pi/4}\cos\left(\Phi_0/2\right)\frac{1}{\sqrt{\gamma}} = (-1)^{n'} \tag{440}$$

5.2.9 Evenness Conditions On Scar

Because we want the normal derivative to vanish as the scarred orbit is approached

$$\lim_{\tau'\to 0}\left\{\mathrm{Re}\left[W'_+\left(\tau',m,s'\right)+e^{i\Phi'_0}W^{*\prime}_+\left(\tau,m,s'\right)\right]-\frac{1}{\tau'}\mathrm{Re}\left[W_+\left(\tau',m,s'\right)+e^{i\Phi'_0}W^*_+\left(\tau',m,s'\right)\right]\right\}\to 0$$

$$\lim_{\tau\to 0}\left\{\mathrm{Re}\left[e^{-i\pi/2}W'_+\left(\tau,m,s\right)+e^{i\pi/2}e^{i\Phi_0}W^{*\prime}_+\left(\tau,m,s\right)\right]-\frac{1}{\tau}\mathrm{Re}\left[e^{-i\pi/2}W_+\left(\tau,m,s\right)+e^{i\pi/2}e^{i\Phi_0}W^*_+\left(\tau,m,s\right)\right]\right\}\to 0$$

where the first is the Region 2 form and the second is the Region 1 form. If we write the real part as one half the sum of the function and its conjugate, we see that these conditions imply

$$e^{i\Phi'_0} = -\lim_{\tau'\to 0}\frac{\left[W'_+\left(\tau',m,s'\right)-\frac{1}{\tau'}W_+\left(\tau',m,s'\right)\right]}{\left[W'_+\left(\tau',m,s'\right)-\frac{1}{\tau'}W_+\left(\tau',m,s'\right)\right]^*}$$

and

$$e^{i\Phi_0} = \lim_{\tau\to 0}\frac{\left[W'_+\left(\tau,m,s\right)-\frac{1}{\tau}W_+\left(\tau,m,s\right)\right]}{\left[W'_+\left(\tau,m,s\right)-\frac{1}{\tau}W_+\left(\tau,m,s\right)\right]^*}$$

Using the properties of the Whittaker functions and $s=-s'$ we can evaluate these. The properties of the functions we desire are

$$W_+\left(\tau,m,s\right)\sim\left(-i\tau^2/2\right)^{1/2-m/2}\frac{(m-1)!}{\Gamma\left(1/2+m/2-is/2\right)}\ ,\ \tau\to 0\ ,\ m\neq 0$$

$$\sim -\left(-i\tau^2/2\right)^{1/2}\frac{1}{\Gamma\left(1/2-is/2\right)}\left[\ln\left(-i\tau^2/2\right)\right.$$

$$\left.+\left\{\psi\left(1/2-is/2\right)-2\psi\left(1\right)\right\}\right]\ ,\ \tau\to 0\ ,\ m=0$$

and

$$W'_+\left(\tau,m,s\right)\sim -i\tau\left(-i\tau^2/2\right)^{-1/2-m/2}\frac{\left(1/2-m/2\right)(m-1)!}{\Gamma\left(1/2+m/2-is/2\right)}\ ,\ \tau\to 0\ ,\ m\neq 0$$

$$\sim i\tau\frac{1}{2}\left(-i\tau^2/2\right)^{-1/2}\frac{1}{\Gamma\left(1/2-is/2\right)}\left[\ln\left(-i\tau^2/2\right)+\left\{\psi\left(1/2-is/2\right)-2\psi\left(1\right)\right\}+2\right]\ ,\ \tau\to 0\ ,\ m=0$$

and thus (this also works for $m=0$)

$$e^{i\Phi_0} = e^{i\pi(m-1)/2}\frac{\Gamma\left(1/2+m/2+is/2\right)}{\Gamma\left(1/2+m/2-is/2\right)}$$

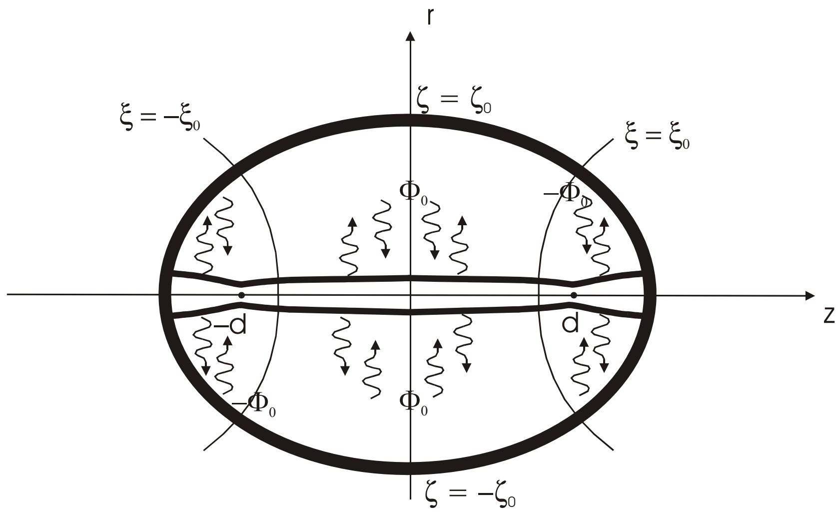

Figure 32. Illustration depicting the reflection of the waves of the scarred orbit from the outer chaotic region, with an inversion of the reflection phase on the other side of the focal point.

and

$$e^{i\Phi_0'} = -e^{i\pi(m-1)/2}\frac{\Gamma\left(1/2 + m/2 + is'/2\right)}{\Gamma\left(1/2 + m/2 - is'/2\right)}$$

or

$$e^{-i\Phi_0'} = -e^{-i\pi(m-1)/2}\frac{\Gamma\left(1/2 + m/2 + is/2\right)}{\Gamma\left(1/2 + m/2 - is/2\right)} = -e^{i\Phi_0 - i(m-1)\pi} = (-1)^m\, e^{i\Phi_0}$$

Thus

$$\Phi_0' = -\Phi_0 + m\pi \tag{441}$$

The single remaining condition then determines $s' = -s$ given a value of the reflection phase Φ_0. Note that this choice of reflection phase conjugate implies that the incoming wave from the outer region travels toward the scarred orbit in one region, but on the other side of the focus travels away from the scarred orbit as illustrated in Figure 32. This construction of the transverse dependence has thus allowed a consistent solution between the two regions to be found.

5.2.10 Summary Of Conditions

The summary of conditions is now given. The first is the vanishing of the normal derivative as the orbit is approached

$$e^{i\Phi_0} = e^{i\pi(m-1)/2}\frac{\Gamma\left(1/2 + m/2 + is/2\right)}{\Gamma\left(1/2 + m/2 - is/2\right)}$$

$$e^{i\Phi_0'} = -e^{i\pi(m-1)/2}\frac{\Gamma\left(1/2 + m/2 + is'/2\right)}{\Gamma\left(1/2 + m/2 - is'/2\right)}$$

or

$$\Phi_0' = -\Phi_0 + m\pi$$

determines s' given Φ_0 or vice-versa

$$e^{-i\Phi_0} = i\frac{\Gamma\left(1/2 + m/2 + is'/2\right)}{\Gamma\left(1/2 + m/2 - is'/2\right)} e^{-i\pi m/2}$$

From the mirror condition

$$\Phi_1/2 = k_p \ell - \pi p$$

where k_p and s' are related by

$$s' \ln\left[\tanh\left(\zeta_0/2\right)\right] = \left(k - k_p\right)\ell$$

This can also be written in terms of the stability exponents as

$$-s' = s = 2\left(k - k_p\right)\ell/\ln\left(\frac{\ell + d}{\ell - d}\right) = 2\left(k - k_p\right) L/\ln\left(\Lambda_+\right)$$

where

$$\sigma_0' = \ln\left[\tanh\left(\zeta_0/2\right)\right] = -\frac{1}{2}\ln\left(\frac{\ell + d}{\ell - d}\right) = -\frac{1}{2}\ln\left(\frac{\lambda_+ + 1}{\lambda_- + 1}\right) = -\frac{1}{2}\ln\left(\lambda_+\right) = -\frac{1}{4}\ln\left(\Lambda_+\right)$$

and

$$\sqrt{d/\ell} = \frac{\sqrt{\Lambda_+ - 1}}{\left(\sqrt{\Lambda_+} + 1\right)}$$

and where $\lambda_\pm^2 = \Lambda_\pm$, $\lambda_+\lambda_- = 1 = \Lambda_+\Lambda_-$ and $\left(\lambda_\pm + 1\right)/2 = \left(\ell \pm d\right)/R$. The phase matching conditions can be written as

$$\Phi_1/2 - \Phi_0'/2 - \Phi_0/2 = \Phi_1/2 - m\pi/2 = \pi\left(n + n'\right)$$

$$k\delta = -\gamma - s' \ln\left(2\sqrt{\gamma}\right) + \Phi_0'/2 + n\pi = -\gamma - s' \ln\left(2\sqrt{\gamma}\right) - \Phi_0/2 + n\pi + m\pi/2$$

or

$$\Phi_1/2 = \pi\left(n + n'\right) + m\pi/2$$

$$k\delta = -\gamma - s' \ln\left(2\sqrt{\gamma}\right) - \Phi_0/2 + n\pi + m\pi/2$$

and the amplitude conditions give the coefficients as

$$C = \left(-1\right)^{n' - n + m} e^{s'\pi/2}$$

$$C_0 = \left(-1\right)^{n'} e^{s'\pi/4}\sqrt{\gamma}\sec\left(\Phi_0/2\right)$$

It appears like the phase $\Phi_1/2$ adds something for odd values of m since it must be a multiple of $\pi/2$ and sign changes in the Region 2 solution due to this phase are accompanied by sign changes in C and in the Region 1 solution (which thus can be absorbed into the amplitude coefficients). Furthermore the factor $\sec(\Phi_0/2)$ only enters because we failed to set the problem up with symmetrical factors $\exp(\pm\Phi_0/2)$ in the combinations of Whittaker functions.

5.2.11 Final Set Of Conditions For Axisymmetric Mode

Thus for the axisymmetric mode $m = 0$ if we set Φ_1 to zero we have the evenness condition across the scar orbit to determine the allowed values of the separation constant s' in terms of the chaotic phase Φ_0

$$e^{-i\Phi_0} = i\frac{\Gamma(1/2 + is'/2)}{\Gamma(1/2 - is'/2)} = \cosh(\pi s'/2)\,\frac{i}{\pi}\Gamma^2(1/2 + is'/2) \tag{442}$$

We have the mirror conditions which connect the separation constant values and the resonant frequencies k

$$k_p \ell = \pi p \tag{443}$$

$$s' \ln[\tanh(\zeta_0/2)] = (k - k_p)\,\ell \tag{444}$$

We also have the focal point shift δ

$$k\delta = -\gamma - s' \ln(2\sqrt{\gamma}) - \Phi_0/2 + n\pi \tag{445}$$

and the amplitude constants (here we note that if n and n' are both even or odd C is one, but if they have opposite parities, then $C = -1$ which cancels the phase shift $\Phi_1/2$, then an odd multiple of π)

$$C = e^{s'\pi/2} \tag{446}$$

$$C_0 = (-1)^n\, e^{s'\pi/4}\sqrt{\gamma}\sec(\Phi_0/2) \tag{447}$$

To get a feel for the connection with Φ_0 for small s' we can expand as

$$e^{-i\Phi_0} = i\frac{\Gamma(1/2 + is'/2)}{\Gamma(1/2 - is'/2)} \sim i\frac{1 + (is'/2)\,\psi(1/2) + (is'/2)^2\left[\psi^2(1/2) + \psi'(1/2)\right]/2}{1 - (is'/2)\,\psi(1/2) + (is'/2)^2\left[\psi^2(1/2) + \psi'(1/2)\right]/2}$$

$$\sim i\frac{1 - (is'/2)(\gamma' + 2\ln 2) + (is'/2)^2\left\{(\gamma' + 2\ln 2)^2 + \pi^2/2\right\}/2}{1 + (is'/2)(\gamma' + 2\ln 2) + (is'/2)^2\left\{(\gamma' + 2\ln 2)^2 + \pi^2/2\right\}/2}$$

$$\sim i\left[1 - is'(\gamma' + 2\ln 2) - s'^2(\gamma' + 2\ln 2)^2/2\right]$$

where $\psi(z)$ is the digamma function [11] and $\gamma' \approx 0.5772$ is Euler's constant. Thus we have

$$\Phi_0 \to -\pi/2 \text{ as } s' \to 0 \tag{448}$$

5.2.12 Focal Shift In Axisymmetric Calculations

In the calculations of the focal point shift we use

$$k_p \ell = \pi p \tag{449}$$

$$s' = 2\left(k_p - k\right)\ell / \ln\left(\frac{\ell + d}{\ell - d}\right) \tag{450}$$

and the focal point shift δ

$$k\left(d + \delta\right) = -s' \ln\left(2\sqrt{kd}\right) - \left(\Phi_0/2 + \pi/4\right) + \left(n + 1/4\right)\pi \tag{451}$$

Note that for $s = 0$ and $k = k_p$, we have δ_0 where

$$k_p\left(d + \delta_0\right) = \left(n + 1/4\right)\pi$$

The transcendental equation for s' can be written as

$$e^{-i(\Phi_0 + \pi/2)} = \frac{\Gamma\left(is'/2 + 1/2\right)}{\Gamma\left(-is'/2 + 1/2\right)} \tag{452}$$

giving

$$k\left(d + \delta\right) = -s' \ln\left(2\sqrt{kd}\right) + \arg\Gamma\left(1/2 + is'/2\right) + \left(n + 1/4\right)\pi \tag{453}$$

Noting that

$$kd = \left(k - k_p\right)\ell\left(d/\ell\right) + k_p\ell\left(d/\ell\right) = p\pi\left(d/\ell\right) - \frac{1}{2}s'\left(d/\ell\right)\ln\left(\frac{\ell + d}{\ell - d}\right)$$

gives

$$k\delta = -s'\left\{\ln\left(2\sqrt{kd}\right) - \frac{1}{2}\left(d/\ell\right)\ln\left(\frac{\ell + d}{\ell - d}\right)\right\} + \arg\Gamma\left(1/2 + is'/2\right) + \left\{\left(n + 1/4\right) - p\left(d/\ell\right)\right\}\pi$$

or ignoring terms of order $\left(k - k_p\right)\delta$

$$k_p\delta \sim -s'\left\{\ln\left(2\sqrt{kd}\right) - \frac{1}{2}\left(d/\ell\right)\ln\left(\frac{\ell + d}{\ell - d}\right)\right\} + \arg\Gamma\left(1/2 + is'/2\right) + \left\{\left(n + 1/4\right) - p\left(d/\ell\right)\right\}\pi$$

or

$$k_p\left(d + \delta\right) \sim -s'\left\{\ln\left(2\sqrt{k_p d}\right) + \frac{1}{2}\left(k - k_p\right)/k_p - \frac{1}{2}\left(d/\ell\right)\ln\left(\frac{\ell + d}{\ell - d}\right)\right\} + \arg\Gamma\left(1/2 + is'/2\right) + \left(n + 1/4\right)\pi$$

or

$$\left(d + \delta\right)/\ell \sim \frac{1}{k_p\ell}\left[s'\left\{\frac{1}{2}\left(d/\ell\right)\ln\left(\frac{\ell + d}{\ell - d}\right) + \frac{1}{4}s'\ln\left(\frac{\ell + d}{\ell - d}\right)/\left(k_p\ell\right) - \ln\left(2\sqrt{k_p d}\right)\right\} + \arg\Gamma\left(1/2 + is'/2\right)\right]$$

$$+ (n + 1/4)/p$$

Dropping the quadratic term in $1/k_p^2$

$$(d+\delta)/\ell \sim \frac{1}{k_p \ell} \left[s' \left\{ \frac{1}{2} (d/\ell) \ln \left(\frac{\ell + d}{\ell - d} \right) - \ln \left(2\sqrt{k_p d} \right) \right\} + \arg \Gamma (1/2 + is'/2) \right] + (n + 1/4)/p$$

Expansion For Small Separation If the value of $s' \to 0$ we can expand the gamma function as

$$\Gamma (1/2 + is'/2) \sim \sqrt{\pi} \left[1 + \psi (1/2) is'/2 - \psi^2 (1/2) s'^2/8 - \psi' (1/2) s'^2/8 \right]$$

$$\sim \sqrt{\pi} \left[1 - (\gamma' + 2\ln 2) is'/2 - \left\{ (\gamma' + 2\ln 2)^2 + \pi^2/2 \right\} s'^2/8 \right]$$

and

$$\arg \Gamma (1/2 + is'/2) \sim - (\gamma' + 2\ln 2) s'/2 \ , \ s' \to 0$$

to find

$$k(d+\delta) = -s' \left[\ln \left(4\sqrt{kd} \right) + \gamma'/2 \right] + (n + 1/4)\pi$$

or

$$(d+\delta)/\ell \sim \frac{1}{k_p \ell} \left[s' \left\{ \frac{1}{2} (d/\ell) \ln \left(\frac{\ell + d}{\ell - d} \right) - \ln \left(4\sqrt{k_p d} \right) - \gamma'/2 \right\} \right] + (n + 1/4)/p$$

5.2.13 Average Focal Shift

The normalized shift function, without corrections ($s' = 0$) is

$$k_p (d + \delta_0) = (n + 1/4)\pi \tag{454}$$

and using

$$k_p \ell = p\pi \tag{455}$$

is

$$(d + \delta_0)/\ell = (n + 1/4)/p \tag{456}$$

This can be written as

$$k_p \delta_0/\pi = n - n_c \geq 0 \tag{457}$$

where

$$n_c = pd/\ell - 1/4$$

and the inequality displaces the focus to the right of the geometrical focal point (as we found in the 2D problem [2]). Now for for large values of p (with $d/\ell < 1$) and n, we assume that $n - n_c$ is between 0 and 1 (we might take it to be uniformly distributed) and examine the average

$$k_p \langle \delta_0 \rangle / \pi \approx 1/2 \tag{458}$$

For example with the frequency range

$$209.71 \text{ m}^{-1} \leq k \leq 628.7 \text{ m}^{-1}$$

with $\ell = 0.11176$ m, $d = 0.0336969$ m we find that

$$p_1 = 8 \leq p \leq 22 = p_2$$

Then we find

$$\langle \delta_0 \rangle / \ell = \frac{1}{p_2 - p_1} \int_{p_1}^{p_2} \frac{dp}{2p} = \frac{1/2}{p_2 - p_1} \ln (p_2/p_1) = 0.0361286$$

and

$$d_e = d + \langle \delta \rangle \approx d + \langle \delta_0 \rangle \approx 0.0377346 \text{ m} \tag{459}$$

5.3 Stadium Normalization For Axisymmetric Mode

The preceding acoustic case of scalar normalization gives the normalization condition

$$\oint_{S_{scar}} \left[\frac{\partial u}{\partial \omega} \left(\frac{1}{\omega} \frac{\partial u^*}{\partial n} \right) - u^* \frac{\partial}{\partial \omega} \left(\frac{1}{\omega} \frac{\partial u}{\partial n} \right) \right] dS$$

$$= 2 \frac{k^2}{\omega^2} \int_V |u|^2 \, dV \tag{460}$$

or for a real function with unit volume integral

$$\oint_{S_{scar}} \left[\frac{\partial u}{\partial \omega} \left(\frac{1}{\omega} \frac{\partial u}{\partial n} \right) - u \frac{\partial}{\partial \omega} \left(\frac{1}{\omega} \frac{\partial u}{\partial n} \right) \right] dS = 2 \frac{k^2}{\omega^2} \tag{461}$$

If we set

$$\frac{\partial u}{\partial n} = 0 \text{ , on } S_{scar}$$

then

$$2 \frac{k^2}{\omega} = - \int_{S_{scar}} u \frac{\partial^2 u}{\partial \omega \partial n} dS = \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} u \frac{\partial^2 u}{\partial \omega h_\zeta \partial \zeta} h_\varphi d\varphi h_\xi d\xi$$

$$+\left[\int_0^{\zeta_0}(\xi=\pi/2)+\int_0^{\zeta_0}(\xi=-\pi/2)\right]\int_0^{2\pi}u\frac{\partial^2 u}{\partial\omega h_\xi\partial\xi'}h_\varphi d\varphi h_\zeta d\zeta$$

$$\sim d\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}u\frac{\partial^2 u}{\partial\omega\partial\zeta}\zeta d\varphi\cos\xi d\xi=2d\int_0^{\pi/2}\int_0^{2\pi}u\frac{\partial^2 u}{\partial\omega\partial\tau}\tau d\varphi\cos\xi d\xi$$

$$+2d\int_0^{\zeta_0}\int_0^{2\pi}u\frac{\partial^2 u}{\partial\omega\partial\xi'}\xi' d\varphi\sinh\zeta d\zeta$$

$$\sim d\int_{-\xi_0}^{\xi_0}\int_0^{2\pi}u\frac{\partial^2 u}{\partial\omega\partial\zeta}\zeta d\varphi\cos\xi d\xi=2d\int_0^{\pi/2}\int_0^{2\pi}u\frac{\partial^2 u}{\partial\omega\partial\tau}\tau d\varphi\cos\xi d\xi$$

$$+2d\int_0^{\zeta_0}\int_0^{2\pi}u\frac{\partial^2 u}{\partial\omega\partial\tau'}\tau' d\varphi\sinh\zeta d\zeta$$

with metric coefficients

$$h_\zeta=h_\xi=d\sqrt{\sinh^2\zeta+\cos^2\xi}$$

$$h_\varphi=d\sinh\zeta\cos\xi$$

Thus

$$\frac{k^2}{\omega}=2\pi d\int_0^{\pi/2}u\frac{\partial^2 u}{\partial\omega\partial\tau}\tau\cos\xi d\xi+2\pi d\int_0^{\zeta_0}u\frac{\partial^2 u}{\partial\omega\partial\tau'}\tau'\sinh\zeta d\zeta \tag{462}$$

The solution in Region 1 is

$$u_p=2C\frac{\psi_0(\tau,s)}{\cos\xi}\cos(\gamma\sin\xi-s\sigma)$$

$$=2e^{s'\pi/2}\frac{\psi_0(\tau,s)}{\sin\xi'}\cos\left(\gamma\cos\xi'-s\sigma\right)\ ,\ \xi'=\pi/2-\xi \tag{463}$$

where

$$\psi_0(\tau,s)=c_0\frac{\sqrt{2}}{\tau}\operatorname{Re}\left[e^{-i\pi/2}W_+(\tau,0,s)+e^{i\pi/2}e^{i\Phi_0}W_+^*(\tau,0,s)\right] \tag{464}$$

$$\sigma=\int_0^{\xi}\frac{d\xi}{\cos\xi}=\operatorname{arcsinh}(\tan\xi)=\int_{\xi'}^{\pi/2}\frac{d\xi'}{\sin\xi'}=-\ln\left\{\tan\left(\xi'/2\right)\right\} \tag{465}$$

$$\tau=\sqrt{2\gamma}\zeta \tag{466}$$

$$W_+(\tau,0,s)=W_{is/2,0}\left(-i\tau^2/2\right) \tag{467}$$

$$k_p \ell = \pi p \tag{468}$$

The solution in Region 2 is

$$u_p = \frac{2}{\sinh \zeta} \psi_0 \left(\tau', s'\right) \cos \left(\gamma \cosh \zeta - s' \sigma' - \pi/2\right) \tag{469}$$

where

$$s' = -s \tag{470}$$

$$\psi_0 \left(\tau', s'\right) = c_0 \frac{\sqrt{2}}{\tau'} \operatorname{Re} \left[W_+ \left(\tau', 0, -s\right) + e^{-i\Phi_0} W_+^* \left(\tau', 0, -s\right)\right] \tag{471}$$

$$\tau' = \sqrt{2\gamma} \xi' \tag{472}$$

$$\xi' = \pm \pi/2 - \xi \tag{473}$$

$$2s' \sigma_0' = 2s' \ln \left[\tanh \left(\zeta_0/2\right)\right] = \left(k - k_p\right) L \tag{474}$$

$$\sigma' = \int_\infty^\zeta \frac{d\zeta}{\sinh \zeta} = \ln \left[\tanh \left(\zeta/2\right)\right] \tag{475}$$

The solution in Region 3 is

$$u_p = C_0 c_0 \frac{1}{\xi' \sqrt{\gamma}} \operatorname{Re} \left[W_+ \left(\xi' \sqrt{2\gamma}, 0, s'\right) + e^{-i\Phi_0} W_+^* \left(\xi' \sqrt{2\gamma}, 0, s'\right)\right]$$

$$\frac{1}{\zeta \sqrt{\gamma}} \operatorname{Re} \left[e^{-i\pi/2} W_+ \left(\zeta \sqrt{2\gamma}, 0, -s'\right) + e^{i\pi/2} e^{i\Phi_0} W_+^* \left(\zeta \sqrt{2\gamma}, 0, -s'\right)\right]$$

or

$$u_p = (-1)^n e^{s' \pi/4} \sqrt{\gamma} \sec \left(\Phi_0/2\right) c_0 \frac{1}{\xi' \sqrt{\gamma}} \operatorname{Re} \left[W_+ \left(\xi' \sqrt{2\gamma}, 0, s'\right) + e^{-i\Phi_0} W_+^* \left(\xi' \sqrt{2\gamma}, 0, s'\right)\right]$$

$$\frac{1}{\zeta \sqrt{\gamma}} \operatorname{Re} \left[e^{-i\pi/2} W_+ \left(\zeta \sqrt{2\gamma}, 0, -s'\right) + e^{i\pi/2} e^{i\Phi_0} W_+^* \left(\zeta \sqrt{2\gamma}, 0, -s'\right)\right] \tag{476}$$

Here we apply the normalization with only Region 1 and Region 2 solutions separated by the coordinate value $\xi = \pi/2$ ($\xi' = 0$) and $\cosh \zeta_0 = \ell/d$. The next subsection discusses the Region 3 contribution, where the principal value interpretation used here is shown to be justified (the next subsection is restricted to the $s = 0$ limit for simplicity). To carry out the integration we introduce a slight displacement Δ on either side of the focus, so that the integration range is $\xi = 0$ to

$$\xi = \mathrm{Arcsin}\,(1-\Delta/d) \approx \frac{\pi}{2} - \sqrt{2\Delta/d} \tag{477}$$

and from ζ equal to

$$\zeta = \mathrm{Arccosh}\,(1+\Delta/d) \approx \sqrt{2\Delta/d} \tag{478}$$

to ζ_0. Thus the integral is

$$\frac{k^2}{\omega} = 2\pi d \int_0^{\pi/2-\sqrt{2\Delta/d}} u \frac{\partial^2 u}{\partial\omega\partial\tau}\tau\cos\xi d\xi$$

$$+2\pi d \int_{\sqrt{2\Delta/d}}^{\zeta_0} u \frac{\partial^2 u}{\partial\omega\partial\tau'}\tau'\sinh\zeta d\zeta \tag{479}$$

Then in Region 1 we have

$$\frac{\partial^2 u}{\partial\omega\partial\tau}\tau = -2e^{s'\pi/2}\frac{1}{\cos\xi}\cos\left(\gamma\sin\xi - s\sigma\right)c_0\sqrt{2}\frac{\partial\Phi_0}{\partial\omega}\,\mathrm{Re}\left[e^{i\Phi_0}\left\{W_+'\left(\tau,0,s\right) - \frac{1}{\tau}W_+\left(\tau,0,s\right)\right\}^*\right]$$

$$e^{i\Phi_0} = -i\frac{\Gamma\left(1/2+is/2\right)}{\Gamma\left(1/2-is/2\right)}$$

$$\psi_0\left(0,s\right) = c_0\,\mathrm{Re}\left[e^{-i\pi/4}e^{\pi s/2}\Gamma\left(1/2+is/2\right)\right]$$

Thus

$$\frac{\partial^2 u}{\partial\omega\partial\tau}\tau = 4c_0e^{s'\pi/2}\frac{1}{\cos\xi}\cos\left(\gamma\sin\xi - s\sigma\right)\frac{\partial\Phi_0}{\partial\omega}\,\mathrm{Re}\left[\frac{e^{-i\pi/4}}{\Gamma\left(1/2-is/2\right)}\right]$$

$$u = 2e^{s'\pi/2}\frac{\psi_0\left(\tau,s\right)}{\cos\xi}\cos\left(\gamma\cos\xi' - s\sigma\right)$$

Now taking the limit as $\tau \to 0$

$$u\frac{\partial^2 u}{\partial\omega\partial\tau}\tau\cos\xi = \frac{8}{\pi}e^{s'\pi/2}\frac{c_0^2}{\cos\xi}\cos^2\left(\gamma\sin\xi - s\sigma\right)\frac{\partial\Phi_0}{\partial\omega}\cosh\left(\pi s/2\right)\mathrm{Re}^2\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]$$

$$= \frac{8}{\pi}e^{s'\pi/2}\frac{c_0^2}{\cos\xi}\cos^2\left(\gamma\sin\xi - s\sigma\right)\frac{\partial k^2}{\partial\omega}\frac{\partial\Phi_0}{\partial k^2}\cosh\left(\pi s/2\right)\mathrm{Re}^2\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]$$

$$= \frac{16}{\pi}\frac{k^2}{\omega}e^{s'\pi/2}\frac{c_0^2}{\cos\xi}\cos^2\left(\gamma\sin\xi - s\sigma\right)\frac{d\Phi_0}{dk^2}\cosh\left(\pi s/2\right)\mathrm{Re}^2\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]$$

In region 2 we write

$$\frac{\partial^2 u}{\partial\omega\partial\tau'}\tau'$$

$$= \frac{2}{\sinh\zeta} c_0 \sqrt{2}\,\mathrm{Re}\left[-ie^{-i\Phi_0}\left\{W_+'(\tau',0,-s) - \frac{1}{\tau'}W_+(\tau',0,-s)\right\}^*\right]\frac{\partial\Phi_0}{\partial\omega}\cos(\gamma\cosh\zeta - s'\sigma' - \pi/2)$$

$$u = \frac{2}{\sinh\zeta}\psi_0(\tau', s')\cos(\gamma\cosh\zeta - s'\sigma' - \pi/2)$$

and as $\tau' \to 0$

$$\frac{\partial^2 u}{\partial\omega\partial\tau'}\tau' = -\frac{2}{\sinh\zeta}c_0\sqrt{2}\,\mathrm{Re}\left[\frac{\Gamma(1/2 - is/2)}{\Gamma(1/2 + is/2)}\frac{e^{i\pi/4}\sqrt{2}}{\Gamma(1/2 - is/2)}\right]\frac{\partial\Phi_0}{\partial\omega}\cos(\gamma\cosh\zeta - s'\sigma' - \pi/2)$$

$$= -\frac{4}{\sinh\zeta}c_0\,\mathrm{Re}\left[\frac{e^{i\pi/4}}{\Gamma(1/2 + is/2)}\right]\frac{\partial\Phi_0}{\partial\omega}\cos(\gamma\cosh\zeta - s'\sigma' - \pi/2)$$

$$u = \frac{2}{\sinh\zeta}\psi_0(0, s')\cos(\gamma\cosh\zeta - s'\sigma' - \pi/2)$$

$$= \frac{2}{\sinh\zeta}c_0 e^{-\pi s/2}\,\mathrm{Re}\left[e^{-i\pi/4}\Gamma(1/2 + is/2)\right]\cos(\gamma\cosh\zeta - s'\sigma' - \pi/2)$$

$$\psi_0(\tau', s') = c_0\frac{\sqrt{2}}{\tau'}\,\mathrm{Re}\left[W_+(\tau',0,-s) + e^{-i\Phi_0}W_+^*(\tau',0,-s)\right]$$

$$\psi_0(0, s') = c_0 e^{-\pi s/2}\,\mathrm{Re}\left[e^{-i\pi/4}\Gamma(1/2 + is/2)\right]$$

$$u\frac{\partial^2 u}{\partial\omega\partial\tau'}\tau'\sinh\zeta$$

$$= -\frac{8}{\sinh\zeta}c_0^2 e^{-\pi s/2}\,\mathrm{Re}\left[e^{-i\pi/4}\Gamma(1/2 + is/2)\right]\mathrm{Re}\left[\frac{e^{-i\pi/4}}{\Gamma(1/2 - is/2)}\right]\frac{\partial\Phi_0}{\partial\omega}\cos^2(\gamma\cosh\zeta - s'\sigma' - \pi/2)$$

$$= -\frac{8/\pi}{\sinh\zeta}c_0^2 e^{-\pi s/2}\,\mathrm{Re}^2\left[e^{-i\pi/4}\Gamma(1/2 + is/2)\right]\frac{\partial k^2}{\partial\omega}\frac{\partial\Phi_0}{\partial k^2}\cosh(\pi s/2)\cos^2(\gamma\cosh\zeta - s'\sigma' - \pi/2)$$

$$= -\frac{16/\pi}{\sinh\zeta}\frac{k^2}{\omega}c_0^2 e^{-\pi s/2}\,\mathrm{Re}^2\left[e^{-i\pi/4}\Gamma(1/2 + is/2)\right]\frac{d\Phi_0}{dk^2}\cosh(\pi s/2)\cos^2(\gamma\cosh\zeta - s'\sigma' - \pi/2)$$

Now assembling the normalization condition

$$\frac{k^2}{\omega} = 32d\frac{k^2}{\omega}e^{-\pi s/2}c_0^2\frac{d\Phi_0}{dk^2}\cosh(\pi s/2)\,\mathrm{Re}^2\left[e^{-i\pi/4}\Gamma(1/2 + is/2)\right]\int_0^{\pi/2 - \sqrt{2\Delta/d}}\cos^2(\gamma\sin\xi - s\sigma)\frac{d\xi}{\cos\xi}$$

$$-32d\frac{k^2}{\omega}e^{-\pi s/2}c_0^2\frac{d\Phi_0}{dk^2}\cosh\left(\pi s/2\right)\mathrm{Re}^2\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]\int_{\sqrt{2\Delta/d}}^{\zeta_0}\cos^2\left(\gamma\cosh\zeta-s'\sigma'-\pi/2\right)\frac{d\zeta}{\sinh\zeta}$$

Changing variables to $\sigma=\int_0^\xi\frac{d\xi}{\cos\xi}$ =arcsinh($\tan\xi$) in the first integral with $\sin\xi=\tanh\sigma$ and to $\sigma'=\int_\infty^\zeta\frac{d\zeta}{\sinh\zeta}=\ln\left[\tanh\left(\zeta/2\right)\right]$ in the second integral with $\cosh\left(\zeta\right)=-\coth\left(\sigma'\right)$ gives

$$\frac{k^2}{\omega}=32d\frac{k^2}{\omega}e^{-\pi s/2}c_0^2\frac{d\Phi_0}{dk^2}\cosh\left(\pi s/2\right)\mathrm{Re}^2\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]$$

$$\left[\int_0^{\mathrm{arcsinh}\left\{\tan\left(\pi/2-\sqrt{2\Delta/d}\right)\right\}}\cos^2\left(\gamma\tanh\sigma-s\sigma\right)d\sigma-\int_{\ln\left\{\tanh\sqrt{\Delta/(2d)}\right\}}^{\ln\{\tanh(\zeta_0/2)\}}\cos^2\left(\gamma\coth\sigma'+s'\sigma'+\pi/2\right)d\sigma'\right]$$

Changing to $\sigma''=-\sigma'$ in the second integral

$$\frac{k^2}{\omega}=32d\frac{k^2}{\omega}e^{-\pi s/2}c_0^2\frac{d\Phi_0}{dk^2}\cosh\left(\pi s/2\right)\mathrm{Re}^2\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]$$

$$\left[\int_0^{\mathrm{arcsinh}\left\{\cot\sqrt{2\Delta/d}\right\}}\cos^2\left(\gamma\tanh\sigma-s\sigma\right)d\sigma+\int_{-\ln\left\{\tanh\sqrt{\Delta/(2d)}\right\}}^{-\ln\{\tanh(\zeta_0/2)\}}\cos^2\left(\gamma\coth\sigma''-s'\sigma''-\pi/2\right)d\sigma''\right]$$

or approximating the limits

$$\frac{k^2}{\omega}=32d\frac{k^2}{\omega}e^{-\pi s/2}c_0^2\frac{d\Phi_0}{dk^2}\cosh\left(\pi s/2\right)\mathrm{Re}^2\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]$$

$$\left[\int_0^{-\ln\left\{\sqrt{\Delta/(2d)}\right\}}\cos^2\left(\gamma\tanh\sigma-s\sigma\right)d\sigma+\int_{-\ln\left\{\sqrt{\Delta/(2d)}\right\}}^{-\sigma_0'}\sin^2\left(\gamma\coth\sigma''-s'\sigma''\right)d\sigma''\right]$$

Averaging over the rapidly varying sinusoids gives

$$\frac{k^2}{\omega}\approx16d\frac{k^2}{\omega}e^{-\pi s/2}c_0^2\frac{d\Phi_0}{dk^2}\cosh\left(\pi s/2\right)\mathrm{Re}^2\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]$$

$$\left[\int_0^{-\ln\left\{\sqrt{\Delta/(2d)}\right\}}d\sigma+\int_{-\ln\left\{\sqrt{\Delta/(2d)}\right\}}^{-\sigma_0'}d\sigma''\right]$$

or

$$1\approx-16de^{-\pi s/2}c_0^2\frac{d\Phi_0}{dk^2}\cosh\left(\pi s/2\right)\mathrm{Re}^2\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]\sigma_0'$$

or

$$1\approx4de^{-\pi s/2}c_0^2\frac{d\Phi_0}{dk^2}\cosh\left(\pi s/2\right)\mathrm{Re}^2\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]\ln\left(\Lambda_+\right)$$

where $\sigma_0' = \ln\{\tanh(\zeta_0/2)\} = \frac{1}{2}\ln\left(\frac{\ell-d}{\ell+d}\right) = -\frac{1}{4}\ln(\Lambda_+) < 0$. This evaluation made use of a principal value interpretation of the energy theorem integration (in $\zeta$and ξ') at the focal point, which can be more rigorously justified by considering the contribution from the focal region [2] in the next subsection.

We now adopt Antonsen's conjecture [1] that

$$v^2 = \frac{A}{8}\left|\frac{d\Phi_0}{dk^2}\right|^{-1} \tag{480}$$

is the square of a unit Gaussian random variable and that v is a Gaussian random variable with zero mean and unit variance. This is the same as in the preceding scalar bowtie cavity and assumes the eigenfunction is even along the orbit. Thus we find

$$c_0^2 = v^2 \frac{2e^{\pi s/2}}{Ad\ln(\Lambda_+)\cosh(\pi s/2)\,\mathrm{Re}^2\left[e^{-i\pi/4}\Gamma(1/2+is/2)\right]}$$

or

$$c_0 = v\frac{2}{\sqrt{Ad\ln(\Lambda_+)(1+e^{-\pi s})}\,\mathrm{Re}\left[e^{-i\pi/4}\Gamma(1/2+is/2)\right]} \tag{481}$$

5.3.1 Focal Region Contribution To Normalization

The focal region can be included in the energy theorem integration. The solution in region 3 is

$$u = C_0 c_0 \frac{1}{\xi'\sqrt{\gamma}}\,\mathrm{Re}\left[W_+\left(\xi'\sqrt{2\gamma},0,s'\right) + e^{i\Phi_0'}W_+^*\left(\xi'\sqrt{2\gamma},0,s'\right)\right]$$

$$\frac{1}{\zeta\sqrt{\gamma}}\,\mathrm{Re}\left[e^{-i\pi/2}W_+\left(\zeta\sqrt{2\gamma},0,-s'\right) + e^{i\pi/2}e^{i\Phi_0}W_+^*\left(\zeta\sqrt{2\gamma},0,-s'\right)\right] \tag{482}$$

$$= C_0 c_0 \frac{\sqrt{2}}{\tau'}\,\mathrm{Re}\left[W_+(\tau',0,s') + e^{i\Phi_0'}W_+^*(\tau',0,s')\right]$$

$$\frac{\sqrt{2}}{\tau}\,\mathrm{Re}\left[e^{-i\pi/2}W_+(\tau,0,-s') + e^{i\pi/2}e^{i\Phi_0}W_+^*(\tau,0,-s')\right] \tag{483}$$

$$= \frac{C_0}{c_0}\psi_0(\tau',s')\,\psi_0(\tau,s)$$

Expanding as we leave the focal region using the asymptotic form

$$W_+(\tau,m,s) \sim e^{i\tau^2/4}\left(-i\tau^2/2\right)^{is/2} = e^{i\tau^2/4+s\pi/4+is\ln\left(\tau/\sqrt{2}\right)}\ ,\ \tau\to\infty$$

gives

$$u \sim 2C_0 e^{s'\pi/4}\cos(\Phi_0'/2)\,c_0\frac{1}{\zeta\sqrt{\gamma}}\,\mathrm{Re}\left[e^{-i\pi/2}W_+\left(\zeta\sqrt{2\gamma},0,-s'\right) + e^{i\pi/2}e^{i\Phi_0}W_+^*\left(\zeta\sqrt{2\gamma},0,-s'\right)\right]$$

$$\frac{1}{\xi'\sqrt{\gamma}}\cos\left[\xi'^2\gamma/2+s'\ln\left(\xi'\sqrt{\gamma}\right)-\Phi_0'/2\right]\text{ , Region }3\rightarrow 1$$

$$u\sim 2C_0e^{-s'\pi/4}\cos\left(\Phi_0/2\right)c_0\frac{1}{\xi'\sqrt{\gamma}}\operatorname{Re}\left[W_+\left(\xi'\sqrt{2\gamma},0,s'\right)+e^{i\Phi_0'}W_+^*\left(\xi'\sqrt{2\gamma},0,s'\right)\right]$$

$$\frac{1}{\zeta\sqrt{\gamma}}\cos\left[\zeta^2\gamma/2-s'\ln\left(\zeta\sqrt{\gamma}\right)-\pi/2-\Phi_0/2\right]\text{ , Region }3\rightarrow 2$$

The idea behind a correction is to integrate the exact form minus the two asymptotic limits (one on each side of the focal point) in the energy theorem.

$$\frac{\partial u}{\partial\omega}\sim C_0c_0\frac{1}{\xi'\sqrt{\gamma}}\operatorname{Re}\left[i\frac{\partial\Phi_0'}{\partial\omega}e^{i\Phi_0'}W_+^*\left(\xi'\sqrt{2\gamma},0,s'\right)\right]$$

$$\frac{1}{\zeta\sqrt{\gamma}}\operatorname{Re}\left[e^{-i\pi/2}W_+\left(\zeta\sqrt{2\gamma},0,-s'\right)+e^{i\pi/2}e^{i\Phi_0}W_+^*\left(\zeta\sqrt{2\gamma},0,-s'\right)\right]$$

$$+C_0c_0\frac{1}{\xi'\sqrt{\gamma}}\operatorname{Re}\left[W_+\left(\xi'\sqrt{2\gamma},0,s'\right)+e^{i\Phi_0'}W_+^*\left(\xi'\sqrt{2\gamma},0,s'\right)\right]$$

$$\frac{1}{\zeta\sqrt{\gamma}}\operatorname{Re}\left[i\frac{\partial\Phi_0}{\partial\omega}e^{i\pi/2}e^{i\Phi_0}W_+^*\left(\zeta\sqrt{2\gamma},0,-s'\right)\right]\tag{484}$$

or

$$\frac{\partial u}{\partial\omega}\sim C_0c_0\frac{2}{\tau'}\operatorname{Re}\left[i\frac{\partial\Phi_0'}{\partial\omega}e^{i\Phi_0'}W_+^*\left(\tau',0,s'\right)\right]$$

$$\frac{1}{\tau}\operatorname{Re}\left[e^{-i\pi/2}W_+\left(\tau,0,-s'\right)+e^{i\pi/2}e^{i\Phi_0}W_+^*\left(\tau,0,-s'\right)\right]$$

$$+C_0c_0\frac{2}{\tau'}\operatorname{Re}\left[W_+\left(\tau',0,s'\right)+e^{i\Phi_0'}W_+^*\left(\tau',0,s'\right)\right]$$

$$\frac{1}{\tau}\operatorname{Re}\left[i\frac{\partial\Phi_0}{\partial\omega}e^{i\pi/2}e^{i\Phi_0}W_+^*\left(\tau,0,-s'\right)\right]\tag{485}$$

Now taking the normal derivatives

$$\frac{\partial^2u}{\partial\tau\partial\omega}\sim C_0c_0\frac{\sqrt{2}}{\tau'}\operatorname{Re}\left[i\frac{\partial\Phi_0'}{\partial\omega}e^{i\Phi_0'}W_+^*\left(\tau',0,s'\right)\right]$$

$$\frac{\sqrt{2}}{\tau}\operatorname{Re}\left[\left\{e^{-i\pi/2}W_+'\left(\tau,0,-s'\right)+e^{i\pi/2}e^{i\Phi_0}W_+^{*\prime}\left(\tau,0,-s'\right)\right\}-\frac{1}{\tau}\left\{e^{-i\pi/2}W_+\left(\tau,0,-s'\right)+e^{i\pi/2}e^{i\Phi_0}W_+^*\left(\tau,0,-s'\right)\right\}\right]$$

$$+C_0c_0\frac{\sqrt{2}}{\tau'}\operatorname{Re}\left[W_+\left(\tau',0,s'\right)+e^{i\Phi_0'}W_+^*\left(\tau',0,s'\right)\right]$$

$$\frac{\sqrt{2}}{\tau}\operatorname{Re}\left[i\frac{\partial\Phi_0}{\partial\omega}e^{i\pi/2}e^{i\Phi_0}\left\{W_+^{*\prime}(\tau,0,-s')-\frac{1}{\tau}W_+^*(\tau,0,-s')\right\}\right] \tag{486}$$

$$\sim C_0\frac{\sqrt{2}}{\tau'}\operatorname{Re}\left[i\frac{\partial\Phi_0'}{\partial\omega}e^{i\Phi_0'}W_+^*(\tau',0,s')\right]\frac{\partial}{\partial\tau}\psi_0(\tau,s)$$

$$+\frac{C_0}{c_0}\psi_0(\tau',s')\frac{\partial^2}{\partial\tau\partial\omega}\psi_0(\tau,s)$$

$$\frac{\partial^2 u}{\partial\tau'\partial\omega}\sim C_0c_0\frac{\sqrt{2}}{\tau'}\operatorname{Re}\left[i\frac{\partial\Phi_0'}{\partial\omega}e^{i\Phi_0'}\left\{W_+^{*\prime}(\tau',0,s')-\frac{1}{\tau'}W_+^*(\tau',0,s')\right\}\right]$$

$$\frac{\sqrt{2}}{\tau}\operatorname{Re}\left[e^{-i\pi/2}W_+(\tau,0,-s')+e^{i\pi/2}e^{i\Phi_0}W_+^*(\tau,0,-s')\right]$$

$$+C_0c_0\frac{\sqrt{2}}{\tau'}\operatorname{Re}\left[\left\{W_+'(\tau',0,s')+e^{i\Phi_0'}W_+^{*\prime}(\tau',0,s')\right\}-\frac{1}{\tau'}\left\{W_+(\tau',0,s')+e^{i\Phi_0'}W_+^*(\tau',0,s')\right\}\right]$$

$$\frac{\sqrt{2}}{\tau}\operatorname{Re}\left[i\frac{\partial\Phi_0}{\partial\omega}e^{i\pi/2}e^{i\Phi_0}W_+^*(\tau,0,-s')\right] \tag{487}$$

$$\sim\frac{C_0}{c_0}\psi_0(\tau,s)\frac{\partial^2}{\partial\omega\partial\tau'}\psi_0(\tau',s')$$

$$+C_0\frac{\sqrt{2}}{\tau}\operatorname{Re}\left[i\frac{\partial\Phi_0}{\partial\omega}e^{i\pi/2}e^{i\Phi_0}W_+^*(\tau,0,-s')\right]\frac{\partial}{\partial\tau'}\psi_0(\tau',s')$$

Letting $\tau\to 0$ in the first expression times τ gives

$$\left(\tau\frac{\partial^2 u}{\partial\tau\partial\omega}\right)_{\tau\to 0}\sim C_0\psi_0(\tau',s')\,2\operatorname{Re}\left[\frac{e^{-i\pi/4}}{\Gamma(1/2-is/2)}\right]\frac{\partial\Phi_0}{\partial\omega}$$

and

$$u(\tau=0)=\frac{C_0}{c_0}\psi_0(\tau',s')\,\psi_0(0,s)=C_0\operatorname{Re}\left[e^{\pi s/2}e^{-i\pi/4}\Gamma(1/2+is/2)\right]\psi_0(\tau',s')$$

$$\left(u\tau\frac{\partial^2 u}{\partial\tau\partial\omega}\right)_{\tau\to 0}\sim 2C_0^2e^{\pi s/2}\frac{\operatorname{Re}^2\left[e^{-i\pi/4}\Gamma(1/2+is/2)\right]}{|\Gamma(1/2-is/2)|^2}\psi_0^2(\tau',s')\frac{\partial\Phi_0}{\partial\omega}$$

Letting $\tau'\to 0$ in the second expression times τ' gives

$$\left(\tau'\frac{\partial^2 u}{\partial\tau'\partial\omega}\right)_{\tau'\to 0}\sim C_0\psi_0(\tau,s)\,2\operatorname{Re}\left[\frac{e^{i\pi/4}}{\Gamma(1/2-is'/2)}\right]\frac{\partial\Phi_0'}{\partial\omega}$$

and

$$u\left(\tau'=0\right)=\frac{C_0}{c_0}\psi_0\left(0,s'\right)\psi_0\left(\tau,s\right)=C_0\,\mathrm{Re}\left[e^{\pi s'/2}e^{i\pi/4}\Gamma\left(1/2+is'/2\right)\right]\psi_0\left(\tau,s\right)$$

$$\left(u\tau'\frac{\partial^2 u}{\partial\tau'\partial\omega}\right)_{\tau'\to 0}\sim 2C_0^2e^{\pi s'/2}\frac{\mathrm{Re}^2\left[e^{i\pi/4}\Gamma\left(1/2+is'/2\right)\right]}{\left|\Gamma\left(1/2-is'/2\right)\right|^2}\psi_0^2\left(\tau,s\right)\frac{\partial\Phi_0'}{\partial\omega}$$

The normalization condition is

$$\frac{k^2}{\omega}=2\pi d\int_0^{\pi/2}u\frac{\partial^2 u}{\partial\omega\partial\tau}\tau\cos\xi d\xi+2\pi d\int_0^{\zeta_0}u\frac{\partial^2 u}{\partial\omega\partial\tau'}\tau'\sinh\zeta d\zeta$$

$$=2\pi d\int_0^{\pi/2}u\frac{\partial^2 u}{\partial\omega\partial\tau}\tau\sin\xi' d\xi'+2\pi d\int_0^{\zeta_0}u\frac{\partial^2 u}{\partial\omega\partial\tau'}\tau'\sinh\zeta d\zeta$$

$$=2\pi\int_0^{d}u\frac{\partial^2 u}{\partial\omega\partial\tau}\tau dz+2\pi\int_d^{\ell}u\frac{\partial^2 u}{\partial\omega\partial\tau'}\tau' dz \tag{488}$$

However, in 2D we found it simpler to examine this correction at the peak $s=0$. Let us follow the same procedure here. The region 3 solution becomes

$$-\Phi_0'=\Phi_0\to-\pi/2 \text{ as } s'\to 0 \tag{489}$$

$$u=C_0c_0\frac{\sqrt{2}}{\xi'\sqrt{2\gamma}}\,\mathrm{Re}\left[W_+\left(\xi'\sqrt{2\gamma},0,0\right)+e^{i\pi/2}W_+^*\left(\xi'\sqrt{2\gamma},0,0\right)\right]$$

$$\frac{\sqrt{2}}{\zeta\sqrt{2\gamma}}\,\mathrm{Re}\left[e^{-i\pi/2}W_+\left(\zeta\sqrt{2\gamma},0,0\right)+W_+^*\left(\zeta\sqrt{2\gamma},0,0\right)\right]$$

$$=\frac{C_0}{c_0}\psi_0\left(\tau',0\right)\psi_0\left(\tau,0\right) \tag{490}$$

where the Whittaker function takes the form

$$W_+\left(\tau,0,0\right)=\frac{1}{2}\tau\sqrt{\pi/2}e^{i\pi/4}H_0^{(1)}\left(\tau^2/4\right) \tag{491}$$

Thus the scalar solution in Region 3 is

$$u=C_0c_0\frac{\pi}{2}J_0\left(\xi'^2\gamma/2\right)J_0\left(\zeta^2\gamma/2\right) \tag{492}$$

where

$$\psi_0\left(\tau',0\right)=c_0\sqrt{\frac{\pi}{2}}J_0\left(\tau'^2/4\right)$$

$$\psi_0\left(\tau,0\right)=c_0\sqrt{\frac{\pi}{2}}J_0\left(\tau^2/4\right)$$

$$c_0\left(0\right)=v\frac{2}{\sqrt{\pi Ad\ln\left(\Lambda_+\right)}} \tag{493}$$

$$C_0\left(0\right)=\left(-1\right)^{n'}\sqrt{2\gamma} \tag{494}$$

and

$$k_p\left(d+\delta_0\right)=\left(n+1/4\right)\pi$$

$$\left(u\tau\frac{\partial^2 u}{\partial\tau\partial\omega}\right)_{\tau\to 0}\sim\gamma c_0^2\pi J_0^2\left(\tau'^2/4\right)\frac{\partial\Phi_0}{\partial\omega}$$

$$\tau'^2/4=k_p\left(d+\delta_0-z\right)$$

$$\left(u\tau'\frac{\partial^2 u}{\partial\tau'\partial\omega}\right)_{\tau'\to 0}\sim-\gamma c_0^2\pi J_0^2\left(\tau^2/4\right)\frac{\partial\Phi_0}{\partial\omega}$$

$$\tau^2/4=k_p\left(z-d-\delta_0\right)$$

Note that this function has a change in sign at $\tau=\tau'=0$ due to the outer phase derivative (the function itself should be continuous). Expanding for large arguments gives

$$\left(u\tau\frac{\partial^2 u}{\partial\tau\partial\omega}\right)_{\tau\to 0}\sim\gamma c_0^2\frac{4}{\tau'^2/2}\cos^2\left(\tau'^2/4-\pi/4\right)\frac{\partial\Phi_0}{\partial\omega}$$

$$\left(u\tau'\frac{\partial^2 u}{\partial\tau'\partial\omega}\right)_{\tau'\to 0}\sim-\gamma c_0^2\frac{4}{\tau^2/2}\cos^2\left(\tau^2/4-\pi/4\right)\frac{\partial\Phi_0}{\partial\omega}$$

Averaging over the rapidly varying trigonometric functions for high frequencies gives

$$\left(u\tau\frac{\partial^2 u}{\partial\tau\partial\omega}\right)_{\tau\to 0}\sim\gamma c_0^2\frac{4}{\tau'^2}\frac{\partial\Phi_0}{\partial\omega}$$

$$\left(u\tau'\frac{\partial^2 u}{\partial\tau'\partial\omega}\right)_{\tau'\to 0}\sim-\gamma c_0^2\frac{4}{\tau^2}\frac{\partial\Phi_0}{\partial\omega}$$

To be consistent with the original treatment, where we did not employ the focal shift for continuity, we use

$$\tau^2/4=\tau'^2/4=k_p\left(\left|z-d\right|\right)$$

in the asymptotic forms. The correction we are after uses the difference of the exact form and these asymptotic forms

$$\Delta\left(u\tau'\frac{\partial^2 u}{\partial\tau'\partial\omega}\right)_{\tau,\tau'\to 0}\sim-\mathrm{sgn}\left(z-d-\delta_0\right)\gamma c_0^2\pi J_0^2\left(k_p\left(\left|z-d-\delta_0\right|\right)\right)\frac{\partial\Phi_0}{\partial\omega}$$

$$+\mathrm{sgn}\left(z-d\right)\gamma c_0^2\frac{1}{k_p\left(\left|z-d\right|\right)}\frac{\partial\Phi_0}{\partial\omega}$$

to give $(R \to \infty)$

$$N_3\left(s=0\right) \sim 2\pi \int_{-R}^{d} \Delta\left(u\frac{\partial^2 u}{\partial\omega\partial\tau}\tau\right)_{\tau\to 0} dz + 2\pi \int_{d}^{d+\delta_0} \Delta\left(u\frac{\partial^2 u}{\partial\omega\partial\tau}\tau\right)_{\tau\to 0} dz + 2\pi \int_{d+\delta_0}^{R} \Delta\left(u\frac{\partial^2 u}{\partial\omega\partial\tau'}\tau'\right)_{\tau'\to 0} dz$$

$$\sim 2\pi\gamma c_0^2 \frac{\partial\Phi_0}{\partial\omega} \int_{-R}^{d} \left[\pi J_0^2\left(k_p\left(|z-d-\delta_0|\right)\right) - \frac{1}{k_p\left(|z-d|\right)}\right] dz$$

$$+2\pi\gamma c_0^2 \frac{\partial\Phi_0}{\partial\omega} \int_{d}^{d+\delta_0} \left[\pi J_0^2\left(k_p\left(|z-d-\delta_0|\right)\right) + \frac{1}{k_p\left(|z-d|\right)}\right] dz$$

$$+2\pi\gamma c_0^2 \frac{\partial\Phi_0}{\partial\omega} \int_{d+\delta_0}^{R} \left[-\pi J_0^2\left(k_p\left(|z-d-\delta_0|\right)\right) + \frac{1}{k_p\left(|z-d|\right)}\right] dz$$

$$\sim 2\pi^2\gamma c_0^2 \frac{\partial\Phi_0}{\partial\omega} \int_{-R}^{d+\delta_0} \left[J_0^2\left(k_p\left(|z-d-\delta_0|\right)\right)\right] dz$$

$$+2\pi^2\gamma c_0^2 \frac{\partial\Phi_0}{\partial\omega} \int_{d+\delta_0}^{R} \left[-J_0^2\left(k_p\left(|z-d-\delta_0|\right)\right)\right] dz$$

$$+\frac{2\pi}{k_p}\gamma c_0^2 \frac{\partial\Phi_0}{\partial\omega} \int_{-R}^{R} \frac{dz}{z-d}$$

$$\sim 2\pi^2\gamma c_0^2 \frac{\partial\Phi_0}{\partial\omega} \int_{-R}^{d+\delta_0} \left[J_0^2\left(k_p\left(|d+\delta_0-z|\right)\right)\right] dz$$

$$+2\pi^2\gamma c_0^2 \frac{\partial\Phi_0}{\partial\omega} \int_{d+\delta_0}^{R} \left[-J_0^2\left(k_p\left(|z-d-\delta_0|\right)\right)\right] dz$$

$$+\frac{2\pi}{k_p}\gamma c_0^2 \frac{\partial\Phi_0}{\partial\omega} \int_{-R}^{R} \frac{dz}{z-d}$$

$$\sim 2\pi^2\gamma c_0^2 \frac{\partial\Phi_0}{\partial\omega} \left(\int_0^{R+d+\delta_0} - \int_0^{R-d-\delta_0}\right) J_0^2\left(k_p u\right) du$$

$$+\frac{2\pi}{k_p}\gamma c_0^2 \frac{\partial\Phi_0}{\partial\omega} \int_{-R-d}^{R-d} \frac{du}{u}$$

$$\sim 2\pi^2\gamma c_0^2 \frac{\partial\Phi_0}{\partial\omega} \int_{R-d-\delta_0}^{R+d+\delta_0} J_0^2\left(k_p u\right) du$$

$$+\frac{2\pi}{k_p}\gamma c_0^2 \frac{\partial\Phi_0}{\partial\omega} \ln\left|\frac{R-d}{-R-d}\right|$$

$$\sim 2\pi^2\gamma c_0^2 \frac{\partial \Phi_0}{\partial \omega} 2\left(d+\delta_0\right) J_0^2\left(k_p\left(R+d\right)\right)$$

$$+\frac{2\pi}{k_p}\gamma c_0^2 \frac{\partial \Phi_0}{\partial \omega} \ln\left|\frac{R-d}{-R-d}\right| \rightarrow 0$$

The Region 3 correction thus vanishes as the overall interval of integration becomes large compared to the focal region $R >> d$. The principal value definition is therefore OK. Note that the above integration of the Bessel function assumes that $k_p\delta_0 \rightarrow 0$, but the order of the result is valid even if this is not true.

5.4 Scalar Stadium Projections

The projections of the high frequency rotationally symmetric stadium cavity field and of the axisymmetric random plane wave field are now discussed.

5.4.1 Trigonometric Scar Projection

The trigonometric projection of the high frequency scar solution in the stadium is now discussed. The projection kernel in this case is motivated by the $s = 0$ limit of the scar solution

$$V_p = 2\int_0^d u\left(0,0,z\right)\cos\left(k_p z\right) dz + 2\int_d^\ell u\left(0,0,z\right)\cos\left(k_p z - \pi/2\right) dz \tag{495}$$

The cylindrical form of the solution connected by

$$r = d\sinh\zeta\cos\xi = d\sinh\zeta\sin\xi'$$

$$z = d\cosh\zeta\sin\xi = d\cosh\zeta\cos\xi'$$

$$0 < \zeta < \infty$$

$$-\pi/2 < \xi < \pi/2\ ,\ \ 0 < \varphi < 2\pi$$

in Region 1 is

$$u_p\left(0,0,z\right) = 2e^{-s\pi/2}\frac{\psi_0\left(0,s\right)}{\sqrt{1-z^2/d^2}}\cos\left[k_p z + p_0\left(z\right)\right]$$

$$p_0\left(z\right) = \frac{s}{2}\left\{\left(z/\ell\right)\ln\left(\frac{\ell+d}{\ell-d}\right) - \ln\left(\frac{d+z}{d-z}\right)\right\}$$

$$\psi_0\left(0,s\right) = c_0 \operatorname{Re}\left[e^{-i\pi/4}e^{\pi s/2}\Gamma\left(1/2+is/2\right)\right]$$

$$c_0 = v\frac{2}{\sqrt{Ad\ln\left(\Lambda_+\right)\left(1+e^{-\pi s}\right)}\operatorname{Re}\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]}$$

where

$$s = 2\,(k - k_p)\,\ell / \ln\left(\frac{\ell + d}{\ell - d}\right)$$

$$k_p \ell = \pi p$$

The cylindrical form of the solution in Region 2 is

$$u_p\,(0,0,z) = \frac{2}{\sqrt{z^2/d^2 - 1}} \psi_0\,(0,s')\cos\left[k_p z + p_0\,(z) - \pi/2\right]$$

$$p_0\,(z) = -\frac{s'}{2}\left\{(z/\ell)\ln\left(\frac{\ell+d}{\ell-d}\right) - \ln\left(\frac{z+d}{z-d}\right)\right\}$$

$$\psi_0\,(0,s') = c_0 e^{-\pi s/2}\,\mathrm{Re}\left[e^{-i\pi/4}\Gamma\,(1/2 + is/2)\right]$$

where

$$s' = -s$$

Then the trigonometric projection is

$$V_p / \left\{4c_0\,\mathrm{Re}\left[e^{-i\pi/4}\Gamma\,(1/2 + is/2)\right]\right\} =$$

$$\int_0^d \cos\left[k_p z + p_0\,(z)\right]\cos\left(k_p z\right)\frac{dz}{\sqrt{1 - z^2/d^2}} + e^{-\pi s/2}\int_d^\ell \cos\left[k_p z - \pi/2 + p_0\,(z)\right]\cos\left(k_p z - \pi/2\right)\frac{dz}{\sqrt{z^2/d^2 - 1}}$$

where

$$p_0\,(z) = \frac{s}{2}\left\{(z/\ell)\ln\left(\frac{\ell+d}{\ell-d}\right) - \ln\left|\frac{z+d}{z-d}\right|\right\}$$

and the normalization is

$$c_0\,\mathrm{Re}\left[e^{-i\pi/4}\Gamma\,(1/2 + is/2)\right] = v\frac{2}{\sqrt{Ad\ln\left(\Lambda_+\right)\left(1 + e^{-\pi s}\right)}}$$

Retaining only the leading terms by averaging over the rapidly varying cosines gives

$$V_p / \left\{v\frac{4}{\sqrt{Ad\ln\left(\Lambda_+\right)\left(1 + e^{-\pi s}\right)}}\right\} \sim$$

$$\int_0^d \cos\left[p_0\,(z)\right]\frac{dz}{\sqrt{1 - z^2/d^2}} + e^{-\pi s/2}\int_d^\ell \cos\left[p_0\,(z)\right]\frac{dz}{\sqrt{z^2/d^2 - 1}}$$

$$p_0\,(z) = \frac{s}{2}\left\{(z/\ell)\ln\left|\frac{\ell+d}{\ell-d}\right| - \ln\left|\frac{z+d}{z-d}\right|\right\}$$

Thus we finally have

$$\left\langle LV_p^2\right\rangle = L^2 G_1\left(s\right)/A \tag{496}$$

where

$$G_1\left(s\right) = \frac{8\left(d/\ell\right)}{\ln\left(\Lambda_+\right)\left(1+e^{-\pi s}\right)}$$

$$\left[\int_0^1 \cos\left\{p_0\left(zd\right)\right\}\frac{dz}{\sqrt{1-z^2}} + e^{-\pi s/2}\int_1^{\ell/d}\cos\left\{p_0\left(zd\right)\right\}\frac{dz}{\sqrt{z^2-1}}\right]^2 \tag{497}$$

and

$$p_0\left(zd\right) = \frac{s}{2}\left\{\frac{1}{2}\left(zd/\ell\right)\ln\left(\Lambda_+\right) - \ln\left|\frac{z+1}{z-1}\right|\right\} \tag{498}$$

$$d/\ell = \frac{\left(\Lambda_+-1\right)}{\left(\sqrt{\Lambda_+}+1\right)^2} = \frac{\sqrt{\Lambda_+}-1}{\sqrt{\Lambda_+}+1} \tag{499}$$

$$\Lambda_+ = \left(\frac{\ell+d}{\ell-d}\right)^2 \tag{500}$$

$$\left(\ell/d+\sqrt{\ell^2/d^2-1}\right) = \frac{\left(\sqrt{\Lambda_+}+1\right)^2}{\left(\Lambda_+-1\right)}\left[1+\sqrt{1-\left(\frac{\sqrt{\Lambda_+}-1}{\sqrt{\Lambda_+}+1}\right)^2}\right]$$

$$= \frac{\left(\sqrt{\Lambda_+}+1\right)}{\left(\Lambda_+-1\right)}\left(\Lambda_+^{1/4}+1\right)^2 = \frac{\Lambda_+^{1/4}+1}{\Lambda_+^{1/4}-1}$$

For $s=0$ we have

$$G_1\left(0\right) = \frac{4\left(d/\ell\right)}{\ln\left(\Lambda_+\right)}\left[\int_0^1\frac{dz}{\sqrt{1-z^2}} + \int_1^{\ell/d}\frac{dz}{\sqrt{z^2-1}}\right]^2$$

$$= \frac{4\left(d/\ell\right)}{\ln\left(\Lambda_+\right)}\left[\pi/2+\operatorname{arccosh}\left(\ell/d\right)\right]^2$$

$$= \frac{4\left(d/\ell\right)}{\ln\left(\Lambda_+\right)}\left[\pi/2+\ln\left(\ell/d+\sqrt{\ell^2/d^2-1}\right)\right]^2$$

$$= \frac{4}{\ln\left(\Lambda_+\right)}\frac{\left(\Lambda_+-1\right)}{\left(\sqrt{\Lambda_+}+1\right)^2}\left[\pi/2+\ln\left\{\frac{\left(\sqrt{\Lambda_+}+1\right)}{\left(\Lambda_+-1\right)}\left(\Lambda_+^{1/4}+1\right)^2\right\}\right]^2 \tag{501}$$

Note that for $\Lambda_+\to 1$

$$G_1(0) \sim \left[\pi/2 + \ln\left(\frac{8}{\Lambda_+ - 1}\right)\right]^2 \tag{502}$$

5.4.2 Focal Region Contribution To Trigonometric Scar Projection

It is instructive to examine the rough size of the correction to the trigonometric projection would arise from the focal region. The solution in region 3 is

$$u = C_0 c_0 \frac{1}{\xi'\sqrt{\gamma}} \operatorname{Re}\left[W_+\left(\xi'\sqrt{2\gamma}, 0, s'\right) + e^{i\Phi_0'} W_+^*\left(\xi'\sqrt{2\gamma}, 0, s'\right)\right]$$

$$\frac{1}{\zeta\sqrt{\gamma}} \operatorname{Re}\left[e^{-i\pi/2} W_+\left(\zeta\sqrt{2\gamma}, 0, -s'\right) + e^{i\pi/2} e^{i\Phi_0} W_+^*\left(\zeta\sqrt{2\gamma}, 0, -s'\right)\right]$$

$$= C_0 c_0 \frac{\sqrt{2}}{\tau'} \operatorname{Re}\left[W_+(\tau', 0, s') + e^{i\Phi_0'} W_+^*(\tau', 0, s')\right]$$

$$\frac{\sqrt{2}}{\tau} \operatorname{Re}\left[e^{-i\pi/2} W_+(\tau, 0, -s') + e^{i\pi/2} e^{i\Phi_0} W_+^*(\tau, 0, -s')\right]$$

$$= \frac{C_0}{c_0} \psi_0(\tau', s')\, \psi_0(\tau, s)$$

Expanding as we leave the focal region using the asymptotic form

$$W_+(\tau, m, s) \sim e^{i\tau^2/4}\left(-i\tau^2/2\right)^{is/2} = e^{i\tau^2/4 + s\pi/4 + is\ln\left(\tau/\sqrt{2}\right)}\ ,\ \tau \to \infty$$

gives

$$u \sim 2C_0 e^{s'\pi/4} \cos(\Phi_0'/2)\, c_0 \frac{1}{\zeta\sqrt{\gamma}} \operatorname{Re}\left[e^{-i\pi/2} W_+\left(\zeta\sqrt{2\gamma}, 0, -s'\right) + e^{i\pi/2} e^{i\Phi_0} W_+^*\left(\zeta\sqrt{2\gamma}, 0, -s'\right)\right]$$

$$\frac{1}{\xi'\sqrt{\gamma}} \cos\left[\xi'^2\gamma/2 + s'\ln\left(\xi'\sqrt{\gamma}\right) - \Phi_0'/2\right]\ ,\ \text{Region } 3 \to 1$$

$$u \sim 2C_0 e^{-s'\pi/4} \cos(\Phi_0/2)\, c_0 \frac{1}{\xi'\sqrt{\gamma}} \operatorname{Re}\left[W_+\left(\xi'\sqrt{2\gamma}, 0, s'\right) + e^{i\Phi_0'} W_+^*\left(\xi'\sqrt{2\gamma}, 0, s'\right)\right]$$

$$\frac{1}{\zeta\sqrt{\gamma}} \cos\left[\zeta^2\gamma/2 - s'\ln\left(\zeta\sqrt{\gamma}\right) - \pi/2 - \Phi_0/2\right]\ ,\ \text{Region } 3 \to 2$$

The idea behind a correction is to integrate the exact form minus the two asymptotic limits (one on each side of the focal point) in the projection. Letting $\tau \to 0$ in the first expression times τ gives

$$u(\tau = 0) = \frac{C_0}{c_0} \psi_0(\tau', s')\, \psi_0(0, s) = C_0 \operatorname{Re}\left[e^{\pi s/2} e^{-i\pi/4} \Gamma(1/2 + is/2)\right] \psi_0(\tau', s')$$

Letting $\tau' \to 0$ in the second expression times τ' gives

$$u\left(\tau'=0\right)=\frac{C_0}{c_0}\psi_0\left(0,s'\right)\psi_0\left(\tau,s\right)=C_0\,\mathrm{Re}\left[e^{\pi s'/2}e^{i\pi/4}\Gamma\left(1/2+is'/2\right)\right]\psi_0\left(\tau,s\right)$$

However, in 2D [2] we found it simpler to examine this correction at the peak $s=0$. Let us follow the same procedure here. The region 3 solution becomes

$$-\Phi_0'=\Phi_0\to-\pi/2 \text{ as } s'\to0$$

$$u=C_0c_0\frac{\sqrt{2}}{\xi'\sqrt{2\gamma}}\,\mathrm{Re}\left[W_+\left(\xi'\sqrt{2\gamma},0,0\right)+e^{i\pi/2}W_+^*\left(\xi'\sqrt{2\gamma},0,0\right)\right]$$

$$\frac{\sqrt{2}}{\zeta\sqrt{2\gamma}}\,\mathrm{Re}\left[e^{-i\pi/2}W_+\left(\zeta\sqrt{2\gamma},0,0\right)+W_+^*\left(\zeta\sqrt{2\gamma},0,0\right)\right]$$

$$=\frac{C_0}{c_0}\psi_0\left(\tau',0\right)\psi_0\left(\tau,0\right)$$

$$W_+\left(\tau,0,0\right)=\frac{1}{2}\tau\sqrt{\pi/2}e^{i\pi/4}H_0^{(1)}\left(\tau^2/4\right)$$

Thus

$$u=C_0c_0\frac{\pi}{2}J_0\left(\xi'^2\gamma/2\right)J_0\left(\zeta^2\gamma/2\right)$$

$$\psi_0\left(\tau',0\right)=c_0\sqrt{\frac{\pi}{2}}J_0\left(\tau'^2/4\right)$$

$$\psi_0\left(\tau,0\right)=c_0\sqrt{\frac{\pi}{2}}J_0\left(\tau^2/4\right)$$

$$c_0\left(0\right)=v\frac{2}{\sqrt{\pi Ad\ln\left(\Lambda_+\right)}}$$

$$C_0\left(0\right)=\left(-1\right)^{n'}\sqrt{2\gamma}$$

$$k_p\left(d+\delta_0\right)=\left(n+1/4\right)\pi$$

Here we employ the focal shift δ_0 for continuity of the kernel function and therefore take

$$\tau'^2/4=k_p\left(d+\delta_0-z\right)$$

$$\tau^2/4=k_p\left(z-d-\delta_0\right)$$

or

$$u = v\left(-1\right)^{n'}\sqrt{\frac{2\pi k}{A\ln\left(\Lambda_+\right)}}J_0\left(\tau'^2/4\right)J_0\left(\tau^2/4\right)$$

Expanding for large arguments gives

$$u \sim v\left(-1\right)^{n'}\sqrt{\frac{k}{A\ln\left(\Lambda_+\right)}}\frac{4}{\tau'}\cos\left(\tau'^2/4-\pi/4\right)J_0\left(\tau^2/4\right)\ \ ,\ \tau'\to\infty$$

$$\sim v\left(-1\right)^{n'}\sqrt{\frac{k}{A\ln\left(\Lambda_+\right)}}\frac{4}{\tau}\cos\left(\tau^2/4-\pi/4\right)J_0\left(\tau'^2/4\right)\ \ ,\ \tau\to\infty$$

Thus we can write the correction to the scar trigonometric projection for $s=0$ as

$$\Delta V_p = 2v\left(-1\right)^{n'}\sqrt{\frac{2\pi k}{A\ln\left(\Lambda_+\right)}}$$

$$\left[\int_0^d\left\{J_0\left(k_p\left(d+\delta_0-z\right)\right)-\frac{\sqrt{2}}{\sqrt{\pi k_p\left(d+\delta_0-z\right)}}\cos\left(k_p\left(d+\delta_0-z\right)-\pi/4\right)\right\}\cos\left(k_pz\right)dz\right.$$

$$+\int_d^{d+\delta_0}\left\{J_0\left(k_p\left(d+\delta_0-z\right)\right)-\frac{\sqrt{2}}{\sqrt{\pi k_p\left(d+\delta_0-z\right)}}\cos\left(k_p\left(d+\delta_0-z\right)-\pi/4\right)\right\}\cos\left(k_pz-\pi/2\right)dz$$

$$\left.+\int_{d+\delta_0}^{\ell}\left\{J_0\left(k_p\left(z-d-\delta_0\right)\right)-\frac{\sqrt{2}}{\sqrt{\pi k_p\left(z-d-\delta_0\right)}}\cos\left(k_p\left(z-d-\delta_0\right)-\pi/4\right)\right\}\cos\left(k_pz-\pi/2\right)dz\right]$$

Extending these limits to infinity

$$\Delta V_p = 2v\left(-1\right)^{n'}\sqrt{\frac{2\pi k}{A\ln\left(\Lambda_+\right)}}$$

$$\left[\int_{-\infty}^d\left\{J_0\left(k_p\left(d+\delta_0-z\right)\right)-\frac{\sqrt{2}}{\sqrt{\pi k_p\left(d+\delta_0-z\right)}}\cos\left(k_p\left(d+\delta_0-z\right)-\pi/4\right)\right\}\cos\left(k_pz\right)dz\right.$$

$$+\int_d^{d+\delta_0}\left\{J_0\left(k_p\left(d+\delta_0-z\right)\right)-\frac{\sqrt{2}}{\sqrt{\pi k_p\left(d+\delta_0-z\right)}}\cos\left(k_p\left(d+\delta_0-z\right)-\pi/4\right)\right\}\cos\left(k_pz-\pi/2\right)dz$$

$$\left.+\int_{d+\delta_0}^{\infty}\left\{J_0\left(k_p\left(z-d-\delta_0\right)\right)-\frac{\sqrt{2}}{\sqrt{\pi k_p\left(z-d-\delta_0\right)}}\cos\left(k_p\left(z-d-\delta_0\right)-\pi/4\right)\right\}\cos\left(k_pz-\pi/2\right)dz\right]$$

and changing variables

$$\Delta V_p = 2v\left(-1\right)^{n'}\sqrt{\frac{2\pi k}{A\ln\left(\Lambda_+\right)}}$$

$$\left[\int_0^\infty \left\{J_0\left(k_p z\right) - \frac{\sqrt{2}}{\sqrt{\pi k_p z}}\cos\left(k_p z - \pi/4\right)\right\}\cos\left(k_p\left(-z+d+\delta_0\right)\right)dz\right.$$

$$+\int_d^{d+\delta_0}\left\{J_0\left(k_p\left(d+\delta_0-z\right)\right) - \frac{\sqrt{2}}{\sqrt{\pi k_p\left(d+\delta_0-z\right)}}\cos\left(k_p\left(d+\delta_0-z\right)-\pi/4\right)\right\}\left\{\cos\left(k_p z-\pi/2\right)-\cos\left(k_p z\right)\right\}dz$$

$$\left.+\int_0^\infty \left\{J_0\left(k_p z\right) - \frac{\sqrt{2}}{\sqrt{\pi k_p z}}\cos\left(k_p z - \pi/4\right)\right\}\cos\left(k_p\left(z+d+\delta_0\right)-\pi/2\right)dz\right]$$

or

$$\Delta V_p = \frac{2}{k_p}v\left(-1\right)^{n'}\sqrt{\frac{2\pi k}{A\ln\left(\Lambda_+\right)}}$$

$$\left[\int_0^\infty \left\{J_0\left(u\right) - \frac{\sqrt{2}}{\sqrt{\pi u}}\cos\left(u - \pi/4\right)\right\}\cos\left(-u+k_p\left(d+\delta_0\right)\right)du\right.$$

$$+\int_0^{k_p\delta_0}\left\{J_0\left(u\right) - \frac{\sqrt{2}}{\sqrt{\pi u}}\cos\left(u - \pi/4\right)\right\}\left\{\cos\left(u-k_p\left(d+\delta_0\right)+\pi/2\right)-\cos\left(u-k_p\left(d+\delta_0\right)\right)\right\}du$$

$$\left.+\int_0^\infty \left\{J_0\left(u\right) - \frac{\sqrt{2}}{\sqrt{\pi u}}\cos\left(u - \pi/4\right)\right\}\cos\left(u+k_p\left(d+\delta_0\right)-\pi/2\right)du\right]$$

or

$$\Delta V_p = \frac{4}{k_p}v\left(-1\right)^{n'}\sqrt{\frac{2\pi k}{A\ln\left(\Lambda_+\right)}}$$

$$\left[\cos\left(k_p\left(d+\delta_0\right)-\pi/4\right)\int_0^\infty \left\{J_0\left(u\right) - \frac{\sqrt{2}}{\sqrt{\pi u}}\cos\left(u - \pi/4\right)\right\}\cos\left(u-\pi/4\right)du\right.$$

$$\left.-\sin\left(\pi/4\right)\int_0^{k_p\delta_0}\left\{J_0\left(u\right) - \frac{\sqrt{2}}{\sqrt{\pi u}}\cos\left(u - \pi/4\right)\right\}\sin\left(u-k_p\left(d+\delta_0\right)+\pi/4\right)du\right]$$

Noting that [20]

$$\int_0^\infty J_0\left(u\right)\cos\left(\beta u-\pi/4\right)du = \cos\left(\pi/4\right)\int_0^\infty J_0\left(u\right)\cos\left(\beta u\right)du + \sin\left(\pi/4\right)\int_0^\infty J_0\left(u\right)\sin\left(\beta u\right)du$$

$$= \frac{\cos(\pi/4)}{\sqrt{1-\beta^2}} \ , \ \beta < 1$$

and

$$\sqrt{2/\pi}\int_0^\infty \cos(u-\pi/4)\cos(\beta u-\pi/4)\frac{du}{\sqrt{u}} = \frac{1}{2}\int_0^\infty \cos\{(1-\beta)u\}\frac{du}{\sqrt{u}} + \frac{1}{2}\int_0^\infty \cos\{(1+\beta)u\}\frac{du}{\sqrt{u}}$$

$$= \frac{1}{2}\sqrt{2/\pi}\sqrt{\pi}\cos(\pi/4)\left[\frac{1}{\sqrt{1-\beta}} + \frac{1}{\sqrt{1+\beta}}\right] = \frac{\cos(\pi/4)}{\sqrt{1-\beta^2}}\frac{1}{\sqrt{2}}\left(\sqrt{1+\beta}+\sqrt{1-\beta}\right)$$

Combining gives

$$\int_0^\infty \left\{J_0(u) - \frac{\sqrt{2}}{\sqrt{\pi u}}\cos(u-\pi/4)\right\}\cos(u-\pi/4)\,du = \lim_{\beta\to 1}\frac{\cos(\pi/4)}{\sqrt{1-\beta^2}}\left[1-\frac{1}{\sqrt{2}}\left(\sqrt{1+\beta}+\sqrt{1-\beta}\right)\right]$$

$$= -\cos(\pi/4)/2$$

Thus we can write the correction as

$$\Delta V_p = \frac{4}{k_p}v(-1)^{n'+n}\sqrt{\frac{2\pi k}{A\ln(\Lambda_+)}}$$

$$[-\cos(k_p(d+\delta_0)-\pi/4)\cos(\pi/4)/2$$

$$-\sin(\pi/4)\int_0^{k_p\delta_0}\left\{J_0(u) - \frac{\sqrt{2}}{\sqrt{\pi u}}\cos(u-\pi/4)\right\}\sin(u)\,du\Bigg]$$

$$= \frac{4}{k_p}v(-1)^{n'+n}\sqrt{\frac{2\pi k}{A\ln(\Lambda_+)}}$$

$$[-(-1)^n\cos(\pi/4)/2$$

$$-\sin(\pi/4)\int_0^{k_p\delta_0}\left\{J_0(u) - \frac{\sqrt{2}}{\sqrt{\pi u}}\cos(u-\pi/4)\right\}\sin(u)\,du\Bigg]$$

Using the expansion

$$J_0(u)\sin(u) = u\sum_{m=0}^\infty \frac{1}{2^{2m}m!\,(m+1)!}\sum_{n=0}^\infty \frac{1}{(2n+1)!}\left(-u^2\right)^{n+m}$$

$$= u\sum_{n=0}^\infty\left[\sum_{m=0}^n \frac{1}{2^{2m}m!\,(m+1)!}\frac{1}{(2n-2m+1)!}\right]\left(-u^2\right)^n$$

yields

$$\int_0^{k_p\delta_0} J_0(u)\sin(u)\,du = \frac{1}{2}(k_p\delta_0)^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}(k_p\delta_0)^{2n}\left[\sum_{m=0}^{n} \frac{1}{2^{2m}m!\,(m+1)!}\frac{1}{(2n-2m+1)!}\right]$$

where the series in brackets for each of the first few n's is

$$\sum_{m=0}^{n} \frac{1}{2^{2m}m!\,(m+1)!}\frac{1}{(2n-2m+1)!} = 1, n=0$$

$$= \frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}\right) = \frac{7}{24}\ ,\ n=1$$

$$= \frac{1}{24}\left(\frac{1}{5}+\frac{1}{2}+\frac{1}{8}\right) = \frac{11}{320}\ ,\ n=2$$

$$= \left(\frac{1}{7!}+\frac{1}{8}\frac{1}{120}+\frac{1}{32\,(6)}\frac{1}{6}+\frac{1}{2^6 3!4!}\right) = \frac{143}{64512}\ ,\ n=3$$

so that

$$\int_0^{k_p\delta_0} J_0(u)\sin(u)\,du = \frac{1}{2}(k_p\delta_0)^2\left[1-\frac{7}{48}(k_p\delta_0)^2+\frac{11}{960}(k_p\delta_0)^4-\frac{143}{258048}(k_p\delta_0)^6+\cdots\right]$$

and similarly

$$\int_0^{k_p\delta_0}\left\{\frac{\sqrt{2}}{\sqrt{\pi u}}\cos(u-\pi/4)\right\}\sin(u)\,du = \frac{1}{2}\sqrt{1/\pi}\int_0^{k_p\delta_0}\left[\sin(2u)+1-\cos(2u)\right]\frac{du}{\sqrt{u}} =$$

$$= \frac{1}{2}\sqrt{1/\pi}\left\{2\sqrt{k_p\delta_0}+\sum_{n=0}^{\infty}(-1)^n\int_0^{k_p\delta_0}\left[\frac{2^{2n+1}}{(2n+1)!}u^{2n+1/2}-\frac{2^{2n}}{(2n)!}u^{2n-1/2}\right]du\right\}$$

$$= \sqrt{k_p\delta_0/\pi}\left\{1+\sum_{n=0}^{\infty}\frac{2^{2n}(-1)^n}{(2n)!}(k_p\delta_0)^{2n}\left[\frac{(k_p\delta_0)}{(2n+3/2)(2n+1)}-\frac{1/2}{(2n+1/2)}\right]\right\}$$

Then with

$$k_p\langle\delta_0\rangle/\pi \approx 1/2$$

$$\sin(\pi/4)\int_0^{k_p\langle\delta_0\rangle} J_0(u)\sin(u)\,du \approx 0.496$$

$$\sin(\pi/4)\int_0^{k_p\langle\delta_0\rangle}\left\{\frac{\sqrt{2}}{\sqrt{\pi u}}\cos(u-\pi/4)\right\}\sin(u)\,du \approx 0.565$$

we find

$$\Delta V_p \approx \frac{4}{k_p} v \left(-1\right)^{n'+n} \sqrt{\frac{2\pi k}{A \ln\left(\Lambda_+\right)}}$$

$$\left[-\left(-1\right)^n 0.35355 + 0.069\right]$$

Note that the original value for $s = 0$ is

$$V_p \sim \frac{4}{\sqrt{k}} v \frac{\sqrt{kd/2}}{\sqrt{A \ln\left(\Lambda_+\right)}} \left[\frac{\pi}{2} + \operatorname{arccosh}\left(\ell/d\right)\right]$$

$$\sim \frac{4}{\sqrt{k}} v \frac{\sqrt{kd/2}}{\sqrt{A \ln\left(\Lambda_+\right)}} \left[\frac{\pi}{2} + \ln\left(\ell/d + \sqrt{\left(\ell/d\right)^2 - 1}\right)\right]$$

$$\sim \frac{4}{\sqrt{k}} v \frac{\sqrt{kd/2}}{\sqrt{A \ln\left(\Lambda_+\right)}} \left[\frac{\pi}{2} + \ln\left(\frac{\Lambda_+^{1/4} + 1}{\Lambda_+^{1/4} - 1}\right)\right]$$

$$\sim \frac{4}{\sqrt{k}} v \frac{\sqrt{kd/2}}{\sqrt{A \ln\left(\Lambda_+\right)}} \left[\frac{\pi}{2} + \ln\left(\frac{\sqrt{\ell + d} + \sqrt{\ell - d}}{\sqrt{\ell + d} - \sqrt{\ell - d}}\right)\right]$$

and therefore

$$\Delta V_p / V_p = O\left[\frac{\sqrt{\pi k d} / \left(k_p d\right)}{\frac{\pi}{2} + \ln\left(\frac{\Lambda_+^{1/4}+1}{\Lambda_+^{1/4}-1}\right)}\right] = O\left(\frac{1}{2\sqrt{k_p d}}\right)$$

Using

$$k_p \ell = \pi p$$

with $\ell = 0.11176$ m, $d = 0.0336969$ m, and

$$p_1 = 8 \leq p \leq 22 = p_2$$

we see that this is less than a 20% error term for $p = 8$ and less than a 11% error for $p = 22$.

5.4.3 Random Plane Wave Trigonometric Projection

The projection is again taken as

$$V_{pr}^T = \int_{-\ell}^{\ell} u_r\left(0,0,z\right) \left[\begin{array}{c} \cos\left(k_p z\right) \ , \ \ |z| < d \\ \cos\left(k_p \left|z\right| - \pi/2\right) \ , \ \ |z| > d \end{array}\right] dz \tag{503}$$

In this case let us take the random plane wave to be symmetrized along the orbit (we will take only the even part of the random plane wave below)

$$V_{pr}^T = 2\int_0^d u_r\left(0,0,z\right) \cos\left(k_p z\right) dz + 2\int_d^{\ell} u_r\left(0,0,z\right) \cos\left(k_p z - \pi/2\right) dz$$

where the axisymmetric component of the random plane wave is

$$u_r(\rho, 0, z) = \lim_{N\to\infty} \sqrt{4k/(\pi AN)}\,\mathrm{Re}\left[\sum_{j=1}^{N} a_j J_0(k\rho\sin\theta_j)\, e^{i\alpha_j + ikz\cos\theta_j}\right] \tag{504}$$

Thus (the factor of 2 is introduced here to include the negative half of the integration interval)

$$V_{pr}^{T} = 2\lim_{N\to\infty} \sqrt{4k/(\pi AN)} \sum_{j=1}^{N} a_j \int_0^d \cos(\alpha_j + kz\cos\theta_j)\cos(k_p z)\, dz$$

$$+2\lim_{N\to\infty} \sqrt{4k/(\pi AN)} \sum_{j=1}^{N} a_j \int_d^\ell \cos(\alpha_j + kz\cos\theta_j)\cos(k_p z - \pi/2)\, dz$$

Taking the variance and averaging over the random amplitudes a_j with $\langle a_j a_{j'}\rangle_{a_j} = \delta_{jj'}$

$$\left\langle V_{pr}^{T2}\right\rangle_{a_j} = 4\lim_{N\to\infty} \frac{4k}{\pi AN} \sum_{j=1}^{N} \left[\int_0^d \int_0^d \cos(k_p z)\cos(k_p z')\cos(\alpha_j + kz\cos\theta_j)\cos(\alpha_j + kz'\cos\theta_j)\, dzdz'\right.$$

$$+2\int_0^d \int_d^\ell \cos(k_p z)\cos(k_p z' - \pi/2)\cos(\alpha_j + kz\cos\theta_j)\cos(\alpha_j + kz'\cos\theta_j)\, dzdz'$$

$$\left.+\int_d^\ell \int_d^\ell \cos(k_p z - \pi/2)\cos(k_p z' - \pi/2)\cos(\alpha_j + kz\cos\theta_j)\cos(\alpha_j + kz'\cos\theta_j)\, dzdz'\right]$$

Averaging over the random phases using

$$\langle\cos(\alpha_j + kz\cos\theta_j)\cos(\alpha_j + kz'\cos\theta_j)\rangle_{\alpha_j} = \frac{1}{2\pi}\frac{1}{2}\int_0^{2\pi} \left[\cos(k(z - z')\cos\theta) + \cos(2\alpha_j + k(z + z')\cos\theta)\right] d\alpha_j$$

$$= \frac{1}{2}\cos(k(z - z')\cos\theta) = \frac{1}{2}\cos(kz\cos\theta)\cos(kz'\cos\theta) + \frac{1}{2}\sin(kz\cos\theta)\sin(kz'\cos\theta)$$

and taking only the even part (the odd part will vanish if the integration is carried out over both halves of the stadium) gives

$$\left\langle V_{pr}^{T2}\right\rangle_{a_j,\alpha_j} = 2\lim_{N\to\infty} \frac{4k}{\pi AN} \sum_{j=1}^{N} \left[\int_0^d \cos(k_p z)\cos(kz\cos\theta)\, dz \int_0^d \cos(k_p z')\cos(kz'\cos\theta)\, dz'\right.$$

$$+2\int_0^d \cos(k_p z)\cos(kz\cos\theta)\, dz \int_d^\ell \cos(k_p z' - \pi/2)\cos(kz'\cos\theta)\, dz'$$

$$\left.+\int_d^\ell \cos(k_p z - \pi/2)\cos(kz\cos\theta)\, dz \int_d^\ell \cos(k_p z' - \pi/2)\cos(kz'\cos\theta)\, dz'\right]$$

or

$$\left\langle V_{pr}^{T2}\right\rangle_{a_j,\alpha_j} = 2\lim_{N\to\infty}\frac{k}{\pi AN}\sum_{j=1}^{N}$$

$$\left[\int_0^d \left\{\cos\left(\left(k_p - k\cos\theta\right)z\right) + \cos\left(\left(k_p + k\cos\theta\right)z\right)\right\}dz \int_0^d \left\{\cos\left(\left(k_p - k\cos\theta\right)z'\right) + \cos\left(\left(k_p + k\cos\theta\right)z'\right)\right\}dz'\right.$$

$$+2\int_0^d \left\{\cos\left(\left(k_p - k\cos\theta\right)z\right) + \cos\left(\left(k_p + k\cos\theta\right)z\right)\right\}dz \int_d^\ell \left\{\sin\left(\left(k_p - k\cos\theta\right)z'\right) + \sin\left(\left(k_p + k\cos\theta\right)z'\right)\right\}dz'$$

$$\left. + \int_d^\ell \left\{\sin\left(\left(k_p - k\cos\theta\right)z\right) + \sin\left(\left(k_p + k\cos\theta\right)z\right)\right\}dz \int_d^\ell \left\{\sin\left(\left(k_p - k\cos\theta\right)z'\right) + \sin\left(\left(k_p + k\cos\theta\right)z'\right)\right\}dz'\right]$$

or

$$\left\langle V_{pr}^{T2}\right\rangle_{a_j,\alpha_j} = 2\lim_{N\to\infty}\frac{k}{\pi AN}\sum_{j=1}^{N}$$

$$\left[\frac{\sin\left(\left(k_p - k\cos\theta\right)d\right)}{\left(k_p - k\cos\theta\right)} + \frac{\cos\left(\left(k_p - k\cos\theta\right)d\right)}{\left(k_p - k\cos\theta\right)} - \frac{\cos\left(\left(k_p - k\cos\theta\right)\ell\right)}{\left(k_p - k\cos\theta\right)}\right.$$

$$\left. + \frac{\sin\left(\left(k_p + k\cos\theta\right)d\right)}{\left(k_p + k\cos\theta\right)} + \frac{\cos\left(\left(k_p + k\cos\theta\right)d\right)}{\left(k_p + k\cos\theta\right)} - \frac{\cos\left(\left(k_p + k\cos\theta\right)\ell\right)}{\left(k_p + k\cos\theta\right)}\right]^2$$

Now averaging over the angle θ gives

$$\left\langle V_p^{T2}\right\rangle_r = \frac{2k}{\pi A}\frac{1}{2}\int_0^\pi$$

$$\left[\frac{\sin\left(\left(k_p - k\cos\theta\right)d\right)}{\left(k_p - k\cos\theta\right)} + \frac{\cos\left(\left(k_p - k\cos\theta\right)d\right)}{\left(k_p - k\cos\theta\right)} - \frac{\cos\left(\left(k_p - k\cos\theta\right)\ell\right)}{\left(k_p - k\cos\theta\right)}\right.$$

$$\left. + \frac{\sin\left(\left(k_p + k\cos\theta\right)d\right)}{\left(k_p + k\cos\theta\right)} + \frac{\cos\left(\left(k_p + k\cos\theta\right)d\right)}{\left(k_p + k\cos\theta\right)} - \frac{\cos\left(\left(k_p + k\cos\theta\right)\ell\right)}{\left(k_p + k\cos\theta\right)}\right]^2 \sin\theta d\theta$$

Now when $k \to k_p$ the first terms peak for $\theta \to 0$ and the second terms peak for $\theta \to \pi$ with $\cos\theta = -\cos\left(\theta - \pi\right)$ and $\sin\theta = -\sin\left(\theta - \pi\right)$. Thus we find (and $\theta = \zeta/\sqrt{k\ell/2}$)

$$\left\langle V_p^{T2}\right\rangle_r =$$

$$\frac{2k}{\pi A}\frac{1}{2}\int_0^\infty \left[\frac{\sin\left(k_p - k + k\theta^2/2\right)d}{k_p - k + k\theta^2/2} + \frac{\cos\left(k_p - k + k\theta^2/2\right)d}{k_p - k + k\theta^2/2} - \frac{\cos\left(k_p - k + k\theta^2/2\right)\ell}{k_p - k + k\theta^2/2}\right]^2 \theta d\theta$$

$$+\frac{2k}{\pi A}\frac{1}{2}\int_{-\infty}^{\pi}$$

$$\left[\frac{\sin\left(k_p-k+k\left(\theta-\pi\right)^2/2\right)d}{k_p-k+k\left(\theta-\pi\right)^2/2}+\frac{\cos\left(k_p-k+k\left(\theta-\pi\right)^2/2\right)d}{k_p-k+k\left(\theta-\pi\right)^2/2}-\frac{\cos\left(k_p-k+k\left(\theta-\pi\right)^2/2\right)\ell}{k_p-k+k\left(\theta-\pi\right)^2/2}\right]^2\left(\pi-\theta\right)d\theta$$

$$=\frac{2k}{\pi A}\int_0^{\infty}\left[\frac{\sin\left(k_p-k+k\theta^2/2\right)d}{k_p-k+k\theta^2/2}+\frac{\cos\left(k_p-k+k\theta^2/2\right)d}{k_p-k+k\theta^2/2}-\frac{\cos\left(k_p-k+k\theta^2/2\right)\ell}{k_p-k+k\theta^2/2}\right]^2\theta d\theta$$

$$=\frac{4\ell}{\pi A}\int_0^{\infty}\left[\frac{\sin\left(\left(k_p-k\right)\ell+\zeta^2\right)d/\ell}{\left(k_p-k\right)\ell+\zeta^2}+\frac{\cos\left(\left(k_p-k\right)\ell+\zeta^2\right)d/\ell}{\left(k_p-k\right)\ell+\zeta^2}-\frac{\cos\left(\left(k_p-k\right)\ell+\zeta^2\right)}{\left(k_p-k\right)\ell+\zeta^2}\right]^2\zeta d\zeta$$

Now letting

$$\lambda=2\left(k-k_p\right)L$$

and

$$\left\langle LV_p^{T2}\right\rangle_r=L^2G\left(\lambda\right)/A$$

gives

$$\left\langle V_p^{T2}\right\rangle_r=\frac{2L}{\pi A}\int_0^{\infty}$$

$$\left[\frac{\sin\left(\lambda/4-\zeta^2\right)d/\ell}{\lambda/4-\zeta^2}-\frac{\cos\left(\lambda/4-\zeta^2\right)d/\ell}{\lambda/4-\zeta^2}+\frac{\cos\left(\lambda/4-\zeta^2\right)}{\lambda/4-\zeta^2}\right]^2\zeta d\zeta$$

or

$$G\left(\lambda\right)=\frac{2}{\pi}\int_0^{\infty}\frac{\left[\sin\left(\lambda/4-\zeta^2\right)d/\ell-\cos\left(\lambda/4-\zeta^2\right)d/\ell+\cos\left(\lambda/4-\zeta^2\right)\right]^2}{\left(\lambda/4-\zeta^2\right)^2}\zeta d\zeta$$

$$=\frac{1}{\pi}\int_0^{\infty}\frac{\left[\sin\left(\lambda/4-\xi\right)d/\ell-\cos\left(\lambda/4-\xi\right)d/\ell+\cos\left(\lambda/4-\xi\right)\right]^2}{\left(\lambda/4-\xi\right)^2}d\xi$$

$$=\frac{1}{\pi}\int_{-\lambda/4}^{\infty}\frac{\left[\sin\left(\xi d/\ell\right)+\cos\left(\xi d/\ell\right)-\cos\xi\right]^2}{\xi^2}d\xi$$

Note that

$$G\left(\infty\right)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\left[\sin\left(\xi d/\ell\right)+\cos\left(\xi d/\ell\right)-\cos\xi\right]^2}{\xi^2}d\xi$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\left[1+\cos^2\xi + \sin\left(2\xi d/\ell\right) - 2\sin\left(\xi d/\ell\right)\cos\xi - 2\cos\left(\xi d/\ell\right)\cos\xi\right]}{\xi^2} d\xi$$

$$= \frac{2}{\pi}\int_{0}^{\infty} \frac{\left[\frac{1}{2}\cos\left(2\xi\right) - \frac{1}{2} + 1 - \cos\left(\left(1+d/\ell\right)\xi\right) + 1 - \cos\left(\left(1-d/\ell\right)\xi\right)\right]}{\xi^2} d\xi$$

$$= 1 + d/\ell + 1 - d/\ell - 2/2 = 1$$

Integration by parts gives

$$G\left(\lambda\right) = -\frac{1}{\pi}\frac{\left[-\sin\left(\frac{\lambda}{4}d/\ell\right) + \cos\left(\frac{\lambda}{4}d/\ell\right) - \cos\left(\lambda/4\right)\right]^2}{\lambda/4}$$

$$+\frac{2}{\pi}\int_{-\lambda/4}^{\infty} \frac{\left[\sin\left(\xi d/\ell\right) + \cos\left(\xi d/\ell\right) - \cos\xi\right]\left[\left(d/\ell\right)\cos\left(\xi d/\ell\right) - \left(d/\ell\right)\sin\left(\xi d/\ell\right) + \sin\xi\right]}{\xi} d\xi$$

$$= -\frac{1}{\pi}\frac{\left[-\sin\left(\frac{\lambda}{4}d/\ell\right) + \cos\left(\frac{\lambda}{4}d/\ell\right) - \cos\left(\lambda/4\right)\right]^2}{\lambda/4}$$

$$+\frac{2}{\pi}\int_{-\lambda/4}^{\infty} \frac{\left[\left(d/\ell\right)\cos\left(2\xi d/\ell\right) + \sin\xi\left\{\sin\left(\xi d/\ell\right) + \cos\left(\xi d/\ell\right)\right\} + \left(d/\ell\right)\cos\xi\left\{\sin\left(\xi d/\ell\right) - \cos\left(\xi d/\ell\right)\right\} - \frac{1}{2}\sin\left(2\xi\right)\right]}{\xi} d\xi$$

or

$$G\left(\lambda\right) + \frac{1}{\pi}\frac{\left[-\sin\left(\frac{\lambda}{4}d/\ell\right) + \cos\left(\frac{\lambda}{4}d/\ell\right) - \cos\left(\lambda/4\right)\right]^2}{\lambda/4} = \frac{2}{\pi}\int_{-\lambda/4}^{\infty}$$

$$+\frac{\left[\frac{1}{2}\left\{\sin\left(\left(1-d/\ell\right)\xi\right) + \sin\left(\left(1+d/\ell\right)\xi\right)\right\} + \frac{1}{2}\left(d/\ell\right)\left\{\sin\left(\left(1+d/\ell\right)\xi\right) - \sin\left(\left(1-d/\ell\right)\xi\right)\right\} - \frac{1}{2}\sin\left(2\xi\right)\right]}{\xi} d\xi$$

$$+\frac{2}{\pi}\int_{-\lambda/4}^{\infty}$$

$$\frac{\left[\frac{1}{2}\left\{\cos\left(\left(1-d/\ell\right)\xi\right) - \cos\left(\left(1+d/\ell\right)\xi\right)\right\} - \frac{1}{2}\left(d/\ell\right)\left\{\cos\left(\left(1-d/\ell\right)\xi\right) + \cos\left(\left(1+d/\ell\right)\xi\right)\right\} + \left(d/\ell\right)\cos\left(2\xi d/\ell\right)\right]}{\xi} d\xi$$

or

$$G\left(\lambda\right) + \frac{1}{\pi}\frac{\left[-\sin\left(\frac{\lambda}{4}d/\ell\right) + \cos\left(\frac{\lambda}{4}d/\ell\right) - \cos\left(\lambda/4\right)\right]^2}{\lambda/4} =$$

$$\frac{1}{2} + \frac{1}{\pi}\left[\left(1-d/\ell\right)\mathrm{Si}\left(\left(1-d/\ell\right)\lambda/4\right) + \left(1+d/\ell\right)\mathrm{Si}\left(\left(1+d/\ell\right)\lambda/4\right) - \mathrm{Si}\left(\lambda/2\right)\right]$$

$$+\frac{1}{\pi}\left[(1+d/\ell)\,\mathrm{Ci}\left(-(1+d/\ell)\,\lambda/4\right)-(1-d/\ell)\,\mathrm{Ci}\left(-(1-d/\ell)\,\lambda/4\right)-2\,(d/\ell)\,\mathrm{Ci}\left(-(d/\ell)\,\lambda/2\right)\right]$$

$$=\frac{1}{2}+\frac{1}{\pi}\left[(1-d/\ell)\,\mathrm{Si}\left((1-d/\ell)\,\lambda/4\right)+(1+d/\ell)\,\mathrm{Si}\left((1+d/\ell)\,\lambda/4\right)-\mathrm{Si}\left(\lambda/2\right)\right]$$

$$-\frac{1}{\pi}\left[(1+d/\ell)\,\mathrm{Cin}\left((1+d/\ell)\,\lambda/4\right)-(1-d/\ell)\,\mathrm{Cin}\left((1-d/\ell)\,\lambda/4\right)-2\,(d/\ell)\,\mathrm{Cin}\left((d/\ell)\,\lambda/2\right)\right]$$

$$+\frac{1}{\pi}\left[(1+d/\ell)\ln(1+d/\ell)-(1-d/\ell)\ln(1-d/\ell)-2\,(d/\ell)\ln(2\,(d/\ell))\right] \tag{505}$$

Note that

$$G(0)=\frac{1}{2}+\frac{1}{\pi}\left[(1+d/\ell)\ln(1+d/\ell)-(1-d/\ell)\ln(1-d/\ell)-2\,(d/\ell)\ln(2\,(d/\ell))\right] \tag{506}$$

$$G(-\infty)=0 \tag{507}$$

$$G(\infty)=1 \tag{508}$$

Thus taking into account the even symmetry along the orbit

$$G_s(\lambda)=2G(\lambda) \tag{509}$$

A comparison of the trigonometric projections for the cavity scar field and axisymmetric random plane wave field are given in Figure 33. Notice that in the stadium there is a large enhancement in the cavity scar field relative to the random plane wave field near $s_p=0$, where the cavity modal frequency aligns with the scar frequency $k\to k_p$. There is in addition the axisymmetric enhancement due to the symmetry of the field $\left\langle LV_p^2\right\rangle/\left\langle LV_{pr}^2\right\rangle=O(kV/A)>>1$ (analogous to the factor of two increase in the 2D even geometry).

5.4.4 Elliptic System Scar Projection

The elliptic projection operator is now discussed. The high frequency scar constructions in prolate spheroidal coordinates

$$r=d\sinh\zeta\cos\xi=d\sinh\zeta\sin\xi'$$

$$z=d\cosh\zeta\sin\xi=d\cosh\zeta\cos\xi'$$

$$0<\zeta<\infty$$

$$-\pi/2<\xi<\pi/2\ ,\ 0<\varphi<2\pi$$

are given to motivate the projection. The solution in Region 1 is

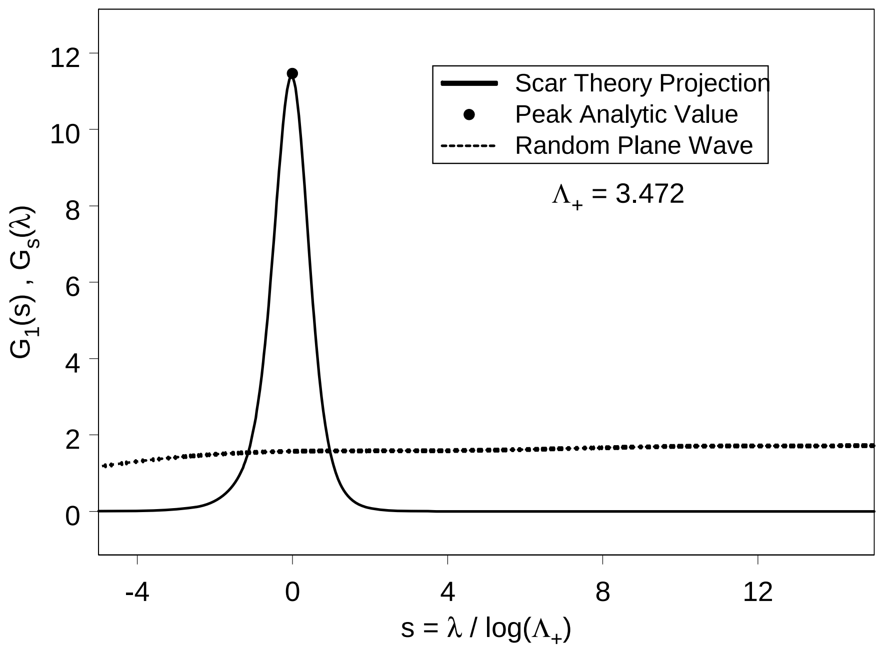

Figure 33. Comparison of trigonometric projections of the rotationally symmetric stadium cavity field and the axisymmetric random plane wave field.

$$u_p = 2e^{s'\pi/2}\frac{\psi_0(\tau,s)}{\sin\xi'}\cos\left(kd\cos\xi' - s\sigma\right)\ ,\ \xi' = \pi/2 - \xi$$

$$\psi_0(0,s) = c_0\,\mathrm{Re}\left[e^{-i\pi/4}e^{\pi s/2}\Gamma(1/2 + is/2)\right]$$

$$c_0 = v\frac{2}{\sqrt{Ad\ln(\Lambda_+)(1+e^{-\pi s})}\,\mathrm{Re}\left[e^{-i\pi/4}\Gamma(1/2+is/2)\right]}$$

where

$$\psi_0(\tau,s) = c_0\frac{\sqrt{2}}{\tau}\,\mathrm{Re}\left[e^{-i\pi/2}W_+(\tau,0,s) + e^{i\pi/2}e^{i\Phi_0}W_+^*(\tau,0,s)\right]$$

$$\sigma = \int_0^{\xi}\frac{d\xi}{\cos\xi} = \mathrm{arcsinh}(\tan\xi) = \int_{\xi'}^{\pi/2}\frac{d\xi'}{\sin\xi'} = -\ln\left\{\tan\left(\xi'/2\right)\right\}$$

$$\tau = \sqrt{2\gamma}\zeta$$

$$W_+(\tau,0,s) = W_{is/2,0}\left(-i\tau^2/2\right)$$

$$k_p\ell = \pi p$$

The solution in Region 2 is

$$u_p = \frac{2}{\sinh\zeta}\psi_0(\tau',s')\cos(kd\cosh\zeta - s'\sigma' - \pi/2)$$

$$\psi_0(0,s') = c_0e^{-\pi s/2}\,\mathrm{Re}\left[e^{-i\pi/4}\Gamma(1/2+is/2)\right]$$

where

$$s' = -s$$

$$\psi_0(\tau',s') = c_0\frac{\sqrt{2}}{\tau'}\,\mathrm{Re}\left[W_+(\tau',0,-s) + e^{-i\Phi_0}W_+^*(\tau',0,-s)\right]$$

$$\tau' = \sqrt{2\gamma}\xi'$$

$$\xi' = \pm\pi/2 - \xi$$

$$2s'\sigma_0' = 2s'\ln\left[\tanh(\zeta_0/2)\right] = (k - k_p)L$$

$$\sigma' = \int_{\infty}^{\zeta} \frac{d\zeta}{\sinh \zeta} = \ln \left[\tanh \left(\zeta/2\right)\right]$$

The projection is taken as

$$V_p = \lim_{\zeta \to 0} \frac{2}{\zeta d} \int_0^{\pi/2} \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos\left[k_p z + p_0\left(z\right)\right]}{\cos \xi} u\left(\zeta, \varphi, \xi\right) h_\varphi d\varphi h_\xi d\xi$$

$$+ \lim_{\xi' \to 0} \frac{2}{\xi' d} \int_0^{\zeta_0} \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos\left[k_p z - \pi/2 + p_0\left(z\right)\right]}{\sinh \zeta} u\left(\zeta, \varphi, \xi\right) h_\varphi d\varphi h_\zeta d\zeta$$

$$= 2 \int_0^d \cos\left[k_p z + p_0\left(z\right)\right] u\left(0, 0, \xi\right) h_\xi \frac{d\xi}{dz} dz$$

$$+2 \int_d^\ell \cos\left[k_p z - \pi/2 + p_0\left(z\right)\right] u\left(\zeta, 0, \pi/2\right) h_\zeta \frac{d\zeta}{dz} dz \tag{510}$$

or (note that this limit definition of the integration around the scar produces a definition without the extra amplitude divergence factors in the kernel of the projection operator)

$$V_p = 2 \int_0^d \cos\left[k_p z + p_0\left(z\right)\right] u\left(0, 0, z\right) dz$$

$$+2 \int_d^\ell \cos\left[k_p z - \pi/2 + p_0\left(z\right)\right] u\left(0, 0, z\right) dz \tag{511}$$

where the metric coefficients are

$$h_\zeta = h_\xi = d\sqrt{\sinh^2 \zeta + \cos^2 \xi}$$

$$h_\varphi = d \sinh \zeta \cos \xi = d \sinh \zeta \sin \xi'$$

$$\frac{dz}{d\zeta} = d \sinh \zeta \sin \xi$$

$$\frac{dz}{d\xi} = d \cosh \zeta \cos \xi$$

and for $\zeta \to 0$

$$h_\xi \frac{d\xi}{dz} \to 1$$

and for $\xi \to \pi/2$

$$h_\zeta \frac{d\zeta}{dz} \to 1$$

where

$$p_0(z) = \frac{s}{2}\left\{(z/\ell)\ln\left|\frac{\ell+d}{\ell-d}\right| - \ln\left|\frac{z+d}{z-d}\right|\right\} \tag{512}$$

The cylindrical form of the solution in Region 1 is

$$u_p(0,0,z) = 2e^{-s\pi/2}\frac{\psi_0(0,s)}{\sqrt{1-z^2/d^2}}\cos\left[k_p z + p_0(z)\right]$$

$$p_0(z) = \frac{s}{2}\left\{(z/\ell)\ln\left(\frac{\ell+d}{\ell-d}\right) - \ln\left(\frac{d+z}{d-z}\right)\right\}$$

$$\psi_0(0,s) = c_0 e^{\pi s/2}\,\mathrm{Re}\left[e^{-i\pi/4}\Gamma(1/2+is/2)\right]$$

$$c_0 = v\frac{2}{\sqrt{Ad\ln(\Lambda_+)(1+e^{-\pi s})}\,\mathrm{Re}\left[e^{-i\pi/4}\Gamma(1/2+is/2)\right]}$$

where

$$s = 2(k-k_p)\,\ell/\ln\left(\frac{\ell+d}{\ell-d}\right)$$

$$k_p\ell = \pi p$$

The cylindrical form of the solution in Region 2 is

$$u_p(0,0,z) = \frac{2}{\sqrt{z^2/d^2-1}}\psi_0(0,s')\cos\left[k_p z + p_0(z) - \pi/2\right]$$

$$p_0(z) = -\frac{s'}{2}\left\{(z/\ell)\ln\left(\frac{\ell+d}{\ell-d}\right) - \ln\left(\frac{z+d}{z-d}\right)\right\}$$

$$\psi_0(0,s') = c_0 e^{-\pi s/2}\,\mathrm{Re}\left[e^{-i\pi/4}\Gamma(1/2+is/2)\right]$$

where

$$s' = -s$$

We now include the exponential ratio for $s \neq 0$ between region 1 to region 2 in redefining the elliptic projection operator. As in the 2D case [2] we take the operator to have the proper exponential weights relative to the scar solution, but we remove the growing weight as $s \to \pm\infty$

$$\exp(\pi|s|/4)\,V_p = 2e^{s\pi/4}\int_0^d \cos\left[k_p z + p_0(z)\right]u(0,0,z)\,dz$$

$$+2e^{-s\pi/4}\int_d^\ell \cos\left[k_p z - \pi/2 + p_0(z)\right]u(0,0,z)\,dz \tag{513}$$

Then the elliptic projection is

$$\exp\left(\pi\left|s\right|/4\right)V_p/\left\{4c_0\,\mathrm{Re}\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]\right\}=$$

$$e^{s\pi/4}\int_0^d\cos^2\left[k_pz+p_0\left(z\right)\right]\frac{dz}{\sqrt{1-z^2/d^2}}+e^{-s\pi/4}e^{-\pi s/2}\int_d^\ell\cos^2\left[k_pz-\pi/2+p_0\left(z\right)\right]\frac{dz}{\sqrt{z^2/d^2-1}}$$

where

$$p_0\left(z\right)=\frac{s}{2}\left\{\left(z/\ell\right)\ln\left(\frac{\ell+d}{\ell-d}\right)-\ln\left|\frac{z+d}{z-d}\right|\right\}$$

$$c_0\,\mathrm{Re}\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]=v\frac{2}{\sqrt{Ad\ln\left(\Lambda_+\right)\left(1+e^{-\pi s}\right)}}$$

Averaging over the rapidly varying cosines gives

$$\exp\left(\pi\left|s\right|/4\right)V_p/\left\{v\frac{4}{\sqrt{Ad\ln\left(\Lambda_+\right)\left(1+e^{-\pi s}\right)}}\right\}\sim$$

$$e^{s\pi/4}\int_0^d\frac{dz}{\sqrt{1-z^2/d^2}}+e^{-s\pi/4}e^{-\pi s/2}\int_d^\ell\frac{dz}{\sqrt{z^2/d^2-1}}$$

Thus we finally have

$$\left\langle LV_p^2\right\rangle=L^2G_1\left(s\right)/A \tag{514}$$

where

$$\exp\left(\pi\left|s\right|/2\right)G_1\left(s\right)=\frac{8\left(d/\ell\right)}{\ln\left(\Lambda_+\right)\left(1+e^{-\pi s}\right)}\left[e^{s\pi/4}\pi/2+e^{-s\pi/4}e^{-\pi s/2}\mathrm{arccosh}\left(\ell/d\right)\right]^2$$

$$=\frac{8\left(d/\ell\right)}{\ln\left(\Lambda_+\right)\left(1+e^{-\pi s}\right)}\left[e^{s\pi/4}\pi/2+e^{-s\pi/4}e^{-\pi s/2}\ln\left(\ell/d+\sqrt{\ell^2/d^2-1}\right)\right]^2$$

$$=\frac{4}{\cosh\left(\pi s/2\right)\ln\left(\Lambda_+\right)}\frac{\left(\Lambda_+-1\right)}{\left(\sqrt{\Lambda_+}+1\right)^2}\left[e^{s\pi/2}\pi/2+e^{-\pi s/2}\ln\left\{\frac{\left(\sqrt{\Lambda_+}+1\right)}{\left(\Lambda_+-1\right)}\left(\Lambda_+^{1/4}+1\right)^2\right\}\right]^2 \tag{515}$$

where

$$d/\ell=\frac{\left(\Lambda_+-1\right)}{\left(\sqrt{\Lambda_+}+1\right)^2}$$

$$\Lambda_+=\left(\frac{\ell+d}{\ell-d}\right)^2$$

Scar Projection With Amplitude Divergence Factors If we attempt to include the amplitude divergence factors in the scar projection then

$$\exp\left(\pi\left|s\right|/4\right)V_{pr}=2e^{s\pi/4}\int_0^d\frac{\cos\left[k_pz+p_0\left(z\right)\right]}{\sqrt{1-z^2/d^2}}u\left(0,0,z\right)dz$$

$$+2e^{-s\pi/4}\int_d^\ell\frac{\cos\left[k_pz-\pi/2+p_0\left(z\right)\right]}{\sqrt{z^2/d^2-1}}u\left(0,0,z\right)dz \tag{516}$$

The solution in Region 1 is

$$u_p\left(0,0,z\right)=2e^{-s\pi/2}\frac{\psi_0\left(0,s\right)}{\sqrt{1-z^2/d^2}}\cos\left[k_pz+p_0\left(z\right)\right]$$

$$p_0\left(z\right)=\frac{s}{2}\left\{\left(z/\ell\right)\ln\left(\frac{\ell+d}{\ell-d}\right)-\ln\left(\left|\frac{d+z}{d-z}\right|\right)\right\}$$

$$\psi_0\left(0,s\right)=c_0\operatorname{Re}\left[e^{-i\pi/4}e^{\pi s/2}\Gamma\left(1/2+is/2\right)\right]$$

$$c_0=v\frac{2}{\sqrt{Ad\ln\Lambda_+\left(1+e^{-\pi s}\right)}\operatorname{Re}\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]}$$

where

$$s=2\left(k-k_p\right)\ell/\ln\left(\frac{\ell+d}{\ell-d}\right)$$

$$k_p\ell=\pi p$$

The solution in Region 2 is

$$u_p\left(0,0,z\right)=\frac{2}{\sqrt{z^2/d^2-1}}\psi_0\left(0,s'\right)\cos\left[k_pz+p_0\left(z\right)-\pi/2\right]$$

$$p_0\left(z\right)=-\frac{s'}{2}\left\{\left(z/\ell\right)\ln\left(\frac{\ell+d}{\ell-d}\right)-\ln\left(\left|\frac{z+d}{z-d}\right|\right)\right\}$$

$$\psi_0\left(0,s'\right)=c_0e^{-\pi s/2}\operatorname{Re}\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]$$

where

$$s'=-s$$

Thus

$$\exp\left(\pi\left|s\right|/4\right)V_{pr}=4e^{-s\pi/4}\psi_0\left(0,s\right)\int_0^d\cos^2\left[k_pz+p_0\left(z\right)\right]\frac{dz}{1-z^2/d^2}$$

$$+4e^{-s\pi/4}\psi_0\left(0,s'\right)\int_d^\ell\cos^2\left[k_pz+p_0\left(z\right)-\pi/2\right]\frac{dz}{z^2/d^2-1} \tag{517}$$

This appears to be divergent without considering the transition of the solution to Region 3 at the focal point (which would then become a major contribution to the projection rather than a small correction, and represent a poor choice of the projection operator)! Alternatively the projection could be interpreted as a principal value if we introduced a minus sign into the projection operator definition between the two regions. The corresponding integration in the energy theorem was evaluated as a principal value limit because it included the change in sign of the outer phase derivative factor which multiplied the normal derivative of the solution on the orbit between the two regions. Such new definitions of the projection might be worth future consideration, but in this section we will not include the amplitude divergence factors in the kernel.

5.4.5 Random Plane Wave Elliptic System Projection

Here we again take the projection to be

$$\exp\left(\pi\left|s\right|/4\right)V_{pr}=2e^{s\pi/4}\int_0^d\cos\left[k_pz+p_0\left(z\right)\right]u_r\left(0,0,z\right)dz$$

$$+2e^{-s\pi/4}\int_d^\ell\cos\left[k_pz-\pi/2+p_0\left(z\right)\right]u_r\left(0,0,z\right)dz \tag{518}$$

$$p_0\left(z\right)=\frac{s}{2}\left\{\left(z/\ell\right)\ln\left(\frac{\ell+d}{\ell-d}\right)-\ln\left|\frac{z+d}{z-d}\right|\right\}$$

with the axisymmetric random plane wave representation

$$u_r\left(\rho,0,z\right)=\lim_{N\to\infty}\sqrt{4k/\left(\pi AN\right)}\,\mathrm{Re}\left[\sum_{j=1}^N a_jJ_0\left(k\rho\sin\theta_j\right)e^{i\alpha_j+ikz\cos\theta_j}\right] \tag{519}$$

Then

$$V_{pr}=2\lim_{N\to\infty}\sqrt{4k/\left(\pi AN\right)}\sum_{j=1}^N a_j\left\{e^{-\left(|s|-s\right)\pi/4}\int_0^d\cos\left[k_pz+p_0\left(z\right)\right]\cos\left(\alpha_j+kz\cos\theta_j\right)dz\right.$$

$$\left.+e^{-\left(|s|+s\right)\pi/4}\int_d^\ell\cos\left[k_pz-\pi/2+p_0\left(z\right)\right]\cos\left(\alpha_j+kz\cos\theta_j\right)dz\right\}$$

and averaging over the amplitudes a_j

$$\left\langle V_{pr}^2\right\rangle_{a_j}=4\lim_{N\to\infty}\frac{4k}{\pi AN}\sum_{j=1}^N\left\{e^{-\left(|s|-s\right)\pi/4}\int_0^d\cos\left[k_pz+p_0\left(z\right)\right]\cos\left(\alpha_j+kz\cos\theta_j\right)dz\right.$$

$$\left.+e^{-\left(|s|+s\right)\pi/4}\int_d^\ell\cos\left[k_pz-\pi/2+p_0\left(z\right)\right]\cos\left(\alpha_j+kz\cos\theta_j\right)dz\right\}^2$$

Next averaging over the phases α_j, using

$$\left\langle\cos\left(\alpha_j+kz\cos\theta_j\right)\cos\left(\alpha_j+kz'\cos\theta_j\right)\right\rangle_{\alpha_j}=\frac{1}{2\pi}\frac{1}{2}\int_0^{2\pi}\left[\cos\left(k\left(z-z'\right)\cos\theta_j\right)+\cos\left(2\alpha_j+k\left(z+z'\right)\cos\theta_j\right)\right]d\alpha_j$$

$$= \frac{1}{2}\cos\left(k\left(z-z'\right)\cos\theta_j\right) = \frac{1}{2}\cos\left(kz\cos\theta_j\right)\cos\left(kz'\cos\theta_j\right) + \frac{1}{2}\sin\left(kz\cos\theta_j\right)\sin\left(kz'\cos\theta_j\right)$$

and taking only the even part (the odd part will vanish if the integration is carried out over both halves of the stadium) gives

$$\left\langle V_{pr}^2\right\rangle_{a_j} = 2\lim_{N\to\infty}\frac{4k}{\pi AN}\sum_{j=1}^{N}\left\{e^{-(|s|-s)\pi/4}\int_0^d \cos\left[k_p z + p_0\left(z\right)\right]\cos\left(kz\cos\theta_j\right)dz\right.$$

$$\left.+e^{-(|s|+s)\pi/4}\int_d^\ell \cos\left[k_p z - \pi/2 + p_0\left(z\right)\right]\cos\left(kz\cos\theta_j\right)dz\right\}^2$$

$$= \lim_{N\to\infty}\frac{2k}{\pi AN}\sum_{j=1}^{N}\left[e^{-(|s|-s)\pi/4}\int_0^d \left\{\cos\left(\left(k_p - k\cos\theta_j\right)z + p_0\left(z\right)\right) + \cos\left(\left(k_p + k\cos\theta_j\right)z + p_0\left(z\right)\right)\right\}dz\right.$$

$$\left.+e^{-(|s|+s)\pi/4}\int_d^\ell \left\{\cos\left(\left(k_p - k\cos\theta_j\right)z - \pi/2 + p_0\left(z\right)\right) + \cos\left(\left(k_p + k\cos\theta_j\right)z - \pi/2 + p_0\left(z\right)\right)\right\}dz\right]^2$$

Now averaging over the angle θ_j gives

$$\left\langle V_p^2\right\rangle_r = \frac{2k}{\pi A}\frac{1}{2}\int_0^\pi$$

$$\left[e^{-(|s|-s)\pi/4}\int_0^d \left\{\cos\left(\left(k_p - k\cos\theta_j\right)z + p_0\left(z\right)\right) + \cos\left(\left(k_p + k\cos\theta_j\right)z + p_0\left(z\right)\right)\right\}dz\right.$$

$$\left.+e^{-(|s|+s)\pi/4}\int_d^\ell \left\{\cos\left(\left(k_p - k\cos\theta_j\right)z - \pi/2 + p_0\left(z\right)\right) + \cos\left(\left(k_p + k\cos\theta_j\right)z - \pi/2 + p_0\left(z\right)\right)\right\}dz\right]^2 \sin\theta_j d\theta_j$$

Now when $k \to k_p$ the first terms peak for $\theta_j \to 0$ and the second terms peak for $\theta_j \to \pi$ with $\cos\theta_j = -\cos\left(\theta_j - \pi\right)$ and $\sin\theta_j = -\sin\left(\theta_j - \pi\right)$. Thus we find (and $\theta_j = \zeta/\sqrt{k\ell/2}$)

$$\left\langle V_p^2\right\rangle_r \approx \frac{2k}{\pi A}\frac{1}{2}\int_0^\infty \left[e^{-(|s|-s)\pi/4}\int_0^d \cos\left(\left(k_p - k + k\theta_j^2/2\right)z + p_0\left(z\right)\right)dz\right.$$

$$\left.+e^{-(|s|+s)\pi/4}\int_d^\ell \cos\left(\left(k_p - k + k\theta_j^2/2\right)z - \pi/2 + p_0\left(z\right)\right)dz\right]^2 \theta_j d\theta_j$$

$$+\frac{2k}{\pi A}\frac{1}{2}\int_{-\infty}^\pi \left[e^{-(|s|-s)\pi/4}\int_0^d \cos\left(\left(k_p - k + k\left(\theta_j - \pi\right)^2/2\right)z + p_0\left(z\right)\right)dz\right.$$

$$+e^{-(|s|+s)\pi/4}\int_d^\ell \cos\left(\left(k_p-k+k\left(\theta_j-\pi\right)^2/2\right)z-\pi/2+p_0\left(z\right)\right)dz\Bigg]^2\left(\pi-\theta_j\right)d\theta_j$$

$$\approx\frac{2k}{\pi A}\int_0^\infty\Bigg[e^{-(|s|-s)\pi/4}\int_0^d \cos\left(\left(k_p-k+k\theta_j^2/2\right)z+p_0\left(z\right)\right)dz$$

$$+e^{-(|s|+s)\pi/4}\int_d^\ell \cos\left(\left(k_p-k+k\theta_j^2/2\right)z-\pi/2+p_0\left(z\right)\right)dz\Bigg]^2\theta_j d\theta_j$$

$$\approx\frac{4}{\pi A\ell}\int_0^\infty\Bigg[e^{-(|s|-s)\pi/4}\int_0^d \cos\left(\left(\left(k_p-k\right)\ell+\zeta^2\right)z/\ell+p_0\left(z\right)\right)dz$$

$$+e^{-(|s|+s)\pi/4}\int_d^\ell \cos\left(\left(\left(k_p-k\right)\ell+\zeta^2\right)z/\ell-\pi/2+p_0\left(z\right)\right)dz\Bigg]^2\zeta d\zeta$$

Now setting

$$\lambda=2\left(k-k_p\right)L$$

$$\left\langle LV_p^2\right\rangle_r=L^2G\left(\lambda\right)/A \tag{520}$$

gives

$$G\left(\lambda\right)=\frac{2}{\pi}\int_0^\infty\Bigg[e^{-(|s|-s)\pi/4}\int_0^{d/\ell}\cos\left(\left(-\lambda/4+\zeta^2\right)z+p_0\left(z\ell\right)\right)dz$$

$$+e^{-(|s|+s)\pi/4}\int_{d/\ell}^1\cos\left(\left(-\lambda/4+\zeta^2\right)z/\ell-\pi/2+p_0\left(z\ell\right)\right)dz\Bigg]^2\zeta d\zeta$$

$$p_0\left(z\ell\right)=\frac{s}{2}\left\{z\ln\left(\frac{1+d/\ell}{1-d/\ell}\right)-\ln\left|\frac{z+d/\ell}{z-d/\ell}\right|\right\}$$

$$s=\lambda/\ln\left(\Lambda_+\right)$$

To account for the evenness along the orbit this is doubled

$$G_s\left(\lambda\right)=2G\left(\lambda\right)=\frac{2}{\pi}\int_{-\lambda/4}^\infty\Bigg[e^{-(|s|-s)\pi/4}\int_0^{d/\ell}\cos\left(\xi z+p_0\left(z\ell\right)\right)dz$$

$$+e^{-(|s|+s)\pi/4}\int_{d/\ell}^1\sin\left(\xi z+p_0\left(z\ell\right)\right)dz\Bigg]^2d\xi$$

For purposes of numerical integration

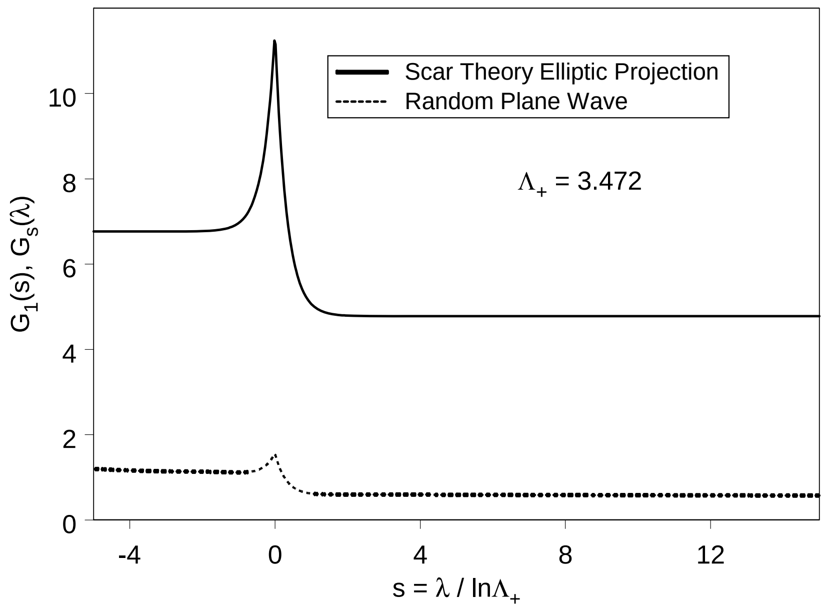

Figure 34. Comparison of elliptic projections of the rotationally symmetric stadium cavity field and the axisymmetric random plane wave field.

$$G_s(\lambda) = 2G(\lambda) = \frac{2}{\pi}\int_{-\lambda/4}^{\infty}\left[e^{-(|s|-s)\pi/4}\int_0^{d/\ell}\{\cos(\xi z)\cos(p_0(z\ell)) - \sin(\xi z)\sin(p_0(z\ell))\}\,dz\right.$$

$$\left. + e^{-(|s|+s)\pi/4}\int_{d/\ell}^{1}\{\cos(\xi z)\sin(p_0(z\ell)) + \sin(\xi z)\cos(p_0(z\ell))\}\,dz\right]^2 d\xi \qquad (521)$$

The comparison of the elliptic projections in the stadium cavity is shown in Figure 34.

6 VECTOR TREATMENT OF AXISYMMETRIC STADIUM

We now consider the electromagnetic vector problem in the axisymmetric stadium cavity.

6.1 Modal Solution In Region One: Between Foci

The parabolic equation in quasirectangular coordinates for the Hertz potential component Π_{ev} or $\Pi_{e\eta}$ (or W) resulting from the vector wave equation is the same as that arising from the three dimensional scalar Helmholtz equation. Hence we can transform the parabolic equation for the Hertz potential from the quasirectangular system back to prolate spheroidal coordinates.

Because the solution must be even in ξ we can write in Region 1

$$\Pi_{ev} = C\left[W(\zeta,\varphi,\xi)\, e^{i\gamma\sin\xi} + W(\zeta,\varphi,-\xi)\, e^{-i\gamma\sin\xi}\right] \ , \ \ |\xi| < \xi_0$$

$$= C\frac{1}{\cos\xi}\left[\Psi(\tau,\varphi,\sigma)\, e^{i\gamma\sin\xi} + \Psi(\tau,\varphi,-\sigma)\, e^{-i\gamma\sin\xi}\right]$$

$$= 2C\frac{\psi_m(\tau,s)}{\cos\xi}\cos(\gamma\sin\xi - s\sigma)\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}$$

$$= 2C\frac{\psi_m(\tau,s)}{\sin\xi'}\cos\left(\gamma\cos\xi' - s\sigma\right)\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}, \ \xi' = \pi/2 - \xi$$

where in the even stadium case

$$k_p\ell = p\pi$$

For matching purposes (with Region 2) it is convenient to take

$$\psi_m(\tau,s) = c_m\frac{\sqrt{2}}{\tau}\,\mathrm{Re}\left[e^{-i\pi/2}W_+(\tau,m,s) + e^{i\pi/2}e^{i\Phi_0}W_+^*(\tau,m,s)\right]$$

$$\sigma = \int_0^\xi \frac{d\xi}{\cos\xi} = \mathrm{arcsinh}(\tan\xi)$$

$$= \ln(\tan\xi + \sec\xi) = \ln\left\{\tan\frac{1}{2}(\xi + \pi/2)\right\} = -\ln\left\{\tan\left(\xi'/2\right)\right\}$$

$$\tau = \sqrt{2\gamma}\zeta$$

$$W_+(\tau,m,s) = W_{is/2,m/2}\left(-i\tau^2/2\right)$$

For the case where the mode is odd along the orbit

$$\Pi_{ev} = C\left[W(\zeta,\varphi,\xi)\, e^{i\gamma\sin\xi} - W(\zeta,\varphi,-\xi)\, e^{-i\gamma\sin\xi}\right] \ , \ \ |\xi| < \xi_0$$

$$= C\frac{1}{\cos\xi}\left[\Psi(\tau,\varphi,\sigma)\, e^{i\gamma\sin\xi} - \Psi(\tau,\varphi,-\sigma)\, e^{-i\gamma\sin\xi}\right]$$

$$= 2C\frac{\psi_m(\tau,s)}{\cos\xi}\sin(\gamma\sin\xi - s\sigma)\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}$$

$$= 2C\frac{\psi_m(\tau,s)}{\sin\xi'}\sin\left(\gamma\cos\xi' - s\sigma\right)\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}, \ \xi' = \pi/2 - \xi$$

where in the odd stadium case

$$k_p \ell = (p - 1/2)\,\pi$$

6.1.1 Behavior Of Zero Mode Near Orbit For Stadium Solutions

The Region 1 behavior of the zero mode $m = 0$

$$\psi_0 = c_0 \frac{\sqrt{2}}{\tau} \operatorname{Re}\left[e^{-i\pi/2} W_+(\tau, 0, s) + e^{i\pi/2} e^{i\Phi_0} W_+^*(\tau, 0, s)\right]$$

is now discussed near $\tau \to 0$. Again using

$$W_+(\tau, 0, s) \sim -\left(-i\tau^2/2\right)^{1/2} \frac{1}{\Gamma(1/2 - is/2)} \left[\ln\left(-i\tau^2/2\right)\left(1 - s\tau^2/4\right)\right.$$

$$\left. + \psi(1/2 - is/2) - 2\psi(1) + \left(s\tau^2/4\right)\{-\psi(1/2 - is/2) + 2\psi(1) + 2\} + O\left(\tau^4 \ln\tau\right)\right]$$

$$W_+'(\tau, 0, s) \sim i\tau \frac{1}{2}\left(-i\tau^2/2\right)^{-1/2} \frac{1}{\Gamma(1/2 - is/2)}$$

$$\left[\ln\left(-i\tau^2/2\right) + \{\psi(1/2 - is/2) - 2\psi(1)\} + 2 + O\left(\tau^2 \ln\tau\right)\right] \ , \ \tau \to 0 \ , \ m = 0$$

and taking

$$\tau \frac{\partial \psi_0}{\partial \tau} = c_0 \sqrt{2} \operatorname{Re}\left[e^{-i\pi/2} W_+' + e^{i\pi/2} e^{i\Phi_0} W_+^{*\prime}\right] - \psi_0$$

to vanish as $\tau \to 0$ gives the reflection phase

$$e^{i\Phi_0} = -\frac{e^{-i\pi/2}\left(W_+' - \frac{1}{\tau} W_+\right)}{e^{i\pi/2}\left(W_+' - \frac{1}{\tau} W_+\right)^*} = -i \frac{\Gamma(1/2 + is/2)}{\Gamma(1/2 - is/2)} + O\left(\tau^2\right)$$

We also have

$$\psi_0(0, s) = c_0 \operatorname{Re}\left[e^{\pi s/2} e^{-i\pi/4} \Gamma(1/2 + is/2)\right] \ , \ \tau \to 0 \tag{522}$$

which is again $-i$ inside the real part times the preceding bowtie result. In general

$$\psi_0(\tau, s) \sim c_0 \operatorname{Re}\left[e^{-i\pi/4} \Gamma(1/2 + is/2)\right] e^{\pi s/2}\left(1 - s\tau^2/4\right) + O\left(\tau^4 \ln\tau\right)$$

$$\tau \frac{\partial}{\partial \tau} \psi_0(\tau, s) \sim -c_0 \operatorname{Re}\left[e^{-i\pi/4} \Gamma(1/2 + is/2)\right] e^{\pi s/2}\left(s\tau^2/2\right) + O\left(\tau^4 \ln\tau\right) \sim -\left(s\tau^2/2\right)\psi_0$$

$$\frac{1}{\tau} \frac{\partial}{\partial \tau} \psi_0(\tau, s) \sim -c_0 \operatorname{Re}\left[e^{-i\pi/4} \Gamma(1/2 + is/2)\right] e^{\pi s/2}(s/2) + O\left(\tau^4 \ln\tau\right) \sim -\frac{s}{2}\psi_0$$

$$\frac{1}{\tau} \frac{\partial}{\partial \tau}\left(\tau \frac{\partial}{\partial \tau} \psi_0\right) \sim -c_0 \operatorname{Re}\left[e^{-i\pi/4} \Gamma(1/2 + is/2)\right] e^{\pi s/2} s + O\left(\tau^2 \ln\tau\right) \sim -s\psi_0 + O\left(\tau^2 \ln\tau\right)$$

In addition

$$\tau\frac{\partial^2\psi_0}{\partial\omega\partial\tau}(\tau,s)\approx c_0\sqrt{2}\,\mathrm{Re}\left[i\frac{\partial\Phi_0}{\partial\omega}e^{i\Phi_0}e^{i\pi/2}\left(W_+'-\frac{1}{\tau}W_+\right)^*\right]$$

and

$$\tau\frac{\partial^2\psi_0}{\partial\omega\partial\tau}(0,s)\approx 2c_0\,\mathrm{Re}\left[\frac{e^{-i\pi/4}}{\Gamma\left(1/2-is/2\right)}\right]\frac{\partial\Phi_0}{\partial\omega}$$

6.2 Modal Description In Region Two: Outside Foci

We first modify the quasirectangular coordinate system to the region outside the foci.

6.2.1 Quasirectangular Coordinates Outside Foci

It will be useful in the vector problem to introduce a quasirectangular system of coordinates (υ',η',ζ)

$$\upsilon'=\xi'\cos\varphi \tag{523}$$

$$\eta'=\xi'\sin\varphi \tag{524}$$

where

$$\xi=\pm\pi/2-\xi'$$

$$\xi'=\sqrt{\upsilon'^2+\eta'^2} \tag{525}$$

$$\cos\varphi=\upsilon'/\sqrt{\upsilon'^2+\eta'^2} \tag{526}$$

$$\sin\varphi=\eta'/\sqrt{\upsilon'^2+\eta'^2} \tag{527}$$

We can write

$$x=d\sinh\zeta\cos\xi\cos\varphi=\pm d\sinh\zeta\sin\xi'\cos\varphi=\pm d\sinh\zeta\frac{\sin\sqrt{\upsilon'^2+\eta'^2}}{\sqrt{\upsilon'^2+\eta'^2}}\upsilon' \tag{528}$$

$$y=d\sinh\zeta\cos\xi\sin\varphi=\pm d\sinh\zeta\sin\xi'\sin\varphi=\pm d\sinh\zeta\frac{\sin\sqrt{\upsilon'^2+\eta'^2}}{\sqrt{\upsilon'^2+\eta'^2}}\eta' \tag{529}$$

$$z=d\cosh\zeta\sin\xi=\pm d\cosh\zeta\cos\xi'=\pm d\cosh\zeta\cos\sqrt{\upsilon'^2+\eta'^2} \tag{530}$$

Thus the position vector

$$\underline{r} = x\underline{e}_x + y\underline{e}_y + z\underline{e}_z \tag{531}$$

can be used to define the unit vectors [10]

$$|\partial\underline{r}/\partial\upsilon'|\,\underline{e}_{\upsilon'} = \partial\underline{r}/\partial\upsilon' =$$

$$\frac{\pm d\sinh\zeta}{\sqrt{\upsilon'^2+\eta'^2}}\left[\frac{\upsilon'^2}{\sqrt{\upsilon'^2+\eta'^2}}\cos\sqrt{\upsilon'^2+\eta'^2}+\frac{\eta'^2}{\upsilon'^2+\eta'^2}\sin\sqrt{\upsilon'^2+\eta'^2}\right]\underline{e}_x$$

$$+\frac{\pm d\sinh\zeta}{\sqrt{\upsilon'^2+\eta'^2}}\left[\frac{1}{\sqrt{\upsilon'^2+\eta'^2}}\cos\sqrt{\upsilon'^2+\eta'^2}-\frac{1}{\upsilon'^2+\eta'^2}\sin\sqrt{\upsilon'^2+\eta'^2}\right]\upsilon'\eta'\underline{e}_y$$

$$-\frac{\pm d\cosh\zeta}{\sqrt{\upsilon'^2+\eta'^2}}\upsilon'\sin\sqrt{\upsilon'^2+\eta'^2}\underline{e}_z \tag{532}$$

$$|\partial\underline{r}/\partial\eta'|\,\underline{e}_{\eta'} = \partial\underline{r}/\partial\eta' =$$

$$\frac{\pm d\sinh\zeta}{\sqrt{\upsilon'^2+\eta'^2}}\left[\frac{1}{\sqrt{\upsilon'^2+\eta'^2}}\cos\sqrt{\upsilon'^2+\eta'^2}-\frac{1}{\upsilon'^2+\eta'^2}\sin\sqrt{\upsilon'^2+\eta'^2}\right]\upsilon'\eta'\underline{e}_x$$

$$+\frac{\pm d\sinh\zeta}{\sqrt{\upsilon'^2+\eta'^2}}\left[\frac{\eta'^2}{\sqrt{\upsilon'^2+\eta'^2}}\cos\sqrt{\upsilon'^2+\eta'^2}+\frac{\upsilon'^2}{\upsilon'^2+\eta'^2}\sin\sqrt{\upsilon'^2+\eta'^2}\right]\underline{e}_y$$

$$-\frac{\pm d\cosh\zeta}{\sqrt{\upsilon'^2+\eta'^2}}\eta'\sin\sqrt{\upsilon'^2+\eta'^2}\underline{e}_z \tag{533}$$

We can write that

$$\underline{e}_{\upsilon'}\cdot\underline{e}_{\upsilon'} = \underline{e}_{\eta'}\cdot\underline{e}_{\eta'} = 1 \tag{534}$$

and the cross term is

$$\frac{\partial\underline{r}}{\partial\upsilon'}\cdot\frac{\partial\underline{r}}{\partial\eta'} = \frac{\upsilon'\eta' d^2}{\upsilon'^2+\eta'^2}\left[\sinh^2\zeta\left(\cos^2\sqrt{\upsilon'^2+\eta'^2}-\frac{\sin^2\sqrt{\upsilon'^2+\eta'^2}}{\upsilon'^2+\eta'^2}\right)+\cosh^2\zeta\sin^2\sqrt{\upsilon'^2+\eta'^2}\right]$$

$$= \frac{\upsilon'\eta' d^2}{\upsilon'^2+\eta'^2}\left[\sinh^2\zeta\left(1-\frac{\sin^2\sqrt{\upsilon'^2+\eta'^2}}{\upsilon'^2+\eta'^2}\right)+\sin^2\sqrt{\upsilon'^2+\eta'^2}\right] \tag{535}$$

Alternatively we can begin in the prolate spheroidal system with metric coefficients

$$h_\zeta = h_\xi = d\sqrt{\sinh^2\zeta+\cos^2\xi} = d\sqrt{\sinh^2\zeta+\sin^2\xi'} = d\sqrt{\sinh^2\zeta+\sin^2\sqrt{\upsilon'^2+\eta'^2}}$$

$$h_\varphi = d\sinh\zeta\cos\xi = \pm d\sinh\zeta\sin\xi' = \pm d\sinh\zeta\sin\sqrt{\upsilon'^2+\eta'^2}$$

and differential position vector [10]

$$d\underline{r} = h_\xi d\xi \underline{e}_\xi + h_\varphi d\varphi \underline{e}_\varphi + h_\zeta d\zeta \underline{e}_\zeta \tag{536}$$

$$\tan\varphi = \eta'/\upsilon' \tag{537}$$

$$\xi' = \sqrt{\upsilon'^2 + \eta'^2}$$

$$= -\frac{\eta'}{\upsilon'^2\left(1 + \eta'^2/\upsilon'^2\right)}$$

Therefore

$$\left|\frac{\partial \underline{r}}{\partial \upsilon'}\right| \underline{e}_{\upsilon'} = \frac{\partial \underline{r}}{\partial \upsilon'} = h_\xi \frac{\partial \xi}{\partial \upsilon'}\underline{e}_\xi + h_\varphi \frac{\partial \varphi}{\partial \upsilon'}\underline{e}_\varphi + h_\zeta \frac{\partial \zeta}{\partial \upsilon'}\underline{e}_\zeta = h_\xi \frac{\partial \xi}{\partial \upsilon'}\underline{e}_\xi + h_\varphi \frac{\partial \varphi}{\partial \upsilon'}\underline{e}_\varphi$$

$$= -h_\xi \frac{\upsilon'}{\sqrt{\upsilon'^2 + \eta'^2}}\underline{e}_\xi - h_\varphi \frac{\eta'}{\upsilon'^2 + \eta'^2}\underline{e}_\varphi \tag{538}$$

$$\left|\frac{\partial \underline{r}}{\partial \upsilon'}\right| = \frac{1}{\sqrt{\upsilon'^2 + \eta'^2}}\sqrt{h_\varphi^2 \frac{\eta'^2}{\upsilon'^2 + \eta'^2} + h_\xi^2 \upsilon'^2} = \sqrt{g_{\upsilon'\upsilon'}}$$

$$= \frac{1}{\xi'}\sqrt{h_\varphi^2 \sin^2\varphi + h_\xi^2 \xi'^2 \cos^2\varphi} = \frac{d}{\xi'}\sqrt{\sinh^2\zeta \sin^2\xi' \sin^2\varphi + \left(\sinh^2\zeta + \sin^2\xi'\right)\xi'^2\cos^2\varphi} \tag{539}$$

As a check, if we transform the metric coefficients and the prolate spheroidal unit vectors in this expression to the Cartesian system by means of the results in the preceding subsection we end up with the same unit vector as the preceding expression. Nevertheless these prolate spheroidal expressions are somewhat simpler. Thus the other direction becomes

$$\left|\frac{\partial \underline{r}}{\partial \eta'}\right| \underline{e}_{\eta'} = \frac{\partial \underline{r}}{\partial \eta'} = h_\xi \frac{\partial \xi}{\partial \eta'}\underline{e}_\xi + h_\varphi \frac{\partial \varphi}{\partial \eta'}\underline{e}_\varphi + h_\zeta \frac{\partial \zeta}{\partial \eta'}\underline{e}_\zeta = h_\xi \frac{\partial \xi}{\partial \eta'}\underline{e}_\xi + h_\varphi \frac{\partial \varphi}{\partial \eta'}\underline{e}_\varphi$$

$$= -h_\xi \frac{\eta'}{\sqrt{\upsilon'^2 + \eta'^2}}\underline{e}_\xi + h_\varphi \frac{\upsilon'}{\upsilon'^2 + \eta'^2}\underline{e}_\varphi \tag{540}$$

$$\left|\frac{\partial \underline{r}}{\partial \eta'}\right| = \frac{1}{\sqrt{\upsilon'^2 + \eta'^2}}\sqrt{h_\varphi^2 \frac{\upsilon'^2}{\upsilon'^2 + \eta'^2} + h_\xi^2 \eta'^2} = \sqrt{g_{\eta'\eta'}}$$

$$= \frac{1}{\xi'}\sqrt{h_\varphi^2 \cos^2\varphi + h_\xi^2 \xi'^2 \sin^2\varphi} = \frac{d}{\xi'}\sqrt{\sinh^2\zeta \sin^2\xi' \cos^2\varphi + \left(\sinh^2\zeta + \sin^2\xi'\right)\xi'^2\sin^2\varphi} \tag{541}$$

Taking the dot product gives

$$\left|\frac{\partial \underline{r}}{\partial \upsilon'}\right|\left|\frac{\partial \underline{r}}{\partial \eta'}\right| \underline{e}_{\upsilon'} \cdot \underline{e}_{\eta'} = \frac{\partial \underline{r}}{\partial \upsilon'} \cdot \frac{\partial \underline{r}}{\partial \eta'} = \frac{\upsilon'\eta'}{\upsilon'^2 + \eta'^2}\left(h_\xi^2 - \frac{h_\varphi^2}{\upsilon'^2 + \eta'^2}\right)$$

$$= \frac{\upsilon'\eta'}{\upsilon'^2+\eta'^2} d^2 \left(\sinh^2\zeta + \sin^2\xi' - \frac{\sinh^2\zeta \sin^2\sqrt{\upsilon'^2+\eta'^2}}{\upsilon'^2+\eta'^2} \right) = \sqrt{g_{\upsilon'\eta'}}$$

or

$$\underline{e}_{\upsilon'} \cdot \underline{e}_{\eta'} = \frac{\upsilon'\eta'\xi'^2\left(h_\xi^2 - h_\varphi^2/\xi'^2\right)}{\sqrt{h_\varphi^2\eta'^2 + h_\xi^2\upsilon'^2\xi'^2}\sqrt{h_\varphi^2\upsilon'^2 + h_\xi^2\eta'^2\xi'^2}}$$

$$= \frac{\upsilon'\eta'\left(\upsilon'^2+\eta'^2\right)\left(h_\xi^2 - h_\varphi^2/\xi'^2\right)}{\sqrt{h_\varphi^2\eta'^2 + h_\xi^2\upsilon'^2\left(\upsilon'^2+\eta'^2\right)}\sqrt{h_\varphi^2\upsilon'^2 + h_\xi^2\eta'^2\left(\upsilon'^2+\eta'^2\right)}}$$

$$= \frac{\xi'^2\left(h_\xi^2 - h_\varphi^2/\xi'^2\right)\cos\varphi\sin\varphi}{\sqrt{h_\varphi^2\sin^2\varphi + h_\xi^2\xi'^2\cos^2\varphi}\sqrt{h_\varphi^2\cos^2\varphi + h_\xi^2\xi'^2\sin^2\varphi}}$$

$$= \frac{\xi'^2\left(\sinh^2\zeta + \sin^2\xi' - \sinh^2\zeta\sin^2\xi'/\xi'^2\right)\cos\varphi\sin\varphi}{\sqrt{\sinh^2\zeta\sin^2\xi'\sin^2\varphi + \left(\sinh^2\zeta + \sin^2\xi'\right)\xi'^2\cos^2\varphi}\sqrt{\sinh^2\zeta\sin^2\xi'\cos^2\varphi + \left(\sinh^2\zeta + \sin^2\xi'\right)\xi'^2\sin^2\varphi}}$$

$$\sqrt{g_{\upsilon'\upsilon'}g_{\eta'\eta'}}\underline{e}_{\upsilon'} \cdot \underline{e}_{\eta'} = d^2\left(\sinh^2\zeta + \sin^2\xi' - \sinh^2\zeta\sin^2\xi'/\xi'^2\right)\cos\varphi\sin\varphi \qquad (542)$$

where the metric coefficients are [10]

$$g_{\upsilon'\upsilon'} = \frac{\partial x}{\partial \upsilon'}\frac{\partial x}{\partial \upsilon'} + \frac{\partial y}{\partial \upsilon'}\frac{\partial y}{\partial \upsilon'} + \frac{\partial z}{\partial \upsilon'}\frac{\partial z}{\partial \upsilon'} \qquad (543)$$

$$g_{\upsilon'\eta'} = \frac{\partial x}{\partial \upsilon'}\frac{\partial x}{\partial \eta'} + \frac{\partial y}{\partial \upsilon'}\frac{\partial y}{\partial \eta'} + \frac{\partial z}{\partial \upsilon'}\frac{\partial z}{\partial \eta'} \qquad (544)$$

$$g_{\eta'\eta'} = \frac{\partial x}{\partial \eta'}\frac{\partial x}{\partial \eta'} + \frac{\partial y}{\partial \eta'}\frac{\partial y}{\partial \eta'} + \frac{\partial z}{\partial \eta'}\frac{\partial z}{\partial \eta'} \qquad (545)$$

Of course in the system $(\upsilon', \eta', \zeta)$ it is immediately clear from these representations and the orthogonality of the prolate spheroidal system that

$$\underline{e}_{\upsilon'} \cdot \underline{e}_\zeta = \underline{e}_{\eta'} \cdot \underline{e}_\zeta = 0 \qquad (546)$$

6.2.2 Near The Axis (Orbit)

Note that we expect ξ' to have small values on the orbit. Orthogonality approximately holds in the coordinate system $(\upsilon', \eta', \zeta)$ if $h_\xi \approx \left|h_\varphi/\xi'\right|$. Note that to have $h_\xi = d\sqrt{\sinh^2\zeta + \sin^2\xi'} \sim \left|h_\varphi/\xi'\right| = \left(d/\xi'\right)\sinh\zeta\sin\xi'$ we must have $\xi' << 1$, consistent with our expected value $\xi' \to 0$ on the orbit.)

Thus if orthogonality approximately holds, then we can write [10]

$$\underline{dr} = h_\zeta d\zeta \underline{e}_\zeta + h_\varphi d\varphi \underline{e}_\varphi + h_\xi d\xi \underline{e}_\xi \approx h_{\upsilon'} d\upsilon' \underline{e}_{\upsilon'} + h_{\eta'} d\eta' \underline{e}_{\eta'} + h_\zeta d\zeta \underline{e}_\zeta \tag{547}$$

$$\frac{1}{\xi'^2}\sqrt{h_\varphi^2 \eta'^2 + h_\xi^2 \upsilon'^2 \xi'^2} = \frac{1}{\sqrt{\upsilon'^2 + \eta'^2}}\sqrt{h_\varphi^2 \frac{\eta'^2}{\upsilon'^2 + \eta'^2} + h_\xi^2 \upsilon'^2} = \sqrt{g_{\upsilon'\upsilon'}} \approx h_{\upsilon'} \tag{548}$$

$$\frac{1}{\xi'^2}\sqrt{h_\varphi^2 \upsilon'^2 + h_\xi^2 \eta'^2 \xi'^2} = \frac{1}{\sqrt{\upsilon'^2 + \eta'^2}}\sqrt{h_\varphi^2 \frac{\upsilon'^2}{\upsilon'^2 + \eta'^2} + h_\xi^2 \eta'^2} = \sqrt{g_{\eta'\eta'}} \approx h_{\eta'} \tag{549}$$

$$g_{\upsilon'\eta'} \approx 0 \tag{550}$$

The coordinate relations in this limit $\xi'^2 = \upsilon'^2 + \eta'^2 \to 0$ (with $0 < \zeta < \infty$) become

$$x = \pm d \sinh\zeta \sin\xi' \cos\varphi = \pm d \sinh\zeta \frac{\sin\sqrt{\upsilon'^2 + \eta'^2}}{\sqrt{\upsilon'^2 + \eta'^2}} \upsilon' \sim \pm d\upsilon' \sinh\zeta \tag{551}$$

$$y = \pm d \sinh\zeta \sin\xi' \sin\varphi = \pm d \sinh\zeta \frac{\sin\sqrt{\upsilon'^2 + \eta'^2}}{\sqrt{\upsilon'^2 + \eta'^2}} \eta' \sim \pm d\eta' \sinh\zeta \tag{552}$$

$$r = \pm d \sinh\zeta \sin\xi' \sim d\xi' \sinh\zeta \tag{553}$$

$$z = \pm d \cosh\zeta \cos\xi' = \pm d \cosh\zeta \cos\sqrt{\upsilon'^2 + \eta'^2} \sim \pm d \cosh\zeta \tag{554}$$

Notice that for the positive side or top sign of $\xi = \pm\pi/2 - \xi'$ the positive coordinate υ' points in the direction of positive x. The metric coefficients become

$$h_{\upsilon'} \sim d \sinh\zeta \sim h_{\eta'} \tag{555}$$

where

$$h_\zeta = h_\xi = d\sqrt{\sinh^2\zeta + \sin^2\xi'} \sim d\sqrt{\sinh^2\zeta + \xi'^2} \sim d\sinh\zeta \tag{556}$$

$$h_\varphi = \pm d \sinh\zeta \sin\xi' \sim \pm d\xi' \sinh\zeta \tag{557}$$

Thus in this approximate limit we can regard them as all equal (except h_φ)

$$h_{\upsilon'} \sim h_{\eta'} \sim h_\zeta = h_\xi \sim h = d\sqrt{\sinh^2\zeta + \xi'^2} \sim d\sinh\zeta \tag{558}$$

Note also that

$$\underline{e}_{\upsilon'} \cdot \underline{e}_\xi = \frac{-h_\xi \xi' \upsilon'}{\sqrt{h_\varphi^2 \eta'^2 + h_\xi^2 \upsilon'^2 \xi'^2}} \sim -\cos\varphi \tag{559}$$

$$\underline{e}_{\upsilon'} \cdot \underline{e}_{\varphi} = \frac{-h_{\varphi}\eta'}{\sqrt{h_{\varphi}^2\eta'^2 + h_{\xi}^2\upsilon'^2\xi'^2}} \sim \mp \sin\varphi \tag{560}$$

$$\underline{e}_{\eta'} \cdot \underline{e}_{\xi} = \frac{-h_{\xi}\xi'\eta'}{\sqrt{h_{\varphi}^2\upsilon'^2 + h_{\xi}^2\eta'^2\xi'^2}} \sim -\sin\varphi \tag{561}$$

$$\underline{e}_{\eta'} \cdot \underline{e}_{\varphi} = \frac{h_{\varphi}\upsilon'}{\sqrt{h_{\varphi}^2\upsilon'^2 + h_{\xi}^2\eta'^2\xi'^2}} \sim \pm \cos\varphi \tag{562}$$

$$\underline{e}_{\upsilon'} \sim -\underline{e}_{\xi}\cos\varphi \mp \underline{e}_{\varphi}\sin\varphi \tag{563}$$

$$\underline{e}_{\eta'} \sim -\underline{e}_{\xi}\sin\varphi \pm \underline{e}_{\varphi}\cos\varphi \tag{564}$$

$$-\underline{e}_{\xi} \sim \underline{e}_{\upsilon'}\cos\varphi + \underline{e}_{\eta'}\sin\varphi \tag{565}$$

$$\pm\underline{e}_{\varphi} \sim -\underline{e}_{\upsilon'}\sin\varphi + \underline{e}_{\eta'}\cos\varphi \tag{566}$$

Cross Products From the preceding results the cross products of the prolate spheroidal unit vectors are

$$\underline{e}_{\xi} \times \underline{e}_{\zeta} = \underline{e}_{\varphi}$$

$$\underline{e}_{\zeta} \times \underline{e}_{\varphi} = \underline{e}_{\xi}$$

$$\underline{e}_{\varphi} \times \underline{e}_{\xi} = \underline{e}_{\zeta}$$

and the cross products of the quasirectangular unit vectors near the axis are

$$\underline{e}_{\upsilon'} \times \underline{e}_{\eta'} \sim \pm\underline{e}_{\zeta} \tag{567}$$

$$\underline{e}_{\zeta} \times \underline{e}_{\upsilon'} \sim \pm\underline{e}_{\eta'} \tag{568}$$

$$\underline{e}_{\eta'} \times \underline{e}_{\zeta} \sim \pm\underline{e}_{\upsilon'} \tag{569}$$

6.2.3 Transformation of Scalar Parabolic Equation to Quasirectangular System

The parabolic equation in the scalar three-dimensional axisymmetric case is (for region 2)

$$\frac{1}{\cos\xi}\frac{\partial}{\partial\xi}\left(\cos\xi\frac{\partial W}{\partial\xi}\right)+\frac{1}{\cos^2\xi}\frac{\partial^2 W}{\partial\varphi^2}$$

$$+i2\gamma\sinh\zeta\frac{\partial W}{\partial\zeta}+\left(i2\gamma\cosh\zeta+\gamma^2\cos^2\xi\right)W=0$$

or approximately

$$\frac{1}{\xi'}\frac{\partial}{\partial\xi'}\left(\xi'\frac{\partial W}{\partial\xi'}\right)+\frac{1}{\xi'^2}\frac{\partial^2 W}{\partial\varphi^2}$$

$$+i2\gamma\sinh\zeta\frac{\partial W}{\partial\zeta}+\left(i2\gamma\cosh\zeta+\gamma^2\xi'^2\right)W=0 \tag{570}$$

can be transformed from the prolate spheroidal system to this quasirectangular system using

$$\xi'\frac{\partial W}{\partial\xi'}\sim\xi'\frac{\partial W}{\partial\upsilon'}\frac{\partial\upsilon'}{\partial\xi'}+\xi'\frac{\partial W}{\partial\eta'}\frac{\partial\eta'}{\partial\xi'}=\frac{\partial W}{\partial\upsilon'}\upsilon'+\frac{\partial W}{\partial\eta'}\eta'$$

$$\frac{\partial W}{\partial\varphi}\sim\frac{\partial W}{\partial\upsilon'}\frac{\partial\upsilon'}{\partial\varphi}+\frac{\partial W}{\partial\eta'}\frac{\partial\eta'}{\partial\varphi}=-\frac{\partial W}{\partial\upsilon'}\eta'+\frac{\partial W}{\partial\eta'}\upsilon'$$

$$\xi'\frac{\partial}{\partial\xi'}\left(\xi'\frac{\partial W}{\partial\xi'}\right)\sim\frac{\partial}{\partial\upsilon'}\left(\frac{\partial W}{\partial\upsilon'}\upsilon'+\frac{\partial W}{\partial\eta'}\eta'\right)\upsilon'+\frac{\partial}{\partial\eta'}\left(\frac{\partial W}{\partial\upsilon'}\upsilon'+\frac{\partial W}{\partial\eta'}\eta'\right)\eta'$$

$$\sim\frac{\partial^2 W}{\partial\upsilon'^2}\upsilon'^2+\frac{\partial W}{\partial\upsilon'}\upsilon'+2\frac{\partial^2 W}{\partial\eta'\partial\upsilon'}\eta'\upsilon'+\frac{\partial W}{\partial\eta'}\eta'+\frac{\partial^2 W}{\partial\eta'^2}\eta'^2$$

$$\frac{\partial^2 W}{\partial\varphi^2}\sim-\frac{\partial}{\partial\upsilon'}\left(-\frac{\partial W}{\partial\upsilon'}\eta'+\frac{\partial W}{\partial\eta'}\upsilon'\right)\eta'+\frac{\partial}{\partial\eta'}\left(-\frac{\partial W}{\partial\upsilon'}\eta'+\frac{\partial W}{\partial\eta'}\upsilon'\right)\upsilon'$$

$$\sim\frac{\partial^2 W}{\partial\upsilon'^2}\eta'^2-\frac{\partial W}{\partial\eta'}\eta'-2\frac{\partial^2 W}{\partial\eta'\partial\upsilon'}\upsilon'\eta'-\frac{\partial W}{\partial\upsilon'}\upsilon'+\frac{\partial^2 W}{\partial\eta'^2}\upsilon'^2$$

and

$$\xi'\frac{\partial}{\partial\xi'}\left(\xi'\frac{\partial W}{\partial\xi'}\right)+\frac{\partial^2 W}{\partial\varphi^2}\sim\left(\frac{\partial^2 W}{\partial\upsilon'^2}+\frac{\partial^2 W}{\partial\eta'^2}\right)\left(\upsilon'^2+\eta'^2\right)=\xi'^2\left(\frac{\partial^2 W}{\partial\upsilon'^2}+\frac{\partial^2 W}{\partial\eta'^2}\right)$$

to give

$$\frac{\partial^2 W}{\partial\upsilon'^2}+\frac{\partial^2 W}{\partial\eta'^2}+i2\gamma\sinh\zeta\frac{\partial W}{\partial\zeta}$$

$$+\left[i2\gamma\cosh\zeta+\gamma^2\left(\upsilon'^2+\eta'^2\right)\right]W=0 \tag{571}$$

where W is now a function of υ', η', and ζ. This is the same leading order parabolic equation as we find in the vector case below.

Note that we will later need

$$\xi' \frac{\partial W}{\partial \xi'} \sim \frac{\partial W}{\partial \upsilon'} \upsilon' + \frac{\partial W}{\partial \eta'} \eta' = \frac{\partial W}{\partial \upsilon'} \xi' \cos\varphi + \frac{\partial W}{\partial \eta'} \xi' \sin\varphi$$

$$\frac{\partial W}{\partial \varphi} \sim -\frac{\partial W}{\partial \upsilon'} \eta' + \frac{\partial W}{\partial \eta'} \upsilon' = -\frac{\partial W}{\partial \upsilon'} \xi' \sin\varphi + \frac{\partial W}{\partial \eta'} \xi' \cos\varphi$$

or

$$\xi' \frac{\partial W}{\partial \upsilon'} \sim \xi' \frac{\partial W}{\partial \xi'} \cos\varphi - \frac{\partial W}{\partial \varphi} \sin\varphi$$

$$\xi' \frac{\partial W}{\partial \eta'} \sim \xi' \frac{\partial W}{\partial \xi'} \sin\varphi + \frac{\partial W}{\partial \varphi} \cos\varphi$$

6.2.4 Hertz Potentials

The electromagnetic field is a vector field satisfying in source free homogeneous regions

$$\nabla \times \underline{H} = -i\omega\varepsilon \underline{E}$$

$$\nabla \times \underline{E} = i\omega\mu \underline{H}$$

$$\nabla \cdot \underline{H} = 0$$

$$\nabla \cdot \underline{E} = 0$$

and the boundary conditions on the walls (in our case these hold on the end mirrors $\zeta = \zeta_0$)

$$E_\xi = E_\varphi = H_\zeta = 0$$

or in the quasi-rectangular system

$$E_{\upsilon'} = E_{\eta'} = H_\zeta = 0 \tag{572}$$

We use the electric Hertz potential $\underline{\Pi}_e$ (we can alternatively use the magnetic Hertz potential $\underline{\Pi}_m$), with the fields given by

$$\underline{E} = \nabla \times \nabla \times \underline{\Pi}_e \tag{573}$$

$$\underline{H} = -i\omega\varepsilon \nabla \times \underline{\Pi}_e$$

where the Hertz vector satisfies the vector wave equation

$$-\nabla \times \nabla \times \underline{\Pi}_e + \nabla \left(\nabla \cdot \underline{\Pi}_e\right) + k^2 \underline{\Pi}_e = \left(\nabla^2 + k^2\right) \underline{\Pi}_e = 0 \tag{574}$$

$$k^2 = \omega^2 \mu \varepsilon$$

and the electric field is thus given by either (262) or, using (263), by

$$\underline{E} = \nabla \left(\nabla \cdot \underline{\Pi}_e \right) + k^2 \underline{\Pi}_e \tag{575}$$

At high frequencies we can make one of the following sets of approximations [5]

$$\Pi_{e\upsilon'} = \Phi \tag{576}$$

$$\Pi_{e\eta'} = 0 = \Pi_{e\zeta} \tag{577}$$

or

$$\Pi_{e\eta'} = \Phi \tag{578}$$

$$\Pi_{e\upsilon'} = 0 = \Pi_{e\zeta} \tag{579}$$

We cannot satisfy (263) exactly with this approximate choice, so instead we require that the potential $\Pi_{e\upsilon'}$ (or in the second case $\Pi_{e\eta'}$) satisfies the equation resulting from equating $E_{\upsilon'}$ (or in the second case $E_{\eta'}$) from (573) and (575). Thus the equation for the potential is [5]

$$\left[-\nabla \times \nabla \times \left(\underline{\Pi}_{e\upsilon'} \underline{e}_{\upsilon'} \right) + \nabla \left\{ \nabla \cdot \left(\underline{\Pi}_{e\upsilon'} \underline{e}_{\upsilon'} \right) \right\} + k^2 \left(\Pi_{e\upsilon'} \underline{e}_{\upsilon'} \right) \right] \cdot \underline{e}_{\upsilon'} = 0 \tag{580}$$

or

$$\left[-\nabla \times \nabla \times \left(\Pi_{e\eta'} \underline{e}_{\eta'} \right) + \nabla \left\{ \nabla \cdot \left(\Pi_{e\eta'} \underline{e}_{\eta'} \right) \right\} + k^2 \left(\Pi_{e\eta'} \underline{e}_{\eta'} \right) \right] \cdot \underline{e}_{\eta'} = 0 \tag{581}$$

6.2.5 Approximate Orthogonality And Fields

When we assume approximate orthogonality the equations simplify (as given in Vaynshteyn [5]). We can derive the following equations by using the orthogonal curvilinear coordinate results for gradient, divergence and curl [10], using the same metric coefficient h for all three coordinates

$$h_\zeta = h_\xi = d\sqrt{\sinh^2 \zeta + \cos^2 \xi} = d\sqrt{\sinh^2 \zeta + \sin^2 \xi'}$$

$$h_\varphi = d \sinh \zeta \cos \xi = \pm d \sinh \zeta \sin \xi'$$

$$\xi' = \pm \pi/2 - \xi$$

$$h_\zeta = h_\xi = h \sim d \sinh \zeta \sim h_{\upsilon'} \sim h_{\eta'} \tag{582}$$

$$h_\varphi \sim \pm d\xi' \sinh \zeta \sim \pm h\xi' \sim \pm h\sqrt{\upsilon'^2 + \eta'^2} \tag{583}$$

Thus the equation for the potential, assuming approximate orthogonality and equal metric coefficients (for the case where $\Pi_{e\upsilon'} = \Phi$), is

$$\left\{-\nabla \times \left[\frac{1}{h^2}\underline{e}_{\eta'}\frac{\partial}{\partial\zeta}\left(h\Pi_{e\upsilon'}\right) - \frac{1}{h^2}\underline{e}_{\zeta}\frac{\partial}{\partial\eta'}\left(h\Pi_{e\upsilon'}\right)\right]\right\}\cdot\underline{e}_{\upsilon'}$$

$$+\left\{\nabla\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon'}\left(h^2\Pi_{e\upsilon'}\right)\right]\right\}\cdot\underline{e}_{\upsilon'} + k^2\Pi_{e\upsilon'} = 0 \tag{584}$$

or

$$\frac{\partial}{\partial\zeta}\left\{\frac{1}{h}\frac{\partial}{\partial\zeta}\left(h\Pi_{e\upsilon'}\right)\right\} + \frac{\partial}{\partial\eta'}\left\{\frac{1}{h}\frac{\partial}{\partial\eta'}\left(h\Pi_{e\upsilon'}\right)\right\}$$

$$+h\frac{\partial}{\partial\upsilon'}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon'}\left(h^2\Pi_{e\upsilon'}\right)\right] + k^2h^2\Pi_{e\upsilon'} = 0 \tag{585}$$

Note that the first and second terms are from $(-\nabla\times\nabla\times\underline{\Pi}_e)\cdot\underline{e}_{\upsilon'}$ and the third term results from $(\nabla\nabla\cdot\underline{\Pi}_e)\cdot\underline{e}_{\upsilon'}$. The fields are

$$h^2E_{\upsilon'} = h\frac{\partial}{\partial\upsilon'}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon'}\left(h^2\Pi_{e\upsilon'}\right)\right] + k^2h^2\Pi_{e\upsilon'} \tag{586}$$

$$h^2E_{\eta'} = h\frac{\partial}{\partial\eta'}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon'}\left(h^2\Pi_{e\upsilon'}\right)\right] \tag{587}$$

$$h^2E_{\zeta} = h\frac{\partial}{\partial\zeta}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon'}\left(h^2\Pi_{e\upsilon'}\right)\right] \tag{588}$$

and

$$h^2H_{\zeta} = i\omega\varepsilon_0\frac{\partial}{\partial\eta'}\left(h\Pi_{e\upsilon'}\right) \tag{589}$$

$$h^2H_{\eta'} = -i\omega\varepsilon_0\frac{\partial}{\partial\zeta}\left(h\Pi_{e\upsilon'}\right) \tag{590}$$

Alternatively the potential $\Pi_{e\eta'} = \Phi$ satisfies

$$\frac{\partial}{\partial\zeta}\left\{\frac{1}{h}\frac{\partial}{\partial\zeta}\left(h\Pi_{e\eta'}\right)\right\} + \frac{\partial}{\partial\upsilon'}\left\{\frac{1}{h}\frac{\partial}{\partial\upsilon'}\left(h\Pi_{e\eta'}\right)\right\}$$

$$+h\frac{\partial}{\partial\eta'}\left[\frac{1}{h^3}\frac{\partial}{\partial\eta'}\left(h^2\Pi_{e\eta'}\right)\right] + k^2h^2\Pi_{e\eta'} = 0 \tag{591}$$

The fields are

$$h^2E_{\upsilon'} = h\frac{\partial}{\partial\upsilon'}\left[\frac{1}{h^3}\frac{\partial}{\partial\eta'}\left(h^2\Pi_{e\eta'}\right)\right] \tag{592}$$

$$h^2 E_{\eta'} = h\frac{\partial}{\partial \eta'}\left[\frac{1}{h^3}\frac{\partial}{\partial \eta'}\left(h^2 \Pi_{e\eta'}\right)\right] + k^2 h^2 \Pi_{e\eta'} \tag{593}$$

$$h^2 E_{\zeta} = h\frac{\partial}{\partial \zeta}\left[\frac{1}{h^3}\frac{\partial}{\partial \eta'}\left(h^2 \Pi_{e\eta'}\right)\right] \tag{594}$$

and

$$h^2 H_{\zeta} = -i\omega\varepsilon_0 \frac{\partial}{\partial \upsilon'}\left(h\Pi_{e\eta'}\right) \tag{595}$$

$$h^2 H_{\upsilon'} = i\omega\varepsilon_0 \frac{\partial}{\partial \zeta}\left(h\Pi_{e\eta'}\right) \tag{596}$$

6.2.6 Asymptotic Solution Of Quasirectangular Equations

We seek a high frequency region 2 solution of the form

$$\Pi_{e\upsilon'} = \Phi = W\left(\upsilon', \eta', \zeta\right) e^{i\gamma\cosh\zeta} + W\left(\upsilon', \eta', -\zeta\right) e^{-i\gamma\cosh\zeta}\ ,\ \ |\xi| > \xi_0 \tag{597}$$

and insert this into

$$\frac{\partial}{\partial \zeta}\left\{\frac{1}{h}\frac{\partial}{\partial \zeta}\left(h\Pi_{e\upsilon'}\right)\right\} + \frac{\partial}{\partial \eta'}\left\{\frac{1}{h}\frac{\partial}{\partial \eta'}\left(h\Pi_{e\upsilon'}\right)\right\}$$

$$+h\frac{\partial}{\partial \upsilon'}\left[\frac{1}{h^3}\frac{\partial}{\partial \upsilon'}\left(h^2\Pi_{e\upsilon'}\right)\right] + k^2 h^2 \Pi_{e\upsilon'} = 0$$

or

$$\frac{\partial^2}{\partial \zeta^2}\Pi_{e\upsilon'} + \left(\frac{1}{h}\frac{\partial h}{\partial \zeta}\right)\frac{\partial}{\partial \zeta}\Pi_{e\upsilon'} + \Pi_{e\upsilon'}\frac{\partial}{\partial \zeta}\left(\frac{1}{h}\frac{\partial h}{\partial \zeta}\right) + \frac{\partial}{\partial \eta'}\left\{\frac{1}{h}\frac{\partial}{\partial \eta'}\left(h\Pi_{e\upsilon'}\right)\right\}$$

$$+h\frac{\partial}{\partial \upsilon'}\left[\frac{1}{h^3}\frac{\partial}{\partial \upsilon'}\left(h^2\Pi_{e\upsilon'}\right)\right] + k^2 h^2 \Pi_{e\upsilon'} = 0 \tag{598}$$

then W satisfies

$$\frac{\partial^2}{\partial \zeta^2}W + \left(i2\gamma\sinh\zeta + \frac{1}{h}\frac{\partial h}{\partial \zeta}\right)\frac{\partial}{\partial \zeta}W$$

$$+h\frac{\partial}{\partial \upsilon'}\left[\frac{1}{h^3}\frac{\partial}{\partial \upsilon'}\left(h^2 W\right)\right] + \frac{\partial}{\partial \eta'}\left\{\frac{1}{h}\frac{\partial}{\partial \eta'}\left(hW\right)\right\}$$

$$+\left[k^2h^2 + i\gamma\cosh\zeta - \gamma^2\sinh^2\zeta + \left(i\gamma\sinh\zeta + \frac{\partial}{\partial \zeta}\right)\left(\frac{1}{h}\frac{\partial h}{\partial \zeta}\right)\right]W = 0$$

or

$$\frac{1}{h}\frac{\partial}{\partial\zeta}\left(h\frac{\partial}{\partial\zeta}W\right)+i2\gamma\sinh\zeta\frac{\partial}{\partial\zeta}W$$

$$+\frac{1}{h}\frac{\partial}{\partial\upsilon'}\left(h\frac{\partial}{\partial\upsilon'}W\right)+\frac{1}{h}\frac{\partial}{\partial\eta'}\left(h\frac{\partial}{\partial\eta'}W\right)$$

$$+\left[k^2h^2-\gamma^2\sinh^2\zeta+i\gamma\cosh\zeta+\left(i\gamma\sinh\zeta+\frac{\partial}{\partial\zeta}\right)\left(\frac{1}{h}\frac{\partial h}{\partial\zeta}\right)\right.$$

$$\left.+\frac{\partial}{\partial\eta'}\left(\frac{1}{h}\frac{\partial h}{\partial\eta'}\right)+h\frac{\partial}{\partial\upsilon'}\left(\frac{2}{h^2}\frac{\partial h}{\partial\upsilon'}\right)\right]W=0$$

or

$$\frac{1}{h}\frac{\partial}{\partial\zeta}\left(h\frac{\partial}{\partial\zeta}W\right)+i2\gamma\sinh\zeta\frac{\partial}{\partial\zeta}W$$

$$+\frac{1}{h}\frac{\partial}{\partial\upsilon'}\left(h\frac{\partial}{\partial\upsilon'}W\right)+\frac{1}{h}\frac{\partial}{\partial\eta'}\left(h\frac{\partial}{\partial\eta'}W\right)$$

$$+\left[\gamma^2\xi'^2+i\gamma\cosh\zeta+\left(i\gamma\sinh\zeta+\frac{\partial}{\partial\zeta}\right)\left(\frac{1}{h}\frac{\partial h}{\partial\zeta}\right)+\frac{\partial}{\partial\eta'}\left(\frac{1}{h}\frac{\partial h}{\partial\eta'}\right)+h\frac{\partial}{\partial\upsilon'}\left(\frac{2}{h^2}\frac{\partial h}{\partial\upsilon'}\right)\right]W=0$$

We now neglect the first term $\partial^2W/\partial\zeta^2$ and let (except in the k^2h^2 term)

$$h\sim d\sinh\zeta$$

giving

$$(i2\gamma\sinh\zeta+\coth\zeta)\frac{\partial W}{\partial\zeta}+\frac{\partial^2W}{\partial\upsilon'^2}+\frac{\partial^2W}{\partial\eta'^2}$$

$$+\left[\gamma^2\left(\upsilon'^2+\eta'^2\right)+i2\gamma\cosh\zeta-\operatorname{csch}^2\zeta\right]W=0 \tag{599}$$

Now take

$$W=f(\zeta)\,\Psi(\upsilon',\eta',\sigma') \tag{600}$$

where we want

$$\frac{f'}{f}=-\frac{i2\gamma\cosh\zeta-\operatorname{csch}^2\zeta}{i2\gamma\sinh\zeta+\coth\zeta}$$

$$1/f=\sinh\zeta+\frac{1}{i2\gamma}\coth\zeta \tag{601}$$

Using this along with the transformation

$$\sigma' = \int_{\infty}^{\zeta} \frac{d\zeta}{\sinh\zeta} = \ln\left[\tanh\left(\zeta/2\right)\right] \tag{602}$$

gives

$$\left(i2\gamma + \frac{\coth\zeta}{\sinh\zeta}\right)\frac{\partial\Psi}{\partial\sigma'} + \frac{\partial^2\Psi}{\partial\upsilon'^2} + \frac{\partial^2\Psi}{\partial\eta'^2}$$

$$+\left[\gamma^2\left(\upsilon'^2 + \eta'^2\right)\right]\Psi = 0$$

Note that

$$\sigma' = \int_{\infty}^{\zeta - i\pi} \frac{d\zeta}{\sinh\zeta} = -\int_{\infty}^{\zeta} \frac{d\zeta}{\sinh\zeta}$$

$$\sigma_0' = \ln\left[\tanh\left(\zeta_0/2\right)\right]$$

$$\sinh\left(\zeta - i\pi\right) \to e^{-i\pi}\sinh\zeta \tag{603}$$

We could also have dropped the terms $\coth\zeta$ and $\operatorname{csch}^2\zeta$ terms compared to the $\gamma\sinh\zeta$ and the $\gamma\cosh\zeta$ term (these terms could be accounted for by higher order terms in the asymptotic series just like the $\partial^2 W/\partial\zeta^2$ term and the higher order terms $\partial h/\partial\upsilon'$ and $\partial h/\partial\eta'$) to obtain directly

$$i2\gamma\sinh\zeta\frac{\partial W}{\partial\zeta} + \frac{\partial^2 W}{\partial\upsilon'^2} + \frac{\partial^2 W}{\partial\eta'^2}$$

$$+\left[\gamma^2\left(\upsilon'^2 + \eta'^2\right) + i2\gamma\cosh\zeta\right]W = 0 \tag{604}$$

which is the same as the preceding scalar case. For this leading term approach we would take

$$W = \frac{1}{\sinh\zeta}\Psi\left(\upsilon', \eta', \sigma'\right) \tag{605}$$

and obtain the same equation for Ψ. Scaling the independent variables by letting

$$\tau_x' = \sqrt{2\gamma}\upsilon' \tag{606}$$

$$\tau_y' = \sqrt{2\gamma}\eta' \tag{607}$$

gives

$$i\frac{\partial\Psi}{\partial\sigma'} + \frac{\partial^2\Psi}{\partial\tau_x'^2} + \frac{\partial^2\Psi}{\partial\tau_y'^2} + \frac{1}{4}\left(\tau_x'^2 + \tau_y'^2\right)\Psi = 0$$

Separating the variables

$$\Psi\left(\tau_x, \tau_y, \sigma\right) = \psi_a\left(\tau_x'\right)\psi_b\left(\tau_y'\right)\psi_c\left(\sigma'\right)$$

gives

$$\left(\frac{i}{\psi_c}\frac{\partial \psi_c}{\partial \sigma'}\right)+\left(\frac{1}{\psi_a}\frac{\partial^2 \psi_a}{\partial \tau_x'^2}+\tau_x'^2/4\right)+\left(\frac{1}{\psi_b}\frac{\partial^2 \psi_b}{\partial \tau_y'^2}+\tau_y'^2/4\right)=0$$

Each parenthetical term is independent of the variables associated with the others and therefore each equals a constant. We take the first term as s

$$\frac{\partial \psi_c}{\partial \sigma'}=-is'\psi_c$$

or

$$\psi_c=e^{-is'\sigma'} \tag{608}$$

Then we can write

$$\frac{\partial^2 \psi_a}{\partial \tau_x'^2}+\left(s_a'+\tau_x'^2/4\right)\psi_a=0$$

$$\frac{\partial^2 \psi_b}{\partial \tau_y'^2}+\left(s_b'+\tau_y'^2/4\right)\psi_b=0$$

$$s_a'+s_b'=s' \tag{609}$$

$$\Psi\left(\tau_x',\tau_y',\sigma'\right)=e^{-is_a'\sigma'}\psi_a\left(s_a',\tau_x'\right)e^{-is_b'\sigma'}\psi_b\left(s_b',\tau_y'\right)=e^{-is'\sigma'}\psi_a\left(s_a',\tau_x'\right)\psi_b\left(s_b',\tau_y'\right)$$

This is a form of the equation of the parabolic cylinder functions

$$\frac{\partial^2 \psi}{\partial \tau'^2}+\left(\frac{\tau'^2}{4}+s'\right)\psi=0 \tag{610}$$

The solution that is outgoing in τ is [1]

$$U_+\left(s,\tau\right)=e^{-\pi(s+i/2)/4}U\left(-is,\tau e^{-i\pi/4}\right) \tag{611}$$

where $U\left(a,z\right)$ is the standard solution [11]. Following [1] the total transverse solution is taken as the incident plus reflected form

$$\psi\left(s',\tau'\right)=c\,\mathrm{Re}\left[U_+\left(s',\tau'\right)+e^{i\Phi_0'}U_+^*\left(s',\tau'\right)\right] \tag{612}$$

where the constant c is used for normalization. The transverse boundary condition in τ' is a reflection with a random phase $\Phi_0'\left(k^2\right)$ which was introduced by Antonsen to match to the chaotic region of the cavity; it describes the phase relation between a wave leaving the vicinity of the unstable periodic orbit and one returning [1] with the variation of the pth component along the orbit. Figure 9 schematically illustrates a wave bouncing back and forth between mirrors in the region of the scarred orbit; it leaves the vicinity of the orbit and eventually returns from the outer chaotic region with transverse reflection phase $\Phi_0'\left(k^2\right)$. (For purposes of simplification this figure does not include the vertical evenness of the cavity, which confines the wave leaving and the wave reflected to either the upper half or lower half of the cavity.) Thus $\psi_a=\psi_a\left(s_a',\tau_x'\right)$ and $\psi_b=\psi_b\left(s_b',\tau_y'\right)$ are elliptic cylinder functions like in the two-dimensional case [6], [1].

On the mirror we have

$$0 = h^2 E_{\upsilon'} = h\frac{\partial}{\partial \upsilon'}\left[\frac{1}{h^3}\frac{\partial}{\partial \upsilon'}\left(h^2 \Pi_{e\upsilon'}\right)\right] + k^2 h^2 \Pi_{e\upsilon'}$$

$$0 = h^2 E_{\eta'} = h\frac{\partial}{\partial \eta'}\left[\frac{1}{h^3}\frac{\partial}{\partial \upsilon'}\left(h^2 \Pi_{e\upsilon'}\right)\right]$$

and

$$0 = h^2 H_{\zeta} = i\omega\varepsilon_0 \frac{\partial}{\partial \eta'}\left(h \Pi_{e\upsilon'}\right)$$

which are satisfied if

$$\Pi_{e\upsilon'}\left(\zeta = \zeta_0\right) = 0 \tag{613}$$

6.2.7 Asymptotic Solution Of Parabolic Equation In Prolate Spheroidal System

The parabolic equation in quasirectangular coordinates for the Hertz potential component $\Pi_{e\upsilon'}$ or $\Pi_{e\eta'}$ (or W) resulting from the vector wave equation is the same as that arising from the three dimensional scalar Helmholtz equation. Hence we can transform the parabolic equation for the Hertz potential from the quasirectangular system back to prolate spheroidal coordinates. In the second region outside the foci we assume that $0 < \zeta < \zeta_0$ and that $\cos^2 \xi << 1$. The potential in the even case is then

$$\Pi_{e\upsilon'} = e^{-i\pi/2}\left[W\left(\zeta, \varphi, \xi\right) e^{i\gamma\cosh\zeta} + W\left(\zeta - i\pi, \varphi, \xi\right) e^{-i\gamma\cosh\zeta}\right] \;,\;\; |\xi| > \xi_0$$

$$= \frac{1}{\sinh\zeta}\left[\Psi_m\left(\tau', \sigma'\right) e^{i\gamma\cosh\zeta - i\pi/2} + \Psi_m\left(\tau', -\sigma'\right) e^{-i\gamma\cosh\zeta + i\pi/2}\right]\left\{\begin{array}{c}\cos\left(m\varphi\right)\\ \sin\left(m\varphi\right)\end{array}\right\}$$

$$= \frac{2}{\sinh\zeta}\psi_m\left(\tau', s'\right)\cos\left(\gamma\cosh\zeta - s'\sigma' - \pi/2\right)\left\{\begin{array}{c}\cos\left(m\varphi\right)\\ \sin\left(m\varphi\right)\end{array}\right\} \tag{614}$$

where

$$\psi_m\left(\tau', s'\right) = c_m \frac{\sqrt{2}}{\tau'}\,\mathrm{Re}\left[W_+\left(\tau', m, s'\right) + e^{i\Phi_0'} W_+^*\left(\tau', m, s'\right)\right] \tag{615}$$

$$W_+\left(\tau', m, s'\right) = W_{is'/2, m/2}\left(-i\tau'^2/2\right)$$

$$\tau' = \sqrt{2\gamma}\xi'$$

$$\xi' = \pm\pi/2 - \xi$$

$$2s'\sigma_0' = 2s'\ln\left[\tanh\left(\zeta_0/2\right)\right] = \left(k - k_p\right) L \tag{616}$$

$$\sigma' = \int_{\infty}^{\zeta} \frac{d\zeta}{\sinh\zeta} = \ln\left[\tanh\left(\zeta/2\right)\right] \tag{617}$$

The mirror condition $\Pi_{ev'} = 0$ gives

$$\cos\left(\gamma\cosh\zeta_0 - s'\sigma_0' - \pi/2\right) = \cos\left(k_p\ell - \pi/2\right) = 0$$

or

$$k_p\ell = \pi p \tag{618}$$

In the odd case the potential is

$$\Pi_{ev'} = e^{-i\pi/2}\left[W\left(\zeta,\varphi,\xi\right)e^{i\gamma\cosh\zeta} - W\left(\zeta - i\pi,\varphi,\xi\right)e^{-i\gamma\cosh\zeta}\right] \ , \ \ |\xi| > \xi_0$$

$$= \frac{1}{\sinh\zeta}\left[\Psi_m\left(\tau',\sigma'\right)e^{i\gamma\cosh\zeta - i\pi/2} - \Psi_m\left(\tau',-\sigma'\right)e^{-i\gamma\cosh\zeta + i\pi/2}\right]\left\{\begin{array}{c}\cos\left(m\varphi\right)\\ \sin\left(m\varphi\right)\end{array}\right\}$$

$$= \frac{2i}{\sinh\zeta}\psi_m\left(\tau',s'\right)\sin\left(\gamma\cosh\zeta - s'\sigma' - \pi/2\right)\left\{\begin{array}{c}\cos\left(m\varphi\right)\\ \sin\left(m\varphi\right)\end{array}\right\}$$

The mirror condition $\Pi_{ev'} = 0$ gives

$$\sin\left(\gamma\cosh\zeta_0 - s'\sigma_0' - \pi/2\right) = \sin\left(k_p\ell - \pi/2\right) = 0$$

or

$$k_p\ell = \pi\left(p - 1/2\right)$$

Behavior Of Zero Mode Near Orbit For Stadium Solutions The Region 2 behavior of the zero mode

$$\psi_0\left(\tau',s'\right) = c_0\frac{\sqrt{2}}{\tau'}\,\mathrm{Re}\left[W_+\left(\tau',0,s'\right) + e^{i\Phi_0'}W_+^*\left(\tau',0,s'\right)\right]$$

$$W_+\left(\tau',0,s'\right) = W_{is'/2,0}\left(-i\tau'^2/2\right)$$

$$\tau' = \sqrt{2\gamma}\xi'$$

$$\xi' = \pm\pi/2 - \xi$$

Now taking

$$\tau'\frac{\partial\psi_0}{\partial\tau'} = c_0\sqrt{2}\,\mathrm{Re}\left[W_+' + e^{i\Phi_0'}W_+^{*\prime}\right] - \psi_0$$

which when taken to vanish as $\tau' \to 0$ gives

$$e^{i\Phi_0'} = -\frac{\left(W_+' - \frac{1}{\tau'}W_+\right)}{\left(W_+' - \frac{1}{\tau'}W_+\right)^*} = i\frac{\Gamma\left(1/2 + is'/2\right)}{\Gamma\left(1/2 - is'/2\right)} + O\left(\tau'^2\right)$$

The value as $\tau' \to 0$ is

$$\psi_0\left(0, s'\right) = c_0 \operatorname{Re}\left[e^{\pi s'/2} e^{i\pi/4} \Gamma\left(1/2 + is'/2\right)\right] \ , \ \tau' \to 0 \tag{619}$$

and in general

$$\psi_0\left(\tau', s'\right) \sim c_0 \operatorname{Re}\left[e^{i\pi/4}\Gamma\left(1/2 + is'/2\right)\right] e^{\pi s'/2}\left(1 - s'\tau'^2/4\right) + O\left(\tau'^4 \ln \tau'\right)$$

$$\tau' \frac{\partial}{\partial \tau'}\psi_0\left(\tau', s'\right) \sim -c_0 \operatorname{Re}\left[e^{i\pi/4}\Gamma\left(1/2 + is'/2\right)\right] e^{\pi s'/2}\left(s'\tau'^2/2\right) + O\left(\tau'^4 \ln \tau'\right) \sim -\left(s'\tau'^2/2\right)\psi_0$$

$$\frac{1}{\tau'}\frac{\partial}{\partial \tau'}\psi_0\left(\tau', s'\right) \sim -c_0 \operatorname{Re}\left[e^{i\pi/4}\Gamma\left(1/2 + is'/2\right)\right] e^{\pi s'/2}\left(s'/2\right) + O\left(\tau'^4 \ln \tau'\right) \sim -\frac{s'}{2}\psi_0$$

$$\frac{1}{\tau'}\frac{\partial}{\partial \tau'}\left(\tau'\frac{\partial}{\partial \tau'}\psi_0\right) \sim -c_0 \operatorname{Re}\left[e^{i\pi/4}\Gamma\left(1/2 + is'/2\right)\right] e^{\pi s'/2} s' + O\left(\tau'^2 \ln \tau'\right) \sim -s'\psi_0 + O\left(\tau'^2 \ln \tau'\right)$$

In addition

$$\tau' \frac{\partial^2 \psi_0}{\partial \omega \partial \tau'}\left(\tau', s'\right) \approx c_0\sqrt{2} \operatorname{Re}\left[i\frac{\partial \Phi_0'}{\partial \omega} e^{i\Phi_0'}\left(W_+' - \frac{1}{\tau'}W_+\right)^*\right]$$

$$\approx -c_0\sqrt{2} \operatorname{Re}\left[\frac{\partial \Phi_0'}{\partial \omega}\frac{\Gamma\left(1/2 + is'/2\right)}{\Gamma\left(1/2 - is'/2\right)}\left(W_+' - \frac{1}{\tau'}W_+\right)^*\right]$$

and

$$\tau' \frac{\partial^2 \psi_0}{\partial \omega \partial \tau'}\left(0, s'\right) \approx 2c_0 \operatorname{Re}\left[\frac{e^{i\pi/4}}{\Gamma\left(1/2 - is'/2\right)}\right]\frac{\partial \Phi_0'}{\partial \omega}$$

6.2.8 Approach Of Focal Point

We first take the limits of the outer two regions as the focal region is approached. However to allow flexibility in phase matching through the focus, we will allow the focal region to shift by a small amount in the Region 1 and Region 2 solutions by adding a horizontal x shift of δ, where $\delta << d$ is a geometrically small shift relative to the Region 3 solution. (This can also be viewed as distorting the coordinate with $d \to d + \delta$, which corresponds to a slight reduction of the elliptic cylinder radius of curvature on axis, since $d = \ell\sqrt{1 - R/\ell}$. Because the elliptic cylinder has increasing radii of curvature off axis, such a reduction on axis, might actually represent the stadium mirrors better at a finite wavelength.) Therefore

$$d \to d + \delta \tag{620}$$

$$\gamma \to \gamma + k\delta \tag{621}$$

where $\delta << d$ is a geometrically small shift. Thus in Region 1 we find

$$\Pi_{ev} \sim 2Cc_m \frac{1}{\sqrt{\gamma}\zeta} \operatorname{Re}\left[e^{-i\pi/2} W_+\left(\sqrt{2\gamma}\zeta, m, s\right) + e^{i\pi/2} e^{i\Phi_0} W_+^*\left(\sqrt{2\gamma}\zeta, m, s\right)\right]$$

$$\xi'^{-1} \cos\left[\gamma\left(1 - \xi'^2/2\right) + k\delta + s\ln\left(\xi'/2\right)\right] \left\{ \begin{array}{c} \cos(m\varphi) \\ \sin(m\varphi) \end{array} \right\} \text{ , } \xi' \to 0$$

In Region 2 we find

$$\Pi_{ev'} \sim 2c_m \frac{1}{\sqrt{\gamma}\xi'} \operatorname{Re}\left[W_+\left(\sqrt{2\gamma}\xi', m, s'\right) + e^{i\Phi_0'} W_+^*\left(\sqrt{2\gamma}\xi', m, s'\right)\right]$$

$$\zeta^{-1} \cos\left[\gamma\left(1 + \zeta^2/2\right) + k\delta - s'\ln(\zeta/2) - \pi/4\right] \left\{ \begin{array}{c} \cos(m\varphi) \\ \sin(m\varphi) \end{array} \right\} \text{ , } \zeta \to 0$$

where

$$s' \ln\left[\tanh\left(\zeta_0/2\right)\right] = (k - k_p)\,\ell$$

$$k_p \ell = \pi p$$

Let us restrict attention to the $m = 0$ azimuthal mode (keep in mind here that the quasirectangular unit vectors are still present since we only have the υ and υ' components of the potential).

Thus in Region 1 we find

$$\Pi_{ev} \sim 2Cc_0 \frac{1}{\sqrt{\gamma}\zeta} \operatorname{Re}\left[e^{-i\pi/2} W_+\left(\sqrt{2\gamma}\zeta, 0, s\right) + e^{i\pi/2} e^{i\Phi_0} W_+^*\left(\sqrt{2\gamma}\zeta, 0, s\right)\right]$$

$$\xi'^{-1} \cos\left[\gamma\left(1 - \xi'^2/2\right) + k\delta + s\ln\left(\xi'/2\right)\right] \text{ , } \xi' \to 0$$

In Region 2 we find

$$\Pi_{ev'} \sim 2c_0 \frac{1}{\sqrt{\gamma}\xi'} \operatorname{Re}\left[W_+\left(\sqrt{2\gamma}\xi', 0, s'\right) + e^{i\Phi_0'} W_+^*\left(\sqrt{2\gamma}\xi', 0, s'\right)\right]$$

$$\zeta^{-1} \cos\left[\gamma\left(1 + \zeta^2/2\right) + k\delta - s'\ln(\zeta/2) - \pi/4\right] \text{ , } \zeta \to 0$$

Small Shift In Focal Position We can also view this shift as a change in the original coordinate system (r, z) or (ζ, ξ) or (ζ, ξ')

$$r = d\sinh\zeta\cos\xi = d\sinh\zeta\sin\xi' \sim d\zeta\xi'$$

$$z = d\cosh\zeta\sin\xi = d\cosh\zeta\cos\xi' \sim d\left(1+\zeta^2/2-\xi'^2/2\right)$$

to the focal coordinate system (r', z') or $\left(\widehat{\zeta}, \widehat{\xi}'\right)$ with

$$z-\delta = z' = d\cosh\widehat{\zeta}\cos\widehat{\xi}' \sim d\left(1+\widehat{\zeta}^2/2-\widehat{\xi}'^2/2\right)$$

$$r = r' = d\sinh\widehat{\zeta}\sin\widehat{\xi}' \sim d\widehat{\zeta}\widehat{\xi}'$$

Thus the small shift δ enters as an additive correction as in the preceding section.

6.2.9 Focal Region Three

In region 1, but near the focus $\xi', \zeta \to 0$, we approximate the metric coefficients

$$\sqrt{g_{\upsilon\upsilon}} = \frac{d}{\zeta}\sqrt{\sinh^2\zeta\sin^2\xi'\sin^2\varphi + \left(\sinh^2\zeta+\sin^2\xi'\right)\zeta^2\cos^2\varphi}$$

$$\sim d\sqrt{\xi'^2+\zeta^2\cos^2\varphi} \sim h_\upsilon$$

$$\sqrt{g_{\eta\eta}} = \frac{d}{\zeta}\sqrt{\sinh^2\zeta\sin^2\xi'\cos^2\varphi + \left(\sinh^2\zeta+\sin^2\xi'\right)\zeta^2\sin^2\varphi}$$

$$\sim d\sqrt{\xi'^2+\zeta^2\sin^2\varphi} \sim h_\eta$$

$$\sqrt{g_{\upsilon\upsilon}g_{\eta\eta}}\underline{e}_\upsilon\cdot\underline{e}_\eta = d^2\left(\sinh^2\zeta+\sin^2\xi'-\sinh^2\zeta\sin^2\xi'/\zeta^2\right)\cos\varphi\sin\varphi$$

$$\sim d^2\zeta^2\cos\varphi\sin\varphi$$

We see that for $\zeta, \xi' \to 0$, but $\xi' >> \zeta$ that $\underline{e}_\upsilon\cdot\underline{e}_\eta = O\left(\zeta^2/\xi'^2\right) \to 0$. However for $\xi' = O(\zeta)$ the dot product is not small, and the coordinates are not orthogonal. Similarly, in region 2, but near the focus $\zeta, \xi' \to 0$, we approximate the metric coefficients

$$\sqrt{g_{\upsilon'\upsilon'}} = \frac{d}{\xi'}\sqrt{\sinh^2\zeta\sin^2\xi'\sin^2\varphi + \left(\sinh^2\zeta+\sin^2\xi'\right)\xi'^2\cos^2\varphi}$$

$$\sim d\sqrt{\zeta^2+\xi'^2\cos^2\varphi} \sim h_{\upsilon'}$$

$$\sqrt{g_{\eta'\eta'}} = \frac{d}{\xi'}\sqrt{\sinh^2\zeta\sin^2\xi'\cos^2\varphi + \left(\sinh^2\zeta+\sin^2\xi'\right)\xi'^2\sin^2\varphi}$$

$$\sim d\sqrt{\zeta^2+\xi'^2\sin^2\varphi} \sim h_{\eta'}$$

$$\sqrt{g_{\upsilon'\upsilon'}g_{\eta'\eta'}}\underline{e}_{\upsilon'}\cdot\underline{e}_{\eta'}=d^2\left(\sinh^2\zeta+\sin^2\xi'-\sinh^2\zeta\sin^2\xi'/\xi'^2\right)\cos\varphi\sin\varphi$$

$$\sim d^2\xi'^2\cos\varphi\sin\varphi$$

We see that for $\zeta,\xi'\to 0$, but $\zeta >> \xi'$ that $\underline{e}_{\upsilon'}\cdot\underline{e}_{\eta'}=O\left(\xi'^2/\zeta^2\right)\to 0$. However for $\zeta=O\left(\xi'\right)$ the dot product is not small, and the coordinates are not orthogonal.

We anticipate that if $\zeta=O\left(\xi'\right)$ there may be issues with the accuracy of the approximate solutions in the quasi-rectangular coordinate system. One way to deal with this would be to use a different vector approach, like spherical coordinates with Debeye potentials (or spherical vector wave functions) near one focal point

$$\underline{E}=\nabla\times\nabla\times(\underline{r}\psi_e)+i\omega\mu_0\nabla\times(\underline{r}\psi_m)=-\nabla\times(\underline{r}\times\nabla\psi_e)+i\omega\mu_0\nabla\times(\underline{r}\psi_m)$$

$$\underline{H}=\nabla\times\nabla\times(\underline{r}\psi_m)+i\omega\varepsilon_0\nabla\times(\underline{r}\psi_e)=-\nabla\times(\underline{r}\times\nabla\psi_m)+i\omega\varepsilon_0\nabla\times(\underline{r}\psi_e)$$

with

$$\left(\nabla^2+k^2\right)\psi=0$$

Another approach would be to transition to Cartesian unit vectors as the focal point is approached

$$\Pi_{e\upsilon}\,(\text{Region 1})\sim\Pi_{ex}\sim\Pi_{e\upsilon'}\,(\text{Region 2})\tag{622}$$

and use

$$\underline{\Pi}_e=\Pi_{ex}\underline{e}_x$$

with the scalar Helmholtz equation solved in spheroidal coordinates

$$\left(\nabla^2+k^2\right)\Pi_{ex}=0$$

in the focal region 3.

However, for the moment, we try restricting ourselves so that we have simplified the geometry near one focus by taking $\zeta,\xi'<<1$, but still maintain $\xi'>>\zeta$ on the region 1 side of the focal region, or alternatively, $\zeta>>\xi'$ on the region 2 side of the focal region. Such an approach should allow us to connect the region 1 and region 2 solutions, but may be of questionable accuracy at the focal point; however on the axis with $\zeta=0$ in region 1 and $\xi'=0$ in region 2, which is the path of greatest interest, it may remain valid all the way to the focal point given the above considerations on orthogonality. Thus we will simply make the replacement $u\to\Pi_{e\upsilon}$ or $u\to\Pi_{e\upsilon'}$ depending on which side we are on, with u being the solution of

$$\frac{1}{\cos\xi}\frac{\partial}{\partial\xi}\left(\cos\xi\frac{\partial u}{\partial\xi}\right)+\left(\frac{1}{\cos^2\xi}+\frac{1}{\sinh^2\zeta}\right)\frac{\partial^2u}{\partial\varphi^2}$$

$$+\frac{1}{\sinh\zeta}\frac{\partial}{\partial\zeta}\left(\sinh\zeta\frac{\partial u}{\partial\zeta}\right)+\gamma^2\left(\cosh^2\zeta-\sin^2\xi\right)u=0$$

or

$$\frac{1}{\cos\xi}\frac{\partial}{\partial\xi}\left(\cos\xi\frac{\partial u}{\partial\xi}\right)+\left(\frac{1}{\cos^2\xi}+\frac{1}{\sinh^2\zeta}\right)\frac{\partial^2 u}{\partial\varphi^2}$$

$$+\frac{1}{\sinh\zeta}\frac{\partial}{\partial\zeta}\left(\sinh\zeta\frac{\partial u}{\partial\zeta}\right)+\gamma^2\left(\sinh^2\zeta+\cos^2\xi\right)u=0$$

or

$$\frac{1}{\sin\xi'}\frac{\partial}{\partial\xi'}\left(\sin\xi'\frac{\partial u}{\partial\xi'}\right)+\left(\frac{1}{\sin^2\xi'}+\frac{1}{\sinh^2\zeta}\right)\frac{\partial^2 u}{\partial\varphi^2}$$

$$+\frac{1}{\sinh\zeta}\frac{\partial}{\partial\zeta}\left(\sinh\zeta\frac{\partial u}{\partial\zeta}\right)+\gamma^2\left(\sinh^2\zeta+\sin^2\xi'\right)u=0$$

as $\xi'=\pi/2-\xi$ (we are assuming that $\zeta^2<<1$ and that $\xi'^2<<1$)

$$\frac{1}{\xi'}\frac{\partial}{\partial\xi'}\left(\xi'\frac{\partial u}{\partial\xi'}\right)+\left(\frac{1}{\xi'^2}+\frac{1}{\zeta^2}\right)\frac{\partial^2 u}{\partial\varphi^2}$$

$$+\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial u}{\partial\zeta}\right)+\gamma^2\left(\zeta^2+\xi'^2\right)u=0 \tag{623}$$

or taking

$$\Pi_{ev}\sim u=u_m\left(\zeta,\xi'\right)\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\} \tag{624}$$

$$\frac{1}{\xi'}\frac{\partial}{\partial\xi'}\left(\xi'\frac{\partial u_m}{\partial\xi'}\right)+\left(\gamma^2\xi'^2-m^2/\xi'^2\right)u_m$$

$$+\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial u_m}{\partial\zeta}\right)+\left(\gamma^2\zeta^2-m^2/\zeta^2\right)u_m=0$$

Separating variables

$$u_m=R_m(\zeta)\,Z_m\left(\xi'\right) \tag{625}$$

$$\left[\frac{1}{Z_m}\frac{1}{\xi'}\frac{\partial}{\partial\xi'}\left(\xi'\frac{\partial Z_m}{\partial\xi'}\right)+\gamma^2\xi'^2-m^2/\xi'^2\right]$$

$$+\left[\frac{1}{R_m}\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial R_m}{\partial\zeta}\right)+\gamma^2\zeta^2-m^2/\zeta^2\right]=0$$

Taking the separation constant to be $2\gamma g$ we find

$$\frac{1}{\xi'}\frac{\partial}{\partial\xi'}\left(\xi'\frac{\partial Z_m}{\partial\xi'}\right)+\left(-2\gamma g+\gamma^2\xi'^2-m^2/\xi'^2\right)Z_m=0$$

$$\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial R_m}{\partial\zeta}\right)+\left(2\gamma g+\gamma^2\zeta^2-m^2/\zeta^2\right)R_m=0$$

Thus we find Whittaker equations in both directions. Letting

$$\xi'\sqrt{2\gamma}=\tau'$$

in the first and

$$\zeta\sqrt{2\gamma}=\tau$$

in the second, gives

$$\frac{1}{\tau'}\frac{\partial}{\partial\tau'}\left(\tau'\frac{\partial Z_m}{\partial\tau'}\right)+\left(-g+\frac{1}{4}\tau'^2-\frac{m^2}{\tau'^2}\right)Z_m=0$$

$$\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial R_m}{\partial\tau}\right)+\left(g+\frac{1}{4}\tau^2-\frac{m^2}{\tau^2}\right)R_m=0$$

$$u_m=X_mZ_m=C_0c_m\frac{\sqrt{2}}{\tau'}\operatorname{Re}\left[W_+\left(\tau',m,-g\right)+B'W_+^*\left(\tau',m,-g\right)\right]$$

$$\frac{\sqrt{2}}{\tau}\operatorname{Re}\left[W_+\left(\tau,m,g\right)+BW_+^*\left(\tau,m,g\right)\right]$$

$$=C_0c_m\frac{1}{\xi'\sqrt{\gamma}}\operatorname{Re}\left[W_+\left(\xi'\sqrt{2\gamma},m,-g\right)+B'W_+^*\left(\xi'\sqrt{2\gamma},m,-g\right)\right]$$

$$\frac{1}{\zeta\sqrt{\gamma}}\operatorname{Re}\left[e^{-i\pi/2}W_+\left(\zeta\sqrt{2\gamma},m,g\right)+e^{i\pi/2}BW_+^*\left(\zeta\sqrt{2\gamma},m,g\right)\right]$$

For purposes of matching we take $g=s=-s'$ and $B=e^{i\Phi_0}$ and $B'=e^{i\Phi_0'}$

$$\Pi_{ev}\sim u=C_0c_m\frac{1}{\xi'\sqrt{\gamma}}\operatorname{Re}\left[W_+\left(\xi'\sqrt{2\gamma},m,s'\right)+e^{i\Phi_0'}W_+^*\left(\xi'\sqrt{2\gamma},m,s'\right)\right]$$

$$\frac{1}{\zeta\sqrt{\gamma}}\operatorname{Re}\left[e^{-i\pi/2}W_+\left(\zeta\sqrt{2\gamma},m,-s'\right)+e^{i\pi/2}e^{i\Phi_0}W_+^*\left(\zeta\sqrt{2\gamma},m,-s'\right)\right]\left\{\begin{array}{c}\cos\left(m\varphi\right)\\ \sin\left(m\varphi\right)\end{array}\right\}\tag{626}$$

Expanding as we leave the focal region using the asymptotic form

$$W_+\left(\tau,m,s\right)\sim e^{i\tau^2/4}\left(-i\tau^2/2\right)^{is/2}=e^{i\tau^2/4+s\pi/4+is\ln\left(\tau/\sqrt{2}\right)}\ ,\ \tau\to\infty$$

gives

$$\Pi_{ev}\sim u\sim 2C_0e^{s'\pi/4}\cos\left(\Phi_0'/2\right)c_m\frac{1}{\zeta\sqrt{\gamma}}\operatorname{Re}\left[e^{-i\pi/2}W_+\left(\zeta\sqrt{2\gamma},m,-s'\right)+e^{i\pi/2}e^{i\Phi_0}W_+^*\left(\zeta\sqrt{2\gamma},m,-s'\right)\right]$$

$$\frac{1}{\xi'\sqrt{\gamma}}\cos\left[\xi'^2\gamma/2+s'\ln\left(\xi'\sqrt{\gamma}\right)-\Phi_0'/2\right]\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}, \text{Region } 3\to 1$$

$$\Pi_{ev'}\sim u\sim 2C_0e^{-s'\pi/4}\cos(\Phi_0/2)\,c_m\frac{1}{\xi'\sqrt{\gamma}}\,\mathrm{Re}\left[W_+\left(\xi'\sqrt{2\gamma},m,s'\right)+e^{i\Phi_0'}W_+^*\left(\xi'\sqrt{2\gamma},m,s'\right)\right]$$

$$\frac{1}{\zeta\sqrt{\gamma}}\cos\left[\zeta^2\gamma/2-s'\ln(\zeta\sqrt{\gamma})-\pi/2-\Phi_0/2\right]\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}, \text{Region } 3\to 2$$

These must match to the limiting forms of the outer solutions from the preceding sections

$$\Pi_{ev}\sim u=2C\frac{\psi_m(\tau,s)}{\sin\xi'}\cos\left[\gamma\cos\xi'+s\ln\left\{\tan\left(\xi'/2\right)\right\}\right]\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}, \ \xi'=\pi/2-\xi$$

$$\psi_m(\tau,s)=c_m\frac{\sqrt{2}}{\tau}\,\mathrm{Re}\left[e^{-i\pi/2}W_+(\tau,m,s)+e^{i\pi/2}e^{i\Phi_0}W_+^*(\tau,m,s)\right]$$

or

$$\Pi_{ev}\sim u\sim 2C\frac{1}{\xi'}c_m\frac{1}{\zeta\sqrt{\gamma}}\,\mathrm{Re}\left[e^{-i\pi/2}W_+\left(\zeta\sqrt{2\gamma},m,s\right)+e^{i\pi/2}e^{i\Phi_0}W_+^*\left(\zeta\sqrt{2\gamma},m,s\right)\right]$$

$$\cos\left[\gamma\left(1-\xi'^2/2\right)+k\delta+s\ln\left(\xi'/2\right)\right]\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}, \ \xi'\to 0$$

in Region 1, and to

$$\Pi_{ev'}\sim u=\frac{2}{\sinh\zeta}\psi_m(\tau',s')\cos\left[\gamma\cosh\zeta-s'\ln\left\{\tanh(\zeta/2)\right\}-\pi/2-\Phi_1/2\right]\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}$$

$$\psi_m(\tau',s')=c_m\frac{\sqrt{2}}{\tau'}\,\mathrm{Re}\left[W_+(\tau',m,s')+e^{i\Phi_0'}W_+^*(\tau',m,s')\right]$$

or

$$\Pi_{ev'}\sim u\sim\frac{2}{\zeta}c_m\frac{1}{\xi'\sqrt{\gamma}}\,\mathrm{Re}\left[W_+\left(\xi'\sqrt{2\gamma},m,s'\right)+e^{i\Phi_0'}W_+^*\left(\xi'\sqrt{2\gamma},m,s'\right)\right]$$

$$\cos\left[\gamma\left(1+\zeta^2/2\right)+k\delta-s'\ln(\zeta/2)-\pi/2-\Phi_1/2\right]\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}, \ \zeta\to 0$$

in Region 2, where

$$s'\ln\left[\tanh(\zeta_0/2)\right]=(k-k_p)\,\ell$$

$$k_p\ell-\pi/2-\Phi_1/2=\pi(p-1/2)$$

The functional behaviors are identical, but the phases only match if

$$s'\ln(2\sqrt{\gamma})-\Phi_0'/2=-\gamma-k\delta+n\pi \tag{627}$$

$$-s'\ln\left(2\sqrt{\gamma}\right)-\pi/2-\Phi_0/2=\gamma+k\delta-\pi/2-\Phi_1/2+n'\pi \tag{628}$$

and the amplitudes match if

$$C_0 e^{s'\pi/4}\cos\left(\Phi_0'/2\right)\frac{1}{\sqrt{\gamma}}=C\left(-1\right)^n \tag{629}$$

$$C_0 e^{-s'\pi/4}\cos\left(\Phi_0/2\right)\frac{1}{\sqrt{\gamma}}=\left(-1\right)^{n'} \tag{630}$$

6.2.10 Evenness Conditions On Scar

Because we want the normal derivative to vanish as the scarred orbit is approached

$$\lim_{\tau'\to 0}\left\{\mathrm{Re}\left[W_+'\left(\tau',m,s'\right)+e^{i\Phi_0'}W_+^{*\prime}\left(\tau,m,s'\right)\right]-\frac{1}{\tau'}\mathrm{Re}\left[W_+\left(\tau',m,s'\right)+e^{i\Phi_0'}W_+^{*}\left(\tau',m,s'\right)\right]\right\}\to 0$$

$$\lim_{\tau\to 0}\left\{\mathrm{Re}\left[e^{-i\pi/2}W_+'\left(\tau,m,s\right)+e^{i\pi/2}e^{i\Phi_0}W_+^{*\prime}\left(\tau,m,s\right)\right]-\frac{1}{\tau}\mathrm{Re}\left[e^{-i\pi/2}W_+\left(\tau,m,s\right)+e^{i\pi/2}e^{i\Phi_0}W_+^{*}\left(\tau,m,s\right)\right]\right\}\to 0$$

where the first is the Region 2 form and the second is the Region 1 form. If we write the real part as one half the sum of the function and its conjugate, we see that these conditions imply

$$e^{i\Phi_0'}=-\lim_{\tau'\to 0}\frac{\left[W_+'\left(\tau',m,s'\right)-\frac{1}{\tau'}W_+\left(\tau',m,s'\right)\right]}{\left[W_+'\left(\tau',m,s'\right)-\frac{1}{\tau'}W_+\left(\tau',m,s'\right)\right]^*}$$

and

$$e^{i\Phi_0}=\lim_{\tau\to 0}\frac{\left[W_+'\left(\tau,m,s\right)-\frac{1}{\tau}W_+\left(\tau,m,s\right)\right]}{\left[W_+'\left(\tau,m,s\right)-\frac{1}{\tau}W_+\left(\tau,m,s\right)\right]^*}$$

Using the properties of the Whittaker functions and $s=-s'$ we can evaluate these. The properties of the functions we desire are

$$W_+\left(\tau,m,s\right)\sim\left(-i\tau^2/2\right)^{1/2-m/2}\frac{(m-1)!}{\Gamma\left(1/2+m/2-is/2\right)}\ ,\ \tau\to 0\ ,\ m\neq 0$$

$$\sim-\left(-i\tau^2/2\right)^{1/2}\frac{1}{\Gamma\left(1/2-is/2\right)}\left[\ln\left(-i\tau^2/2\right)\right.$$

$$\left.+\left\{\psi\left(1/2-is/2\right)-2\psi\left(1\right)\right\}\right]\ ,\ \tau\to 0\ ,\ m=0$$

and

$$W_+'\left(\tau,m,s\right)\sim-i\tau\left(-i\tau^2/2\right)^{-1/2-m/2}\frac{\left(1/2-m/2\right)(m-1)!}{\Gamma\left(1/2+m/2-is/2\right)}\ ,\ \tau\to 0\ ,\ m\neq 0$$

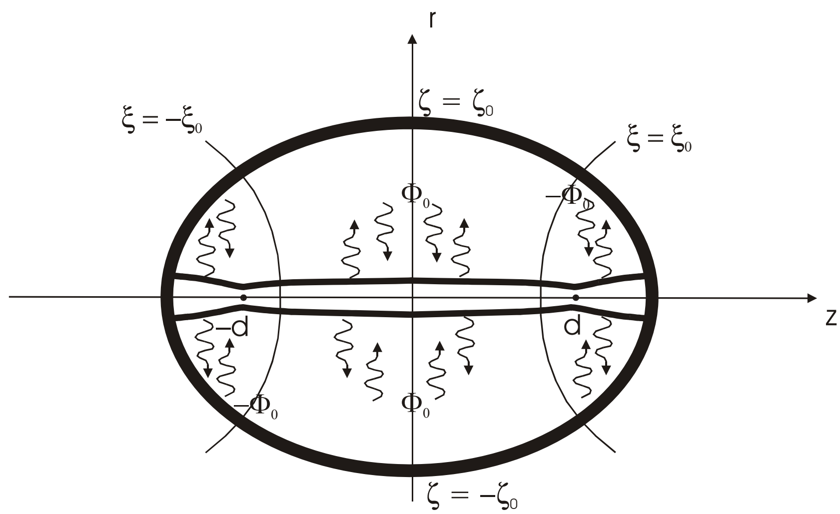

Figure 35. Sketch of reflection phase meaning in stadium geometry with interior foci.

$$\sim i\tau\frac{1}{2}\left(-i\tau^2/2\right)^{-1/2}\frac{1}{\Gamma\left(1/2-is/2\right)}\left[\ln\left(-i\tau^2/2\right)+\left\{\psi\left(1/2-is/2\right)-2\psi\left(1\right)\right\}+2\right]\ ,\ \tau\to 0\ ,\ m=0$$

and thus (this also works for $m=0$)

$$e^{i\Phi_0}=e^{i\pi(m-1)/2}\frac{\Gamma\left(1/2+m/2+is/2\right)}{\Gamma\left(1/2+m/2-is/2\right)}$$

and

$$e^{i\Phi_0'}=-e^{i\pi(m-1)/2}\frac{\Gamma\left(1/2+m/2+is'/2\right)}{\Gamma\left(1/2+m/2-is'/2\right)}$$

or

$$e^{-i\Phi_0'}=-e^{-i\pi(m-1)/2}\frac{\Gamma\left(1/2+m/2+is/2\right)}{\Gamma\left(1/2+m/2-is/2\right)}=-e^{i\Phi_0-i(m-1)\pi}=(-1)^m\,e^{i\Phi_0}$$

Thus

$$\Phi_0'=-\Phi_0+m\pi \tag{631}$$

The single remaining condition then determines $s'=-s$ given a value of the reflection phase Φ_0. Note that this choice of reflection phase conjugate implies that the incoming wave from the outer region travels toward the scarred orbit in one region, but on the other side of the focus travels away from the scarred orbit as illustrated in Figure 35. This construction of the transverse dependence has thus allowed a consistent solution between the two regions to be found.

6.2.11 Summary Of Conditions

The summary of conditions is now given. The first is the vanishing of the normal derivative as the orbit is approached

$$e^{i\Phi_0} = e^{i\pi(m-1)/2}\frac{\Gamma\left(1/2+m/2+is/2\right)}{\Gamma\left(1/2+m/2-is/2\right)}$$

$$e^{i\Phi_0'} = -e^{i\pi(m-1)/2}\frac{\Gamma\left(1/2+m/2+is'/2\right)}{\Gamma\left(1/2+m/2-is'/2\right)}$$

or

$$\Phi_0' = -\Phi_0 + m\pi$$

determines s' given Φ_0 or vice-versa

$$e^{-i\Phi_0} = i\frac{\Gamma\left(1/2+m/2+is'/2\right)}{\Gamma\left(1/2+m/2-is'/2\right)}e^{-i\pi m/2}$$

From the mirror boundary condition we have

$$\Phi_1/2 = k_p\ell - \pi p$$

where k_p and s' are related by

$$s'\ln\left[\tanh\left(\zeta_0/2\right)\right] = \left(k-k_p\right)\ell$$

This can also be written in terms of the stability exponents as

$$-s' = s = 2\left(k-k_p\right)\ell/\ln\left(\frac{\ell+d}{\ell-d}\right) = 2\left(k-k_p\right)L/\ln\left(\Lambda_+\right)$$

where

$$\sigma_0' = \ln\left[\tanh\left(\zeta_0/2\right)\right] = -\frac{1}{2}\ln\left(\frac{\ell+d}{\ell-d}\right) = -\frac{1}{2}\ln\left(\frac{\lambda_++1}{\lambda_-+1}\right) = -\frac{1}{2}\ln\left(\lambda_+\right) = -\frac{1}{4}\ln\left(\Lambda_+\right)$$

and

$$\sqrt{d/\ell} = \frac{\sqrt{\Lambda_+-1}}{\left(\sqrt{\Lambda_+}+1\right)}$$

and where $\lambda_\pm^2 = \Lambda_\pm$, $\lambda_+\lambda_- = 1 = \Lambda_+\Lambda_-$ and $\left(\lambda_\pm+1\right)/2 = \left(\ell \pm d\right)/R$. The phase matching conditions can be written as

$$\Phi_1/2 - \Phi_0'/2 - \Phi_0/2 = \Phi_1/2 - m\pi/2 = \pi\left(n+n'\right)$$

$$k\delta = -\gamma - s'\ln\left(2\sqrt{\gamma}\right) + \Phi_0'/2 + n\pi = -\gamma - s'\ln\left(2\sqrt{\gamma}\right) - \Phi_0/2 + n\pi + m\pi/2$$

or

$$\Phi_1/2 = \pi\left(n+n'\right) + m\pi/2$$

$$k\delta = -\gamma - s' \ln\left(2\sqrt{\gamma}\right) - \Phi_0/2 + n\pi + m\pi/2$$

and the amplitude conditions give the coefficients as

$$C = (-1)^{n'-n+m}\, e^{s'\pi/2}$$

$$C_0 = (-1)^{n'}\, e^{s'\pi/4}\sqrt{\gamma}\sec\left(\Phi_0/2\right)$$

It appears like the phase $\Phi_1/2$ adds something for odd values of m since it must be a multiple of $\pi/2$ and sign changes in the Region 2 solution due to this phase are accompanied by sign changes in C and in the Region 1 solution (which thus can be absorbed into the amplitude coefficients). Furthermore the factor $\sec\left(\Phi_0/2\right)$ only enters because we failed to set the problem up with symmetrical factors $\exp\left(\pm\Phi_0/2\right)$ in the combinations of Whittaker functions.

6.2.12 Final Set Of Conditions For Axisymmetric Mode

Thus for the axisymmetric mode $m = 0$ if we set Φ_1 to zero we have the evenness condition across the scar orbit to determine the allowed values of the separation constant s' in terms of the chaotic phase Φ_0

$$e^{-i\Phi_0} = i\frac{\Gamma\left(1/2 + is'/2\right)}{\Gamma\left(1/2 - is'/2\right)} = \cosh\left(\pi s'/2\right)\frac{i}{\pi}\Gamma^2\left(1/2 + is'/2\right) \tag{632}$$

We have the mirror conditions which connect the separation constant values and the resonant frequencies k

$$k_p\ell = \pi p \tag{633}$$

$$s' \ln\left[\tanh\left(\zeta_0/2\right)\right] = \left(k - k_p\right)\ell \tag{634}$$

We also have the focal point shift δ

$$k\delta = -\gamma - s' \ln\left(2\sqrt{\gamma}\right) - \Phi_0/2 + n\pi \tag{635}$$

and the amplitude constants (here we note that if n and n' are both even or odd C is one, but if they have opposite parities, then $C = -1$ which cancels the phase shift $\Phi_1/2$ then an odd multiple of π)

$$C = e^{s'\pi/2} \tag{636}$$

$$C_0 = (-1)^n\, e^{s'\pi/4}\sqrt{\gamma}\sec\left(\Phi_0/2\right) \tag{637}$$

To get a feel for the connection with Φ_0 for small s' we can expand as

$$e^{-i\Phi_0} = i\frac{\Gamma\left(1/2 + is'/2\right)}{\Gamma\left(1/2 - is'/2\right)} \sim i\frac{1 + \left(is'/2\right)\psi\left(1/2\right) + \left(is'/2\right)^2\left[\psi^2\left(1/2\right) + \psi'\left(1/2\right)\right]/2}{1 - \left(is'/2\right)\psi\left(1/2\right) + \left(is'/2\right)^2\left[\psi^2\left(1/2\right) + \psi'\left(1/2\right)\right]/2}$$

$$\sim i \frac{1 - (is'/2)(\gamma' + 2\ln 2) + (is'/2)^2 \left\{(\gamma' + 2\ln 2)^2 + \pi^2/2\right\}/2}{1 + (is'/2)(\gamma' + 2\ln 2) + (is'/2)^2 \left\{(\gamma' + 2\ln 2)^2 + \pi^2/2\right\}/2}$$

$$\sim i \left[1 - is'(\gamma' + 2\ln 2) - s'^2 (\gamma' + 2\ln 2)^2 /2\right]$$

where $\psi(z)$ is the digamma function [11] and $\gamma' \approx 0.5772$ is Euler's constant. Thus we have

$$\Phi_0 \to -\pi/2 \text{ as } s' \to 0 \tag{638}$$

6.2.13 Focal Shift In Axisymmetric Calculations

In the calculations of the focal point shift we use

$$k_p \ell = \pi p \tag{639}$$

$$s' = 2(k_p - k)\ell / \ln\left(\frac{\ell + d}{\ell - d}\right) \tag{640}$$

and the focal point shift δ

$$k(d + \delta) = -s' \ln\left(2\sqrt{kd}\right) - (\Phi_0/2 + \pi/4) + (n + 1/4)\pi \tag{641}$$

The transcendental equation for s' can be written as

$$e^{-i(\Phi_0 + \pi/2)} = \frac{\Gamma(is'/2 + 1/2)}{\Gamma(-is'/2 + 1/2)} \tag{642}$$

giving

$$k(d + \delta) = -s' \ln\left(2\sqrt{kd}\right) + \arg\Gamma(1/2 + is'/2) + (n + 1/4)\pi \tag{643}$$

Noting that

$$kd = (k - k_p)\ell(d/\ell) + k_p\ell(d/\ell) = p\pi(d/\ell) - \frac{1}{2}s'(d/\ell)\ln\left(\frac{\ell + d}{\ell - d}\right)$$

gives

$$k\delta = -s'\left\{\ln\left(2\sqrt{kd}\right) - \frac{1}{2}(d/\ell)\ln\left(\frac{\ell + d}{\ell - d}\right)\right\} + \arg\Gamma(1/2 + is'/2) + \{(n + 1/4) - p(d/\ell)\}\pi$$

or ignoring terms of order $(k - k_p)\delta$

$$k_p\delta \sim -s'\left\{\ln\left(2\sqrt{kd}\right) - \frac{1}{2}(d/\ell)\ln\left(\frac{\ell + d}{\ell - d}\right)\right\} + \arg\Gamma(1/2 + is'/2) + \{(n + 1/4) - p(d/\ell)\}\pi$$

or

$$k_p\left(d+\delta\right)\sim -s'\left\{\ln\left(2\sqrt{k_pd}\right)+\frac{1}{2}\left(k-k_p\right)/k_p-\frac{1}{2}\left(d/\ell\right)\ln\left(\frac{\ell+d}{\ell-d}\right)\right\}+\arg\Gamma\left(1/2+is'/2\right)+\left(n+1/4\right)\pi$$

or

$$\left(d+\delta\right)/\ell\sim\frac{1}{k_p\ell}\left[s'\left\{\frac{1}{2}\left(d/\ell\right)\ln\left(\frac{\ell+d}{\ell-d}\right)+\frac{1}{4}s'\ln\left(\frac{\ell+d}{\ell-d}\right)/\left(k_p\ell\right)-\ln\left(2\sqrt{k_pd}\right)\right\}+\arg\Gamma\left(1/2+is'/2\right)\right]$$

$$+\left(n+1/4\right)/p$$

Dropping the quadratic term in $1/k_p^2$

$$\left(d+\delta\right)/\ell\sim\frac{1}{k_p\ell}\left[s'\left\{\frac{1}{2}\left(d/\ell\right)\ln\left(\frac{\ell+d}{\ell-d}\right)-\ln\left(2\sqrt{k_pd}\right)\right\}+\arg\Gamma\left(1/2+is'/2\right)\right]+\left(n+1/4\right)/p$$

Expansion For Small Separation If the value of $s'\to 0$ we can expand the gamma function as

$$\Gamma\left(1/2+is'/2\right)\sim\sqrt{\pi}\left[1+\psi\left(1/2\right)is'/2-\psi^2\left(1/2\right)s'^2/8-\psi'\left(1/2\right)s'^2/8\right]$$

$$\sim\sqrt{\pi}\left[1-\left(\gamma'+2\ln 2\right)is'/2-\left\{\left(\gamma'+2\ln 2\right)^2+\pi^2/2\right\}s'^2/8\right]$$

and

$$\arg\Gamma\left(1/2+is'/2\right)\sim-\left(\gamma'+2\ln 2\right)s'/2\ ,\ \ s'\to 0$$

and

$$k\left(d+\delta\right)=-s'\left[\ln\left(4\sqrt{kd}\right)+\gamma'/2\right]+\left(n+1/4\right)\pi$$

or

$$\left(d+\delta\right)/\ell\sim\frac{1}{k_p\ell}\left[s'\left\{\frac{1}{2}\left(d/\ell\right)\ln\left(\frac{\ell+d}{\ell-d}\right)-\ln\left(4\sqrt{k_pd}\right)-\gamma'/2\right\}\right]+\left(n+1/4\right)/p$$

6.2.14 Average Focal Shift

The normalized shift function, without corrections ($s'=0$) is

$$k_p\left(d+\delta_0\right)=\left(n+1/4\right)\pi$$

and using

$$k_p\ell=p\pi$$

is

$$(d+\delta_0)/\ell = (n+1/4)/p$$

This can be written as

$$k_p\delta_0/\pi = n - n_c \geq 0$$

where

$$n_c = pd/\ell - 1/4$$

and the inequality displaces the focus to the right of the geometrical focal point (as we found in the 2D problem). Now for for large values of p (with $d/\ell < 1$) and n, we assume that $n - n_c$ is between 0 and 1 (we might take it to be uniformly distributed) and examine the average

$$k_p\langle\delta_0\rangle/\pi \approx 1/2$$

For example with the frequency range

$$209.71 \text{ m}^{-1} \leq k \leq 628.7 \text{ m}^{-1}$$

with $\ell = 0.11176$ m, $d = 0.0336969$ m we find that

$$p_1 = 8 \leq p \leq 22 = p_2$$

Then we find

$$\langle\delta_0\rangle/\ell = \frac{1}{p_2 - p_1}\int_{p_1}^{p_2}\frac{dp}{2p} = \frac{1/2}{p_2 - p_1}\ln(p_2/p_1) = 0.0361286$$

and

$$d_e = d + \langle\delta\rangle \approx d + \langle\delta_0\rangle \approx 0.0377346 \text{ m}$$

6.3 Vector Normalization Condition For Stadium Cavity

The method used for normalization of the eigenfunction components by Antonsen [1] is now put into the framework of the electromagnetic energy theorem [13]

$$\nabla\cdot\left(\frac{\partial\underline{E}}{\partial\omega}\times\underline{H}^* + \underline{E}^*\times\frac{\partial\underline{H}}{\partial\omega}\right) = i\left[\frac{\partial(\omega\mu)}{\partial\omega}\underline{H}\cdot\underline{H}^* + \frac{\partial(\omega\varepsilon)}{\partial\omega}\underline{E}\cdot\underline{E}^*\right] - \frac{\partial\underline{E}}{\partial\omega}\cdot\underline{J}^* - \underline{E}^*\cdot\frac{\partial\underline{J}}{\partial\omega}$$

Integrating over the cavity volume and using the divergence theorem (and inserting the electrical properties of free space)

$$\oint_S\left(\frac{\partial\underline{E}}{\partial\omega}\times\underline{H}^* + \underline{E}^*\times\frac{\partial\underline{H}}{\partial\omega}\right)\cdot\underline{n}dS = i\int_V\left(\mu_0\underline{H}\cdot\underline{H}^* + \varepsilon_0\underline{E}\cdot\underline{E}^*\right)dV - \int_V\left(\frac{\partial\underline{E}}{\partial\omega}\cdot\underline{J}^* + \underline{E}^*\cdot\frac{\partial\underline{J}}{\partial\omega}\right)dV$$

where the unit vector $\underline{n}$ in the divergence theorem points out of the cavity region.

6.3.1 Source Free Form Of Theorem

The source free form is thus

$$\oint_S \left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega} \right) \cdot \underline{n} dS = i \int_V \left(\mu_0 \underline{H} \cdot \underline{H}^* + \varepsilon_0 \underline{E} \cdot \underline{E}^* \right) dV$$

Using $\underline{n} \times \underline{E} = 0$ on the cavity walls, the surface integral on the cavity boundary vanishes

$$\int_{S_{cavity}} \left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega} \right) \cdot \underline{n} dS = \int_{S_{cavity}} \left[\left(\underline{n} \times \frac{\partial \underline{E}}{\partial \omega} \right) \cdot \underline{H}^* + \left(\underline{n} \times \underline{E}^* \right) \cdot \frac{\partial \underline{H}}{\partial \omega} \right] dS = 0$$

However a part of the closed surface S_{scar} is taken to surround the scarred orbit $\zeta = \sqrt{\upsilon^2 + \eta^2} \to 0$

$$\int_{S_{scar}} \left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega} \right) \cdot \underline{n} dS = i \int_V \left(\mu_0 \underline{H} \cdot \underline{H}^* + \varepsilon_0 \underline{E} \cdot \underline{E}^* \right) dV$$

$$= \left(\int_{S_{scar}^{(1)}} + \int_{S_{scar}^{(2)}} \right) \left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega} \right) \cdot \underline{n} dS$$

where the unit normal here $\underline{n}$ points into the scarred region. We take the fields to be [17], [18]

$$\underline{E} = \nabla \times \nabla \times \underline{\Pi}_e = \nabla \left(\nabla \cdot \underline{\Pi}_e \right) + k^2 \underline{\Pi}_e$$

$$\underline{H} = -i\omega\varepsilon \nabla \times \underline{\Pi}_e$$

6.3.2 Region One

For the first polarization state [5]

$$\Pi_{e\upsilon} = \Phi$$

$$\Pi_{e\eta} = \Pi_{e\xi} = 0$$

Note that as $\zeta = \sqrt{\upsilon^2 + \eta^2} \to 0$ the fields become $E_\upsilon \neq 0$, $E_\eta = 0$, $E_\xi = 0$, $H_\xi = 0$, $H_\eta \neq 0$ at $\upsilon = \eta = 0$, where

$$h^2 E_\upsilon = h \frac{\partial}{\partial \upsilon} \left[\frac{1}{h^3} \frac{\partial}{\partial \upsilon} \left(h^2 \Pi_{e\upsilon} \right) \right] + k^2 h^2 \Pi_{e\upsilon} \sim \left(\frac{\partial^2}{\partial \upsilon^2} + k^2 h^2 \right) \Pi_{e\upsilon} = \left(2\gamma \frac{\partial^2}{\partial \tau_x^2} + k^2 h^2 \right) \Pi_{e\upsilon}$$

$$h^2 E_\eta = h \frac{\partial}{\partial \eta} \left[\frac{1}{h^3} \frac{\partial}{\partial \upsilon} \left(h^2 \Pi_{e\upsilon} \right) \right] \sim \frac{\partial^2}{\partial \eta \partial \upsilon} \Pi_{e\upsilon} = 2\gamma \frac{\partial^2}{\partial \tau_y \partial \tau_x} \Pi_{e\upsilon}$$

$$h^2 E_\xi = h \frac{\partial}{\partial \xi} \left[\frac{1}{h^3} \frac{\partial}{\partial \upsilon} \left(h^2 \Pi_{e\upsilon} \right) \right] \sim \frac{\partial^2}{\partial \xi \partial \upsilon} \Pi_{e\upsilon} = \sqrt{2\gamma} \frac{\partial^2}{\partial \xi \partial \tau_x} \Pi_{e\upsilon}$$

$$h^2 H_\xi = i\omega\varepsilon_0 \frac{\partial}{\partial\eta}\left(h\Pi_{ev}\right) \sim i\omega\varepsilon_0 h \frac{\partial}{\partial\eta}\Pi_{ev} = i\omega\varepsilon_0 \sqrt{2\gamma} h \frac{\partial}{\partial\tau_y}\Pi_{ev}$$

$$h^2 H_\eta = -i\omega\varepsilon_0 \frac{\partial}{\partial\xi}\left(h\Pi_{ev}\right) \sim -i\omega\varepsilon_0 h \frac{\partial}{\partial\xi}\Pi_{ev} = -i\omega\varepsilon_0 h \frac{\partial}{\partial\xi}\Pi_{ev}$$

Thus in the boundary integral with (note that the inward unit normal is oppositely directed from the prolate spheroidal unit vector) $\underline{n} = -\underline{e}_\zeta$ and

$$\underline{e}_\upsilon \cdot \underline{e}_\zeta = \frac{h_\zeta \zeta \upsilon}{\sqrt{h_\varphi^2 \eta^2 + h_\zeta^2 \upsilon^2 \zeta^2}} \sim \cos\varphi$$

$$\underline{e}_\eta \cdot \underline{e}_\zeta = \frac{h_\zeta \zeta \eta}{\sqrt{h_\varphi^2 \upsilon^2 + h_\zeta^2 \eta^2 \zeta^2}} \sim \sin\varphi$$

$$\underline{e}_\upsilon \times \underline{e}_\eta \sim \underline{e}_\xi$$

$$\underline{e}_\xi \times \underline{e}_\upsilon \sim \underline{e}_\eta$$

$$\underline{e}_\eta \times \underline{e}_\xi \sim \underline{e}_\upsilon$$

we note that $\underline{E} \times \underline{H}^*$ captures $E_\upsilon \underline{e}_\upsilon \times H_\xi^* \underline{e}_\xi = -E_\upsilon H_\xi^* \underline{e}_\eta$, $E_\upsilon \underline{e}_\upsilon \times H_\eta^* \underline{e}_\eta = E_\upsilon H_\eta^* \underline{e}_\xi$, $E_\eta \underline{e}_\eta \times H_\xi^* \underline{e}_\xi = E_\eta H_\xi^* \underline{e}_\upsilon$ and $E_\xi \underline{e}_\xi \times H_\eta^* \underline{e}_\eta = -E_\xi H_\eta^* \underline{e}_\upsilon$. But at the center $E_\eta = 0$, $E_\xi = 0$, $H_\xi = 0$ and only $E_\upsilon H_\eta^* \underline{e}_\xi$ survives, but is orthogonal to $\underline{n} = \underline{e}_\zeta$. Thus the ω derivative is responsible for a contribution

$$\left(\frac{\partial \underline{E}}{\partial\omega} \times \underline{H}^*\right) \cdot \underline{e}_\zeta = -\frac{\partial E_\xi}{\partial\omega} H_\eta^* \underline{e}_\upsilon \cdot \underline{e}_\zeta \sim -\frac{\partial E_\xi}{\partial\omega} H_\eta^* \cos\varphi = -\frac{\partial E_\xi}{\partial\omega} H_\eta^* \frac{\upsilon}{\sqrt{\upsilon^2+\eta^2}}$$

$$\left(\underline{E}^* \times \frac{\partial \underline{H}}{\partial\omega}\right) \cdot \underline{e}_\zeta = -E_\upsilon^* \frac{\partial H_\xi}{\partial\omega} \underline{e}_\eta \cdot \underline{e}_\zeta \sim -E_\upsilon^* \frac{\partial H_\xi}{\partial\omega} \sin\varphi = -E_\upsilon^* \frac{\partial H_\xi}{\partial\omega} \frac{\eta}{\sqrt{\upsilon^2+\eta^2}}$$

Noting that E_ξ is odd in $\tau_x = \sqrt{2\gamma}\upsilon = \sqrt{2\gamma}\zeta\cos\varphi$ and that H_ξ is odd in $\tau_y = \sqrt{2\gamma}\eta = \sqrt{2\gamma}\zeta\sin\varphi$ and it appears that these will contribute. The ω derivatives may not have even or odd behavior anyway. Thus

$$\left(\frac{\partial \underline{E}}{\partial\omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial\omega}\right) \cdot \underline{n} \sim \left(\frac{\partial E_\xi}{\partial\omega} H_\eta^* \frac{\upsilon}{\sqrt{\upsilon^2+\eta^2}} + E_\upsilon^* \frac{\partial H_\xi}{\partial\omega} \frac{\eta}{\sqrt{\upsilon^2+\eta^2}}\right)$$

$$\sim i\omega\varepsilon_0 \frac{1}{h} \frac{\sqrt{2\gamma}}{\sqrt{\tau_x^2+\tau_y^2}} \left[\frac{1}{h^2} \frac{\partial^3 \Pi_{ev}}{\partial\omega\partial\xi\partial\tau_x} \frac{\partial \Pi_{ev}^*}{\partial\xi}\tau_x + \frac{\partial^2 \Pi_{ev}}{\partial\omega\partial\tau_y}\tau_y \left(\frac{2\gamma}{h^2}\frac{\partial^2}{\partial\tau_x^2} + k^2\right)\Pi_{ev}^*\right]$$

Transforming back to prolate spheroidal coordinates

$$\left(\frac{\partial \underline{E}}{\partial\omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial\omega}\right) \cdot \underline{n}$$

$$\sim i\omega\varepsilon_0 \frac{1}{h}\left[\frac{1}{h^2}\frac{\partial^3\Pi_{ev}}{\partial\omega\partial\xi\partial\upsilon}\frac{\partial\Pi_{ev}^*}{\partial\xi}\cos\varphi + \frac{\partial^2\Pi_{ev}}{\partial\omega\partial\eta}\sin\varphi\left(\frac{1}{h^2}\frac{\partial^2}{\partial\upsilon^2}+k^2\right)\Pi_{ev}^*\right]$$

Noting the identities

$$\frac{\partial W}{\partial\upsilon} \sim \frac{\partial W}{\partial\zeta}\cos\varphi - \frac{1}{\zeta}\frac{\partial W}{\partial\varphi}\sin\varphi$$

$$\frac{\partial^2 W}{\partial\upsilon^2} \sim \frac{\partial^2 W}{\partial\zeta^2}\cos^2\varphi + \frac{1}{\zeta}\frac{\partial W}{\partial\zeta}\sin^2\varphi - \frac{1}{\zeta}\frac{\partial^2 W}{\partial\zeta\partial\varphi}2\sin\varphi\cos\varphi + \frac{1}{\zeta^2}\frac{\partial W}{\partial\varphi}2\sin\varphi\cos\varphi + \frac{1}{\zeta^2}\frac{\partial^2 W}{\partial\varphi^2}\sin^2\varphi$$

$$\frac{\partial W}{\partial\eta} \sim \frac{\partial W}{\partial\zeta}\sin\varphi + \frac{1}{\zeta}\frac{\partial W}{\partial\varphi}\cos\varphi$$

and for the axisymmetric case $m = 0$

$$\frac{\partial W}{\partial\upsilon} \sim \frac{\partial W}{\partial\zeta}\cos\varphi$$

$$\frac{\partial^2 W}{\partial\upsilon^2} \sim \frac{\partial^2 W}{\partial\zeta^2}\cos^2\varphi + \frac{1}{\zeta}\frac{\partial W}{\partial\zeta}\sin^2\varphi$$

$$\frac{\partial W}{\partial\eta} \sim \frac{\partial W}{\partial\zeta}\sin\varphi$$

and

$$\int_0^{2\pi}\frac{\partial W}{\partial\upsilon}\cos\varphi d\varphi \sim \frac{\partial W}{\partial\zeta}\int_0^{2\pi}\cos^2\varphi d\varphi = \pi\frac{\partial W}{\partial\zeta}$$

$$\int_0^{2\pi}\frac{\partial W}{\partial\eta}\frac{\partial^2 W}{\partial\upsilon^2}\sin\varphi d\varphi \sim \frac{\partial W}{\partial\zeta}\frac{\partial^2 W}{\partial\zeta^2}\int_0^{2\pi}\cos^2\varphi\sin^2\varphi d\varphi + \frac{1}{\zeta}\left(\frac{\partial W}{\partial\zeta}\right)^2\int_0^{2\pi}\sin^4\varphi d\varphi$$

$$\sim \frac{1}{4}\frac{\partial W}{\partial\zeta}\frac{\partial^2 W}{\partial\zeta^2}\int_0^{2\pi}(1+\cos 2\varphi)(1-\cos 2\varphi)\,d\varphi + \frac{1}{4}\frac{1}{\zeta}\left(\frac{\partial W}{\partial\zeta}\right)^2\int_0^{2\pi}(1-\cos 2\varphi)^2 d\varphi$$

$$\sim \frac{1}{4}\frac{\partial W}{\partial\zeta}\frac{\partial^2 W}{\partial\zeta^2}\int_0^{2\pi}\left(1-\cos^2 2\varphi\right)d\varphi + \frac{1}{4}\frac{1}{\zeta}\left(\frac{\partial W}{\partial\zeta}\right)^2\int_0^{2\pi}\left(1-2\cos 2\varphi+\cos^2 2\varphi\right)d\varphi$$

$$\sim \frac{\pi}{4}\frac{\partial W}{\partial\zeta}\left(\frac{\partial^2 W}{\partial\zeta^2}+3\frac{1}{\zeta}\frac{\partial W}{\partial\zeta}\right) = \frac{\pi}{4}\frac{\partial W}{\partial\zeta}\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial W}{\partial\zeta}+2W\right)$$

$$\int_0^{2\pi}\frac{\partial W}{\partial\eta}\sin\varphi d\varphi \sim \frac{\partial W}{\partial\zeta}\int_0^{2\pi}\sin^2\varphi d\varphi = \pi\frac{\partial W}{\partial\zeta}$$

or

$$\int_0^{2\pi}\left(\frac{\partial \underline{E}}{\partial\omega}\times\underline{H}^*+\underline{E}^*\times\frac{\partial \underline{H}}{\partial\omega}\right)\cdot\underline{n}d\varphi$$

$$\sim i\omega\varepsilon_0\frac{\pi}{h}\left[\frac{1}{h^2}\frac{\partial^3\Pi_{ev}}{\partial\omega\partial\xi\partial\zeta}\frac{\partial\Pi^*_{ev}}{\partial\xi}+\frac{\partial^2\Pi_{ev}}{\partial\omega\partial\zeta}\left(\frac{1}{4}\frac{1}{h^2}\frac{\partial^2}{\partial\zeta^2}+\frac{3}{4}\frac{1}{h^2}\frac{1}{\zeta}\frac{\partial}{\partial\zeta}+k^2\right)\Pi^*_{ev}\right]$$

Therefore we find (noting that $h_\xi\sim h\sim d\cos\xi$ and $h_\varphi\sim h\zeta$)

$$\int_{S^{(1)}_{scar}}\left(\frac{\partial \underline{E}}{\partial\omega}\times\underline{H}^*+\underline{E}^*\times\frac{\partial \underline{H}}{\partial\omega}\right)\cdot\underline{n}dS$$

$$=\int_{-\xi_1}^{\xi_1}\left(\frac{\partial \underline{E}}{\partial\omega}\times\underline{H}^*+\underline{E}^*\times\frac{\partial \underline{H}}{\partial\omega}\right)\cdot\underline{n}h_\xi d\xi h_\varphi d\varphi$$

$$\sim i\omega\varepsilon_0\pi\int_{-\xi_1}^{\xi_1}\zeta\left[\frac{1}{h^2}\frac{\partial^3\Pi_{ev}}{\partial\omega\partial\xi\partial\zeta}\frac{\partial\Pi^*_{ev}}{\partial\xi}+\frac{\partial^2\Pi_{ev}}{\partial\omega\partial\zeta}\left(\frac{1}{4}\frac{1}{h^2}\frac{\partial^2}{\partial\zeta^2}+\frac{3}{4}\frac{1}{h^2}\frac{1}{\zeta}\frac{\partial}{\partial\zeta}+k^2\right)\Pi^*_{ev}\right]hd\xi$$

$$\sim i\omega\varepsilon_0\pi\int_{-\xi_1}^{\xi_1}\left[\zeta\frac{\partial^3\Pi_{ev}}{\partial\omega\partial\xi\partial\zeta}\frac{\partial\Pi^*_{ev}}{\partial\xi}+\zeta\frac{\partial^2\Pi_{ev}}{\partial\omega\partial\zeta}\left\{\frac{1}{4}\frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial\Pi^*_{ev}}{\partial\zeta}+2\Pi^*_{ev}\right)+k^2h^2\Pi^*_{ev}\right\}\right]d\xi/h$$

$$\sim i\omega\varepsilon_0\frac{\pi}{d}\int_{-\xi_1}^{\xi_1}\left[\tau\frac{\partial^3\Pi_{ev}}{\partial\omega\partial\xi\partial\tau}\frac{\partial\Pi^*_{ev}}{\partial\xi}+\tau\frac{\partial^2\Pi_{ev}}{\partial\omega\partial\tau}\left\{2\gamma\frac{1}{4}\frac{1}{\tau}\frac{\partial}{\partial\tau}\left(\tau\frac{\partial\Pi^*_{ev}}{\partial\tau}+2\Pi^*_{ev}\right)+\gamma^2\cos^2\xi\Pi^*_{ev}\right\}\right]d\xi/\cos\xi$$

$$\sim i\omega\varepsilon_0\frac{\pi}{d}\int_{-\xi_1}^{\xi_1}\left[\tau\frac{\partial^3\Pi_{ev}}{\partial\omega\partial\xi\partial\tau}\frac{\partial\Pi^*_{ev}}{\partial\xi}+\gamma\tau\frac{\partial^2\Pi_{ev}}{\partial\omega\partial\tau}\Pi^*_{ev}\left(\gamma\cos^2\xi-s\right)\right]d\xi/\cos\xi$$

In the stadium cavity we will take

$$\xi_1=\mathrm{Arcsin}\left(1-\Delta/d\right)\sim\frac{\pi}{2}-\sqrt{2\Delta/d}$$

where $\Delta/d\to 0$. Now from the function

$$\Pi_{ev}=\frac{2C}{\cos\xi}\psi_0\left(\tau,s\right)\cos\left(\gamma\sin\xi-s\sigma\right)$$

$$=2C\psi_0\left(\tau,s\right)\cosh\sigma\cos\left(\gamma\tanh\sigma-s\sigma\right)$$

$$C=e^{s'\pi/2}$$

$$\psi_0\left(0,s\right)=c_0\,\mathrm{Re}\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]e^{\pi s/2}$$

$$\tau\frac{\partial^2\psi_0}{\partial\omega\partial\tau}\left(0,s\right)\approx 2c_0\,\mathrm{Re}\left\{\frac{e^{-i\pi/4}}{\Gamma\left(1/2-is/2\right)}\right\}\frac{\partial\Phi_0}{\partial\omega}$$

$$\frac{\partial \Pi_{ev}}{\partial \xi} = 2C\psi_0(\tau, s)\frac{\partial}{\partial \xi}\left[\frac{\cos(\gamma \sin \xi - s\sigma)}{\cos \xi}\right] = 2C\psi_0(\tau, s)\frac{\partial \sigma}{\partial \xi}\frac{\partial}{\partial \sigma}\left[\frac{\cos(\gamma \sin \xi - s\sigma)}{\cos \xi}\right]$$

$$= 2C\psi_0(\tau, s)\cosh \sigma \frac{\partial}{\partial \sigma}\left\{\cosh \sigma \cos(\gamma \tanh \sigma - s\sigma)\right\}$$

where we have used

$$\sinh \sigma = \tan \xi$$

$$\sigma_1 = \ln(\tan \xi_1 + \sec \xi_1) \sim \ln \sqrt{2d/\Delta}$$

$$\cosh \sigma = \sec \xi$$

$$\tanh \sigma = \sin \xi$$

$$d\sigma = \sec \xi d\xi$$

we find

$$\int_{S_{scar}^{(1)}} \left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega}\right) \cdot \underline{n} dS$$

$$\sim i\omega\varepsilon_0 C^2 \frac{\pi}{d} 4\left\{\tau \frac{\partial^2 \psi_0}{\partial \omega \partial \tau}(0, s)\right\}\psi_0(0, s)$$

$$\int_{-\sigma_1}^{\sigma_1}\left[\left\{\sinh \sigma \cosh \sigma \cos(\gamma \tanh \sigma - s\sigma) - \sin(\gamma \tanh \sigma - s\sigma)\left(\gamma - s\cosh^2 \sigma\right)\right\}^2\right.$$

$$\left.+\gamma \cos^2(\gamma \tanh \sigma - s\sigma)\left(\gamma - s\cosh^2 \sigma\right)\right] d\sigma$$

Averaging over the rapidly varying sinusoids for γ large gives

$$\int_{S_{scar}^{(1)}} \left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega}\right) \cdot \underline{n} dS$$

$$\sim i\omega\varepsilon_0 C^2 \frac{\pi}{d} 2\left\{\tau \frac{\partial^2 \psi_0}{\partial \omega \partial \tau}(0, s)\right\}\psi_0(0, s)$$

$$\int_{-\sigma_1}^{\sigma_1}\left[\left\{\sinh^2 \sigma \cosh^2 \sigma + \left(\gamma - s\cosh^2 \sigma\right)^2\right\}\right.$$

$$\left.+\gamma\left(\gamma - s\cosh^2 \sigma\right)\right] d\sigma$$

For the time being we retain only the leading order γ^2 terms

$$-i\int_{S_{scar}^{(1)}}\left(\frac{\partial \underline{E}}{\partial\omega}\times\underline{H}^{*}+\underline{E}^{*}\times\frac{\partial\underline{H}}{\partial\omega}\right)\cdot\underline{n}dS$$

$$\sim 8\omega\varepsilon_0 C^2\frac{\pi}{d}\left\{\tau\frac{\partial^2\psi_0}{\partial\omega\partial\tau}(0,s)\right\}\psi_0(0,s)\gamma^2\sigma_1$$

6.3.3 Region Two

The surface integral in region 2 uses $h_\zeta = h_\xi = h \sim d\sinh\zeta \sim h_{\upsilon'} \sim h_{\eta'}$, where for the first polarization state [5]

$$\Pi_{e\upsilon'} = \Phi$$

$$\Pi_{e\eta'} = \Pi_{e\zeta} = 0$$

Note that as $\xi' = \sqrt{\upsilon'^2+\eta'^2} \to 0$ the fields become $E_{\upsilon'} \neq 0$, $E_{\eta'} = 0$, $E_\zeta = 0$, $H_\zeta = 0$, $H_{\eta'} \neq 0$ at $\upsilon' = \eta' = 0$, where

$$h^2E_{\upsilon'} = h\frac{\partial}{\partial\upsilon'}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon'}\left(h^2\Pi_{e\upsilon'}\right)\right]+k^2h^2\Pi_{e\upsilon'} \sim \left(\frac{\partial^2}{\partial\upsilon'^2}+k^2h^2\right)\Pi_{e\upsilon'} = \left(2\gamma\frac{\partial^2}{\partial\tau_x'^2}+k^2h^2\right)\Pi_{e\upsilon'}$$

$$h^2E_{\eta'} = h\frac{\partial}{\partial\eta'}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon'}\left(h^2\Pi_{e\upsilon'}\right)\right] \sim \frac{\partial^2}{\partial\eta'\partial\upsilon'}\Pi_{e\upsilon'} = 2\gamma\frac{\partial^2}{\partial\tau_y'\partial\tau_x'}\Pi_{e\upsilon'}$$

$$h^2E_{\zeta} = h\frac{\partial}{\partial\zeta}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon'}\left(h^2\Pi_{e\upsilon'}\right)\right] \sim \frac{\partial^2}{\partial\zeta\partial\upsilon'}\Pi_{e\upsilon'} = \sqrt{2\gamma}\frac{\partial^2}{\partial\zeta\partial\tau_x'}\Pi_{e\upsilon'}$$

and

$$h^2H_\zeta = i\omega\varepsilon_0\frac{\partial}{\partial\eta'}\left(h\Pi_{e\upsilon'}\right) \sim i\omega\varepsilon_0 h\frac{\partial}{\partial\eta'}\Pi_{e\upsilon'} = i\omega\varepsilon_0 h\sqrt{2\gamma}\frac{\partial}{\partial\tau_y'}\Pi_{e\upsilon'}$$

$$h^2H_{\eta'} = -i\omega\varepsilon_0\frac{\partial}{\partial\zeta}\left(h\Pi_{e\upsilon'}\right) \sim -i\omega\varepsilon_0 h\frac{\partial}{\partial\zeta}\Pi_{e\upsilon'}$$

Thus in the boundary integral with (note that the inward unit normal is oppositely directed from the prolate spheroidal unit vector on the negative side of the stadium but in the same direction on the positive side) $\underline{n} = \pm\underline{e}_\xi$ and

$$\underline{e}_{\upsilon'}\cdot\underline{e}_\xi = \frac{-h_\xi\xi'\upsilon'}{\sqrt{h_\varphi^2\eta'^2+h_\xi^2\upsilon'^2\xi'^2}} \sim -\cos\varphi$$

$$\underline{e}_{\eta'}\cdot\underline{e}_\xi = \frac{-h_\xi\xi'\eta'}{\sqrt{h_\varphi^2\upsilon'^2+h_\xi^2\eta'^2\xi'^2}} \sim -\sin\varphi$$

$$\underline{e}_{\upsilon'} \times \underline{e}_{\eta'} \sim \pm \underline{e}_{\zeta}$$

$$\underline{e}_{\zeta} \times \underline{e}_{\upsilon'} \sim \pm \underline{e}_{\eta'}$$

$$\underline{e}_{\eta'} \times \underline{e}_{\zeta} \sim \pm \underline{e}_{\upsilon'}$$

we note that $\underline{E} \times \underline{H}^*$ captures $E_{\upsilon'}\underline{e}_{\upsilon'} \times H^*_{\zeta}\underline{e}_{\zeta} = \mp E_{\upsilon'}H^*_{\zeta}\underline{e}_{\eta'}$, $E_{\upsilon'}\underline{e}_{\upsilon'} \times H^*_{\eta'}\underline{e}_{\eta'} = \pm E_{\upsilon'}H^*_{\eta'}\underline{e}_{\zeta}$, $E_{\eta'}\underline{e}_{\eta'} \times H^*_{\zeta}\underline{e}_{\zeta} = \pm E_{\eta'}H^*_{\zeta}\underline{e}_{\upsilon'}$ and $E_{\zeta}\underline{e}_{\zeta} \times H^*_{\eta'}\underline{e}_{\eta'} = \mp E_{\zeta}H^*_{\eta'}\underline{e}_{\upsilon'}$. But at the center $E_{\eta'} = 0$, $E_{\zeta} = 0$, $H_{\zeta} = 0$ and only $\pm E_{\upsilon'}H^*_{\eta'}\underline{e}_{\zeta}$ survives, but is orthogonal to $\underline{n} = \pm \underline{e}_{\xi}$. Thus the ω derivative is responsible for a contribution

$$\left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^*\right) \cdot \underline{e}_{\xi} = \mp \frac{\partial E_{\zeta}}{\partial \omega} H^*_{\eta'} \underline{e}_{\upsilon'} \cdot \underline{e}_{\xi} \sim \pm \frac{\partial E_{\zeta}}{\partial \omega} H^*_{\eta'} \cos\varphi = \pm \frac{\partial E_{\zeta}}{\partial \omega} H^*_{\eta'} \frac{\upsilon'}{\sqrt{\upsilon'^2 + \eta'^2}}$$

$$\left(\underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega}\right) \cdot \underline{e}_{\xi} = \mp E^*_{\upsilon'} \frac{\partial H_{\zeta}}{\partial \omega} \underline{e}_{\eta'} \cdot \underline{e}_{\xi} \sim \pm E^*_{\upsilon'} \frac{\partial H_{\zeta}}{\partial \omega} \sin\varphi = \pm E^*_{\upsilon'} \frac{\partial H_{\zeta}}{\partial \omega} \frac{\eta'}{\sqrt{\upsilon'^2 + \eta'^2}}$$

Noting that E_{ζ} is odd in $\tau'_x = \sqrt{2\gamma}\upsilon' = \sqrt{2\gamma}\xi' \cos\varphi$ and that H_{ζ} is odd in $\tau'_y = \sqrt{2\gamma}\eta' = \sqrt{2\gamma}\xi' \sin\varphi$ and it appears that these will contribute. The ω derivatives may not have even or odd behavior anyway. Thus

$$\left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega}\right) \cdot \underline{n} \sim \left(\frac{\partial E_{\zeta}}{\partial \omega} H^*_{\eta'} \frac{\upsilon'}{\sqrt{\upsilon'^2 + \eta'^2}} + E^*_{\upsilon'} \frac{\partial H_{\zeta}}{\partial \omega} \frac{\eta'}{\sqrt{\upsilon'^2 + \eta'^2}}\right)$$

$$\sim i\omega\varepsilon_0 \frac{1}{h} \frac{\sqrt{2\gamma}}{\sqrt{\tau'^2_x + \tau'^2_y}} \left[\frac{1}{h^2} \frac{\partial^3 \Pi_{e\upsilon'}}{\partial\omega\partial\zeta\partial\tau'_x} \frac{\partial \Pi^*_{e\upsilon'}}{\partial \zeta} \tau'_x + \frac{\partial^2 \Pi_{e\upsilon'}}{\partial\omega\partial\tau'_y} \tau'_y \left(\frac{2\gamma}{h^2} \frac{\partial^2}{\partial \tau'^2_x} + k^2\right) \Pi^*_{e\upsilon'}\right]$$

Transforming back to prolate spheroidal coordinates

$$\left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega}\right) \cdot \underline{n}$$

$$\sim i\omega\varepsilon_0 \frac{1}{h} \left[\frac{1}{h^2} \frac{\partial^3 \Pi_{e\upsilon'}}{\partial\omega\partial\zeta\partial\upsilon'} \frac{\partial \Pi^*_{e\upsilon'}}{\partial \zeta} \cos\varphi + \frac{\partial^2 \Pi_{e\upsilon'}}{\partial\omega\partial\eta'} \sin\varphi \left(\frac{1}{h^2} \frac{\partial^2}{\partial \upsilon'^2} + k^2\right) \Pi^*_{e\upsilon'}\right]$$

Noting the identities

$$\frac{\partial W}{\partial \upsilon'} \sim \frac{\partial W}{\partial \xi'} \cos\varphi - \frac{1}{\xi'} \frac{\partial W}{\partial \varphi} \sin\varphi$$

$$\frac{\partial^2 W}{\partial \upsilon'^2} \sim \frac{\partial^2 W}{\partial \xi'^2} \cos^2\varphi + \frac{1}{\xi'} \frac{\partial W}{\partial \xi'} \sin^2\varphi - \frac{1}{\xi'} \frac{\partial^2 W}{\partial \xi' \partial \varphi} 2\sin\varphi\cos\varphi + \frac{1}{\xi'^2} \frac{\partial W}{\partial \varphi} 2\sin\varphi\cos\varphi + \frac{1}{\xi'^2} \frac{\partial^2 W}{\partial \varphi^2} \sin^2\varphi$$

$$\frac{\partial W}{\partial \eta'} \sim \frac{\partial W}{\partial \xi'} \sin\varphi + \frac{1}{\xi'} \frac{\partial W}{\partial \varphi} \cos\varphi$$

and for the axisymmetric case $m = 0$

$$\frac{\partial W}{\partial \upsilon'} \sim \frac{\partial W}{\partial \xi'} \cos\varphi$$

$$\frac{\partial^2 W}{\partial \upsilon'^2} \sim \frac{\partial^2 W}{\partial \xi'^2} \cos^2\varphi + \frac{1}{\xi'} \frac{\partial W}{\partial \xi'} \sin^2\varphi$$

$$\frac{\partial W}{\partial \eta'} \sim \frac{\partial W}{\partial \xi'} \sin\varphi$$

and

$$\int_0^{2\pi} \frac{\partial W}{\partial \upsilon'} \cos\varphi d\varphi \sim \frac{\partial W}{\partial \xi'} \int_0^{2\pi} \cos^2\varphi d\varphi = \pi \frac{\partial W}{\partial \xi'}$$

$$\int_0^{2\pi} \frac{\partial W}{\partial \eta'} \frac{\partial^2 W}{\partial \upsilon'^2} \sin\varphi d\varphi \sim \frac{\partial W}{\partial \xi'} \frac{\partial^2 W}{\partial \xi'^2} \int_0^{2\pi} \cos^2\varphi \sin^2\varphi d\varphi + \frac{1}{\xi'} \left(\frac{\partial W}{\partial \xi'}\right)^2 \int_0^{2\pi} \sin^4\varphi d\varphi$$

$$\sim \frac{1}{4} \frac{\partial W}{\partial \xi'} \frac{\partial^2 W}{\partial \xi'^2} \int_0^{2\pi} (1+\cos 2\varphi)(1-\cos 2\varphi)\, d\varphi + \frac{1}{4} \frac{1}{\xi'} \left(\frac{\partial W}{\partial \xi'}\right)^2 \int_0^{2\pi} (1-\cos 2\varphi)^2 d\varphi$$

$$\sim \frac{1}{4} \frac{\partial W}{\partial \xi'} \frac{\partial^2 W}{\partial \xi'^2} \int_0^{2\pi} \left(1-\cos^2 2\varphi\right) d\varphi + \frac{1}{4} \frac{1}{\xi'} \left(\frac{\partial W}{\partial \xi'}\right)^2 \int_0^{2\pi} \left(1 - 2\cos 2\varphi + \cos^2 2\varphi\right) d\varphi$$

$$\sim \frac{\pi}{4} \frac{\partial W}{\partial \xi'} \left(\frac{\partial^2 W}{\partial \xi'^2} + 3\frac{1}{\xi'} \frac{\partial W}{\partial \xi'}\right) = \frac{\pi}{4} \frac{\partial W}{\partial \xi'} \frac{1}{\xi'} \frac{\partial}{\partial \xi'} \left(\xi' \frac{\partial W}{\partial \xi'} + 2W\right)$$

$$\int_0^{2\pi} \frac{\partial W}{\partial \eta'} \sin\varphi d\varphi \sim \frac{\partial W}{\partial \xi'} \int_0^{2\pi} \sin^2\varphi d\varphi = \pi \frac{\partial W}{\partial \xi'}$$

or

$$\int_0^{2\pi} \left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega}\right) \cdot \underline{n} d\varphi$$

$$\sim i\omega\varepsilon_0 \frac{\pi}{h} \left[\frac{1}{h^2} \frac{\partial^3 \Pi_{e\upsilon'}}{\partial\omega\partial\zeta\partial\xi'} \frac{\partial \Pi^*_{e\upsilon'}}{\partial \zeta} + \frac{\partial^2 \Pi_{e\upsilon'}}{\partial\omega\partial\xi'} \left(\frac{1}{4}\frac{1}{h^2}\frac{\partial^2}{\partial \xi'^2} + \frac{3}{4}\frac{1}{h^2}\frac{1}{\xi'}\frac{\partial}{\partial \xi'} + k^2\right) \Pi^*_{e\upsilon'}\right]$$

Now taking the lower limit in region 2 to be

$$\zeta_2 = \mathrm{Arccosh}\,(1 + \Delta/d) \approx \sqrt{2\Delta/d}$$

$$\ell = d\cosh\zeta_0$$

where $\Delta/d \to 0$, then we write the contribution from region 2 as (noting that $h_\zeta \sim h$ and $h_\varphi \sim h\xi'$)

$$\int_{S^{(2)}_{scar}} \left(\frac{\partial \underline{E}}{\partial \omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial \omega}\right) \cdot \underline{n} dS$$

$$= 2\int_{\zeta_2}^{\zeta_0}\left(\frac{\partial \underline{E}}{\partial\omega}\times\underline{H}^* + \underline{E}^*\times\frac{\partial\underline{H}}{\partial\omega}\right)\cdot\underline{n}h_\zeta d\zeta h_\varphi d\varphi$$

$$\sim i\omega\varepsilon_0\pi 2\int_{\zeta_2}^{\zeta_0}\xi'\left[\frac{1}{h^2}\frac{\partial^3\Pi_{ev'}}{\partial\omega\partial\zeta\partial\xi'}\frac{\partial\Pi^*_{ev'}}{\partial\zeta}+\frac{\partial^2\Pi_{ev'}}{\partial\omega\partial\xi'}\left(\frac{1}{4}\frac{1}{h^2}\frac{\partial^2}{\partial\xi'^2}+\frac{3}{4}\frac{1}{h^2}\frac{1}{\xi'}\frac{\partial}{\partial\xi'}+k^2\right)\Pi^*_{ev'}\right]hd\zeta$$

$$\sim i\omega\varepsilon_0\pi 2\int_{\zeta_2}^{\zeta_0}\left[\xi'\frac{\partial^3\Pi_{ev'}}{\partial\omega\partial\zeta\partial\xi'}\frac{\partial\Pi^*_{ev'}}{\partial\zeta}+\xi'\frac{\partial^2\Pi_{ev'}}{\partial\omega\partial\xi'}\left\{\frac{1}{4}\frac{1}{\xi'}\frac{\partial}{\partial\xi'}\left(\xi'\frac{\partial\Pi^*_{ev'}}{\partial\xi'}+2\Pi^*_{ev'}\right)+k^2h^2\Pi^*_{ev'}\right\}\right]d\zeta/h$$

$$\sim i\omega\varepsilon_0\frac{\pi}{d}2\int_{\zeta_2}^{\zeta_0}$$

$$\left[\tau'\frac{\partial^3\Pi_{ev'}}{\partial\omega\partial\zeta\partial\tau'}\frac{\partial\Pi^*_{ev'}}{\partial\zeta}+\tau'\frac{\partial^2\Pi_{ev'}}{\partial\omega\partial\tau'}\left\{2\gamma\frac{1}{4}\frac{1}{\tau'}\frac{\partial}{\partial\tau'}\left(\tau'\frac{\partial\Pi^*_{ev'}}{\partial\tau'}+2\Pi^*_{ev'}\right)+\gamma^2\sinh^2\zeta\Pi^*_{ev'}\right\}\right]d\zeta/\sinh\zeta$$

$$\sim i\omega\varepsilon_0\frac{\pi}{d}2\int_{\zeta_2}^{\zeta_0}\left[\tau'\frac{\partial^3\Pi_{ev'}}{\partial\omega\partial\zeta\partial\tau'}\frac{\partial\Pi^*_{ev'}}{\partial\zeta}+\gamma\tau'\frac{\partial^2\Pi_{ev'}}{\partial\omega\partial\tau'}\Pi^*_{ev'}\left(\gamma\sinh^2\zeta-s'\right)\right]d\zeta/\sinh\zeta$$

where we used

$$\frac{1}{\tau'}\frac{\partial}{\partial\tau'}\psi_0\left(\tau'\right)\sim -c_0\,\mathrm{Re}\left[e^{i\pi/4}\Gamma\left(1/2+is'/2\right)\right]e^{\pi s'/2}\left(s'/2\right)+O\left(\tau'^4\ln\tau'\right)\sim-\frac{s'}{2}\psi_0$$

$$\frac{1}{\tau'}\frac{\partial}{\partial\tau'}\left(\tau'\frac{\partial}{\partial\tau'}\psi_0\right)\sim -c_0\,\mathrm{Re}\left[e^{i\pi/4}\Gamma\left(1/2+is'/2\right)\right]e^{\pi s'/2}s'+O\left(\tau'^2\ln\tau'\right)\sim -s'\psi_0+O\left(\tau'^2\ln\tau'\right)$$

to evaluate the final terms in the brackets. Now from the function

$$\Pi_{ev'}=\frac{2}{\sinh\zeta}\psi_0\left(\tau',s'\right)\cos\left(\gamma\cosh\zeta-s'\sigma'-\pi/2\right)$$

$$=-2\sinh\sigma'\psi_0\left(\tau',s'\right)\cos\left(-\gamma\coth\sigma'-s'\sigma'-\pi/2\right)=2\sinh\sigma'\psi_0\left(\tau',s'\right)\sin\left(\gamma\coth\sigma'+s'\sigma'\right)$$

$$\psi_0\left(0,s'\right)=c_0\,\mathrm{Re}\left[e^{i\pi/4}\Gamma\left(1/2+is'/2\right)\right]e^{\pi s'/2}$$

$$\tau'\frac{\partial^2\psi_0}{\partial\omega\partial\tau'}\left(0,s'\right)\approx 2c_0\,\mathrm{Re}\left\{\frac{e^{i\pi/4}}{\Gamma\left(1/2-is'/2\right)}\right\}\frac{\partial\Phi_0'}{\partial\omega}$$

$$\frac{\partial\Pi_{ev'}}{\partial\zeta}=2\psi_0\left(\tau',s'\right)\frac{\partial}{\partial\zeta}\left[\frac{\cos\left(\gamma\cosh\zeta-s'\sigma'-\pi/2\right)}{\sinh\zeta}\right]$$

$$=2\psi_0\left(\tau',s'\right)\frac{\partial\sigma'}{\partial\zeta}\frac{\partial}{\partial\sigma'}\left[\frac{\cos\left(\gamma\cosh\zeta-s'\sigma'-\pi/2\right)}{\sinh\zeta}\right]$$

$$= 2\psi_0\left(\tau', s'\right)\sinh\sigma'\frac{\partial}{\partial\sigma'}\left[\sinh\sigma'\cos\left(\gamma\coth\sigma' + s'\sigma' + \pi/2\right)\right]$$

$$= -2\psi_0\left(\tau', s'\right)\sinh\sigma'\frac{\partial}{\partial\sigma'}\left[\sinh\sigma'\sin\left(\gamma\coth\sigma' + s'\sigma'\right)\right]$$

$$= -2\psi_0\left(\tau', s'\right)\left[\sinh\sigma'\cosh\sigma'\sin\left(\gamma\coth\sigma' + s'\sigma'\right) - \left(\gamma - s'\sinh^2\sigma'\right)\cos\left(\gamma\coth\sigma' + s'\sigma'\right)\right]$$

we find

$$\int_{S^{(2)}_{scar}}\left(\frac{\partial\underline{E}}{\partial\omega}\times\underline{H}^* + \underline{E}^*\times\frac{\partial\underline{H}}{\partial\omega}\right)\cdot\underline{n}dS$$

$$\sim i\omega\varepsilon_0\frac{\pi}{d}2\int_{\sigma'_2}^{\sigma'_0}\left[\tau'\frac{\partial^3\Pi_{ev'}}{\partial\omega\partial\zeta\partial\tau'}\frac{\partial\Pi^*_{ev'}}{\partial\zeta} + \gamma\tau'\frac{\partial^2\Pi_{ev'}}{\partial\omega\partial\tau'}\Pi^*_{ev'}\left(\gamma\mathrm{csch}^2\sigma' - s'\right)\right]d\sigma'$$

$$\sim i\omega\varepsilon_0\frac{\pi}{d}8\left\{\tau'\frac{\partial^2\psi_0}{\partial\omega\partial\tau'}\left(0, s'\right)\right\}\psi_0\left(0, s'\right)\int_{\sigma'_2}^{\sigma'_0}$$

$$\left[\left\{\sinh\sigma'\cosh\sigma'\sin\left(\gamma\coth\sigma' + s'\sigma'\right) - \left(\gamma - s'\sinh^2\sigma'\right)\cos\left(\gamma\coth\sigma' + s'\sigma'\right)\right\}^2\right.$$

$$\left. + \gamma\left(\gamma - s'\sinh^2\sigma'\right)\sin^2\left(\gamma\coth\sigma' + s'\sigma'\right)\right]d\sigma'$$

where we have used

$$\psi_0\left(\tau', s'\right) = c_0\frac{\sqrt{2}}{\tau'}\mathrm{Re}\left[W_+\left(\tau', 0, s'\right) + e^{i\Phi'_0}W^*_+\left(\tau', 0, s'\right)\right]$$

$$W_+\left(\tau', 0, s'\right) = W_{is'/2,0}\left(-i\tau'^2/2\right)$$

$$\tau' = \sqrt{2\gamma}\xi'$$

$$\xi' = \pm\pi/2 - \xi$$

$$2s'\sigma'_0 = 2s'\ln\left[\tanh\left(\zeta_0/2\right)\right] = \left(k - k_p\right)L$$

$$\sigma' = \int_\infty^\zeta\frac{d\zeta}{\sinh\zeta} = \ln\left[\tanh\left(\zeta/2\right)\right] = -\ln\left[\coth\zeta + \mathrm{csch}\zeta\right] = -\ln\left[\coth\zeta + \sqrt{\coth^2\zeta - 1}\right] = -\mathrm{Arccosh}\left(\coth\zeta\right)$$

$$\cosh\sigma' = \frac{1}{2}\left[\tanh\left(\zeta/2\right) + \coth\left(\zeta/2\right)\right] = \frac{1}{2}\left[\frac{\cosh\zeta - 1}{\sinh\zeta} + \frac{\cosh\zeta + 1}{\sinh\zeta}\right] = \coth\zeta$$

$$\sinh\sigma' = \frac{1}{2}\left[\tanh(\zeta/2) - \coth(\zeta/2)\right] = \frac{1}{2}\left[\frac{\cosh\zeta - 1}{\sinh\zeta} - \frac{\cosh\zeta + 1}{\sinh\zeta}\right] = -\mathrm{csch}\zeta$$

$$\tanh\sigma' = -\mathrm{sech}\zeta$$

$$\coth\sigma' = -\cosh\zeta$$

$$\sigma_2' = \ln\left[\tanh(\zeta_2/2)\right] \sim -\ln\sqrt{2d/\Delta}$$

$$\sigma_0' = \ln\left[\tanh(\zeta_0/2)\right] = \ln\left[\frac{\cosh(\zeta_0) - 1}{\sinh(\zeta_0)}\right] = -\frac{1}{2}\ln\left(\frac{\ell + d}{\ell - d}\right) = -\frac{1}{4}\ln(\Lambda_+)$$

Averaging over the sinusoids gives

$$\int_{S_{scar}^{(2)}} \left(\frac{\partial \underline{E}}{\partial\omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial\omega}\right) \cdot \underline{n} dS$$

$$\sim i\omega\varepsilon_0 \frac{\pi}{d} 4 \left\{\tau' \frac{\partial^2\psi_0}{\partial\omega\partial\tau'}(0, s')\right\} \psi_0(0, s')$$

$$\int_{\sigma_2'}^{\sigma_0'} \left[\cosh^2\sigma' \sinh^2\sigma' + \left(\gamma - s'\sinh^2\sigma'\right)^2 + \gamma\left(\gamma - s'\sinh^2\sigma'\right)\right] d\sigma'$$

Keeping only the $O\left(\gamma^2\right)$ terms gives

$$\int_{S_{scar}^{(2)}} \left(\frac{\partial \underline{E}}{\partial\omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial\omega}\right) \cdot \underline{n} dS$$

$$\sim i\omega\varepsilon_0 \frac{\pi}{d} 8\gamma^2 \left\{\tau' \frac{\partial^2\psi_0}{\partial\omega\partial\tau'}(0, s')\right\} \psi_0(0, s')\left(\sigma_0' - \sigma_2'\right)$$

6.3.4 Combining Two Regions

The preceding two contributions

$$-i\int_{S_{scar}^{(2)}} \left(\frac{\partial \underline{E}}{\partial\omega} \times \underline{H}^* + \underline{E}^* \times \frac{\partial \underline{H}}{\partial\omega}\right) \cdot \underline{n} dS$$

$$\sim 8\omega\varepsilon_0 \frac{\pi}{d}\gamma^2 \left\{\tau' \frac{\partial^2\psi_0}{\partial\omega\partial\tau'}(0, s')\right\} \psi_0(0, s')\left(\sigma_0' - \sigma_2'\right)$$

$$\sigma_2' \sim -\ln\sqrt{2d/\Delta}$$

$$-i\int_{S_{scar}^{(1)}}\left(\frac{\partial \underline{E}}{\partial\omega}\times\underline{H}^{*}+\underline{E}^{*}\times\frac{\partial \underline{H}}{\partial\omega}\right)\cdot\underline{n}dS$$

$$\sim 8\omega\varepsilon_0 C^2\frac{\pi}{d}\gamma^2\left\{\tau\frac{\partial^2\psi_0}{\partial\omega\partial\tau}(0,s)\right\}\psi_0(0,s)\,\sigma_1$$

$$\sigma_1\sim\ln\sqrt{2d/\Delta}$$

will now be combined. Noting that

$$\left\{\tau'\frac{\partial^2\psi_0}{\partial\omega\partial\tau'}(0,s')\right\}\psi_0(0,s')=2c_0^2\frac{\mathrm{Re}^2\left[e^{i\pi/4}\Gamma(1/2+is'/2)\right]}{\left|\Gamma(1/2-is'/2)\right|^2}e^{\pi s'/2}\frac{\partial\Phi_0'}{\partial\omega}$$

$$=2c_0^2\frac{\mathrm{Re}^2\left[e^{-i\pi/4}\Gamma(1/2-is'/2)\right]}{\left|\Gamma(1/2+is'/2)\right|^2}e^{\pi s'/2}\frac{\partial\Phi_0'}{\partial\omega}=-2c_0^2\frac{\mathrm{Re}^2\left[e^{-i\pi/4}\Gamma(1/2+is/2)\right]}{\left|\Gamma(1/2-is/2)\right|^2}e^{-\pi s/2}\frac{\partial\Phi_0}{\partial\omega}$$

$$\left\{\tau\frac{\partial^2\psi_0}{\partial\omega\partial\tau}(0,s)\right\}\psi_0(0,s)=2c_0^2\frac{\mathrm{Re}^2\left[e^{-i\pi/4}\Gamma(1/2+is/2)\right]}{\left|\Gamma(1/2-is/2)\right|^2}e^{\pi s/2}\frac{\partial\Phi_0}{\partial\omega}$$

$$C=e^{s'\pi/2}=e^{-s\pi/2}$$

we find that the σ_1 and σ_2' terms cancel

$$\int_V\left(\mu_0\underline{H}\cdot\underline{H}^{*}+\varepsilon_0\underline{E}\cdot\underline{E}^{*}\right)dV=$$

$$-i\int_{S_{scar}}\left(\frac{\partial \underline{E}}{\partial\omega}\times\underline{H}^{*}+\underline{E}^{*}\times\frac{\partial \underline{H}}{\partial\omega}\right)\cdot\underline{n}dS$$

$$\sim 8\omega\varepsilon_0\frac{\pi}{d}\gamma^2\left\{\tau'\frac{\partial^2\psi_0}{\partial\omega\partial\tau'}(0,s')\right\}\psi_0(0,s')\,\sigma_0'$$

$$\sim -8\omega\varepsilon_0\frac{\pi}{d}\gamma^2 2c_0^2\frac{\mathrm{Re}^2\left[e^{-i\pi/4}\Gamma(1/2+is/2)\right]}{\left|\Gamma(1/2-is/2)\right|^2}e^{-\pi s/2}\frac{\partial\Phi_0}{\partial\omega}\sigma_0'$$

or

$$\int_V\left(\mu_0\underline{H}\cdot\underline{H}^{*}+\varepsilon_0\underline{E}\cdot\underline{E}^{*}\right)dV=$$

$$\sim\varepsilon_0 k^4\pi d8\frac{\mathrm{Re}^2\left[e^{i\pi/4}\Gamma(1/2+is'/2)\right]}{\left|\Gamma(1/2-is'/2)\right|^2}e^{-\pi s/2}\left(\frac{d\Phi_0}{dk^2}\right)\ln(\Lambda_+)\,c_0^2$$

Using

$$\Gamma(z)\,\Gamma(1-z)=\pi\csc(\pi z)$$

this becomes

$$\int_V \left(\mu_0 \underline{H} \cdot \underline{H}^* + \varepsilon_0 \underline{E} \cdot \underline{E}^*\right) dV =$$

$$\sim \varepsilon_0 k^4 d8 \,\mathrm{Re}^2 \left[e^{i\pi/4} \Gamma\left(1/2 + is'/2\right)\right] \cosh\left(\pi s'/2\right) e^{\pi s'/2} \left(\frac{d\Phi_0}{dk^2}\right) \ln\left(\Lambda_+\right) c_0^2$$

$$\sim \varepsilon_0 k^4 d4 \,\mathrm{Re}^2 \left[e^{i\pi/4} \Gamma\left(1/2 + is'/2\right)\right] \left(1 + e^{\pi s'}\right) \left(\frac{d\Phi_0}{dk^2}\right) \ln\left(\Lambda_+\right) c_0^2 \tag{644}$$

The mirror condition $\Pi_{ev'} = 0$ gives

$$k_p \ell = \pi p \tag{645}$$

The outer region reflection phase we take [1] to be (again this results from a modal spacing that arises from a single azimuthal parity in the vector case, since the two parities are degenerate) (this also is for the average even eigenvalue spacing along the orbit)

$$\left(\frac{d\Phi_0}{dk^2}\right)^{-1} = \frac{4}{A} v^2 \tag{646}$$

where A is the cross sectional area of the axisymmetric cavity and where v is the Gaussian random variable with unit variance and density

$$f\left(v\right) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}$$

If we choose

$$\int_V \left(\mu_0 \underline{H} \cdot \underline{H}^* + \varepsilon_0 \underline{E} \cdot \underline{E}^*\right) dV = 2\varepsilon_0$$

then

$$c_0 = \frac{v\sqrt{2}}{k^2 \sqrt{Ad\left(1 + e^{\pi s'}\right) \ln\left(\Lambda_+\right)} \left|\mathrm{Re}\left[e^{i\pi/4} \Gamma\left(1/2 + is'/2\right)\right]\right|} \tag{647}$$

6.4 Summary of Results

A summary of the results for the axisymmetric mode $m = 0$ are (note that this mode is a vector mode, with electric field on axis $\zeta = \sqrt{v^2 + \eta^2} \to 0$ polarized in the $\underline{e}_v$ direction).

6.4.1 Elliptic System (Prolate Spheroidal Coordinates)

First we Summarize the results in the prolate spheroidal System.

Even Case The even case along the orbit is

$$\Pi_{ev} = \frac{2C}{\cos\xi}\psi_0\left(\tau, s\right)\cos\left(\gamma\sin\xi - s\sigma\right)$$

$$= 2C\psi_0\left(\tau, s\right)\cosh\sigma\cos\left(\gamma\tanh\sigma - s\sigma\right)$$

$$\psi_0 = c_0\frac{\sqrt{2}}{\tau}\operatorname{Re}\left[e^{-i\pi/2}W_+\left(\tau, 0, s\right) + e^{i\Phi_0}e^{i\pi/2}W_+^*\left(\tau, 0, s\right)\right]$$

$$W_+\left(\tau, 0, s\right) = W_{is/2,0}\left(-i\tau^2/2\right)$$

$$\tau = \sqrt{2\gamma}\zeta$$

$$\sigma = \int_0^\xi \frac{d\xi}{\cos\xi} = \operatorname{arcsinh}\left(\tan\xi\right)$$

$$= \ln\left(\tan\xi + \sec\xi\right) = \ln\left\{\tan\frac{1}{2}\left(\xi + \pi/2\right)\right\} = -\ln\left\{\tan\left(\xi'/2\right)\right\}$$

where

$$\sigma_0 = \frac{1}{2}\ln\left(\frac{d+\ell}{d-\ell}\right) = \frac{1}{4}\ln\left(\Lambda_+\right)$$

the two exact stability exponents [12] can be written as

$$\Lambda_\pm = \left(\frac{d+\ell}{d-\ell}\right)^{\pm 2}$$

The separation constant s is

$$s = \left(k - k_p\right)\ell/\sigma_0 = 2\left(k - k_p\right)L/\ln\left(\Lambda_+\right)$$

Note that

$$\psi_0\left(0, s\right) = c_0\operatorname{Re}\left[e^{-i\pi/4}\Gamma\left(1/2 + is/2\right)\right]e^{\pi s/2}$$

and near the orbit

$$\psi_0\left(\tau, s\right) \sim c_0\operatorname{Re}\left[e^{-i\pi/4}\Gamma\left(1/2 + is/2\right)\right]e^{\pi s/2}\left(1 - s\tau^2/4\right) + O\left(\tau^4\ln\tau\right)$$

$$\tau\frac{\partial^2\psi_0}{\partial\omega\partial\tau}\left(0, s\right) \approx 2c_0\operatorname{Re}\left\{\frac{e^{-i\pi/4}}{\Gamma\left(1/2 - is/2\right)}\right\}\frac{\partial\Phi_0}{\partial\omega}$$

$$C = e^{s'\pi/2}$$

$$\Pi_{ev'} = \frac{2}{\sinh\zeta}\psi_0\left(\tau', s'\right)\cos\left(\gamma\cosh\zeta - s'\sigma' - \pi/2\right)$$

$$= -2\sinh\sigma'\psi_0\left(\tau', s'\right)\cos\left(-\gamma\coth\sigma' - s'\sigma' - \pi/2\right) = 2\sinh\sigma'\psi_0\left(\tau', s'\right)\sin\left(\gamma\coth\sigma' + s'\sigma'\right)$$

where

$$\psi_0\left(\tau', s'\right) = c_0\frac{\sqrt{2}}{\tau'}\operatorname{Re}\left[W_+\left(\tau', 0, s'\right) + e^{i\Phi_0'}W_+^*\left(\tau', 0, s'\right)\right]$$

$$\Phi_0' = -\Phi_0$$

$$W_+\left(\tau', 0, s'\right) = W_{is'/2,0}\left(-i\tau'^2/2\right)$$

$$\tau' = \sqrt{2\gamma}\xi'$$

$$\xi' = \pm\pi/2 - \xi$$

$$\ell = d\cosh\zeta_0$$

$$z \to \pm d\cosh\zeta$$

$$\psi_0\left(0, s'\right) = c_0\operatorname{Re}\left[e^{i\pi/4}\Gamma\left(1/2 + is'/2\right)\right]e^{\pi s'/2}$$

$$\tau'\frac{\partial^2\psi_0}{\partial\omega\partial\tau'}\left(0, s'\right) \approx 2c_0\operatorname{Re}\left\{\frac{e^{i\pi/4}}{\Gamma\left(1/2 - is'/2\right)}\right\}\frac{\partial\Phi_0'}{\partial\omega}$$

$$2s'\sigma_0' = 2s'\ln\left[\tanh\left(\zeta_0/2\right)\right] = \left(k - k_p\right)L$$

$$\sigma' = \int_\infty^\zeta \frac{d\zeta}{\sinh\zeta} = \ln\left[\tanh\left(\zeta/2\right)\right]$$

The mirror condition $\Pi_{ev'} = 0$ gives

$$\cos\left(\gamma\cosh\zeta_0 - s'\sigma_0' - \pi/2\right) = \cos\left(k_p\ell - \pi/2\right) = 0$$

or

$$k_p\ell = \pi p$$

The outer region reflection phase we take [1] to be

$$\left(\frac{d\Phi_0}{dk^2}\right)^{-1} = \frac{4}{A}v^2$$

where A is the cross sectional area of the axisymmetric cavity and where v is the Gaussian random variable with unit variance and density

$$f(v) = \frac{1}{\sqrt{2\pi}}e^{-v^2/2}$$

The normalization constant c_0 is connected to the volume energy by means of

$$\int_V \left(\mu_0 \underline{H}\cdot\underline{H}^* + \varepsilon_0 \underline{E}\cdot\underline{E}^*\right) dV =$$

$$\sim \varepsilon_0 k^4 d4 \operatorname{Re}^2\left[e^{i\pi/4}\Gamma\left(1/2 + is'/2\right)\right]\left(1 + e^{\pi s'}\right)\left(\frac{d\Phi_0}{dk^2}\right)\ln\left(\Lambda_+\right)c_0^2$$

If we choose

$$\int_V \left(\mu_0 \underline{H}\cdot\underline{H}^* + \varepsilon_0 \underline{E}\cdot\underline{E}^*\right) dV = 2\varepsilon_0$$

then

$$c_0 = \frac{v\sqrt{2}}{k^2\sqrt{Ad\left(1 + e^{\pi s'}\right)\ln\left(\Lambda_+\right)}\left|\operatorname{Re}\left[e^{i\pi/4}\Gamma\left(1/2 + is'/2\right)\right]\right|}$$

Note that the scalar form of the normalization was similar

$$\int_V |u|^2\, dV \sim 4d\pi\left(\frac{d\Phi_0}{dk^2}\right)\frac{\operatorname{Re}^2\left[e^{i\pi/4}\Gamma\left(1/2 + is_p/2\right)\right]}{\left|\Gamma\left(1/2 - is_p/2\right)\right|^2}e^{\pi s_p/2}\ln\left(\Lambda_+\right)c_0^2$$

The fields near the axis are

$$E_v \sim \left(\frac{1}{h^2}\frac{\partial^2}{\partial v^2} + k^2\right)\Pi_{ev} = \left(\frac{2\gamma}{h^2}\frac{\partial^2}{\partial \tau_x^2} + k^2\right)\Pi_{ev}$$

$$\sim \left(\frac{2k}{d\cos^2\xi}\cos^2\varphi\frac{\partial^2}{\partial \tau^2} + k^2\right)\Pi_{ev} \sim k^2\Pi_{ev}$$

$$H_\eta \sim \frac{-i\omega\varepsilon_0}{h}\frac{\partial}{\partial \xi}\Pi_{ev} \sim \frac{-i\omega\varepsilon_0}{d\cos\xi}\frac{\partial}{\partial \xi}\Pi_{ev}$$

where

$$h = d\sqrt{\cos^2\xi + \zeta^2} = d\sqrt{\cos^2\xi + v^2 + \eta^2} \sim d\cos\xi$$

$$\tau_x = \sqrt{2\gamma}v = \sqrt{2\gamma}\zeta\cos\varphi$$

$$\tau_y = \sqrt{2\gamma}\eta = \sqrt{2\gamma}\zeta \sin\varphi$$

$$\tau = \sqrt{\tau_x^2 + \tau_y^2} = \sqrt{2\gamma}\zeta$$

Odd Case In the odd case for the stadium we have

$$k\ell - s_p\sigma_0 = (p - 1/2)\,\pi = k_p\ell\,,\; p = 1, 2, 3, ...$$

$$\Pi_{ev}\left(\tau, \xi\right) = \frac{2}{\cos\xi}\psi_0\left(\tau, s\right)\sin\left(\gamma\sin\xi - s\sigma\right)$$

6.4.2 Cylindrical Form

If we transform back to Cylindrical coordinates on axis, then we rewrite the potential

$$\Pi_{ev}\left(0, \xi\right) = \frac{2C}{\cos\xi}\psi_0\left(0, s\right)\cos\left(\gamma\sin\xi - s\sigma\right)$$

$$\tanh\sigma = \sin\xi$$

$$z \to d\sin\xi$$

$$\sigma = \text{Arctanh}\left(z/d\right) = \frac{1}{2}\ln\left(\frac{d+z}{d-z}\right)$$

as

$$\Pi_{ev}\left(0, z\right) = \frac{2C\psi_0\left(0, s_p\right)}{\sqrt{1 - z^2/d^2}}\cos\left[k_p z + p_0\left(z\right)\right]$$

$$p_0\left(z\right) = \left(k - k_p\right)z - \frac{1}{2}s_p\ln\left|\frac{d+z}{d-z}\right|$$

and

$$\Pi_{ev'}\left(\zeta, \pi/2\right) = \frac{2}{\sinh\zeta}\psi_0\left(0, s'\right)\cos\left(\gamma\cosh\zeta - s'\sigma' - \pi/2\right)$$

$$= -2\sinh\sigma'\psi_0\left(0, s'\right)\cos\left(-\gamma\coth\sigma' - s'\sigma' - \pi/2\right) = 2\sinh\sigma'\psi_0\left(\tau', s'\right)\sin\left(\gamma\coth\sigma' + s'\sigma'\right)$$

$$\Pi_{ev'}\left(0, z\right) = \frac{2}{\sqrt{z^2/d^2 - 1}}\psi_0\left(0, s'\right)\cos\left[k_p\left|z\right| + p_0\left(\left|z\right|\right) - \pi/2\right]$$

In the odd case for the stadium

$$\Pi_{ev}\left(0,z\right)=\frac{2C\psi_0\left(0,s_p\right)}{\sqrt{1-z^2/d^2}}\sin\left[k_pz+p_0\left(z\right)\right]$$

$$k\ell-s_p\sigma_0=\left(p-1/2\right)\pi=k_p\ell\,,\;p=1,2,3,...$$

$$\Pi_{ev'}\left(0,z\right)=\frac{2}{\sqrt{z^2/d^2-1}}\psi_0\left(0,s'\right)\sin\left[k_p\left|z\right|+p_0\left(\left|z\right|\right)-\pi/2\right]\mathrm{sgn}\left(z\right)$$

6.5 Vector Scar Projection

We now discuss the projections of the vector scar solution along the orbit of the stadium cavity.

6.5.1 Trigonometric Projection

The trigonometric projection of the solution in the stadium is now discussed.

$$V_p=2\int_0^d E_\upsilon\left(0,0,z\right)\cos\left(k_pz\right)dz+2\int_d^\ell E_{\upsilon'}\left(0,0,z\right)\cos\left(k_pz-\pi/2\right)dz \tag{648}$$

Note that at the shifted focal point $z=d+\delta_0$

$$k_p\left(d+\delta_0\right)=\left(n+1/4\right)\pi \tag{649}$$

or

$$\left(d+\delta_0\right)/\ell=\left(n+1/4\right)/p$$

we have continuity of the kernel

$$\cos\left(k_pz\right)=\cos\left(k_pz-\pi/2\right) \tag{650}$$

The fields are

$$h^2E_\upsilon=h\frac{\partial}{\partial\upsilon}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon}\left(h^2\Pi_{ev}\right)\right]+k^2h^2\Pi_{ev}\sim\left(\frac{\partial^2}{\partial\upsilon^2}+k^2h^2\right)\Pi_{ev}=\left(2\gamma\frac{\partial^2}{\partial\tau_x^2}+k^2h^2\right)\Pi_{ev}$$

$$\sim k^2h^2\Pi_{ev}$$

$$h^2E_{\upsilon'}=h\frac{\partial}{\partial\upsilon'}\left[\frac{1}{h^3}\frac{\partial}{\partial\upsilon'}\left(h^2\Pi_{ev'}\right)\right]+k^2h^2\Pi_{ev'}\sim\left(\frac{\partial^2}{\partial\upsilon'^2}+k^2h^2\right)\Pi_{ev'}=\left(2\gamma\frac{\partial^2}{\partial\tau_x'^2}+k^2h^2\right)\Pi_{ev'}$$

$$\sim k^2h^2\Pi_{ev'}$$

and thus

$$V_p \sim 2k^2 \int_0^d \Pi_{ev}(0,z)\cos(k_p z)\,dz + 2k^2 \int_d^\ell \Pi_{ev'}(0,z)\cos(k_p z - \pi/2)\,dz \tag{651}$$

$$\sim 2C\psi_0(0,s_p)\,2k^2 \int_0^d \cos[k_p z + p_0(z)]\cos(k_p z)\frac{dz}{\sqrt{1-z^2/d^2}}$$

$$+2\psi_0(0,s')\,2k^2 \int_d^\ell \cos[k_p z + p_0(z) - \pi/2]\cos(k_p z - \pi/2)\frac{dz}{\sqrt{z^2/d^2-1}}$$

$$\sim 2C\psi_0(0,s_p)\,k^2 \int_0^d \{\cos(p_0(z)) + \cos[2k_p z + p_0(z)]\}\frac{dz}{\sqrt{1-z^2/d^2}}$$

$$+2\psi_0(0,s')\,k^2 \int_d^\ell \{\cos(p_0(z)) - \cos[2k_p z + p_0(z)]\}\frac{dz}{\sqrt{z^2/d^2-1}}$$

Averaging over the rapidly varying terms gives

$$V_p \sim 2C\psi_0(0,s)\,k^2 \int_0^d \cos[p_0(z)]\frac{dz}{\sqrt{1-z^2/d^2}}$$

$$+2\psi_0(0,s')\,k^2 \int_d^\ell \cos[p_0(z)]\frac{dz}{\sqrt{z^2/d^2-1}}$$

$$p_0(z) = \frac{s}{2}\left\{(z/\ell)\ln\left|\frac{\ell+d}{\ell-d}\right| - \ln\left|\frac{z+d}{z-d}\right|\right\}$$

$$C\psi_0(0,s) = c_0\,\mathrm{Re}\left[e^{-i\pi/4}\Gamma(1/2+is/2)\right]$$

$$\psi_0(0,s') = c_0\,\mathrm{Re}\left[e^{i\pi/4}\Gamma(1/2+is'/2)\right]e^{\pi s'/2}$$

$$k^2 c_0\,\mathrm{Re}\left[e^{i\pi/4}\Gamma(1/2+is'/2)\right] = \frac{v\sqrt{2}}{\sqrt{Ad\,(1+e^{\pi s'})\ln(\Lambda_+)}}$$

$$s = 2(k-k_p)\,\ell/\ln\left(\frac{\ell+d}{\ell-d}\right) = -s'$$

$$k_p\ell = \pi p$$

Then

$$V_p/\left\{2k^2 c_0\,\mathrm{Re}\left[e^{i\pi/4}\Gamma(1/2+is'/2)\right]\right\}$$

$$\sim \int_0^d \cos[p_0(z)]\frac{dz}{\sqrt{1-z^2/d^2}}$$

$$+e^{\pi s'/2}\int_d^\ell \cos\left[p_0\left(z\right)\right]\frac{dz}{\sqrt{z^2/d^2-1}}$$

$$p_0\left(z\right)=\frac{s}{2}\left\{\left(z/\ell\right)\ln\left|\frac{\ell+d}{\ell-d}\right|-\ln\left|\frac{z+d}{z-d}\right|\right\}$$

Thus we finally have

$$\left\langle LV_p^2\right\rangle=L^2G_1\left(s\right)/A$$

where

$$G_1\left(s\right)\sim\frac{4d/\ell}{\left(1+e^{\pi s'}\right)\ln\left(\Lambda_+\right)}$$

$$\left[\int_0^1\cos\left[p_0\left(zd\right)\right]\frac{dz}{\sqrt{1-z^2}}+e^{\pi s'/2}\int_1^{\ell/d}\cos\left[p_0\left(zd\right)\right]\frac{dz}{\sqrt{z^2-1}}\right]^2$$

$$p_0\left(zd\right)=\frac{s}{2}\left\{\frac{1}{2}\left(zd/\ell\right)\ln\left(\Lambda_+\right)-\ln\left|\frac{z+1}{z-1}\right|\right\}\tag{652}$$

$$d/\ell=\frac{\left(\Lambda_+-1\right)}{\left(\sqrt{\Lambda_+}+1\right)^2}\tag{653}$$

$$\Lambda_+=\left(\frac{\ell+d}{\ell-d}\right)^2\tag{654}$$

For $s=0$ we have

$$G_1\left(0\right)=\frac{\left(2d/\ell\right)}{\ln\left(\Lambda_+\right)}\left[\int_0^1\frac{dz}{\sqrt{1-z^2}}+\int_1^{\ell/d}\frac{dz}{\sqrt{z^2-1}}\right]^2$$

$$=\frac{\left(2d/\ell\right)}{\ln\left(\Lambda_+\right)}\left[\pi/2+\operatorname{arccosh}\left(\ell/d\right)\right]^2$$

$$=\frac{\left(2d/\ell\right)}{\ln\left(\Lambda_+\right)}\left[\pi/2+\ln\left(\ell/d+\sqrt{\ell^2/d^2-1}\right)\right]^2$$

For $\ell=0.11176$ m, $R=0.10160$ m, and $d=\ell\sqrt{1-R/\ell}\approx0.0336969$ m, with $\Lambda_+\approx3.472$

$$G_1\left(0\right)\approx5.73076\tag{655}$$

This can also be written as

$$G_1\left(0\right)=\frac{2}{\ln\left(\Lambda_+\right)}\frac{\left(\Lambda_+-1\right)}{\left(\sqrt{\Lambda_+}+1\right)^2}\left[\pi/2+\ln\left\{\frac{\left(\sqrt{\Lambda_+}+1\right)}{\left(\Lambda_+-1\right)}\left(\Lambda_+^{1/4}+1\right)^2\right\}\right]^2\tag{656}$$

Note that for $\Lambda_+ \to 1$

$$G_1(0) \sim \frac{1}{2}\left[\pi/2 + \ln\left(\frac{8}{\Lambda_+ - 1}\right)\right]^2 \tag{657}$$

The odd trigonometric projection would be taken as

$$V_p = 2\int_0^d E_\upsilon(0,0,z)\sin(k_p z)\,dz + 2\int_d^\ell E_{\upsilon'}(0,0,z)\sin(k_p z - \pi/2)\,dz \tag{658}$$

and noting that

$$k_p \ell = \pi(p - 1/2)$$

continuity at $z = d + \delta_0$

$$\sin(k_p(d+\delta_0)) = \sin(k_p(d+\delta_0) - \pi/2)$$

requires

$$k_p(d+\delta_0) = (n + 3/4)\pi$$

Shifted Focus If we redefine the trigonometric projection to have the integral boundary at d_e

$$V_p = 2\int_0^{d_e} E_\upsilon(0,0,z)\cos(k_p z)\,dz + 2\int_{d_e}^\ell E_{\upsilon'}(0,0,z)\cos(k_p z - \pi/2)\,dz \tag{659}$$

the shifted focal location

$$d_e = d + \langle\delta\rangle \approx d + \langle\delta_0\rangle \approx 0.0377346\text{ m} \tag{660}$$

with $\ell = 0.11176$ m, $d = 0.0336969$ m we find that

$$G_1(0) = (d_e/\ell)\left[\pi/2 + \ln\left(\ell/d_e + \sqrt{\ell^2/d_e^2 - 1}\right)\right]^2 / \ln\left(\frac{\ell + d_e}{\ell - d_e}\right)$$

$$\approx 5.29473 \tag{661}$$

which is not too different from the preceding value.

6.5.2 Elliptic System Projection

How should the field projection in the prolate system be defined in the stadium? We could define it through the limit process we used previously (note that the quasirectangular unit vector $\underline{e}_\upsilon$ is part of this projection operator)

$$V_p = \lim_{\zeta \to 0}\frac{1}{\zeta d}\int_{-\pi/2}^{\pi/2}\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos(\gamma\sin\xi - s\sigma)}{\cos\xi}E_\upsilon(\zeta,\varphi,\xi)\,h_\varphi d\varphi h_\xi d\xi$$

$$+\lim_{\xi'\to 0}\frac{2}{\xi' d}\int_0^{\zeta_0}\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos\left(\gamma\cosh\zeta - s'\sigma' - \pi/2\right)}{\sinh\zeta}E_{\upsilon'}\left(\zeta,\varphi,\xi'\right)h_\varphi d\varphi h_\zeta d\zeta$$

$$=\lim_{\zeta\to 0}\frac{1}{\zeta d}\int_{-\pi/2}^{\pi/2}\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos\left(\gamma\sin\xi - s\sigma\right)}{\cos\xi}k^2\Pi_{e\upsilon}\left(\zeta,\xi\right)h_\varphi d\varphi h_\xi d\xi$$

$$+\lim_{\xi'\to 0}\frac{2}{\xi' d}\int_0^{\zeta_0}\frac{1}{2\pi}\int_0^{2\pi}\frac{\sin\left(\gamma\cosh\zeta - s'\sigma'\right)}{\sinh\zeta}k^2\Pi_{e\upsilon'}\left(\zeta,\xi'\right)h_\varphi d\varphi h_\zeta d\zeta \tag{662}$$

or instead using the cylindrical form

$$V_p=\lim_{\zeta\to 0}\frac{2}{\zeta d}\int_0^{\pi/2}\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos\left[k_p z + p_0\left(z\right)\right]}{\cos\xi}E_\upsilon\left(\zeta,\varphi,\xi\right)h_\varphi d\varphi h_\xi d\xi$$

$$+\lim_{\xi'\to 0}\frac{2}{\xi' d}\int_0^{\zeta_0}\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos\left[k_p z - \pi/2 + p_0\left(z\right)\right]}{\sinh\zeta}E_{\upsilon'}\left(\zeta,\varphi,\xi\right)h_\varphi d\varphi h_\zeta d\zeta$$

$$=2\int_0^d\cos\left[k_p z + p_0\left(z\right)\right]E_\upsilon\left(0,0,\xi\right)h_\xi\frac{d\xi}{dz}dz$$

$$+2\int_d^\ell\cos\left[k_p z - \pi/2 + p_0\left(z\right)\right]E_{\upsilon'}\left(\zeta,0,\pi/2\right)h_\zeta\frac{d\zeta}{dz}dz \tag{663}$$

or (hence the limit definitions of the integrals around the scar eliminate the extra divergence factors in the projection operator kernel)

$$V_p=2\int_0^d\cos\left[k_p z + p_0\left(z\right)\right]E_\upsilon\left(0,0,z\right)dz$$

$$+2\int_d^\ell\cos\left[k_p z - \pi/2 + p_0\left(z\right)\right]E_{\upsilon'}\left(0,0,z\right)dz \tag{664}$$

where the metric coefficients are

$$h_\zeta=h_\xi=d\sqrt{\sinh^2\zeta+\cos^2\xi}$$

$$h_\varphi=d\sinh\zeta\cos\xi=d\sinh\zeta\sin\xi'$$

$$z=d\cosh\zeta\sin\xi$$

$$\frac{dz}{d\zeta}=d\sinh\zeta\sin\xi$$

$$\frac{dz}{d\xi}=d\cosh\zeta\cos\xi$$

and for $\zeta \to 0$

$$h_\xi \frac{d\xi}{dz} \to 1$$

and for $\xi \to \pi/2$

$$h_\zeta \frac{d\zeta}{dz} \to 1$$

where

$$p_0(z) = \frac{s}{2}\left\{(z/\ell)\ln\left|\frac{\ell+d}{\ell-d}\right| - \ln\left|\frac{z+d}{z-d}\right|\right\} \tag{665}$$

Note again that a scar projection, consisting of a kernel from the scar solution, and a simple line integral along the orbit will result in a divergence at the focus (a principle value definition would require the introduction of a minus sign in the region 2 term), unless we introduce the region 3 form of the solution. The solution in Region 1 is

$$\Pi_{ev}(0,0,z) = 2e^{-s\pi/2}\frac{\psi_0(0,s)}{\sqrt{1-z^2/d^2}}\cos\left[k_p z + p_0(z)\right]$$

$$p_0(z) = \frac{s}{2}\left\{(z/\ell)\ln\left(\frac{\ell+d}{\ell-d}\right) - \ln\left(\frac{d+z}{d-z}\right)\right\}$$

$$\psi_0(0,s) = c_0 \operatorname{Re}\left[e^{-i\pi/4}e^{\pi s/2}\Gamma(1/2+is/2)\right]$$

The normalization constant c_0 is connected to the volume energy by means of

$$\int_V \left(\mu_0 \underline{H}\cdot\underline{H}^* + \varepsilon_0 \underline{E}\cdot\underline{E}^*\right) dV =$$

$$\sim \varepsilon_0 k^4 d 4 \operatorname{Re}^2\left[e^{i\pi/4}\Gamma(1/2+is'/2)\right]\left(1+e^{\pi s'}\right)\left(\frac{d\Phi_0}{dk^2}\right)\ln(\Lambda_+)\, c_0^2 \tag{666}$$

If we choose

$$\int_V \left(\mu_0 \underline{H}\cdot\underline{H}^* + \varepsilon_0 \underline{E}\cdot\underline{E}^*\right) dV = 2\varepsilon_0$$

then

$$c_0 = \frac{v\sqrt{2}}{k^2\sqrt{Ad\left(1+e^{\pi s'}\right)\ln(\Lambda_+)}\left|\operatorname{Re}\left[e^{i\pi/4}\Gamma(1/2+is'/2)\right]\right|}$$

where

$$s = 2(k-k_p)\,\ell/\ln\left(\frac{\ell+d}{\ell-d}\right)$$

$$k_p \ell = \pi p$$

The solution in Region 2 is

$$\Pi_{ev'}(0,0,z) = \frac{2}{\sqrt{z^2/d^2-1}} \psi_0(0,s') \cos\left[k_p z + p_0(z) - \pi/2\right]$$

$$p_0(z) = -\frac{s'}{2}\left\{(z/\ell)\ln\left(\frac{\ell+d}{\ell-d}\right) - \ln\left(\frac{z+d}{z-d}\right)\right\}$$

$$\psi_0(0,s') = c_0 \operatorname{Re}\left[e^{\pi s'/2} e^{i\pi/4} \Gamma(1/2 + is'/2)\right] \;,\; \tau' \to 0 \tag{667}$$

$$= c_0 \operatorname{Re}\left[e^{-\pi s/2} e^{-i\pi/4} \Gamma(1/2 + is/2)\right] \;,\; \tau' \to 0$$

where

$$s' = -s$$

We now consider the exponential ratios for $s \neq 0$ from region 1 to region 2 in redefining the elliptic system projection operator. As in the 2D case [2] we take the operator to have the proper exponential weights relative to the scar solution, but we remove the weights for $s \to \pm\infty$

$$\exp(\pi|s|/4) V_p = 2e^{s\pi/4}\int_0^d \cos\left[k_p z + p_0(z)\right] E_v(0,0,z)\, dz$$

$$+2e^{-s\pi/4}\int_d^\ell \cos\left[k_p z - \pi/2 + p_0(z)\right] E_{v'}(0,0,z)\, dz$$

$$\sim 2e^{s\pi/4}\int_0^d \cos\left[k_p z + p_0(z)\right] k^2 \Pi_{ev}(0,0,z)\, dz$$

$$+2e^{-s\pi/4}\int_d^\ell \cos\left[k_p z - \pi/2 + p_0(z)\right] k^2 \Pi_{ev'}(0,0,z)\, dz \tag{668}$$

Then the elliptic system projection is

$$\exp(\pi|s|/4) V_p / \left\{4k^2 c_0 \operatorname{Re}\left[e^{-i\pi/4}\Gamma(1/2+is/2)\right]\right\} =$$

$$e^{s\pi/4}\int_0^d \cos^2\left[k_p z + p_0(z)\right] \frac{dz}{\sqrt{1-z^2/d^2}} + e^{-s\pi/4} e^{-\pi s/2}\int_d^\ell \cos^2\left[k_p z - \pi/2 + p_0(z)\right] \frac{dz}{\sqrt{z^2/d^2-1}}$$

$$p_0(z) = \frac{s}{2}\left\{(z/\ell)\ln\left(\frac{\ell+d}{\ell-d}\right) - \ln\left|\frac{z+d}{z-d}\right|\right\}$$

$$k^2 c_0 \operatorname{Re}\left[e^{-i\pi/4}\Gamma(1/2+is/2)\right] = v\frac{\sqrt{2}}{\sqrt{Ad\ln(\Lambda_+)(1+e^{-\pi s})}}$$

Averaging over the rapidly varying cosines gives

$$\exp\left(\pi\left|s\right|/4\right)V_p/\left\{v\frac{2\sqrt{2}}{\sqrt{Ad\ln\left(\Lambda_+\right)\left(1+e^{-\pi s}\right)}}\right\}\sim$$

$$e^{s\pi/4}\int_0^d\frac{dz}{\sqrt{1-z^2/d^2}}+e^{-s\pi/4}e^{-\pi s/2}\int_d^\ell\frac{dz}{\sqrt{z^2/d^2-1}}$$

Thus we finally have

$$\left\langle LV_p^2\right\rangle=L^2G_1\left(s\right)/A \tag{669}$$

where

$$\exp\left(\pi\left|s\right|/2\right)G_1\left(s\right)=\frac{4\left(d/\ell\right)}{\ln\left(\Lambda_+\right)\left(1+e^{-\pi s}\right)}\left[e^{s\pi/4}\pi/2+e^{-s\pi/4}e^{-\pi s/2}\text{arccosh}\left(\ell/d\right)\right]^2$$

$$=\frac{4\left(d/\ell\right)}{\ln\left(\Lambda_+\right)\left(1+e^{-\pi s}\right)}\left[e^{s\pi/4}\pi/2+e^{-s\pi/4}e^{-\pi s/2}\ln\left(\ell/d+\sqrt{\ell^2/d^2-1}\right)\right]^2 \tag{670}$$

$$d/\ell=\frac{\left(\Lambda_+-1\right)}{\left(\sqrt{\Lambda_+}+1\right)^2}=\frac{\sqrt{\Lambda_+}-1}{\sqrt{\Lambda_+}+1}$$

$$\Lambda_+=\left(\frac{\ell+d}{\ell-d}\right)^2$$

$$\left(\ell/d+\sqrt{\ell^2/d^2-1}\right)=\frac{\left(\sqrt{\Lambda_+}+1\right)^2}{\left(\Lambda_+-1\right)}\left[1+\sqrt{1-\left(\frac{\sqrt{\Lambda_+}-1}{\sqrt{\Lambda_+}+1}\right)^2}\right]$$

$$=\frac{\left(\sqrt{\Lambda_+}+1\right)}{\left(\Lambda_+-1\right)}\left(\Lambda_+^{1/4}+1\right)^2$$

For $s=0$ we have

$$G_1\left(0\right)=\frac{2\left(d/\ell\right)}{\ln\left(\Lambda_+\right)}\left[\int_0^1\frac{dz}{\sqrt{1-z^2}}+\int_1^{\ell/d}\frac{dz}{\sqrt{z^2-1}}\right]^2$$

$$=\frac{2\left(d/\ell\right)}{\ln\left(\Lambda_+\right)}\left[\pi/2+\text{arccosh}\left(\ell/d\right)\right]^2$$

$$=\frac{2\left(d/\ell\right)}{\ln\left(\Lambda_+\right)}\left[\pi/2+\ln\left(\ell/d+\sqrt{\ell^2/d^2-1}\right)\right]^2$$

$$=\frac{2}{\ln\left(\Lambda_+\right)}\frac{\left(\Lambda_+-1\right)}{\left(\sqrt{\Lambda_+}+1\right)^2}\left[\pi/2+\ln\left\{\frac{\left(\sqrt{\Lambda_+}+1\right)}{\left(\Lambda_+-1\right)}\left(\Lambda_+^{1/4}+1\right)^2\right\}\right]^2 \tag{671}$$

Note that for $\Lambda_+ \to 1$

$$G_1(0) \sim \frac{1}{2}\left[\pi/2 + \ln\left(\frac{8}{\Lambda_+ - 1}\right)\right]^2 \tag{672}$$

Notice that the only difference between the 3D scalar projections and these 3D vector projections is that u is replaced by $k^2\Pi_{ev}$. Because the vector normalization condition for k^2c_0 is equal to one half the scalar condition for c_0 (due to the change in modal spacing in the vector case), the statistics of the projection will be similar. The actual potential is made up of a sum of different values of p, but because the pth components are asymptotically orthogonal at high frequencies, if the eigenfunction is made up of a sum

$$\Pi_{ev}(0,z) \sim \sum_p \Pi_{evp}(0,z) \tag{673}$$

this projection will pick out the p term of the sum.

6.6 Vector Random Plane Projection

The random plane wave projections are now discussed in the 3D vector formulation for the stadium. The vector random plane wave representation was constructed in the previous bowtie sections but the projection operator in the stadium is different.

6.6.1 Vector Trigonometric Projection m=1

Taking the limit $\varphi, \rho \to 0$ of the $m = 1$ part of the random plane wave representation we find that

$$\underline{E}_r^{(m=1)}(\rho = 0, \varphi = 0) \cdot \underline{e}_\rho =$$

$$\lim_{N\to\infty} \sqrt{2/(VN)} \sum_{j=1}^{N} a_j \left\{\cos\varphi_{pj}\sin\varphi_j + \sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right\}\cos(kz\cos\theta_j + \alpha_j)$$

The part that is even in z is

$$\underline{E}_r^{(m=1,\mathrm{even})}(\rho = 0, \varphi = 0) \cdot \underline{e}_\rho =$$

$$\lim_{N\to\infty} \sqrt{2/(VN)} \sum_{j=1}^{N} a_j \left\{\cos\varphi_{pj}\sin\varphi_j + \sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right\}\cos\alpha_j\cos(kz\cos\theta_j)$$

The trigonometric projection in the stadium is taken as

$$V_{pr}^T = 2\int_0^d \underline{E}_r^{(m=1)} \cdot \underline{e}_\rho \cos(k_p z)\,dz + 2\int_d^\ell \underline{E}_r^{(m=1)} \cdot \underline{e}_\rho \cos(k_p z - \pi/2)\,dz =$$

$$\lim_{N\to\infty} \sqrt{2/(VN)} \sum_{j=1}^{N} a_j \left\{\cos\varphi_{pj}\sin\varphi_j + \sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right\}\cos\alpha_j$$

$$2\left\{\int_0^d \cos(kz\cos\theta_j)\cos(k_p z)\,dz + \int_d^\ell \cos(kz\cos\theta_j)\cos(k_p z-\pi/2)\,dz\right\}$$

$$= \frac{1}{2}\lim_{N\to\infty}\sqrt{2/(VN)}\sum_{j=1}^{N} a_j\left\{\cos\varphi_{pj}\sin\varphi_j + \sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right\}2\cos\alpha_j$$

$$\left\{\int_0^d \left[\cos(kz\cos\theta_j - k_p z) + \cos(kz\cos\theta_j + k_p z)\right]dz\right.$$

$$\left. + \int_d^\ell \left[-\sin(kz\cos\theta_j - k_p z) + \sin(kz\cos\theta_j + k_p z)\right]dz\right\}$$

$$= \lim_{N\to\infty}\sqrt{2/(VN)}\sum_{j=1}^{N} a_j\left\{\cos\varphi_{pj}\sin\varphi_j + \sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right\}\cos\alpha_j$$

$$\left\{\frac{\sin(kd\cos\theta_j - k_p d)}{k\cos\theta_j - k_p} + \frac{\sin(kd\cos\theta_j + k_p d)}{k\cos\theta_j + k_p}\right.$$

$$\left. + \frac{\cos(k\ell\cos\theta_j - k_p\ell) - \cos(kd\cos\theta_j - k_p d)}{k\cos\theta_j - k_p} - \frac{\cos(k\ell\cos\theta_j + k_p\ell) - \cos(kd\cos\theta_j + k_p d)}{k\cos\theta_j + k_p}\right\}$$

$$= \lim_{N\to\infty}\sqrt{2/(VN)}\sum_{j=1}^{N} a_j\left\{\cos\varphi_{pj}\sin\varphi_j + \sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right\}\cos\alpha_j$$

$$\left\{\frac{\sin(kd\cos\theta_j - k_p d) - \cos(kd\cos\theta_j - k_p d) + \cos(k\ell\cos\theta_j - k_p\ell)}{k\cos\theta_j - k_p}\right.$$

$$\left. + \frac{\sin(kd\cos\theta_j + k_p d) + \cos(kd\cos\theta_j + k_p d) - \cos(k\ell\cos\theta_j + k_p\ell)}{k\cos\theta_j + k_p}\right\}$$

$$= \lim_{N\to\infty}\sqrt{2/(VN)}\sum_{j=1}^{N} a_j\left\{\cos\varphi_{pj}\sin\varphi_j + \sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right\}\cos\alpha_j$$

$$\left\{\frac{\sin(k\cos\theta_j - k_p)\,d - \cos(kd\cos\theta_j - k_p d) + \cos(k\ell\cos\theta_j - k_p\ell)}{k\cos\theta_j - k_p}\right.$$

$$\left. + \frac{\sin(k\cos\theta_j + k_p)\,d + \cos(kd\cos\theta_j + k_p d) - \cos(k\ell\cos\theta_j + k_p\ell)}{k\cos\theta_j + k_p}\right\}$$

Now if we form the square and average over a_j we eliminate the cross terms. Averaging over α_j, and then φ_j, followed by φ_{pj}, gives

$$\left\langle V_{pr}^{T2}\right\rangle_{\alpha_j,\varphi_j,\varphi_{pj}} = \lim_{N\to\infty}\frac{1}{4VN}\sum_{j=1}^{N}\left(1+\cos^2\theta_j\right)$$

$$\left\{\frac{\sin\left(k\cos\theta_j - k_p\right)d - \cos\left(kd\cos\theta_j - k_p d\right) + \cos\left(k\ell\cos\theta_j - k_p\ell\right)}{k\cos\theta_j - k_p}\right.$$

$$\left.+\frac{\sin\left(k\cos\theta_j + k_p\right)d + \cos\left(kd\cos\theta_j + k_p d\right) - \cos\left(k\ell\cos\theta_j + k_p\ell\right)}{k\cos\theta_j + k_p}\right\}^2$$

Now averaging over θ_j

$$\left\langle V_{pr}^{T2}\right\rangle = \left\langle V_{pr}^{T2}\right\rangle_{\alpha_j,\varphi_j,\varphi_{pj},\theta_j} = \frac{1}{8V}\int_0^{\pi}\left(1+\cos^2\theta_j\right)$$

$$\left[\frac{\sin\left(kd\cos\theta_j - k_p d\right) - \cos\left(kd\cos\theta_j - k_p d\right) + \cos\left(k\ell\cos\theta_j - k_p\ell\right)}{k\cos\theta_j - k_p}\right.$$

$$\left.+\frac{\sin\left(kd\cos\theta_j + k_p d\right) + \cos\left(kd\cos\theta_j + k_p d\right) - \cos\left(k\ell\cos\theta_j + k_p\ell\right)}{k\cos\theta_j + k_p}\right]^2 \sin\theta_j d\theta_j$$

or letting $u_j = \cos\theta_j$

$$\left\langle V_{pr}^{T2}\right\rangle = \frac{1}{8V}\int_{-1}^{1}\left(1+u_j^2\right)du_j$$

$$\left[\frac{\sin\left(kdu_j - k_p d\right) - \cos\left(kdu_j - k_p d\right) + \cos\left(k\ell u_j - k_p\ell\right)}{ku_j - k_p} + \frac{\sin\left(kdu_j + k_p d\right) + \cos\left(kdu_j + k_p d\right) - \cos\left(k\ell u_j + k_p\ell\right)}{ku_j + k_p}\right]^2$$

We note that when $k \to k_p$ the first term peaks for $\theta_j \to 0$ and the second term peaks for $\theta_j \to \pi$ with $\cos\theta = -\cos\left(\theta - \pi\right)$ and $\sin\theta = -\sin\left(\theta - \pi\right)$. Thus we can approximately separate the two terms as

$$\left\langle V_{pr}^{T2}\right\rangle \approx \frac{1}{8V}\int_0^{\pi}\left(1+\cos^2\theta_j\right)$$

$$\left[\frac{\sin\left(kd\cos\theta_j - k_p d\right) - \cos\left(kd\cos\theta_j - k_p d\right) + \cos\left(k\ell\cos\theta_j - k_p\ell\right)}{k\cos\theta_j - k_p}\right]^2 \sin\theta_j d\theta_j$$

$$+\frac{1}{8V}\int_0^{\pi}\left(1+\cos^2\left(\pi-\theta_j\right)\right)$$

$$\left[\frac{\sin\left(kd\cos\left(\pi-\theta_j\right) - k_p d\right) - \cos\left(kd\cos\left(\pi-\theta_j\right) - k_p d\right) + \cos\left(k\ell\cos\left(\pi-\theta_j\right) - k_p\ell\right)}{k\cos\left(\pi-\theta_j\right) - k_p}\right]^2 \sin\left(\pi-\theta_j\right)d\theta_j$$

$$\approx \frac{1}{4V}\int_0^{\pi}\left(1+\cos^2\theta_j\right)$$

$$\left[\frac{\sin\left(kd\cos\theta_j - k_p d\right) - \cos\left(kd\cos\theta_j - k_p d\right) + \cos\left(k\ell\cos\theta_j - k_p\ell\right)}{k\cos\theta_j - k_p}\right]^2 \sin\theta_j d\theta_j$$

Thus letting $\theta_j = \zeta/\sqrt{k\ell/2}$ gives (note that $1+\cos^2\theta_j \sim 2$)

$$\left\langle V_{pr}^{T2}\right\rangle \approx \frac{1}{2V}\int_0^\infty \left[\frac{\sin\left(k_p - k + k\theta_j^2/2\right)d}{k_p - k + k\theta_j^2/2} + \frac{\cos\left(k_p - k + k\theta_j^2/2\right)d}{k_p - k + k\theta_j^2/2} - \frac{\cos\left(k_p - k + k\theta_j^2/2\right)\ell}{k_p - k + k\theta_j^2/2}\right]^2 \theta_j d\theta_j$$

$$= \frac{\ell}{kV}\int_0^\infty \left[\frac{\sin\left(\left(k_p - k\right)\ell + \zeta^2\right)d/\ell}{\left(k_p - k\right)\ell + \zeta^2} + \frac{\cos\left(\left(k_p - k\right)\ell + \zeta^2\right)d/\ell}{\left(k_p - k\right)\ell + \zeta^2} - \frac{\cos\left(\left(k_p - k\right)\ell + \zeta^2\right)}{\left(k_p - k\right)\ell + \zeta^2}\right]^2 \zeta d\zeta$$

If we renormalize by

$$\int_V \left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)}\right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j} dV \sim \frac{2\pi}{kV}\int_0^{\rho_{\max}} \ell\left(\rho\right)d\rho = \frac{\pi A}{2kV}$$

we have

$$\left\langle V_{pr}^{T2}\right\rangle = \frac{L}{\pi A}\int_0^\infty \left[\frac{\sin\left(\left(k_p - k\right)\ell + \zeta^2\right)d/\ell}{\left(k_p - k\right)\ell + \zeta^2} + \frac{\cos\left(\left(k_p - k\right)\ell + \zeta^2\right)d/\ell}{\left(k_p - k\right)\ell + \zeta^2} - \frac{\cos\left(\left(k_p - k\right)\ell + \zeta^2\right)}{\left(k_p - k\right)\ell + \zeta^2}\right]^2 \zeta d\zeta$$

Now letting

$$\lambda = 2\left(k - k_p\right)L$$

and

$$\left\langle LV_{pr}^{T2}\right\rangle = L^2 G\left(\lambda\right)/A$$

gives

$$\left\langle V_{pr}^{T2}\right\rangle = \frac{L}{\pi A}\int_0^\infty$$

$$\left[\frac{\sin\left(\lambda/4 - \zeta^2\right)d/\ell}{\lambda/4 - \zeta^2} - \frac{\cos\left(\lambda/4 - \zeta^2\right)d/\ell}{\lambda/4 - \zeta^2} + \frac{\cos\left(\lambda/4 - \zeta^2\right)}{\lambda/4 - \zeta^2}\right]^2 \zeta d\zeta$$

or

$$G\left(\lambda\right) = \frac{1}{\pi}\int_0^\infty \frac{\left[\sin\left(\lambda/4 - \zeta^2\right)d/\ell - \cos\left(\lambda/4 - \zeta^2\right)d/\ell + \cos\left(\lambda/4 - \zeta^2\right)\right]^2}{\left(\lambda/4 - \zeta^2\right)^2}\zeta d\zeta$$

$$= \frac{1}{2\pi}\int_0^\infty \frac{\left[\sin\left(\lambda/4 - \xi\right)d/\ell - \cos\left(\lambda/4 - \xi\right)d/\ell + \cos\left(\lambda/4 - \xi\right)\right]^2}{\left(\lambda/4 - \xi\right)^2}d\xi$$

$$= \frac{1}{2\pi}\int_{-\lambda/4}^{\infty}\frac{\left[\sin\left(\xi d/\ell\right)+\cos\left(\xi d/\ell\right)-\cos\xi\right]^2}{\xi^2}d\xi$$

Note that

$$G\left(\infty\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\left[\sin\left(\xi d/\ell\right)+\cos\left(\xi d/\ell\right)-\cos\xi\right]^2}{\xi^2}d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\left[1+\cos^2\xi+\sin\left(2\xi d/\ell\right)-2\sin\left(\xi d/\ell\right)\cos\xi-2\cos\left(\xi d/\ell\right)\cos\xi\right]}{\xi^2}d\xi$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\frac{\left[\frac{1}{2}\cos\left(2\xi\right)-\frac{1}{2}+1-\cos\left(\left(1+d/\ell\right)\xi\right)+1-\cos\left(\left(1-d/\ell\right)\xi\right)\right]}{\xi^2}d\xi$$

$$= \left(1+d/\ell+1-d/\ell-2/2\right)/2 = 1/2$$

Integration by parts gives

$$G\left(\lambda\right) = -\frac{1}{2\pi}\frac{\left[-\sin\left(\frac{\lambda}{4}d/\ell\right)+\cos\left(\frac{\lambda}{4}d/\ell\right)-\cos\left(\lambda/4\right)\right]^2}{\lambda/4}$$

$$+\frac{1}{\pi}\int_{-\lambda/4}^{\infty}\frac{\left[\sin\left(\xi d/\ell\right)+\cos\left(\xi d/\ell\right)-\cos\xi\right]\left[\left(d/\ell\right)\cos\left(\xi d/\ell\right)-\left(d/\ell\right)\sin\left(\xi d/\ell\right)+\sin\xi\right]}{\xi}d\xi$$

$$= -\frac{1}{2\pi}\frac{\left[-\sin\left(\frac{\lambda}{4}d/\ell\right)+\cos\left(\frac{\lambda}{4}d/\ell\right)-\cos\left(\lambda/4\right)\right]^2}{\lambda/4}$$

$$+\frac{1}{\pi}\int_{-\lambda/4}^{\infty}$$

$$\frac{\left[\left(d/\ell\right)\cos\left(2\xi d/\ell\right)+\sin\xi\left\{\sin\left(\xi d/\ell\right)+\cos\left(\xi d/\ell\right)\right\}+\left(d/\ell\right)\cos\xi\left\{\sin\left(\xi d/\ell\right)-\cos\left(\xi d/\ell\right)\right\}-\frac{1}{2}\sin\left(2\xi\right)\right]}{\xi}d\xi$$

or

$$G\left(\lambda\right)+\frac{1}{2\pi}\frac{\left[-\sin\left(\frac{\lambda}{4}d/\ell\right)+\cos\left(\frac{\lambda}{4}d/\ell\right)-\cos\left(\lambda/4\right)\right]^2}{\lambda/4} =$$

$$\frac{1}{\pi}\int_{-\lambda/4}^{\infty}\frac{\left[\frac{1}{2}\left\{\sin\left(\left(1-d/\ell\right)\xi\right)+\sin\left(\left(1+d/\ell\right)\xi\right)\right\}+\frac{1}{2}\left(d/\ell\right)\left\{\sin\left(\left(1+d/\ell\right)\xi\right)-\sin\left(\left(1-d/\ell\right)\xi\right)\right\}-\frac{1}{2}\sin\left(2\xi\right)\right]}{\xi}d\xi$$

$$+\frac{1}{\pi}\int_{-\lambda/4}^{\infty}$$

$$\frac{\left[\frac{1}{2}\left\{\cos\left(\left(1-d/\ell\right)\xi\right)-\cos\left(\left(1+d/\ell\right)\xi\right)\right\}-\frac{1}{2}\left(d/\ell\right)\left\{\cos\left(\left(1-d/\ell\right)\xi\right)+\cos\left(\left(1+d/\ell\right)\xi\right)\right\}+\left(d/\ell\right)\cos\left(2\xi d/\ell\right)\right]}{\xi}d\xi$$

or

$$G\left(\lambda\right)+\frac{1}{2\pi}\frac{\left[-\sin\left(\frac{\lambda}{4}d/\ell\right)+\cos\left(\frac{\lambda}{4}d/\ell\right)-\cos\left(\lambda/4\right)\right]^{2}}{\lambda/4}=$$

$$\frac{1}{4}+\frac{1}{2\pi}\left[\left(1-d/\ell\right)\mathrm{Si}\left(\left(1-d/\ell\right)\lambda/4\right)+\left(1+d/\ell\right)\mathrm{Si}\left(\left(1+d/\ell\right)\lambda/4\right)-\mathrm{Si}\left(\lambda/2\right)\right]$$

$$+\frac{1}{2\pi}\left[\left(1+d/\ell\right)\mathrm{Ci}\left(-\left(1+d/\ell\right)\lambda/4\right)-\left(1-d/\ell\right)\mathrm{Ci}\left(-\left(1-d/\ell\right)\lambda/4\right)-2\left(d/\ell\right)\mathrm{Ci}\left(-\left(d/\ell\right)\lambda/2\right)\right]$$

$$=\frac{1}{4}+\frac{1}{2\pi}\left[\left(1-d/\ell\right)\mathrm{Si}\left(\left(1-d/\ell\right)\lambda/4\right)+\left(1+d/\ell\right)\mathrm{Si}\left(\left(1+d/\ell\right)\lambda/4\right)-\mathrm{Si}\left(\lambda/2\right)\right]$$

$$-\frac{1}{2\pi}\left[\left(1+d/\ell\right)\mathrm{Cin}\left(\left(1+d/\ell\right)\lambda/4\right)-\left(1-d/\ell\right)\mathrm{Cin}\left(\left(1-d/\ell\right)\lambda/4\right)-2\left(d/\ell\right)\mathrm{Cin}\left(\left(d/\ell\right)\lambda/2\right)\right]$$

$$+\frac{1}{2\pi}\left[\left(1+d/\ell\right)\ln\left(1+d/\ell\right)-\left(1-d/\ell\right)\ln\left(1-d/\ell\right)-2\left(d/\ell\right)\ln\left(2\left(d/\ell\right)\right)\right] \tag{674}$$

Note that

$$G\left(0\right)=\frac{1}{4}+\frac{1}{2\pi}\left[\left(1+d/\ell\right)\ln\left(1+d/\ell\right)-\left(1-d/\ell\right)\ln\left(1-d/\ell\right)-2\left(d/\ell\right)\ln\left(2\left(d/\ell\right)\right)\right] \tag{675}$$

$$G\left(-\infty\right)=0 \tag{676}$$

$$G\left(\infty\right)=1/2 \tag{677}$$

Thus taking into account the even symmetry along the orbit

$$G_{s}\left(\lambda\right)=2G\left(\lambda\right) \tag{678}$$

A comparison of the trigonometric projections for the cavity scar field and axisymmetric random plane wave field are given in Figure 36. Notice that in the stadium there is a large enhancement in the cavity scar field relative to the random plane wave field near $s_p = 0$, where the cavity modal frequency aligns with the scar frequency $k \to k_p$. There is in addition the axisymmetric enhancement due to the symmetry of the field $\left\langle LV_p^2\right\rangle / \left\langle LV_{pr}^2\right\rangle = O\left(kV/A\right) >> 1$ (analogous to the factor of two increase in the 2D even geometry [2]).

Shifted Focus point d_e The comparison when the trigonometric projection is defined using the shifted focal

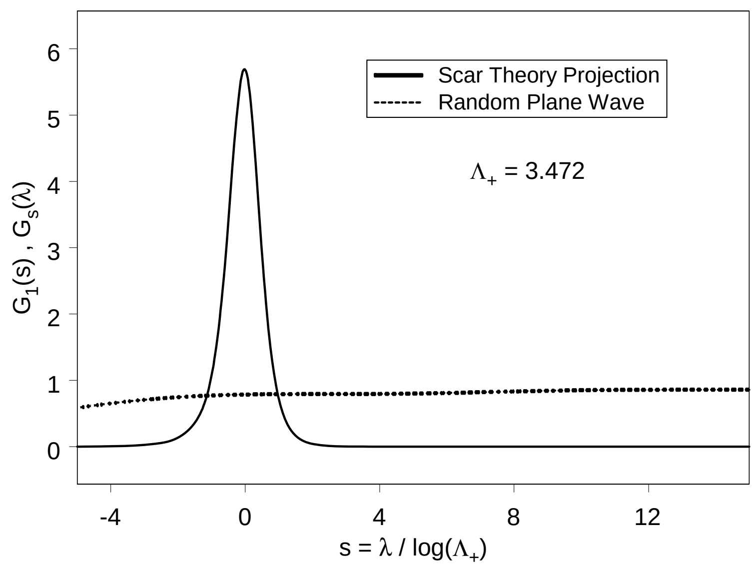

Figure 36. Comparison of trigonometric projections of the rotationally symmetric stadium vector cavity field and the axisymmetric random plane wave field.

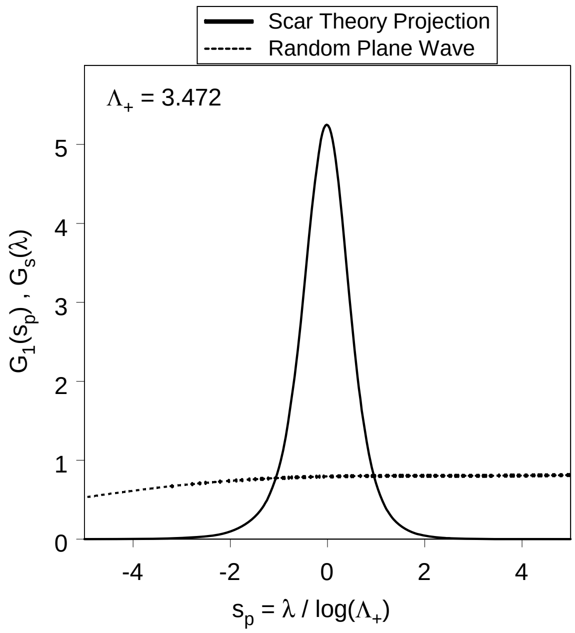

Figure 37. Comparison of trigonometric projections of the rotationally symmetric stadium vector cavity field and the axisymmetric random plane wave field when the trigonometric projection operator is redefined using the shifted focal position d_e.

$$V_{pr}^{T} = 2\int_0^{d_e} \underline{E}_r^{(m=1)} \cdot \underline{e}_\rho \cos\left(k_p z\right) dz + 2\int_{d_e}^{\ell} \underline{E}_r^{(m=1)} \cdot \underline{e}_\rho \cos\left(k_p z - \pi/2\right) dz \tag{679}$$

is shown in Figure 37.

6.6.2 Random Plane Wave Elliptic System Projection

Again taking the even z part of the $m = 1$ plane wave component

$$\underline{E}_r^{(m=1,\text{even})}\left(\rho = 0, \varphi = 0\right) \cdot \underline{e}_\rho =$$

$$\lim_{N\to\infty} \sqrt{2/\left(VN\right)} \sum_{j=1}^{N} a_j \left\{\cos\varphi_{pj}\sin\varphi_j + \sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right\}\cos\alpha_j\cos\left(kz\cos\theta_j\right)$$

$$p_0\left(z\right) = \frac{s}{2}\left\{\left(z/\ell\right)\ln\left(\frac{\ell+d}{\ell-d}\right) - \ln\left|\frac{z+d}{z-d}\right|\right\}$$

and defining the elliptic system projection as we did in the scar section, we find

$$\exp\left(\pi\left|s\right|/4\right)V_{pr}$$

$$= 2e^{s\pi/4}\int_0^d \underline{E}_r^{(m=1)}\cdot\underline{e}_\rho\cos\left[k_p z + p_0(z)\right]dz + 2e^{-s\pi/4}\int_d^\ell \underline{E}_r^{(m=1)}\cdot\underline{e}_\rho\cos\left[k_p z - \pi/2 + p_0(z)\right]dz =$$

$$\lim_{N\to\infty}\sqrt{2/(VN)}\sum_{j=1}^{N} a_j\left\{\cos\varphi_{pj}\sin\varphi_j + \sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right\}\cos\alpha_j$$

$$2\left\{e^{s\pi/4}\int_0^d \cos\left(kz\cos\theta_j\right)\cos\left[k_p z + p_0(z)\right]dz + e^{-s\pi/4}\int_d^\ell \cos\left(kz\cos\theta_j\right)\cos\left[k_p z - \pi/2 + p_0(z)\right]dz\right\}$$

$$= \frac{1}{2}\lim_{N\to\infty}\sqrt{2/(VN)}\sum_{j=1}^{N} a_j\left\{\cos\varphi_{pj}\sin\varphi_j + \sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right\}2\cos\alpha_j$$

$$\left\{e^{s\pi/4}\int_0^d \left[\cos\left(kz\cos\theta_j - k_p z - p_0(z)\right) + \cos\left(kz\cos\theta_j + k_p z + p_0(z)\right)\right]dz\right.$$

$$\left.+e^{-s\pi/4}\int_d^\ell \left[-\sin\left(kz\cos\theta_j - k_p z - p_0(z)\right) + \sin\left(kz\cos\theta_j + k_p z + p_0(z)\right)\right]dz\right\}$$

Then we have

$$\exp\left(\pi|s|/2\right)\left\langle V_{pr}^2\right\rangle_{a_j} = \lim_{N\to\infty}\left\{2/(VN)\right\}\sum_{j=1}^{N}\left\{\cos\varphi_{pj}\sin\varphi_j + \sin\varphi_{pj}\cos\theta_j\cos\varphi_j\right\}^2\cos^2\alpha_j$$

$$\left\{e^{s\pi/4}\int_0^d \left[\cos\left(kz\cos\theta_j - k_p z - p_0(z)\right) + \cos\left(kz\cos\theta_j + k_p z + p_0(z)\right)\right]dz\right.$$

$$\left.+e^{-s\pi/4}\int_d^\ell \left[-\sin\left(kz\cos\theta_j - k_p z - p_0(z)\right) + \sin\left(kz\cos\theta_j + k_p z + p_0(z)\right)\right]dz\right\}^2$$

Averaging over α_j, φ_j, and φ_{pj} gives

$$\exp\left(\pi|s|/2\right)\left\langle V_{pr}^2\right\rangle_{a_j,\alpha_j,\varphi_j,\varphi_{pj}} = \frac{1}{4V}\lim_{N\to\infty}\left\{1/(N)\right\}\sum_{j=1}^{N}\left\{1+\cos^2\theta_j\right\}$$

$$\left\{e^{s\pi/4}\int_0^d \left[\cos\left(kz\cos\theta_j - k_p z - p_0(z)\right) + \cos\left(kz\cos\theta_j + k_p z + p_0(z)\right)\right]dz\right.$$

$$\left.+e^{-s\pi/4}\int_d^\ell \left[-\sin\left(kz\cos\theta_j - k_p z - p_0(z)\right) + \sin\left(kz\cos\theta_j + k_p z + p_0(z)\right)\right]dz\right\}^2$$

Finally averaging over θ_j gives

$$\exp\left(\pi\left|s\right|/2\right)\left\langle V_{pr}^{2}\right\rangle_{a_j,\alpha_j,\varphi_j,\varphi_{pj}} = \frac{1}{8V}\int_0^{\pi}\left\{1+\cos^2\theta_j\right\}$$

$$\left\{e^{s\pi/4}\int_0^d\left[\cos\left(kz\cos\theta_j - k_pz - p_0\left(z\right)\right)+\cos\left(kz\cos\theta_j + k_pz + p_0\left(z\right)\right)\right]dz\right.$$

$$\left.+e^{-s\pi/4}\int_d^{\ell}\left[-\sin\left(kz\cos\theta_j - k_pz - p_0\left(z\right)\right)+\sin\left(kz\cos\theta_j + k_pz + p_0\left(z\right)\right)\right]dz\right\}^2\sin\theta_j d\theta_j$$

Now for large $k\sim k_p$ we expect that the θ_j integration leads to a peaked value of the first terms of the integration for $\theta_j\to 0$ and for the second terms of the integration for $\theta_j\to\pi$. Thus taking $k\to k_p$ with the first term peaks for $\theta_j\to 0$ and the second term peaks for $\theta_j\to\pi$ and using $\cos\theta_j=-\cos\left(\theta_j-\pi\right)$ and $\sin\theta_j=-\sin\left(\theta_j-\pi\right)$

$$\exp\left(\pi\left|s\right|/2\right)\left\langle V_{pr}^{2}\right\rangle_{a_j,\alpha_j,\varphi_j,\varphi_{pj}} \sim \frac{1}{8V}\int_0^{\pi}\left\{1+\cos^2\theta_j\right\}$$

$$\left\{e^{s\pi/4}\int_0^d\left[\cos\left(kz\cos\theta_j - k_pz - p_0\left(z\right)\right)\right]dz\right.$$

$$\left.+e^{-s\pi/4}\int_d^{\ell}\left[-\sin\left(kz\cos\theta_j - k_pz - p_0\left(z\right)\right)\right]dz\right\}^2\sin\theta_j d\theta_j$$

$$+\frac{1}{8V}\int_0^{\pi}\left\{1+\cos^2\theta_j\right\}$$

$$\left\{e^{s\pi/4}\int_0^d\cos\left(-kz\cos\left(\pi-\theta_j\right) + k_pz + p_0\left(z\right)\right)dz\right.$$

$$\left.+e^{-s\pi/4}\int_d^{\ell}\sin\left(-kz\cos\left(\pi-\theta_j\right) + k_pz + p_0\left(z\right)\right)dz\right\}^2\sin\left(\pi-\theta_j\right)d\theta_j$$

$$\exp\left(\pi\left|s\right|/2\right)\left\langle V_{pr}^{2}\right\rangle_{a_j,\alpha_j,\varphi_j,\varphi_{pj}} \sim \frac{1}{4V}\int_0^{\pi}\left\{1+\cos^2\theta_j\right\}$$

$$\left\{e^{s\pi/4}\int_0^d\cos\left(kz\cos\theta_j - k_pz - p_0\left(z\right)\right)dz\right.$$

$$\left.-e^{-s\pi/4}\int_d^{\ell}\sin\left(kz\cos\theta_j - k_pz - p_0\left(z\right)\right)dz\right\}^2\sin\theta_j d\theta_j$$

Letting $\theta_j=\zeta/\sqrt{k\ell/2}$ gives (note that $1+\cos^2\theta_j\sim 2$)

$$\exp\left(\pi\left|s\right|/2\right)\left\langle V_{pr}^{2}\right\rangle_{a_j,\alpha_j,\varphi_j,\varphi_{pj}} = \frac{1}{k\ell V}\int_0^{\infty}$$

$$\left\{ e^{s\pi/4} \int_0^d \cos\left[\left(\lambda/4 - \zeta^2\right) z/\ell - p_0(z)\right] dz \right.$$

$$\left. - e^{-s\pi/4} \int_d^\ell \sin\left[\left(\lambda/4 - \zeta^2\right) z/\ell - p_0(z)\right] dz \right\}^2 \zeta d\zeta$$

where

$$p_0(z) = \frac{s}{2}\left\{(z/\ell)\ln\left(\frac{\ell+d}{\ell-d}\right) - \ln\left|\frac{z+d}{z-d}\right|\right\}$$

$$\lambda = 2(k - k_p) L$$

Renormalizing by

$$\int_V \left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)} \right\rangle_{a_j,\alpha_j,\varphi_{pj},\varphi_j} dV \sim \frac{2\pi}{kV} \int_0^{\rho_{\max}} \ell(\rho)\, d\rho = \frac{\pi A}{2kV}$$

gives

$$\exp(\pi|s|/2) \left\langle V_{pr}^2 \right\rangle_{a_j,\alpha_j,\varphi_j,\varphi_{pj}} = \frac{2}{\ell\pi A} \int_0^\infty$$

$$\left\{ e^{s\pi/4} \int_0^d \cos\left[\left(\lambda/4 - \zeta^2\right) z/\ell - p_0(z)\right] dz \right.$$

$$\left. - e^{-s\pi/4} \int_d^\ell \sin\left[\left(\lambda/4 - \zeta^2\right) z/\ell - p_0(z)\right] dz \right\}^2 \zeta d\zeta$$

Now taking

$$\left\langle L V_{pr}^2 \right\rangle = L^2 G(\lambda)/A$$

gives

$$\exp(\pi|s|/2)\, G(\lambda) = \frac{1}{\pi\ell^2} \int_0^\infty$$

$$\left\{ e^{s\pi/4} \int_0^d \cos\left[\left(\lambda/4 - \zeta^2\right) z/\ell - p_0(z)\right] dz \right.$$

$$\left. - e^{-s\pi/4} \int_d^\ell \sin\left[\left(\lambda/4 - \zeta^2\right) z/\ell - p_0(z)\right] dz \right\}^2 \zeta d\zeta$$

Taking $\zeta^2 - \lambda/4 = \xi$ gives

$$G(\lambda) = \frac{1}{2\pi}\int_{-\lambda/4}^{\infty}$$

$$\left\{e^{-(|s|-s)\pi/4}\int_0^{d/\ell}\cos\left[\xi z + p_0(z\ell)\right]dz\right.$$

$$\left.+e^{-(s+|s|)\pi/4}\int_{d/\ell}^{1}\sin\left[\xi z + p_0(z\ell)\right]dz\right\}^2 d\xi$$

$$p_0(z\ell) = \frac{s}{2}\left\{z\ln\left(\frac{\ell+d}{\ell-d}\right) - \ln\left|\frac{z+d/\ell}{z-d/\ell}\right|\right\}$$

Then because of symmetry along the orbit we define

$$G_s(\lambda) = 2G(\lambda) = \frac{1}{\pi}\int_{-\lambda/4}^{\infty}$$

$$\left\{e^{-(|s|-s)\pi/4}\int_0^{d/\ell}\cos\left[\xi z + p_0(z\ell)\right]dz\right.$$

$$\left.+e^{-(s+|s|)\pi/4}\int_{d/\ell}^{1}\sin\left[\xi z + p_0(z\ell)\right]dz\right\}^2 d\xi$$

where

$$s = \lambda/\ln(\Lambda_+)$$

For purposes of numerical integration

$$G_s(\lambda) = 2G(\lambda) = \frac{1}{\pi}\int_{-\lambda/4}^{\infty}\left[e^{-(|s|-s)\pi/4}\int_0^{d/\ell}\left\{\cos(\xi z)\cos(p_0(z\ell)) - \sin(\xi z)\sin(p_0(z\ell))\right\}dz\right.$$

$$\left.+e^{-(|s|+s)\pi/4}\int_{d/\ell}^{1}\left\{\cos(\xi z)\sin(p_0(z\ell)) + \sin(\xi z)\cos(p_0(z\ell))\right\}dz\right]^2 d\xi \qquad (680)$$

This is exactly half of the scalar case. The comparison of the elliptic system projections in the stadium cavity between the scar construction and the random plane wave is shown in Figure 38.

6.7 Numerical Comparison For Stadium

The trigonometric projection from the numerical solution is (the normalization originally used here is too large by π) shown in the following Figure 39. The additional peaks in this figure are thought to result from the lack of orthogonality associated with the trigonometric projection operator in the stadium cavity.

Figure 40 has the proper normalization value. This is to be compared to the theoretical result in Figure 37. We do not double this result because we used only the even modes in the histogram construction (unlike

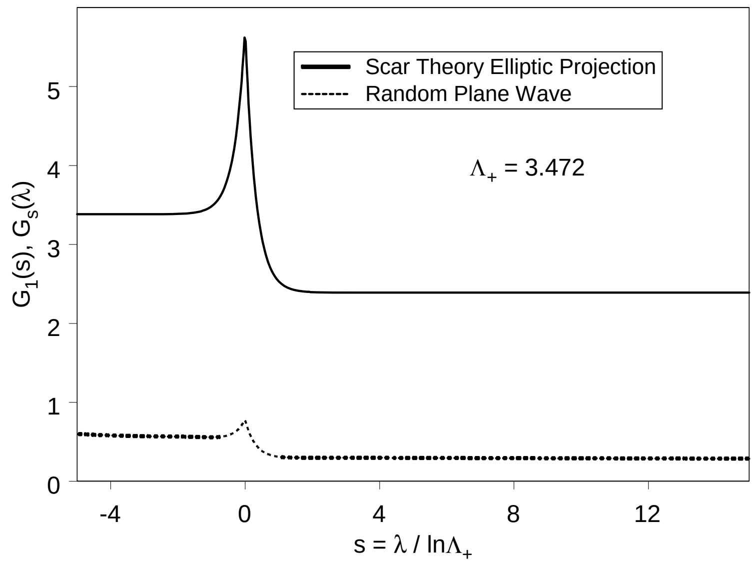

Figure 38. Comparison of elliptic projections of the rotationally symmetric stadium cavity field and the axisymmetric random plane wave field for the vector case.

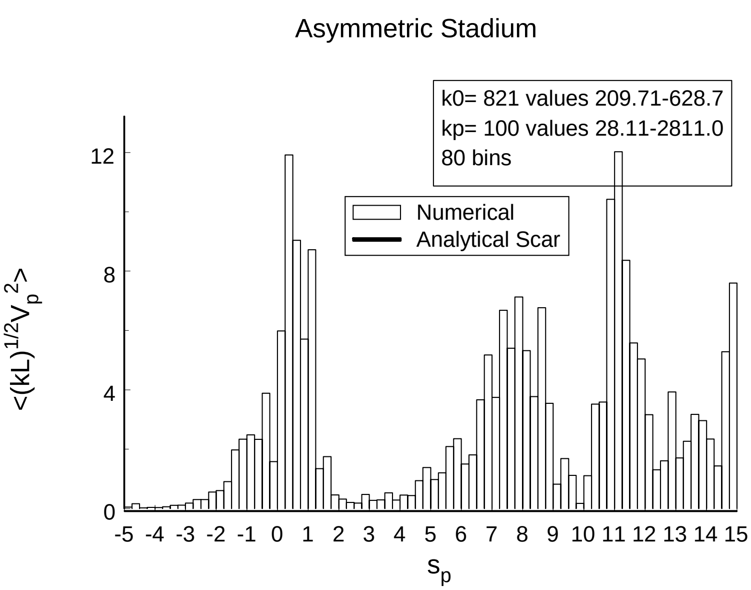

Figure 39. Raw numerical histogram data for the trigonometric projection in the stadium cavity.

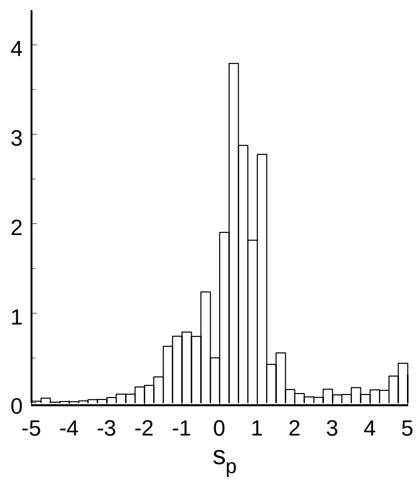

Figure 40. The first peak of the numerical histogram of the trigonometric projection with corrected normalization in the stadium cavity.

the bowtie case).

The modal density for the modes that are even along the orbit are (the modes that are odd along the orbit are expected to have the same density) are expected to be

$$\frac{dN}{dk^2} \sim A/\left(8\pi\right)$$

$$N \sim \left(k_2 - k_1\right)\left(k_2 + k_1\right) A/\left(8\pi\right)$$

$$k_1 \approx 209.586 \text{ m}^{-1}, \;\; k_2 \approx 628.759 \text{ m}^{-1}$$

$$A = \pi R^2 + 2\left(\ell - R\right) 2R \approx 0.0365583 \text{ m}^2$$

$$V = \frac{4}{3}\pi R^3 + 2\left(\ell - R\right)\pi R^2 \approx 0.00505205 \text{ m}^3$$

$$\ell \approx 0.11176 \text{ m}$$

$$R \approx 0.10160 \text{ m}$$

$$N \approx 511$$

The numerical solution for the quarter stadium containing only the odd modes along the orbit (since a perfect electric conductor was used at the symmetry plane) had 478 modes, which is reasonably close to the expected number.

6.8 Point Statistics For Stadium

Finally, the most direct observable in the cavity is the field as a function of location. Thus we intend to examine the field point statistics near the focal point in region 3 where. In region 3 we can write the Hertz potential as

$$\Pi_{ev} \sim u = C_0 c_0 \frac{\sqrt{2}}{\tau'} \operatorname{Re}\left[W_+\left(\tau', 0, s'\right) + e^{i\Phi_0'} W_+^*\left(\tau', 0, s'\right)\right]$$

where

$$\frac{\sqrt{2}}{\tau} \operatorname{Re}\left[e^{-i\pi/2} W_+\left(\tau, 0, s\right) + e^{i\pi/2} e^{i\Phi_0} W_+^*\left(\tau, 0, s\right)\right]$$

$$= \frac{C_0}{c_0} \psi_0\left(\tau', s'\right) \psi_0\left(\tau, s\right) \tag{681}$$

and

$$c_0 = \frac{v\sqrt{2}}{k^2 \sqrt{Ad\left(1 + e^{\pi s'}\right) \ln\left(\Lambda_+\right)} \left|\operatorname{Re}\left[e^{i\pi/4} \Gamma\left(1/2 + is'/2\right)\right]\right|}$$

$$-s' = s = 2\left(k - k_p\right) \ell / \ln\left(\frac{\ell + d}{\ell - d}\right) = 2\left(k - k_p\right) L / \ln\left(\Lambda_+\right)$$

$$s' \ln\left[\tanh\left(\zeta_0/2\right)\right] = \left(k - k_p\right) \ell$$

$$k_p \ell = \pi p$$

$$C_0 = (-1)^{n'} e^{s'\pi/4} \sqrt{\gamma} \sec\left(\Phi_0/2\right)$$

$$\psi_0\left(\tau, s\right) = c_0 \frac{\sqrt{2}}{\tau} \operatorname{Re}\left[e^{-i\pi/2} W_+\left(\tau, 0, s\right) + e^{i\pi/2} e^{i\Phi_0} W_+^*\left(\tau, 0, s\right)\right]$$

$$W_+\left(\tau, 0, s\right) = W_{is/2,0}\left(-i\tau^2/2\right)$$

$$= e^{i\tau^2/4} \left(-i\tau^2/2\right)^{1/2} U\left(1/2 - is/2, 1, -i\tau^2/2\right)$$

$$\tau = \sqrt{2\gamma}\zeta$$

$$\psi_0\left(0,s\right) = c_0 \operatorname{Re}\left[e^{\pi s/2} e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right] \ , \ \tau \to 0$$

$$\frac{1}{\tau}\frac{\partial}{\partial \tau}\psi_0\left(\tau,s\right) \sim -c_0 \operatorname{Re}\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right] e^{\pi s/2}\left(s/2\right) + O\left(\tau^4 \ln \tau\right) \sim -\frac{s}{2}\psi_0$$

$$\frac{1}{\tau}\frac{\partial}{\partial \tau}\left(\tau \frac{\partial}{\partial \tau}\psi_0\right) \sim -c_0 \operatorname{Re}\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right] e^{\pi s/2} s + O\left(\tau^2 \ln \tau\right) \sim -s\psi_0 + O\left(\tau^2 \ln \tau\right)$$

$$\psi_0\left(\tau',s'\right) = c_0 \frac{\sqrt{2}}{\tau'} \operatorname{Re}\left[W_+\left(\tau',0,s'\right) + e^{i\Phi_0'} W_+^*\left(\tau',0,s'\right)\right]$$

$$W_+\left(\tau',0,s'\right) = W_{is'/2,0}\left(-i\tau'^2/2\right)$$

$$\tau' = \sqrt{2\gamma}\xi'$$

$$\xi' = \pm\pi/2 - \xi$$

$$\psi_0\left(0,s'\right) = c_0 \operatorname{Re}\left[e^{\pi s'/2} e^{i\pi/4}\Gamma\left(1/2+is'/2\right)\right] \ , \ \tau' \to 0$$

$$\frac{1}{\tau'}\frac{\partial}{\partial \tau'}\psi_0\left(\tau',s'\right) \sim -c_0 \operatorname{Re}\left[e^{i\pi/4}\Gamma\left(1/2+is'/2\right)\right] e^{\pi s'/2}\left(s'/2\right) + O\left(\tau'^4 \ln \tau'\right) \sim -\frac{s'}{2}\psi_0$$

$$\frac{1}{\tau'}\frac{\partial}{\partial \tau'}\left(\tau' \frac{\partial}{\partial \tau'}\psi_0\right) \sim -c_0 \operatorname{Re}\left[e^{i\pi/4}\Gamma\left(1/2+is'/2\right)\right] e^{\pi s'/2} s' + O\left(\tau'^2 \ln \tau'\right) \sim -s'\psi_0 + O\left(\tau'^2 \ln \tau'\right)$$

The random phase is

$$e^{-i\Phi_0} = i\frac{\Gamma\left(1/2+is'/2\right)}{\Gamma\left(1/2-is'/2\right)} = \cosh\left(\pi s'/2\right)\frac{i}{\pi}\Gamma^2\left(1/2+is'/2\right)$$

$$\sim i\left[1 - is'\left(\gamma'+2\ln 2\right) - s'^2\left(\gamma'+2\ln 2\right)^2/2\right] \ , \ s' \to 0$$

The focal point shift can be written as

$$k\delta = -\gamma - s'\ln\left(2\sqrt{\gamma}\right) - \Phi_0/2 + n\pi$$

On axis we therefore have

$$\Pi_{ev} \sim C_0 \operatorname{Re}\left[e^{\pi s/2} e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]\psi_0\left(\tau',s'\right) \ , \ \text{between foci}$$

$$\sim C_0 \operatorname{Re}\left[e^{\pi s'/2} e^{i\pi/4} \Gamma\left(1/2 + is'/2\right)\right] \psi_0\left(\tau, s\right) \ \text{, outside foci}$$

where the field is

$$h^2 E_\upsilon = h\frac{\partial}{\partial \upsilon}\left[\frac{1}{h^3}\frac{\partial}{\partial \upsilon}\left(h^2 \Pi_{e\upsilon}\right)\right] + k^2 h^2 \Pi_{e\upsilon} \sim \left(\frac{\partial^2}{\partial \upsilon^2} + k^2 h^2\right)\Pi_{e\upsilon} = \left(2\gamma\frac{\partial^2}{\partial \tau_x^2} + k^2 h^2\right)\Pi_{e\upsilon}$$

$$\sim \left(\frac{\partial^2}{\partial \upsilon^2} + k^2 h^2\right)\Pi_{e\upsilon} \sim \frac{\partial^2 \Pi_{e\upsilon}}{\partial \zeta^2}\cos^2\varphi + \frac{1}{\zeta}\frac{\partial \Pi_{e\upsilon}}{\partial \zeta}\sin^2\varphi + k^2 h^2 \Pi_{e\upsilon}$$

$$\sim \frac{1}{d}\left[2\gamma\frac{\partial^2 \Pi_{e\upsilon}}{\partial \tau^2}\cos^2\varphi + 2\gamma\frac{1}{\tau}\frac{\partial \Pi_{e\upsilon}}{\partial \tau}\sin^2\varphi + \gamma^2\left(h/d\right)^2 \Pi_{e\upsilon}\right]$$

$$\gamma = kd$$

$$\tau = \sqrt{2\gamma}\zeta$$

$$h_\zeta = h_\xi = d\sqrt{\sinh^2\zeta + \cos^2\xi} = d\sqrt{\cosh^2\zeta - \sin^2\xi} = d\sqrt{\cosh^2\zeta - \cos^2\xi'} \sim h$$

$$h_\xi \sim h \sim d\cos\xi, \text{ Region 1}$$

$$h_\zeta = h_\xi = h \sim d\sinh\zeta, \text{ Region 2}$$

Thus for large γ, keeping only the leading term, the transverse field on axis is

$$E_\upsilon \sim k^2 \Pi_{e\upsilon}$$

$$\sim k^2 C_0 \operatorname{Re}\left[e^{\pi s/2} e^{-i\pi/4} \Gamma\left(1/2 + is/2\right)\right] \psi_0\left(\tau', s'\right) \ \text{, between foci}$$

$$\sim k^2 C_0 \operatorname{Re}\left[e^{\pi s'/2} e^{i\pi/4} \Gamma\left(1/2 + is'/2\right)\right] \psi_0\left(\tau, s\right) \ \text{, outside foci}$$

and thus the mean square field is

$$\left\langle A E_\upsilon^2 \right\rangle \sim$$

$$\sim A\gamma e^{s'\pi/2} \operatorname{Re}^2\left[e^{\pi s/2} e^{-i\pi/4} \Gamma\left(1/2 + is/2\right)\right] \sec^2\left(\Phi_0/2\right) \left\langle k^4 \psi_0^2\left(\tau', s'\right)\right\rangle \ \text{, between foci}$$

$$\sim A\gamma e^{s'\pi/2} \operatorname{Re}^2\left[e^{\pi s'/2} e^{i\pi/4} \Gamma\left(1/2 + is'/2\right)\right] \sec^2\left(\Phi_0/2\right) \left\langle k^4 \psi_0^2\left(\tau, s\right)\right\rangle \ \text{, outside foci}$$

$$\cos\Phi_0 = 2\cos^2\left(\Phi_0/2\right) - 1$$

$$= i\frac{\left[\Gamma\left(1/2+is'/2\right)+\Gamma\left(1/2-is'/2\right)\right]\left[\Gamma\left(1/2+is'/2\right)-\Gamma\left(1/2-is'/2\right)\right]}{2\Gamma\left(1/2-is'/2\right)\Gamma\left(1/2+is'/2\right)}$$

$$= 2\frac{\mathrm{Re}\left[\Gamma\left(1/2-is'/2\right)\right]\mathrm{Im}\left[\Gamma\left(1/2-is'/2\right)\right]}{\left|\Gamma\left(1/2-is'/2\right)\right|^2} \tag{682}$$

$$\sec^2\left(\Phi_0/2\right) = \frac{2\left|\Gamma\left(1/2-is'/2\right)\right|^2}{2\,\mathrm{Re}\left[\Gamma\left(1/2-is'/2\right)\right]\mathrm{Im}\left[\Gamma\left(1/2-is'/2\right)\right]+\left|\Gamma\left(1/2-is'/2\right)\right|^2}$$

$$= \frac{2\left|\Gamma\left(1/2-is'/2\right)\right|^2}{\left\{\mathrm{Re}\left[\Gamma\left(1/2-is'/2\right)\right]+\mathrm{Im}\left[\Gamma\left(1/2-is'/2\right)\right]\right\}^2} = \frac{2\left|\Gamma\left(1/2-is'/2\right)\right|^2}{2\,\mathrm{Re}^2\left[e^{-i\pi/4}\Gamma\left(1/2-is'/2\right)\right]}$$

$$= \frac{\left|\Gamma\left(1/2+is/2\right)\right|^2}{\mathrm{Re}^2\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]} = \frac{\left|\Gamma\left(1/2+is'/2\right)\right|^2}{\mathrm{Re}^2\left[e^{i\pi/4}\Gamma\left(1/2+is'/2\right)\right]}$$

so that

$$\left\langle AE_\upsilon^2\right\rangle \sim$$

$$\sim A\gamma e^{\pi s/2}\left|\Gamma\left(1/2+is/2\right)\right|^2\left\langle k^4\psi_0^2\left(\tau',s'\right)\right\rangle \ \text{, between foci}$$

$$\sim A\gamma e^{3\pi s'/2}\left|\Gamma\left(1/2+is'/2\right)\right|^2\left\langle k^4\psi_0^2\left(\tau,s\right)\right\rangle \ \text{, outside foci}$$

$$\left\langle k^4c_0^2\right\rangle = \frac{2}{Ad\left(1+e^{\pi s'}\right)\ln\left(\Lambda_+\right)\mathrm{Re}^2\left[e^{i\pi/4}\Gamma\left(1/2+is'/2\right)\right]}$$

or

$$\left\langle AE_\upsilon^2\right\rangle \sim$$

$$\sim \frac{2ke^{\pi s/2}\left|\Gamma\left(1/2+is/2\right)\right|^2}{\left(1+e^{-\pi s}\right)\ln\left(\Lambda_+\right)\mathrm{Re}^2\left[e^{i\pi/4}\Gamma\left(1/2+is'/2\right)\right]}\frac{2}{\tau'^2}\mathrm{Re}^2\left[W_+\left(\tau',0,s'\right)+e^{i\Phi_0'}W_+^*\left(\tau',0,s'\right)\right] \ \text{, between foci}$$

$$\sim \frac{2ke^{\pi s'/2}\left|\Gamma\left(1/2+is'/2\right)\right|^2}{\left(1+e^{-\pi s'}\right)\ln\left(\Lambda_+\right)\mathrm{Re}^2\left[e^{i\pi/4}\Gamma\left(1/2+is'/2\right)\right]}\frac{2}{\tau^2}\mathrm{Re}^2\left[e^{-i\pi/2}W_+\left(\tau,0,s\right)+e^{i\pi/2}e^{i\Phi_0}W_+^*\left(\tau,0,s\right)\right] \ \text{, outside foci}$$

or

$$\left\langle VE_\upsilon^2\right\rangle\left(3\pi A\right)/\left(2kV\right) = \left\langle AE_\upsilon^2\right\rangle\left(3\pi\right)/\left(2k\right) \sim$$

$$\sim \frac{3\pi e^{-\pi s'/2}\left|\Gamma\left(1/2+is'/2\right)\right|^2}{\left(1+e^{\pi s'}\right)\ln\left(\Lambda_+\right)\mathrm{Re}^2\left[e^{i\pi/4}\Gamma\left(1/2+is'/2\right)\right]}\frac{2}{\tau'^2}\mathrm{Re}^2\left[W_+\left(\tau',0,s'\right)+e^{i\Phi_0'}W_+^*\left(\tau',0,s'\right)\right] \ \text{, between foci}$$

$$\sim \frac{3\pi e^{-\pi s/2}\left|\Gamma\left(1/2+is/2\right)\right|^{2}}{\left(1+e^{\pi s}\right)\ln\left(\Lambda_{+}\right)\mathrm{Re}^{2}\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]}\frac{2}{\tau^{2}}\mathrm{Re}^{2}\left[e^{-i\pi/2}W_{+}\left(\tau,0,s\right)+e^{i\pi/2}e^{i\Phi_{0}}W_{+}^{*}\left(\tau,0,s\right)\right]\ ,\ \text{outside foci}$$

$$\left|\Gamma\left(1/2+is/2\right)\right|^{2}=\frac{\pi}{\cosh\left(\pi s/2\right)}=\frac{2\pi e^{\pi s/2}}{e^{\pi s}+1}=\frac{2\pi e^{\pi s'/2}}{e^{\pi s'}+1}=\left|\Gamma\left(1/2+is'/2\right)\right|^{2}$$

and

$$\left\langle VE_{v}^{2}\right\rangle\left(3\pi A\right)/\left(2kV\right)=\left\langle AE_{v}^{2}\right\rangle\left(3\pi\right)/\left(2k\right)\sim$$

$$\sim \frac{6\pi^{2}}{\left(1+e^{\pi s'}\right)^{2}\ln\left(\Lambda_{+}\right)\mathrm{Re}^{2}\left[e^{i\pi/4}\Gamma\left(1/2+is'/2\right)\right]}\frac{2}{\tau'^{2}}\mathrm{Re}^{2}\left[W_{+}\left(\tau',0,s'\right)+e^{i\Phi_{0}'}W_{+}^{*}\left(\tau',0,s'\right)\right]\ ,\ \text{between foci}$$

$$\sim \frac{6\pi^{2}}{\left(1+e^{\pi s}\right)^{2}\ln\left(\Lambda_{+}\right)\mathrm{Re}^{2}\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]}\frac{2}{\tau^{2}}\mathrm{Re}^{2}\left[e^{-i\pi/2}W_{+}\left(\tau,0,s\right)+e^{i\pi/2}e^{i\Phi_{0}}W_{+}^{*}\left(\tau,0,s\right)\right]\ ,\ \text{outside foci}$$

$$e^{-i\Phi_{0}}=i\frac{\Gamma\left(1/2+is'/2\right)}{\Gamma\left(1/2-is'/2\right)}$$

Note that at the point $z=d$, or $\tau'\to 0$ (with $\tau=0$), or $\tau\to 0$ (with $\tau'=0$), we find

$$\left\langle VE_{v}^{2}\right\rangle\left(3\pi A\right)/\left(2kV\right)=\left\langle AE_{v}^{2}\right\rangle\left(3\pi\right)/\left(2k\right)$$

$$\sim \frac{6\pi^{2}e^{\pi s'}}{\left(1+e^{\pi s'}\right)^{2}\ln\left(\Lambda_{+}\right)}\ ,\ \text{between foci}$$

$$\sim \frac{6\pi^{2}e^{\pi s}}{\left(1+e^{\pi s}\right)^{2}\ln\left(\Lambda_{+}\right)}\ ,\ \text{outside foci} \tag{683}$$

In the special case of $s=0$ the field on axis reduces to

$$E_{v}\sim k^{2}\Pi_{ev}$$

$$\sim\left(-1\right)^{n'}v\frac{\sqrt{\pi\gamma}}{\sqrt{Ad\ln\left(\Lambda_{+}\right)}}J_{0}\left(k_{p}\left(d+\delta_{0}-z\right)\right)\ ,\ \text{between foci}$$

$$\sim\left(-1\right)^{n'}v\frac{\sqrt{\pi\gamma}}{\sqrt{Ad\ln\left(\Lambda_{+}\right)}}J_{0}\left(k_{p}\left(z-d-\delta_{0}\right)\right)\ ,\ \text{outside foci}$$

where we have used

$$\psi_{0}\left(\tau',0\right)=c_{0}\sqrt{\frac{\pi}{2}}J_{0}\left(\tau'^{2}/4\right)$$

$$\psi_0(\tau,0)=c_0\sqrt{\frac{\pi}{2}}J_0\left(\tau^2/4\right)$$

$$k^2c_0(0)=v\frac{\sqrt{2}}{\sqrt{\pi Ad\ln(\Lambda_+)}} \tag{684}$$

$$C_0(0)=(-1)^{n'}\sqrt{2\gamma}$$

$$k_p(d+\delta_0)=(n+1/4)\pi$$

$$\tau'^2/4=k_p(d+\delta_0-z)$$

$$\tau^2/4=k_p(z-d-\delta_0)$$

Thus we find

$$\left\langle AE_v^2\right\rangle\sim\frac{k\pi}{\ln(\Lambda_+)}J_0^2\left(k_p(d+\delta_0-z)\right)\ ,\ \text{between foci}$$

$$\sim\frac{k\pi}{\ln(\Lambda_+)}J_0^2\left(k_p(z-d-\delta_0)\right)\ ,\ \text{outside foci}$$

and at $z=d+\delta_0$

$$\left\langle AE_v^2\right\rangle\sim\frac{k\pi}{\ln(\Lambda_+)}$$

6.8.1 Shifted Focus

The normalized shift function, without corrections ($s'=0$) is

$$k_p(d+\delta_0)=(n+1/4)\pi$$

$$k_p\ell=p\pi$$

or

$$(d+\delta_0)/\ell=(n+1/4)/p$$

Noting

$$k_p\delta_0/\pi=n-n_c\geq 0$$

where

$$n_c = pd/\ell - 1/4$$

For large values of p (with $d/\ell < 1$) and n

$$k_p \langle \delta_0 \rangle / \pi \approx 1/2$$

With the frequency range

$$209.71 \text{ m}^{-1} \leq k \leq 628.7 \text{ m}^{-1}$$

and $\ell = 0.11176$ m, $d = 0.0336969$ m we find that

$$p_1 = 8 \leq p \leq 22 = p_2$$

as well as

$$\langle \delta_0 \rangle / \ell = \frac{1}{p_2 - p_1} \int_{p_1}^{p_2} \frac{dp}{2p} = \frac{1/2}{p_2 - p_1} \ln (p_2/p_1) = 0.0361286$$

and thus

$$d_e = d + \langle \delta \rangle \approx d + \langle \delta_0 \rangle \approx 0.0377346 \text{ m}$$

$$\Lambda_+ = \left(\frac{\ell + d}{\ell - d} \right)^2 \approx 3.47198$$

$$\Lambda_+^e = \left(\frac{\ell + d_e}{\ell - d_e} \right)^2 \approx 4.07840$$

$$\ln (\Lambda_+) / \ln (\Lambda_+^e) \approx 0.885481$$

6.8.2 Random Plane Wave Field Statistics

From the random plane wave results with volume normalization for the entire modal field (all m's) we had that

$$\int_V \left\langle \underline{E}_r^{(m=1)} \cdot \underline{E}_r^{(m=1)} \right\rangle_{a_j, \alpha_j, \varphi_{pj}, \varphi_j} dV \sim \frac{2\pi}{kV} \int_0^{\rho_{\max}} \ell(\rho) \, d\rho = \frac{\pi A}{2kV}$$

To make the volume integral for only the $m = 1$ mode equal to unity, we must divide the field square by this result. Having done this, the on-axis random plane wave result is then

$$\left\langle V E_{rx}^{(m=1)2} (\rho = 0) \right\rangle = \frac{2kV}{3\pi A} \tag{685}$$

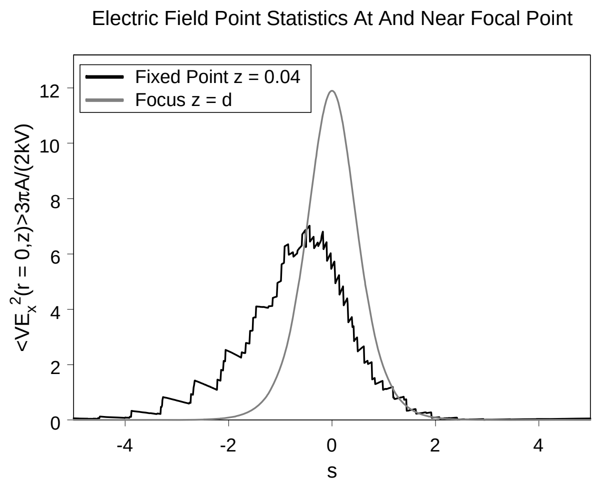

Figure 41. Focal region field square statistic from scar theory. The grey curve is the evaluation of the scar point statistic at $z = d$. The black curve shows the field square statistic at a fixed point $z = 0.04$ m on the scar orbit in the focal region.

6.8.3 Focal Point Comparisons

The preceding focal point result can be written as

$$\left\langle VE_{\upsilon}^{2}\right\rangle \sim \frac{kV}{A}\frac{\pi}{\ln\left(\Lambda_{+}\right)} = \frac{2kV}{3\pi A}\frac{3\pi^{2}/2}{\ln\left(\Lambda_{+}\right)}$$

For the preceding value $\Lambda_{+} \approx 3.472$ we see that with $z = d$

$$\frac{3\pi^{2}/2}{\ln\left(\Lambda_{+}\right)} \approx 11.8937$$

is the enhancement over the $m = 1$ random plane wave result. The result (683) is shown as the grey curve in Figure 41. Note also that at $z = d_e$

$$\frac{3\pi^{2}/2}{\ln\left(\Lambda_{+}^{e}\right)} \approx 10.5317$$

is the enhancement over the $m = 1$ random plane wave result. The result (683) with d replaced by the average focal point d_e is shown as the grey curve in Figure 42.

6.8.4 Fixed Point Field

To evaluate the field squared statistic at a fixed point z we use the general region 3 expression for $m = 0$

$$\left\langle VE_{\upsilon}^{2}\right\rangle (3\pi A)/(2kV) = \left\langle AE_{\upsilon}^{2}\right\rangle (3\pi)/(2k) \sim$$

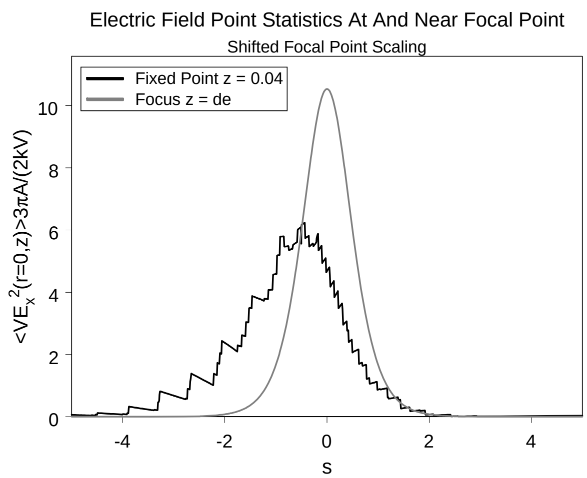

Figure 42. Focal region field square statistic from scar theory. The grey curve is the evaluation of the scar point statistic at $z = d_e$. The black curve shows the field square statistic at a fixed point $z = 0.04$ m on the scar orbit in the focal region. In this case the value of $\ln(\Lambda_+)$ has been replaced by $\ln\left(\Lambda_+^e\right)$ in the theory for both results.

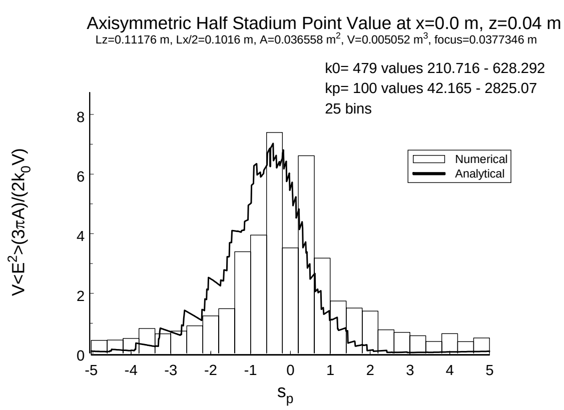

Figure 43. Focal region field square statistic from scar theory. The black curve shows the field square statistic at a fixed point $z = 0.04$ m on the scar orbit in the focal region. The histogram is the numerical simulation data at this same location.

$$\sim \frac{6\pi^2}{\left(1+e^{\pi s'}\right)^2 \ln\left(\Lambda_+\right) \mathrm{Re}^2\left[e^{i\pi/4}\Gamma\left(1/2+is'/2\right)\right]} \frac{2}{\tau'^2} \mathrm{Re}^2\left[W_+\left(\tau',0,s'\right)+e^{-i\Phi_0}W_+^*\left(\tau',0,s'\right)\right] \ , \text{ between foci}$$

$$\sim \frac{6\pi^2}{\left(1+e^{\pi s}\right)^2 \ln\left(\Lambda_+\right) \mathrm{Re}^2\left[e^{-i\pi/4}\Gamma\left(1/2+is/2\right)\right]} \frac{2}{\tau^2} \mathrm{Re}^2\left[e^{-i\pi/2}W_+\left(\tau,0,s\right)+e^{i\pi/2}e^{i\Phi_0}W_+^*\left(\tau,0,s\right)\right] \ , \text{ outside foci}$$

where the outward going solution is given in terms of the Whittaker function as

$$W_+\left(\tau,0,s\right) = W_{is/2,0}\left(-i\tau^2/2\right)$$

the random reflection phase is

$$e^{-i(\Phi_0+\pi/2)} = \frac{\Gamma\left(is'/2+1/2\right)}{\Gamma\left(-is'/2+1/2\right)}$$

the relation between the arguments and the coordinate are

$$\tau^2/4 = k_p\left(z-d-\delta\right)$$

$$\tau'^2/4 = k_p\left(d+\delta-z\right)$$

the separation constants are

$$s' = -s = 4\left(k_p-k\right)\ell/\ln\left(\Lambda_+\right)$$

with stability exponent

$$\Lambda_+ = \left(\frac{\ell+d}{\ell-d}\right)^2$$

and the focal point shift δ is

$$k\left(d+\delta\right) = -s'\ln\left(2\sqrt{kd}\right) + \arg\Gamma\left(1/2+is'/2\right) + \left(n+1/4\right)\pi \tag{686}$$

and in the even case the axial scar wavenumber is

$$k_p\ell = p\pi$$

$$\left(d+\delta\right)/\ell \sim \frac{1}{k_p\ell}\left[s'\left\{\frac{1}{2}\left(d/\ell\right)\ln\left(\frac{\ell+d}{\ell-d}\right) - \ln\left(4\sqrt{k_pd}\right) - \gamma'/2\right\}\right] + \left(n+1/4\right)/p$$

For the choice near the average focal point $d_e \approx 0.0377346$ m (with $\ell = 0.11176$ m, $d = 0.0336969$ m) of $z = 0.04$ m we find the solid black result shown in Figures, 41 and 43. The solid back curve in Figure 42 has used $\ln\left(\Lambda_+^e\right)$ in place of $\ln\left(\Lambda_+\right)$.

6.8.5 Comparison Of Single Mode Solution

A comparison between the scar results and the numerical simulation for

$$k_p \ell = (p - 1/2)\,\pi \tag{687}$$

with $p = 8$ is now given. Because this is an odd case along the orbit we first review the solution for the odd parity. With the parameters $\ell = 0.11176$ m, , $R \approx 0.10160$ m, $d = \ell\sqrt{1 - R/\ell} \approx 0.0336969$ m we note that

$$\Lambda_+ = \left(\frac{\ell + d}{\ell - d}\right)^2 \approx 3.47198 \tag{688}$$

$$k_p = (p - 1/2)\,\pi/\ell \approx 210.826 \text{ m}^{-1} \tag{689}$$

$$k \approx 211.512 \text{ m}^{-1} \tag{690}$$

and therefore

$$-s' = s = 2\,(k - k_p)\,L/\ln(\Lambda_+)$$

$$\approx 0.246375 \tag{691}$$

Relation Of Function Choice To Prior Even Case In the sections above for the even case of the stadium we had taken the functions to be defined by

$$\Pi_{ev} \sim u = \frac{C_0}{c_m}\psi_m(\tau', s')\,\psi_m(\tau, s) = C_0 c_m \frac{\sqrt{2}}{\tau'}\,\mathrm{Re}\left[W_+(\tau', m, s') + e^{i\Phi_0'} W_+^*(\tau', m, s')\right]$$

$$\frac{\sqrt{2}}{\tau}\,\mathrm{Re}\left[e^{-i\pi/2} W_+(\tau, m, s) + e^{i\pi/2} e^{i\Phi_0} W_+^*(\tau, m, s)\right]\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}$$

In the odd case below we are instead using the definitions

$$\Pi_{ev} \sim u = \frac{C_0}{c_m}\psi_m(\tau', s')\,\psi_m(\tau, s) = C_0 c_m \frac{\sqrt{2}}{\tau'}\,\mathrm{Re}\left[e^{-i\pi/2}\left\{W_+(\tau', m, s') - e^{i\Phi_0'} W_+^*(\tau', m, s')\right\}\right]$$

$$\frac{\sqrt{2}}{\tau}\,\mathrm{Re}\left[e^{-i\pi/2}\left\{e^{-i\pi/2} W_+(\tau, m, s) - e^{i\pi/2} e^{i\Phi_0} W_+^*(\tau, m, s)\right\}\right]\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}$$

which can also be rewritten as

$$\Pi_{ev} \sim u = \frac{C_0}{c_m}\psi_m(\tau', s')\,\psi_m(\tau, s) = -C_0 c_m \frac{\sqrt{2}}{\tau'}\,\mathrm{Re}\left[e^{-i\pi/2} W_+(\tau', m, s') + e^{i\pi/2} e^{i\Phi_0'} W_+^*(\tau', m, s')\right]$$

$$\frac{\sqrt{2}}{\tau}\,\mathrm{Re}\left[W_+(\tau, m, s) + e^{i\Phi_0} W_+^*(\tau, m, s)\right]\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}$$

Hence we have in effect flipped the definitions of $\psi_m(\tau', s')$ and $\psi_m(\tau, s)$ from the previous even case and introduced a minus sign in $\psi_m(\tau, s)$.

Matching Of Solutions Across Focal Region In region 3 we take

$$\Pi_{ev} \sim u = C_0 c_m \frac{1}{\xi' \sqrt{\gamma}} \operatorname{Re}\left[e^{-i\pi/2}\left\{W_+\left(\xi'\sqrt{2\gamma}, m, s'\right) - e^{i\Phi_0'} W_+^*\left(\xi'\sqrt{2\gamma}, m, s'\right)\right\}\right]$$

$$\frac{1}{\zeta\sqrt{\gamma}} \operatorname{Re}\left[e^{-i\pi/2}\left\{e^{-i\pi/2} W_+\left(\zeta\sqrt{2\gamma}, m, -s'\right) - e^{i\pi/2} e^{i\Phi_0} W_+^*\left(\zeta\sqrt{2\gamma}, m, -s'\right)\right\}\right]\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\} \tag{692}$$

Expanding as we leave the focal region using the asymptotic form

$$W_+(\tau, m, s) \sim e^{i\tau^2/4}\left(-i\tau^2/2\right)^{is/2} = e^{i\tau^2/4 + s\pi/4 + is\ln\left(\tau/\sqrt{2}\right)}\ ,\ \tau \to \infty$$

gives

$$\frac{c_m}{\xi'\sqrt{\gamma}} \operatorname{Re}\left[e^{-i\pi/2}\left\{W_+\left(\xi'\sqrt{2\gamma}, m, s'\right) - e^{i\Phi_0'} W_+^*\left(\xi'\sqrt{2\gamma}, m, s'\right)\right\}\right]$$

$$\sim 2c_m e^{s'\pi/4}\frac{\sqrt{2}}{\tau'}\cos\left(\Phi_0'/2\right)\sin\left[\tau'^2/4 + s'\ln\left(\tau'/\sqrt{2}\right) - \Phi_0'/2\right]$$

$$\Pi_{ev} \sim 2C_0 e^{s'\pi/4}\cos\left(\Phi_0'/2\right) c_m \frac{1}{\zeta\sqrt{\gamma}} \operatorname{Re}\left[e^{-i\pi/2}\left\{e^{-i\pi/2} W_+\left(\zeta\sqrt{2\gamma}, m, -s'\right) - e^{i\pi/2} e^{i\Phi_0} W_+^*\left(\zeta\sqrt{2\gamma}, m, -s'\right)\right\}\right]$$

$$\frac{1}{\xi'\sqrt{\gamma}}\sin\left[\xi'^2\gamma/2 + s'\ln\left(\xi'\sqrt{\gamma}\right) - \Phi_0'/2\right]\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}\ ,\ \text{Region } 3 \to 1 \tag{693}$$

$$\frac{c_m}{\zeta\sqrt{\gamma}} \operatorname{Re}\left[e^{-i\pi/2}\left\{e^{-i\pi/2} W_+\left(\zeta\sqrt{2\gamma}, m, s\right) - e^{i\pi/2} e^{i\Phi_0} W_+^*\left(\zeta\sqrt{2\gamma}, m, s\right)\right\}\right]$$

$$\sim 2c_m e^{s\pi/4}\frac{\sqrt{2}}{\tau}\cos\left(\Phi_0/2\right)\sin\left[\tau^2/4 + s\ln\left(\tau/\sqrt{2}\right) - \Phi_0/2 - \pi/2\right]$$

$$\Pi_{ev'} \sim 2C_0 e^{-s'\pi/4}\cos\left(\Phi_0/2\right) c_m \frac{1}{\xi'\sqrt{\gamma}} \operatorname{Re}\left[e^{-i\pi/2}\left\{W_+\left(\xi'\sqrt{2\gamma}, m, s'\right) - e^{i\Phi_0'} W_+^*\left(\xi'\sqrt{2\gamma}, m, s'\right)\right\}\right]$$

$$\frac{1}{\zeta\sqrt{\gamma}}\sin\left[\zeta^2\gamma/2 - s'\ln\left(\zeta\sqrt{\gamma}\right) - \pi/2 - \Phi_0/2\right]\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}\ ,\ \text{Region } 3 \to 2 \tag{694}$$

These must match to the limiting forms of the outer solutions from the preceding sections. In region 1 we take

$$\Pi_{ev} = Ce^{-i\pi/2}\left[W(\zeta, \varphi, \xi)\, e^{i\gamma\sin\xi} - W(\zeta, \varphi, -\xi)\, e^{-i\gamma\sin\xi}\right]\ ,\ \ |\xi| < \xi_0$$

$$= C\frac{1}{\cos\xi} e^{-i\pi/2}\left[\Psi(\tau, \varphi, \sigma)\, e^{i\gamma\sin\xi} - \Psi(\tau, \varphi, -\sigma)\, e^{-i\gamma\sin\xi}\right]$$

$$= 2C\frac{\psi_m(\tau,s)}{\cos\xi}\sin(\gamma\sin\xi - s\sigma)\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}$$

$$= 2C\frac{\psi_m(\tau,s)}{\sin\xi'}\sin\left(\gamma\cos\xi' - s\sigma\right)\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\},\ \xi' = \pi/2-\xi$$

$$\sim 2C\frac{\psi_m(\tau,s)}{\sin\xi'}\sin\left[\gamma\cos\xi' + s\ln\left\{\tan\left(\xi'/2\right)\right\}\right]\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\},\ \xi' = \pi/2-\xi$$

with

$$\psi_m(\tau,s) = c_m\frac{\sqrt{2}}{\tau}\,\mathrm{Re}\left[e^{-i\pi/2}\left\{e^{-i\pi/2}W_+(\tau,m,s) - e^{i\pi/2}e^{i\Phi_0}W_+^*(\tau,m,s)\right\}\right]$$

or

$$\Pi_{ev} \sim 2C\frac{1}{\xi'}c_m\frac{1}{\zeta\sqrt{\gamma}}\,\mathrm{Re}\left[e^{-i\pi/2}\left\{e^{-i\pi/2}W_+(\tau,m,s) - e^{i\pi/2}e^{i\Phi_0}W_+^*(\tau,m,s)\right\}\right]$$

$$\sin\left[\gamma\left(1-\xi'^2/2\right) + k\delta + s\ln\left(\xi'/2\right)\right]\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\},\ \xi'\to 0$$

In region 2 we take

$$\Pi_{ev'} = e^{-i\pi - i\Phi_1/2}\left[W(\zeta,\varphi,\xi)\,e^{i\gamma\cosh\zeta} - e^{i\Phi_1}W(\zeta - i\pi,\varphi,\xi)\,e^{-i\gamma\cosh\zeta}\right],\quad |\xi| > \xi_0$$

$$= \frac{1}{\sinh\zeta}e^{-i\pi/2}\left[\Psi_m(\tau',\sigma')\,e^{i\gamma\cosh\zeta - i\pi/2 - i\Phi_1/2} - \Psi_m(\tau',-\sigma')\,e^{-i\gamma\cosh\zeta + i\pi/2 + i\Phi_1/2}\right]\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}$$

$$= \frac{2}{\sinh\zeta}\psi_m(\tau',s')\sin(\gamma\cosh\zeta - s'\sigma' - \pi/2 - \Phi_1/2)\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}$$

or

$$\Pi_{ev'} = \frac{2}{\sinh\zeta}\psi_m(\tau',s')\sin\left[\gamma\cosh\zeta - s'\ln\left\{\tanh(\zeta/2)\right\} - \pi/2 - \Phi_1/2\right]\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\}$$

with

$$\psi_m(\tau',s') = c_m\frac{\sqrt{2}}{\tau'}\,\mathrm{Re}\left[e^{-i\pi/2}\left\{W_+(\tau',m,s') - e^{i\Phi_0'}W_+^*(\tau',m,s')\right\}\right]$$

or

$$\Pi_{ev'} \sim \frac{2}{\zeta}c_m\frac{1}{\xi'\sqrt{\gamma}}\,\mathrm{Re}\left[e^{-i\pi/2}\left\{W_+(\tau',m,s') - e^{i\Phi_0'}W_+^*(\tau',m,s')\right\}\right]$$

$$\sin\left[\gamma\left(1+\zeta^2/2\right) + k\delta - s'\ln(\zeta/2) - \pi/2 - \Phi_1/2\right]\left\{\begin{array}{c}\cos(m\varphi)\\ \sin(m\varphi)\end{array}\right\},\ \zeta\to 0$$

The region 1 and 3 solutions yield

$$\Pi_{ev} \sim 2C_0 e^{s'\pi/4} \cos\left(\Phi_0'/2\right) c_m \frac{1}{\zeta\sqrt{\gamma}} \operatorname{Re}\left[e^{-i\pi/2}\left\{e^{-i\pi/2} W_+\left(\tau, m, -s'\right) - e^{i\pi/2} e^{i\Phi_0} W_+^*\left(\tau, m, -s'\right)\right\}\right]$$

$$\frac{1}{\xi'\sqrt{\gamma}} \sin\left[\xi'^2\gamma/2 + s' \ln\left(\xi'\sqrt{\gamma}\right) - \Phi_0'/2\right] \left\{ \begin{array}{c} \cos\left(m\varphi\right) \\ \sin\left(m\varphi\right) \end{array} \right\}, \text{ Region } 3 \to 1 \tag{695}$$

$$\Pi_{ev} \sim 2C \frac{1}{\xi'} c_m \frac{1}{\zeta\sqrt{\gamma}} \operatorname{Re}\left[e^{-i\pi/2}\left\{e^{-i\pi/2} W_+\left(\tau, m, s\right) - e^{i\pi/2} e^{i\Phi_0} W_+^*\left(\tau, m, s\right)\right\}\right]$$

$$\sin\left[\gamma\left(1 - \xi'^2/2\right) + k\delta + s \ln\left(\xi'/2\right)\right] \left\{ \begin{array}{c} \cos\left(m\varphi\right) \\ \sin\left(m\varphi\right) \end{array} \right\}, \ \xi' \to 0 \tag{696}$$

which match if (we negate one of the sinusoid arguments to match the phases, which introduces an extra minus sign in the amplitudes)

$$-s' \ln\left(2\sqrt{\gamma}\right) + \Phi_0'/2 = \gamma + k\delta + n\pi \tag{697}$$

$$-\left(-1\right)^n C_0 e^{s'\pi/4} \cos\left(\Phi_0'/2\right) \frac{1}{\sqrt{\gamma}} = C \tag{698}$$

The region 2 and 3 solutions yield

$$\Pi_{ev'} \sim 2C_0 e^{-s'\pi/4} \cos\left(\Phi_0/2\right) c_m \frac{1}{\xi'\sqrt{\gamma}} \operatorname{Re}\left[e^{-i\pi/2}\left\{W_+\left(\tau', m, s'\right) - e^{i\Phi_0'} W_+^*\left(\tau', m, s'\right)\right\}\right]$$

$$\frac{1}{\zeta\sqrt{\gamma}} \sin\left[\zeta^2\gamma/2 - s' \ln\left(\zeta\sqrt{\gamma}\right) - \pi/2 - \Phi_0/2\right] \left\{ \begin{array}{c} \cos\left(m\varphi\right) \\ \sin\left(m\varphi\right) \end{array} \right\}, \text{ Region } 3 \to 2 \tag{699}$$

$$\Pi_{ev'} \sim \frac{2}{\zeta} c_m \frac{1}{\xi'\sqrt{\gamma}} \operatorname{Re}\left[e^{-i\pi/2}\left\{W_+\left(\tau', m, s'\right) - e^{i\Phi_0'} W_+^*\left(\tau', m, s'\right)\right\}\right]$$

$$\sin\left[\gamma\left(1 + \zeta^2/2\right) + k\delta - s' \ln\left(\zeta/2\right) - \pi/2 - \Phi_1/2\right] \left\{ \begin{array}{c} \cos\left(m\varphi\right) \\ \sin\left(m\varphi\right) \end{array} \right\}, \ \zeta \to 0 \tag{700}$$

which match if

$$-s' \ln\left(2\sqrt{\gamma}\right) - \Phi_0/2 = \gamma + k\delta - \Phi_1/2 + n'\pi \tag{701}$$

$$C_0 e^{-s'\pi/4} \cos\left(\Phi_0/2\right) \frac{1}{\sqrt{\gamma}} = \left(-1\right)^{n'} \tag{702}$$

Symmetry Conditions On Orbit Region 1 gives

$$\lim_{\tau \to 0} \frac{1}{\tau} \operatorname{Re}\left[e^{-i\pi/2}\left\{e^{-i\pi/2} W_+'\left(\tau, m, s\right) - e^{i\pi/2} e^{i\Phi_0} W_+^{*\prime}\left(\tau, m, s\right)\right\}\right.$$

$$-\frac{1}{\tau}e^{-i\pi/2}\left\{e^{-i\pi/2}W_{+}(\tau,m,s)-e^{i\pi/2}e^{i\Phi_0}W_{+}^{*}(\tau,m,s)\right\}\Bigg]=0$$

$$\lim_{\tau\to 0}\left[\left(1+e^{-i\Phi_0}\right)\left\{W_{+}'(\tau,m,s)-\frac{1}{\tau}W_{+}(\tau,m,s)\right\}+\left(1+e^{i\Phi_0}\right)\left\{W_{+}^{*\prime}(\tau,m,s)-\frac{1}{\tau}W_{+}^{*}(\tau,m,s)\right\}\right]=0$$

Region 2 gives

$$\lim_{\tau'\to 0}\frac{1}{\tau'}\operatorname{Re}\left[e^{-i\pi/2}\left\{W_{+}'(\tau',m,s')-e^{i\Phi_0'}W_{+}^{*\prime}(\tau',m,s')\right\}-\frac{1}{\tau'}e^{-i\pi/2}\left\{W_{+}(\tau',m,s')-e^{i\Phi_0'}W_{+}^{*}(\tau',m,s')\right\}\right]=0$$

$$\lim_{\tau'\to 0}\left[\left(1+e^{-i\Phi_0'}\right)\left\{W_{+}'(\tau',m,s')-\frac{1}{\tau'}W_{+}(\tau',m,s')\right\}-\left(1+e^{i\Phi_0'}\right)\left\{W_{+}^{*\prime}(\tau',m,s')-\frac{1}{\tau'}W_{+}^{*}(\tau',m,s')\right\}\right]=0$$

Thus

$$-\lim_{\tau\to 0}\left[\frac{W_{+}'(\tau,m,s)-\frac{1}{\tau}W_{+}(\tau,m,s)}{W_{+}^{*\prime}(\tau,m,s)-\frac{1}{\tau}W_{+}^{*}(\tau,m,s)}\right]=e^{i\Phi_0}$$

$$\lim_{\tau'\to 0}\left[\frac{W_{+}'(\tau',m,s')-\frac{1}{\tau'}W_{+}(\tau',m,s')}{W_{+}^{*\prime}(\tau',m,s')-\frac{1}{\tau'}W_{+}^{*}(\tau',m,s')}\right]=e^{i\Phi_0'}$$

In the even case we had

$$e^{i\Phi_0}=-\frac{\left(W_{+}'-\frac{1}{\tau}W_{+}\right)}{\left(W_{+}'-\frac{1}{\tau}W_{+}\right)^{*}}$$

and thus to leading order we expect the form of the reflection phase in this odd case to be

$$e^{i\Phi_0}\sim e^{i\pi(1+m)/2}\frac{\Gamma(1/2+m/2+is/2)}{\Gamma(1/2+m/2-is/2)} \tag{703}$$

and from the second equation

$$e^{i\Phi_0'}\sim -e^{i\pi(1+m)/2}\frac{\Gamma(1/2+m/2-is/2)}{\Gamma(1/2+m/2+is/2)}\to e^{-i\Phi_0'}\sim -e^{-i\pi(1+m)/2}\frac{\Gamma(1/2+m/2+is/2)}{\Gamma(1/2+m/2-is/2)}$$

meaning that

$$\Phi_0=-\Phi_0'+\pi m+j2\pi$$

where j is an integer. For m even (including $m=0$) we can take

$$\Phi_0=-\Phi_0' \tag{704}$$

Final Match Conditions If we subtract the two phase match equations from each other

$$\Phi_1/2 = (n' - n)\,\pi$$

and the second equation then yields

$$-s' \ln(2\sqrt{\gamma}) - \Phi_0/2 = \gamma + k\delta + n\pi \tag{705}$$

Taking the ratio of the two amplitude equations gives

$$(-1)^{n-n'+1}\, e^{s'\pi/2} = C \tag{706}$$

and the second equation yields

$$C_0 = e^{s'\pi/4}\sqrt{\gamma}\sec(\Phi_0/2)\,(-1)^{n'} \tag{707}$$

Let us choose $n = n'$ and $\Phi_1 = 0$. For the particular example below we will take $n' = n = 3$ odd and

$$d + \delta \approx 0.04391542 \text{ m} \tag{708}$$

Simplification On Axis For m=0 The electric fields near the axis are approximately taken as

$$E_x \sim E_\upsilon \sim k^2 \Pi_{e\upsilon} \tag{709}$$

$$E_x \sim E_{\upsilon'} \sim k^2 \Pi_{e\upsilon'} \tag{710}$$

Putting $m = 0$ we find

$$\Pi_{e\upsilon} = C\frac{2}{\cos\xi}\psi_0(\tau, s)\sin(\gamma\sin\xi - s\sigma)$$

$$\psi_0(\tau, s) = c_0\frac{\sqrt{2}}{\tau}\,\mathrm{Re}\left[e^{-i\pi/2}\left\{e^{-i\pi/2}W_+(\tau, 0, s) - e^{i\pi/2}e^{i\Phi_0}W_+^*(\tau, 0, s)\right\}\right]$$

$$\sim -c_0\frac{\sqrt{2}}{\tau}\,\mathrm{Re}\left[W_+(\tau, 0, s) + i\frac{\Gamma(1/2 + is/2)}{\Gamma(1/2 - is/2)}W_+^*(\tau, 0, s)\right]$$

and for region 2

$$\Pi_{e\upsilon'} = \frac{2}{\sinh\zeta}\psi_0(\tau', s')\sin(\gamma\cosh\zeta - s'\sigma' - \pi/2)$$

$$\psi_0(\tau', s') = c_0\frac{\sqrt{2}}{\tau'}\,\mathrm{Re}\left[e^{-i\pi/2}\left\{W_+(\tau', 0, s') - e^{i\Phi_0'}W_+^*(\tau', 0, s')\right\}\right]$$

$$= c_0\frac{\sqrt{2}}{\tau'}\,\mathrm{Re}\left[-i\left\{W_+(\tau', 0, s') + i\frac{\Gamma(1/2 + is'/2)}{\Gamma(1/2 - is'/2)}W_+^*(\tau', 0, s')\right\}\right]$$

Finally for region 3

$$\Pi_{ev} \sim -C_0 c_0 \frac{\sqrt{2}}{\tau'} \operatorname{Re}\left[-i\left\{W_+(\tau',0,s') + i\frac{\Gamma(1/2+is'/2)}{\Gamma(1/2-is'/2)} W_+^*(\tau',0,s')\right\}\right]$$

$$\frac{\sqrt{2}}{\tau} \operatorname{Re}\left[W_+(\tau,0,s) + i\frac{\Gamma(1/2+is/2)}{\Gamma(1/2-is/2)} W_+^*(\tau,0,s)\right] \tag{711}$$

and the phase is

$$e^{i\Phi_0} \sim i\frac{\Gamma(1/2+is/2)}{\Gamma(1/2-is/2)} = \frac{e^{i\pi/4}\Gamma(1/2+is/2)}{e^{-i\pi/4}\Gamma(1/2-is/2)} = \frac{f}{f^*} = \left(\frac{f}{|f|}\right)^2$$

or

$$e^{i\Phi_0/2} = \frac{f}{|f|} = \frac{e^{i\pi/4}\Gamma(1/2+is/2)}{\left|e^{i\pi/4}\Gamma(1/2+is/2)\right|}$$

$$\cos(\Phi_0/2) = \frac{\operatorname{Re}(f)}{|f|} = \frac{\operatorname{Re}\left[e^{i\pi/4}\Gamma(1/2+is/2)\right]}{\left|e^{i\pi/4}\Gamma(1/2+is/2)\right|}$$

$$C_0 = -e^{s'\pi/4}\sqrt{\gamma}\sec(\Phi_0/2) = -\frac{e^{-s\pi/4}\sqrt{\gamma}\left|e^{i\pi/4}\Gamma(1/2+is/2)\right|}{\operatorname{Re}\left[e^{i\pi/4}\Gamma(1/2+is/2)\right]}$$

Now going to the axis $\tau \to 0$ gives

$$\Pi_{ev} = C\frac{2}{\cos\xi}\psi_0(0,s)\sin(\gamma\sin\xi - s\sigma)$$

$$\psi_0(0,s) = -c_0 \operatorname{Re}\left[\frac{e^{-i\pi/4}}{\Gamma(1/2-is/2)}\left\{i\pi - \psi(1/2-is/2) + \psi(1/2+is/2)\right\}\right]$$

$$= -c_0 \operatorname{Re}\left[\frac{e^{-i\pi/4}}{\Gamma(1/2-is/2)}\left\{i\pi - \pi\cot\pi(1/2+is/2)\right\}\right]$$

$$= -c_0 \operatorname{Re}\left[e^{i\pi/4}e^{\pi s/2}\Gamma(1/2+is/2)\right]$$

in region 1, and for $\tau' \to 0$

$$\Pi_{ev'} = \frac{2}{\sinh\zeta}\psi_0(0,s')\sin(\gamma\cosh\zeta - s'\sigma' - \pi/2)$$

$$\psi_0(0,s') = c_0 \operatorname{Re}\left[-i\left\{\frac{e^{-i\pi/4}}{\Gamma(1/2-is'/2)}\left\{i\pi - \psi(1/2-is'/2) + \psi(1/2+is'/2)\right\}\right\}\right]$$

$$= c_0 \operatorname{Re}\left[-i\frac{e^{-i\pi/4}}{\Gamma(1/2-is'/2)}\left\{i\pi - \pi\cot\pi(1/2+is'/2)\right\}\right]$$

$$= c_0 \operatorname{Re}\left[e^{-i\pi/4}e^{\pi s'/2}\Gamma(1/2+is'/2)\right] = c_0 \operatorname{Re}\left[e^{i\pi/4}e^{-\pi s/2}\Gamma(1/2+is/2)\right] = -e^{-\pi s}\psi_0(0,s)$$

in region 2.

For $\tau=\zeta\sqrt{2\gamma}$, $\zeta\to 0$

$$\Pi_{ev}\sim C_0\psi_0\left(0,s\right)\frac{\sqrt{2}}{\tau'}\operatorname{Re}\left[-i\left\{W_+\left(\tau',0,s'\right)+i\frac{\Gamma\left(1/2+is'/2\right)}{\Gamma\left(1/2-is'/2\right)}W_+^*\left(\tau',0,s'\right)\right\}\right] \tag{712}$$

$$\sim C_0\psi_0\left(0,s\right)\frac{\sqrt{2}}{\tau'}\operatorname{Im}\left[W_+\left(\tau',0,s'\right)+i\frac{\Gamma\left(1/2+is'/2\right)}{\Gamma\left(1/2-is'/2\right)}W_+^*\left(\tau',0,s'\right)\right]$$

$$\tau'^2/4=k_p\left(d+\delta-z\right)\ ,\ 0<z<d+\delta$$

with the shift δ determined from

$$e^{i\Phi_0}\sim i\frac{\Gamma\left(1/2+is/2\right)}{\Gamma\left(1/2-is/2\right)}$$

$$s\ln\left(2\sqrt{kd}\right)-\Phi_0/2=k\left(d+\delta\right)$$

For $\tau'=\sqrt{2\gamma}\xi'$, $\xi'=\pm\pi/2-\xi\to 0$

$$\Pi_{ev'}\sim\psi_0\left(0,s\right)C_0e^{-\pi s}\frac{\sqrt{2}}{\tau}\operatorname{Re}\left[W_+\left(\tau,0,s\right)+i\frac{\Gamma\left(1/2+is/2\right)}{\Gamma\left(1/2-is/2\right)}W_+^*\left(\tau,0,s\right)\right] \tag{713}$$

$$\tau^2/4=k_p\left(z-d-\delta\right)\ ,\ z>d+\delta$$

Cylindrical Form Of Region 1 And 2 For the region 1 solution $\tau=\zeta\sqrt{2\gamma}$, $\zeta\to 0$

$$\Pi_{ev}=C\frac{2}{\sqrt{1-z^2/d^2}}\psi_0\left(0,s\right)\sin\left[kz-s\frac{1}{2}\ln\left(\frac{d+z}{d-z}\right)\right]$$

$$=-e^{-\pi s/2}\frac{2}{\sqrt{1-z^2/d^2}}\psi_0\left(0,s\right)\sin\left[k_pz+p_0\left(z\right)\right] \tag{714}$$

and

$$p_0\left(z\right)=\frac{s}{2}\left\{\left(z/\ell\right)\ln\left(\frac{\ell+d}{\ell-d}\right)-\ln\left|\frac{d+z}{d-z}\right|\right\}$$

where we used

$$z=d\sin\xi$$

$$\cos\xi=\sqrt{1-z^2/d^2}$$

$$\tanh\sigma=\sin\xi$$

$$\sigma = \mathrm{Arctanh}\,(z/d) = \frac{1}{2}\ln\left(\frac{d+z}{d-z}\right)$$

For the region 2 solution $\tau' = \sqrt{2\gamma}\xi'$, $\xi' = \pm\pi/2 - \xi \to 0$

$$\begin{aligned}
\Pi_{ev'} &= \frac{2}{\sqrt{z^2/d^2-1}}\psi_0\left(0,s'\right)\sin\left[kz + \frac{s'}{2}\ln\left(\frac{z+d}{z-d}\right) - \mathrm{sgn}\,(z)\,\pi/2\right] \\
&= \frac{2}{\sqrt{z^2/d^2-1}}\psi_0\left(0,s'\right)\sin\left[k_p z + p_0\,(z) - \mathrm{sgn}\,(z)\,\pi/2\right] \\
&= -\frac{2}{\sqrt{z^2/d^2-1}}\psi_0\,(0,s)\,e^{-\pi s}\sin\left[k_p z + p_0\,(z) - \mathrm{sgn}\,(z)\,\pi/2\right] \\
&= \frac{2}{\sqrt{z^2/d^2-1}}\psi_0\,(0,s)\,e^{-\pi s}\cos\left[k_p z + p_0\,(z)\right]
\end{aligned} \tag{715}$$

where we used

$$z = \pm d\cosh\zeta$$

$$\sinh\zeta = \sqrt{z^2/d^2-1}$$

$$\sigma' = \ln\left[\tanh\left(\zeta/2\right)\right] = -\frac{1}{2}\ln\left(\frac{z+d}{z-d}\right)$$

Single Scar Mode Comparison Figure 44 shows a comparison of the axisymmetric numerical simulation (solid black curve), the region 1 (714) and region 2 (715) solutions (light grey curves), and the region 3 solution (dark grey curves) given by (712) and (713). The single normalization constant c_0 (the factor in $\psi_0\,(0,s)$) to match the region 2 (grey curve) to the numerical simulation at the point noted on the graph. This normalization is done because we only know the statistics of the scar amplitude construction, and not the actual amplitude for this particular realization of the Gaussian random variable v; but we only match the single constant c_0. The agreement with the simulation looks quite good, including the in the focal region.

7 ELECTROMAGNETIC AXISYMMETRIC CAVITY SIMULATIONS

The numerical simulations to support the theory of axisymmetric cavity scars were performed using a modified version of the body-of-revolution (BOR) radar cross-section code **CICERO,** which was originally written by McDonnell Douglas Research Laboratories [21]. This section describes the modifications made to **CICERO**, the tests we ran to verify that the new code was working correctly, the statistics obtained for a BOR cavity with a cross-section in the shape of a bowtie and finishes with some conclusions.

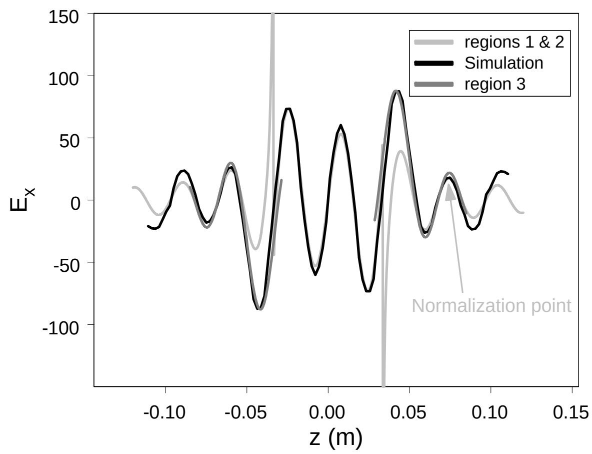

Figure 44. Modal spatial distribution of the transverse electric field along the orbit in the stadium for a particular scar $p = 8$ with $k_p = (p - 1/2)\,\pi/\ell$ at wavenumber $k \approx 211.512$ m^{-1} with half length $\ell = 0.11176$ m, and radius of curvature $R \approx 0.10160$ m, $d \approx 0.0336969$ m, and $d + \delta \approx 0.04391542$ m. The numerical axisymmetric simulation is shown as the solid black curve. The results shown by the light grey curve are a combination of the region 1 ($|z| < d$) and region 2 ($|z| < d$) solutions. The dark grey curves are the results from the region 3 focal solution (the two forms are match at the point $|z| = d + \delta$. The normalization constant c_0 is chosen to match the analytic construction, in particular the region 2 solution (grey curve), to the numerical simulation (black curve) at the point noted on the graph.

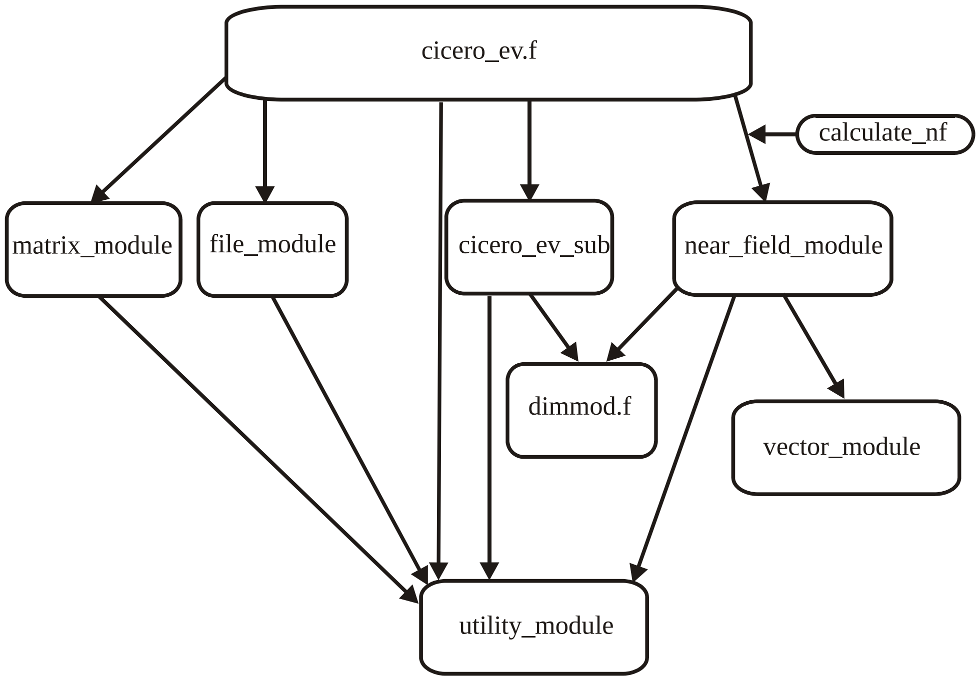

Figure 45. Code diagram for **CICERO_EV**

7.1 CICERO Modifications for Eigenvalue Calculation

CICERO is a frequency-domain method of moments (MOM) code that for each frequency of interest fills an impedance matrix and right hand side and solves for electric and magnetic surface currents. From knowledge of the currents, secondary quantities such as radar cross section are calculated. Our first task was to suppress filling the right hand side and instead calculate the eigenvalues and corresponding eigenvectors of the impedance matrix for a given frequency.

To find the cavity modes numerically, we will sweep over a range of frequencies at a very close frequency spacing, monitoring the ratio of the largest eigenvalue to smallest eigenvalue. At the frequencies where this ratio peaks, the cavity is resonant and this is a frequency where we will calculate the electric field throughout the cavity. **CICERO** automatically calculates secondary quantities for each frequency, but in this case calculating the nearfield is computationally expensive. We therefore have a logical (**calculate_nf)** set in the code prior to compilation that we can use to suppress the electric field calculation during the frequency sweep and enable it when we have a list of resonant frequencies.

The diagram of the resulting code **CICERO_EV** is shown in Figure 45. The code is written in a combination of Fortran90 and Fortran77. The main code is in **cicero_ev.f** which calls the original **CICERO** subroutines found in **CICERO_EV_sub**. The module **dimmod.f** is an old Fortran77 trick to provide proper dimensions to arrays based on the problem being solved through use of an INCLUDE statement. **CICERO_EV** obtains the eigenvalues and eigenvectors of the impedance matrix by calling the subroutine **get_eigenvectors**, which is in the **matrix_module**. Inside **matrix_module**, **get_eigenvectors** in turn calls the LAPACK subroutine **zgeev**.

The subroutines to calculate the near fields are in **near_field_module**, which uses the **vector_module** for vector operations like dot and cross products. The **near_field_module** contains subroutines based on another of our method of moments codes, **Eiger** [23]. In the **near_field_module** we read in the point locations where the electric field is to be calculated in a ***.jfg** file with a call to the subroutine **read_field_elements** from **CICERO_EV**. We then calculate the electric field at each of the points using **nearfield_fill** and write the electric fields and the point locations to files for processing (**write_near_field**) and viewing in **I-deas** (**write_nf_ideas**).

One input file to **CICERO_EV** is **cicero_freq.lst**, which contains the number of frequencies in the first line and a list of frequencies (in Hz) in the subsequent lines. The second input file to **CICERO_EV** is **input**. This is a modification of the **CICERO** input deck. The first line contains two integers:

line 1: **mode ngauss**

The quantity **mode** indicates the ϕ mode that we are interested in. This is designated n in the next section. The ϕ mode is in the form $e^{jn\phi}$. The quantity **ngauss** is the number of gaussian points for the ϕ integration. The second line contains a single integer:

line 2: **nreg**

The quantity **nreg** is the number of regions in the problem. Metal is by default set to be region 0 and is not counted as a region. For all of our cavity problems **nreg** is set to 1 to designate the region inside the cavity. The third line contains four reals, which define the region's material characteristics:

line 3: **epsr, epsi, mur, mui**

where **epsr** and **epsi** are the real and imaginary parts of relative permittivity and **mur** and **mui** are the real and imaginary parts of relative permeability. **CICERO_EV** uses the $e^{+j\omega t}$ time convention. The fourth line contains a single integer, which is the number of shells associated with the region.

line 4: **numshl**

Shells are not handled by **CICERO_EV** so the fourth line should be zero. The fifth line contains a single integer:

line 5: **npi**

where **npi** is the number of points defining a boundary. This must be an odd number. **CICERO_EV** only can solve simple cavities at present (cavities completely filled by one region) so a single boundary can define the cavity. The sixth line contains three integers:

line 6: **nir ner nsh**

The quantity **nir** is the interior region id for this boundary (the outward normal of the boundary is defined as $\widehat{n} = \widehat{\phi} \times \widehat{t}$, where $\widehat{t}$ is the unit vector defined by the increasing node id direction along the boundary), **ner** is the exterior region id for this boundary, and **nsh** is the shell id for this boundary (always set to zero). In **CICERO_EV** we only have one region (plus the 0 metal region) and since we enforce an EFIE, the direction of the normal in the eigenvalue problem is not important. It must be consistent, however, and this is why the direction set by increasing node ids must follow a consistent pattern as seen by the example shown in Figure 46. The next **npi** entries consist of the r value of the points that define the boundary. These are grouped in lines of four reals apiece, with the last line having enough entries to make up the **npi** total entries. This is followed by **npi** entries of the z value of the points that define the boundary, in the same format as the r entries. The last line of **input** is 0 to terminate the reading of the boundaries.

If the logical **compute_nf = .TRUE.** in **CICERO_EV**, the user must input the base name of the files that will contain the electric field. This base file name with different extensions for each frequency (***.nfld00, *.nfld01**, etc...) is generated and filled for use by the statistical processing code. The base file name with the ***.unv** extension is generated and filled for viewing the E field in **I-deas**.

The outputs to **CICERO_EV** are **cic.out**, which contains an echo of the input quantities, **current.out,** which contains the eigenvector associated with the smallest eigenvalue for the frequencies contained in **cicero_freq.lst,** and **condition_number.out**, which contains the condition number of the impedance matrix (largest eigenvalue/smallest eigenvalue) for the frequencies of interest.

7.2 CICERO Modifications for Electric Nearfield Calculation

7.2.1 Direct Derivation of $\overrightarrow{E}^s(r)$

In this section we discuss details of the calculation of the electric near fields at an observation point (r). The scattered E field in terms of vector (A) and scalar (ϕ) potentials is

$$\begin{aligned}\overrightarrow{E}(r)^s &= -j\omega\overrightarrow{A}(r) - \nabla\phi(r)\\ &= -j\omega\mu\int_{s'}\overrightarrow{J}(\overrightarrow{r}')\frac{e^{-jkR}}{4\pi R}da' - \nabla\frac{1}{\varepsilon}\int_{s'}\rho(\overrightarrow{r}')\frac{e^{-jkR}}{4\pi R}da'\end{aligned}$$

The current on bodies of revolution are represented by basis functions oriented in two directions: $\widehat{t}$ and $\widehat{\phi}$ where

$$\widehat{t} = \widehat{x}\sin\theta\cos\phi + \widehat{y}\sin\theta\sin\phi + \widehat{z}\cos\theta$$
$$\widehat{\phi} = -\widehat{x}\sin\phi + \widehat{y}\cos\phi$$

Also, R is the distance between two points: an observation point (ρ, ϕ, z) and a source point (ρ', ϕ', z')

$$R = \sqrt{(\rho' - \rho)^2 + (z' - z)^2 + 2\rho'\rho\left(1 - \cos(\phi' - \phi)\right)}$$

The current continuity equation is

$$\nabla\cdot\overrightarrow{J} = -\frac{\partial\rho}{\partial t}$$

The time-harmonic form of the continuity equation (assuming a $e^{+j\omega t}$ time convention) is:

$$\nabla\cdot\overrightarrow{J} = -j\omega\rho$$

so

$$\rho = -\frac{\nabla\cdot\overrightarrow{J}}{j\omega}$$

Therefore, the scattered field in terms of the electric current J is

$$\overrightarrow{E}^s(r) = -j\omega\mu\int_{s'}\overrightarrow{J}(\overrightarrow{r}')\frac{e^{-jkR}}{4\pi R}da' + \nabla\frac{1}{j\omega\varepsilon}\int_{s'}\nabla'\cdot\overrightarrow{J}(\overrightarrow{r}')\frac{e^{-jkR}}{4\pi R}da'$$

The electric current has two components

$$\begin{aligned}\overrightarrow{J}(\overrightarrow{r}') &= \widehat{t}'J_t(s', \phi') + \widehat{\phi}'J_\phi(s', \phi')\\ &= \widehat{t}'\sum_{n,k}A^t_{nk}\frac{T_k(s')}{\rho(s')}e^{+jn\phi'} - \widehat{\phi}'\sum_{n,k}A^\phi_{nk}\frac{T_{k'}(s')}{\rho(s')}e^{+jn\phi'}\end{aligned}$$

where A^t_{nk} is the coefficient of the $\widehat{t}'$ directed basis, A^ϕ_{nk} is the coefficient of the $\widehat{\phi}'$ directed basis, $T_k(s')$ is a triangle basis function along the curve that generates the body of revolution, and $\rho(s')$ is the radius of the body. In the $\widehat{\phi}$ direction the basis is a Fourier mode $(e^{jn\phi})$. The negative sign in in front of the second series comes from definition.

For a particular Fourier mode (n)

$$\begin{aligned}
\nabla' \cdot \overrightarrow{J}\left(\overrightarrow{r}'\right) &= \frac{1}{\rho\left(s'\right)} \frac{\partial}{\partial s'}\left[\rho\left(s'\right) J_{t'}\left(s', \phi'\right)\right] + \frac{1}{\rho\left(s'\right)} \frac{\partial}{\partial \phi'}\left[J_{\phi'}\left(s', \phi'\right)\right] \\
&= \frac{1}{\rho\left(s'\right)} \frac{\partial}{\partial s'}\left[\rho\left(s'\right) \sum_k A_{nk}^t \frac{T_k\left(s'\right)}{\rho\left(s'\right)} e^{+jn\phi'}\right] - \frac{1}{\rho\left(s'\right)} \frac{\partial}{\partial \phi'}\left[\sum_k A_{nk}^{\phi} \frac{T_k\left(s'\right)}{\rho\left(s'\right)} e^{+jn\phi'}\right] \\
&= \sum_k A_{nk}^t \frac{e^{+jn\phi'}}{\rho\left(s'\right)} \frac{\partial}{\partial s'}\left[T_k\left(s'\right)\right] - \sum_k A_{nk}^{\phi} \frac{1}{\rho\left(s'\right)} \frac{T_k\left(s'\right)}{\rho\left(s'\right)} \frac{\partial}{\partial \phi'}\left[e^{+jn\phi'}\right] \\
&= \frac{e^{+jn\phi'}}{\rho\left(s'\right)}\left[\sum_k A_{nk}^t \frac{\partial T_k\left(s'\right)}{\partial s'} - \sum_k A_{nk}^{\phi} \frac{jn}{\rho\left(s'\right)} T_k\left(s'\right)\right]
\end{aligned}$$

Looking at just the vector potential term for the n^{th} Fourier mode, $\widehat{x}$ component

$$\begin{aligned}
-j\omega A_x &= \widehat{x} \cdot (-j\omega\mu) \int_{s'} \overrightarrow{J}\left(\overrightarrow{r}'\right) \frac{e^{-jkR}}{4\pi R} da' \\
&= -j\omega\mu \int_{s'} \widehat{x} \cdot \widehat{t}' \sum_k A_{nk}^t \frac{T_{k'}\left(s'\right)}{\rho\left(s'\right)} e^{+jn\phi'} \frac{e^{-jkR}}{4\pi R} da' + j\omega\mu \int_{s'} \widehat{x} \cdot \widehat{\phi}' \sum_k A_{nk}^{\phi} \frac{T_{k'}\left(s'\right)}{\rho\left(s'\right)} e^{+jn\phi'} \frac{e^{-jkR}}{4\pi R} da' \\
&= -j\omega\mu \sum_k A_{nk}^t \int_{\Delta_k'} \sin\theta' \frac{T_{k'}\left(s'\right)}{\rho\left(s'\right)} \int_0^{2\pi} \cos\phi' e^{+jn\phi'} \frac{e^{-jkR}}{4\pi R} \rho\left(s'\right) d\phi' ds' \\
&\quad -j\omega\mu \sum_k A_{nk}^{\phi} \int_{\Delta_k'} \frac{T_{k'}\left(s'\right)}{\rho\left(s'\right)} \int_0^{2\pi} \sin\phi' e^{+jn\phi'} \frac{e^{-jkR}}{4\pi R} \rho\left(s'\right) d\phi' ds' \\
&= -\frac{j\omega\mu}{4\pi}\left\{\begin{array}{c} \sum_{k,n} A_{nk}^t \int_{\Delta_k'} \sin\theta' T_{k'}\left(s'\right) \int_0^{2\pi} \cos\phi' \left[\cos\left(n\phi'\right) + j\sin\left(n\phi'\right)\right] \frac{e^{-jkR}}{R} d\phi' ds' \\ + \sum_{k,n} A_{nk}^{\phi} \int_{\Delta_k'} T_{k'}\left(s'\right) \int_0^{2\pi} \sin\phi' \left[\cos\left(n\phi'\right) + j\sin\left(n\phi'\right)\right] \frac{e^{-jkR}}{R} d\phi' ds' \end{array}\right\}
\end{aligned}$$

R is even in ϕ' about the observation angle ϕ, the other quantities under the integration over ϕ are even or odd about $\phi' = 0$. This way of thinking complicates the elimination of certain terms because we are looking at components along x, y, and z rather than components along t and ϕ. Let's specialize our observation plane to $\phi = 0$, then R is even around $\phi' = 0$ just like the other quantities. Some of the terms being integrated are odd around $\phi' = 0$ and don't contribute to the integral while the even terms can be integrated over half the support and multiplied by 2.

$$-j\omega A_x = -\frac{j\omega\mu}{2\pi}\left\{\begin{array}{c} \sum_k A_{nk}^t \int_{\Delta_k'} \sin\theta' T_{k'}\left(s'\right) \int_0^{\pi} \cos\phi' \cos\left(n\phi'\right) \frac{e^{-jkR}}{R} d\phi' ds' \\ +j \sum_k A_{nk}^{\phi} \int_{\Delta_k'} T_{k'}\left(s'\right) \int_0^{\pi} \sin\phi' \sin\left(n\phi'\right) \frac{e^{-jkR}}{R} d\phi' ds' \end{array}\right\} \tag{716}$$

Next look at the vector potential term, $\widehat{y}$ component

$$\begin{aligned}
-j\omega A_y &= \widehat{y} \cdot (-j\omega\mu) \int_{s'} \overrightarrow{J}\left(\overrightarrow{r}'\right) \frac{e^{-jkR}}{4\pi R} da' \\
&= -j\omega\mu \int_{s'} \widehat{y} \cdot \widehat{t}' \sum_k A_{nk}^t \frac{T_{k'}\left(s'\right)}{\rho\left(s'\right)} e^{+jn\phi'} \frac{e^{-jkR}}{4\pi R} da' + j\omega\mu \int_{s'} \widehat{y} \cdot \widehat{\phi}' \sum_k A_{nk}^{\phi} \frac{T_{k'}\left(s'\right)}{\rho\left(s'\right)} e^{+jn\phi'} \frac{e^{-jkR}}{4\pi R} da' \\
&= -j\omega\mu \sum_k A_{nk}^t \int_{\Delta_k'} \sin\theta' \frac{T_{k'}\left(s'\right)}{\rho\left(s'\right)} \int_0^{2\pi} \sin\phi' e^{+jn\phi'} \frac{e^{-jkR}}{4\pi R} \rho\left(s'\right) d\phi' ds' \\
&\quad +j\omega\mu \sum_k A_{nk}^{\phi} \int_{\Delta_k'} \frac{T_{k'}\left(s'\right)}{\rho\left(s'\right)} \int_0^{2\pi} \cos\phi' e^{+jn\phi'} \frac{e^{-jkR}}{4\pi R} \rho\left(s'\right) d\phi' ds' \\
&= -\frac{j\omega\mu}{4\pi}\left\{\begin{array}{c} \sum_k A_{nk}^t \int_{\Delta_k'} \sin\theta' T_{k'}\left(s'\right) \int_0^{2\pi} \sin\phi' \left[\cos\left(n\phi'\right) + j\sin\left(n\phi'\right)\right] \frac{e^{-jkR}}{R} d\phi' ds' \\ - \sum_k A_{nk}^{\phi} \int_{\Delta_k'} T_{k'}\left(s'\right) \int_0^{2\pi} \cos\phi' \left[\cos\left(n\phi'\right) + j\sin\left(n\phi'\right)\right] \frac{e^{-jkR}}{R} d\phi' ds' \end{array}\right\}
\end{aligned}$$

Using the same arguments as above we obtain

$$-j\omega A_y = -\frac{j\omega\mu}{2\pi}\left\{\begin{array}{c} j\sum_k A_{nk}^t \int_{\Delta_k'} \sin\theta' T_{k'}(s') \int_0^\pi \sin\phi' \sin\left(n\phi'\right) \frac{e^{-jkR}}{R} d\phi' ds' \\ -\sum_k A_{nk}^\phi \int_{\Delta_k'} T_{k'}(s') \int_0^\pi \cos\phi' \cos\left(n\phi'\right) \frac{e^{-jkR}}{R} d\phi' ds' \end{array}\right\} \tag{717}$$

Next look at the vector potential term, $\widehat{z}$ component.

$$\begin{aligned} -j\omega A_z &= \widehat{z}\cdot(-j\omega\mu)\int_{s'} \overrightarrow{J}\left(\overrightarrow{r}'\right)\frac{e^{-jkR}}{4\pi R}da' \\ &= -j\omega\mu\int_{s'} \widehat{z}\cdot\widehat{t}'\sum_k A_{nk}^t \frac{T_{k'}(s')}{\rho(s')}e^{+jn\phi'}\frac{e^{-jkR}}{4\pi R}da' + j\omega\mu\int_{s'}\widehat{z}\cdot\widehat{\phi}'\sum_{k,n}\frac{T_{k'}(s')}{\rho(s')}e^{+jn\phi'}\frac{e^{-jkR}}{4\pi R}da' \\ &= -\frac{j\omega\mu}{4\pi}\sum_{k,n}\int_{\Delta_k'}\cos\theta'\frac{T_{k'}(s')}{\rho(s')}\int_0^{2\pi} e^{+jn\phi'}\frac{e^{-jkR}}{R}\rho(s')\,d\phi' ds' \\ &= -\frac{j\omega\mu}{4\pi}\sum_k A_{nk}^t\int_{\Delta_k'}\cos\theta' T_{k'}(s')\int_0^{2\pi}\left[\cos\left(n\phi'\right)+j\sin\left(n\phi'\right)\right]\frac{e^{-jkR}}{R}d\phi' ds' \\ &= -\frac{j\omega\mu}{2\pi}\sum_k A_{nk}^t\int_{\Delta_k'}\cos\theta' T_{k'}(s')\int_0^{\pi}\cos\left(n\phi'\right)\frac{e^{-jkR}}{R}d\phi' ds' \end{aligned}$$

For each case we will do the integration over the support of the triangle by summing 4 pulse contributions multiplied by the support of the pulse. With this approximation, we obtain the following equations for the contribution of the vector potential to the scattered E field.

$$\begin{aligned} -j\omega A_x &= -\frac{j\omega\mu}{2\pi}\left\{\begin{array}{c} \sum_k A_{nk}^t \sum_{q=1}^4 \sin\theta_{qk} T_{qk'}\Delta t_{qk}\int_0^\pi \cos\phi'\cos\left(n\phi'\right)\frac{e^{-jkR}}{R}d\phi' ds' \\ +j\sum_k A_{nk}^\phi \sum_{q=1}^4 T_{qk'}\Delta t_{qk}\int_0^\pi \sin\phi'\sin\left(n\phi'\right)\frac{e^{-jkR}}{R}d\phi' \end{array}\right\} \\ -j\omega A_y &= -\frac{j\omega\mu}{2\pi}\left\{\begin{array}{c} j\sum_k A_{nk}^t \sum_{q=1}^4 \sin\theta_{qk} T_{qk'}\Delta t_{qk}\int_0^\pi \sin\phi'\sin\left(n\phi'\right)\frac{e^{-jkR}}{R}d\phi' ds' \\ -\sum_k A_{nk}^\phi \sum_{q=1}^4 T_{qk'}\Delta t_{qk}\int_0^\pi \cos\phi'\cos\left(n\phi'\right)\frac{e^{-jkR}}{R}d\phi' ds' \end{array}\right\} \\ -j\omega A_z &= -\frac{j\omega\mu}{2\pi}\sum_k A_{nk}^t\sum_{q=1}^4\cos\theta_{qk}T_{qk'}\Delta t_{qk}\int_0^\pi\cos\left(n\phi'\right)\frac{e^{-jkR}}{R}d\phi' ds' \end{aligned} \tag{718}$$

For the scalar potential contribution to the E field we will first calculate the scalar potential

$$
\begin{aligned}
\phi &= \frac{1}{j\omega\varepsilon}\int_{s'} \nabla'\cdot\overrightarrow{J}\left(\overrightarrow{r}'\right)\frac{e^{-jkR}}{4\pi R}da' \\
&= \frac{1}{j\omega\varepsilon}\int_{\Delta'}\int_0^{2\pi}\frac{e^{+jn\phi'}}{\rho\left(s'\right)}\left[\sum_k A_{nk}^t\frac{\partial T_k\left(s'\right)}{\partial s'}-\sum_k A_{nk}^{\phi}\frac{jn}{\rho\left(s'\right)}T_k\left(s'\right)\right]\frac{e^{-jkR}}{4\pi R}\rho\left(s'\right)d\phi' ds' \\
&= \frac{1}{j\omega\varepsilon}\left[\begin{array}{c} \sum_k A_{nk}^t\int_{\Delta'}\frac{\partial T_k\left(s'\right)}{\partial s'}\int_0^{2\pi}e^{+jn\phi'}\frac{e^{-jkR}}{4\pi R}d\phi' ds' \\ -\sum_k A_{nk}^{\phi}jn\int_{\Delta'}\frac{T_k\left(s'\right)}{\rho\left(s'\right)}\int_0^{2\pi}e^{+jn\phi'}\frac{e^{-jkR}}{4\pi R}d\phi' ds' \end{array}\right] \\
&= \frac{1}{j\omega\varepsilon}\left[\begin{array}{c} \sum_k A_{nk}^t\int_{\Delta'}\frac{\partial T_k\left(s'\right)}{\partial s'}\int_0^{2\pi}\left[\cos\left(n\phi'\right)+j\sin\left(n\phi'\right)\right]\frac{e^{-jkR}}{4\pi R}d\phi' ds' \\ -\sum_k A_{nk}^{\phi}jn\int_{\Delta'}\frac{T_k\left(s'\right)}{\rho\left(s'\right)}\int_0^{2\pi}\left[\cos\left(n\phi'\right)+j\sin\left(n\phi'\right)\right]\frac{e^{-jkR}}{4\pi R}d\phi' ds' \end{array}\right] \\
&= \frac{2}{j\omega\varepsilon}\left[\sum_k A_{nk}^t\sum_{q=1}^4\frac{\partial T_{qk}}{\partial s'}\Delta t_{qk}\int_0^{\pi}\cos\left(n\phi'\right)\frac{e^{-jkR}}{4\pi R}d\phi'-\sum_k A_{nk}^{\phi}jn\sum_{q=1}^4\frac{T_{qk}}{\rho_{qk}}\Delta t_{qk}\int_0^{\pi}\cos\left(n\phi'\right)\frac{e^{-jkR}}{4\pi R}d\phi'\right] \\
&= \frac{-j}{\omega\varepsilon 2\pi}\sum_k A_{nk}^t\sum_{q=1}^4\frac{\partial T_{qk}}{\partial s'}\Delta t_{qk}\int_0^{\pi}\cos\left(n\phi'\right)\frac{e^{-jkR}}{R}d\phi'-\frac{1}{\omega\varepsilon 2\pi}n\sum_k A_{nk}^{\phi}\sum_{q=1}^4\frac{T_{qk}}{\rho_{qk}}\Delta t_{qk}\int_0^{\pi}\cos\left(n\phi'\right)\frac{e^{-jkR}}{R}d\phi'
\end{aligned}
$$

The gradient is given by

$$
\nabla\Phi=\widehat{\rho}\frac{\partial\Phi}{\partial\rho}+\widehat{\phi}\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}+\widehat{z}\frac{\partial\Phi}{\partial z}
$$

In the code we implemented the $\widehat{\rho}$ and $\widehat{z}$ derivatives as finite differences

$$
\nabla\Phi_{\rho}=\frac{\Phi\left(\rho+ds/2,z\right)-\Phi\left(\rho-ds/2,z\right)}{ds}
$$

$$
\nabla\Phi_{z}=\frac{\Phi\left(\rho,z+ds/2\right)-\Phi\left(\rho,z-ds/2\right)}{ds}
$$

and the $\widehat{\phi}$ derivative analytically

$$
\nabla\Phi_{\phi}=\frac{jn\Phi\left(\rho,z\right)}{\rho}
$$

which map to the scalar potential contribution to $E\left(r\right)$.

$$
\begin{aligned}
-\nabla_x\Phi &= \frac{+j}{\omega\varepsilon 2\pi}\frac{\partial}{\partial x}\sum_k A_{nk}^{t'}\sum_{q=1}^4\frac{\partial T_{qk}}{\partial s'}\Delta t_{qk}\int_0^{\pi}\cos\left(n\phi'\right)\frac{e^{-jkR}}{R}d\phi' \\
&\quad+\frac{1}{\omega\varepsilon 2\pi}n\frac{\partial}{\partial x}\sum_k A_{nk}^{\phi'}\sum_{q=1}^4\frac{T_{qk}}{\rho_{qk}}\Delta t_{qk}\int_0^{\pi}\cos\left(n\phi'\right)\frac{e^{-jkR}}{R}d\phi'
\end{aligned}
\tag{719}
$$

$$
\begin{aligned}
-\nabla_y\Phi &= \frac{jn}{\rho_p}\left[\begin{array}{c} \frac{-j}{\omega\varepsilon 2\pi}\sum_k A_{nk}^{t'}\sum_{q=1}^4\frac{\partial T_{qk}}{\partial s'}\Delta t_{qk}\int_0^{\pi}\cos\left(n\phi'\right)\frac{e^{-jkR}}{R}d\phi' \\ -\frac{1}{\omega\varepsilon 2\pi}n\sum_k A_{nk}^{\phi'}\sum_{q=1}^4\frac{T_{qk}}{\rho_{qk}}\Delta t_{qk}\int_0^{\pi}\cos\left(n\phi'\right)\frac{e^{-jkR}}{R}d\phi' \end{array}\right] \\
&= \frac{n}{\omega\varepsilon 2\pi\rho_p}\sum_k A_{nk}^{t'}\sum_{q=1}^4\frac{\partial T_{qk}}{\partial s'}\Delta t_{qk}\int_0^{\pi}\cos\left(n\phi'\right)\frac{e^{-jkR}}{R}d\phi' \\
&\quad+\frac{n^2}{j\omega\varepsilon 2\pi\rho_p}\sum_k A_{nk}^{\phi'}\sum_{q=1}^4\frac{T_{qk}}{\rho_{qk}}\Delta t_{qk}\int_0^{\pi}\cos\left(n\phi'\right)\frac{e^{-jkR}}{R}d\phi'
\end{aligned}
\tag{720}
$$

$$-\nabla_z \Phi = \frac{+j}{\omega\varepsilon 2\pi} \frac{\partial}{\partial z} \sum_k A_{nk}^{t'} \sum_{q=1}^{4} \frac{\partial T_{qk}}{\partial s'} \Delta t_{qk} \int_0^{\pi} \cos\left(n\phi'\right) \frac{e^{-jkR}}{R} d\phi' + \frac{1}{\omega\varepsilon 2\pi} n \frac{\partial}{\partial z} \sum_k A_{nk}^{\phi'} \sum_{q=1}^{4} \frac{T_{qk}}{\rho_{qk}} \Delta t_{qk} \int_0^{\pi} \cos\left(n\phi'\right) \frac{e^{-jkR}}{R} d\phi' \quad (721)$$

7.2.2 Alternative Derivation of $\overrightarrow{E}^s(r)$

An alternative method of obtaining the $\overrightarrow{E}^s$ field is to start with the expressions for the impedance matrix of a cavity with PEC walls in terms of the operator $\overline{\overline{L}}(\cdot)$, which is given in [21]. We will change the Galerkin testing procedure to point matching at $\left(\rho_p, 0, z_p\right)$ and get expressions for the x, y, and z components of $\overrightarrow{E}^s$. Using this method gives us a way of checking the direct derivation given above.

We will start with the $\overline{\overline{L}}(\cdot)$ operator and compare the results using this procedure to the $\overrightarrow{E}^s$ field that we derived directly.

$$\overrightarrow{E}^s\left(\rho_p, 0, z_p\right) = -\frac{1}{2\pi} \overline{\overline{L}}\left(\overrightarrow{J_q}\right)$$

where [21]

$$L_{kl}^{tt} = j\omega\mu T_p T_q (\sin\nu_p \sin\nu_q G_{cn} + \cos\nu_p \cos\nu_q G_n) - \frac{j}{\omega\varepsilon} \dot{T}_p \dot{T}_q G_n$$

$$L_{kl}^{\phi t} = -\omega\mu \sin\nu_q T_p T_q G_{sn} - \frac{n}{\omega\varepsilon\rho_p} T_p \dot{T}_q G_n$$
$$L_{kl}^{t\phi} = -\omega\mu \sin\nu_p T_p T_q G_{sn} - \frac{n}{\omega\varepsilon\rho_q} \dot{T}_p T_q G_n$$
$$L_{kl}^{\phi\phi} = -j\omega\mu T_p T_q G_{cn} - \frac{n^2}{j\omega\varepsilon\rho_p\rho_q} T_p T_q G_n$$

and where

$$G_n = \Delta t_q \int_0^{\pi} \cos n\phi \frac{e^{-jkR}}{R} d\phi$$
$$G_{cn} = \Delta t_q \int_0^{\pi} \cos n\phi \cos\phi \frac{e^{-jkR}}{R} d\phi$$
$$G_{sn} = \Delta t_q \int_0^{\pi} \sin n\phi \sin\phi \frac{e^{-jkR}}{R} d\phi$$

In the above expressions, p represents the observation segment $\left(\rho_p, 0, z_p\right)$ and q represents the source segment, ν_q is the same as θ' and Δt_q results from approximating the integration over s' of the source segment as an evaluation of the integrand at the center of the segment and multiplying by the segment support (Δt_q).

For the $\widehat{x}$ component of E^s we set $\nu_p = \pi/2$, and we get contributions from L^{tt} and $L^{t\phi}$.

$$L_{kl}^{xt} = j\omega\mu T_q^t \sin\nu_q G_{cn} - \frac{j}{\omega\varepsilon} \frac{\partial}{\partial x}\left(\dot{T}_q^t G_n\right)$$
$$L_{kl}^{x\phi} = -\omega\mu T_q^{\phi} G_{sn} - \frac{n}{\omega\varepsilon\rho_q} \frac{\partial}{\partial x}\left(T_q^{\phi} G_n\right)$$

$$E_x^s = -\frac{1}{2\pi}\left[j\omega\mu T_q^t \sin\nu_q G_{cn} - \omega\mu T_q^{\phi} G_{sn} - \frac{j}{\omega\varepsilon} \frac{\partial}{\partial x}\left(\dot{T}_q^t G_n\right) - \frac{n}{\omega\varepsilon\rho_q} \frac{\partial}{\partial x}\left(T_q^{\phi} G_n\right)\right] \quad (722)$$

The contributions to E_x^s from the vector potential are

$$\begin{aligned} -j\omega A_x &= -\frac{1}{2\pi}\left[j\omega\mu T_q^t \sin\nu_q G_{cn} - \omega\mu T_q^\phi G_{sn}\right] \\ &= -\frac{j\omega\mu}{2\pi}\left[T_q^t \sin\nu_q \Delta t_q \int_0^\pi \cos n\phi \cos\phi \frac{e^{-jkR}}{R} d\phi + jT_q^\phi \Delta t_q \int_0^\pi \sin n\phi \sin\phi \frac{e^{-jkR}}{R} d\phi\right] \end{aligned}$$

This is identical to what was derived directly in Equation 716. The contributions to E_x^s that arise from the scalar potential

$$\begin{aligned} -\nabla_x\Phi &= -\frac{1}{2\pi}\left[-\frac{j}{\omega\varepsilon}\frac{\partial}{\partial x}\left(\dot{T}_q^t G_n\right) - \frac{n}{\omega\varepsilon\rho_q}\frac{\partial}{\partial x}\left(T_q^\phi G_n\right)\right] \\ &= \frac{j}{\omega\varepsilon 2\pi}\frac{\partial}{\partial x}\left(\dot{T}_q^t \Delta t_q \int_0^\pi \cos n\phi \frac{e^{-jkR}}{R} d\phi\right) + \frac{n}{\omega\varepsilon 2\pi}\frac{\partial}{\partial x}\left(\frac{T_q^\phi}{\rho_q}\Delta t_q \int_0^\pi \cos n\phi \frac{e^{-jkR}}{R} d\phi\right) \end{aligned}$$

Again this is identical to what was derived in Equation 719.

For the $\widehat{z}$ component of E^s we set $v_p = 0$, and we again have contributions from L^{tt} and $L^{t\phi}$.

$$\begin{aligned} L_{kl}^{zt} &= j\omega\mu T_p T_q \cos\nu_q G_n - \frac{j}{\omega\varepsilon}\frac{\partial}{\partial z}\left(\dot{T}_q G_n\right) \\ L_{kl}^{z\phi} &= -\frac{n}{\omega\varepsilon\rho_q}\frac{\partial}{\partial z}\left(T_q G_n\right) \\ E_z^s &= -\frac{1}{2\pi}\left[j\omega\mu T_q^t \cos\nu_q G_n - \frac{j}{\omega\varepsilon}\frac{\partial}{\partial z}\left(\dot{T}_q^t G_n\right) - \frac{n}{\omega\varepsilon\rho_q}\frac{\partial}{\partial z}\left(T_q^\phi G_n\right)\right] \end{aligned}$$

The contributions to E_z^s that arise from the vector potential are

$$\begin{aligned} -j\omega\mu A_z &= -\frac{1}{2\pi}\left[j\omega\mu T_q^t \cos\nu_q G_n\right] \\ &= -\frac{j\omega\mu}{2\pi}\left[T_q^t \cos\nu_q \Delta t_q \int_0^\pi \cos n\phi \frac{e^{-jkR}}{R} d\phi\right] \end{aligned}$$

This is identical to what was derived in Equation 718.

$$-j\omega A_z = -\frac{j\omega\mu}{2\pi}\sum_k A_{nk}^{t'} \sum_{q=1}^{4} \cos\theta_{qk} T_{qk'} \Delta t_{qk} \int_0^\pi \cos\left(n\phi'\right) \frac{e^{-jkR}}{R} d\phi'$$

The contributions to E_z^s that arise from the scalar potential are

$$\begin{aligned} -\nabla_z\Phi &= -\frac{1}{2\pi}\left[-\frac{j}{\omega\varepsilon}\frac{\partial}{\partial z}\left(\dot{T}_q^t G_n\right) - \frac{n}{\omega\varepsilon\rho_q}\frac{\partial}{\partial z}\left(T_q^\phi G_n\right)\right] \\ &= \left[\frac{j}{\omega\varepsilon 2\pi}\frac{\partial}{\partial z}\left(\dot{T}_q^t \Delta t_q \int_0^\pi \cos n\phi \frac{e^{-jkR}}{R} d\phi\right) + \frac{n}{\omega\varepsilon 2\pi}\frac{\partial}{\partial z}\left(\frac{T_q^\phi}{\rho_q}\Delta t_q \int_0^\pi \cos n\phi \frac{e^{-jkR}}{R} d\phi\right)\right] \end{aligned}$$

This is identical to what was derived in Equation 721. For the $\widehat{y}$ component, we again have contributions from $L^{\phi t}$ and $L^{\phi\phi}$.

$$\begin{aligned} L_{kl}^{\phi t} &= -\omega\mu \sin\nu_q T_q^t G_{sn} - \frac{n}{\omega\varepsilon\rho_p}\dot{T}_q^t G_n \\ L_{kl}^{\phi\phi} &= -j\omega\mu T_q^\phi G_{cn} - \frac{n^2}{j\omega\varepsilon\rho_p\rho_q} T_q^\phi G_n \\ E_y^s &= -\frac{1}{2\pi}\left[-\omega\mu \sin\nu_q T_q^t G_{sn} - j\omega\mu T_q^\phi G_{cn} - \frac{n}{\omega\varepsilon\rho_p}\dot{T}_q^t G_n - \frac{n^2}{j\omega\varepsilon\rho_p\rho_q} T_q^\phi G_n\right] \end{aligned}$$

The contributions to E_y^s that arise from the vector potential are

$$
\begin{aligned}
-j\omega A_y &= -\frac{1}{2\pi}\left[-\omega\mu\sin\nu_q T_q^t G_{sn} - j\omega\mu T_q^\phi G_{cn}\right] \\
&\quad -\frac{j\omega\mu}{2\pi}\left[j\sin\nu_q T_q^t \Delta t_q \int_0^\pi \sin n\phi \sin\phi \frac{e^{-jkR}}{R} d\phi - T_q^\phi \Delta t_q \int_0^\pi \cos n\phi\cos\phi \frac{e^{-jkR}}{R} d\phi\right]
\end{aligned}
$$

This is identical to what was derived in Equation 717. The contributions to E_y^s that arise from the scalar potential are

$$
\begin{aligned}
-\nabla_y \Phi &= -\frac{1}{2\pi}\left[-\frac{n}{\omega\varepsilon\rho_p}\dot{T}_q^t G_n - \frac{n^2}{j\omega\varepsilon\rho_p\rho_q} T_q^\phi G_n\right] \\
&= \frac{n}{\omega\varepsilon 2\pi\rho_p}\dot{T}_q^t \Delta t_q \int_0^\pi \cos n\phi \frac{e^{-jkR}}{R} d\phi + \frac{n^2}{j\omega\varepsilon 2\pi\rho_p}\frac{T_q^\phi}{\rho_q}\Delta t_q \int_0^\pi \cos n\phi \frac{e^{-jkR}}{R} d\phi
\end{aligned}
$$

This is identical to what was derived in Equation 720.

The above equations for $\overrightarrow{E}^s$ are implemented in **near_field_module** in the subroutine **lop_nf**.

7.3 CICERO Modifications for Magnetic Nearfield Calculation

7.3.1 Direct Derivation of $\overrightarrow{H}^s(r)$

The scattered $\overrightarrow{H}$ field in terms of vector potential $\overrightarrow{A}$ is

$$
\begin{aligned}
\overrightarrow{H}^s(r) &= \frac{1}{\mu}\nabla\times A(r,r') \\
&= \frac{1}{\mu}\nabla\times\int_{s'}\mu\overrightarrow{J}(\overrightarrow{r}')\frac{e^{-jkR}}{4\pi R}da' \\
&= \int_{s'}\left[\nabla\times\overrightarrow{J}(\overrightarrow{r}')\right]\frac{e^{-jkR}}{4\pi R}da' - \int_{s'}\overrightarrow{J}(\overrightarrow{r}')\times\nabla\frac{e^{-jkR}}{4\pi R}da' \\
&= -\int_{s'}\overrightarrow{J}(\overrightarrow{r}')\times\nabla\frac{e^{-jkR}}{4\pi R}da'
\end{aligned}
$$

where R is the distance between the observation point $\overrightarrow{r} = (\rho,\phi,z)$ and a source point $\overrightarrow{r}' = (\rho',\phi',z')$

$$
\begin{aligned}
R &= |\overrightarrow{r} - \overrightarrow{r}'| \\
&= \sqrt{(\rho'-\rho)^2 + (z'-z)^2 + 2\rho'\rho\left(1-\cos\left(\phi'-\phi\right)\right)}
\end{aligned}
\tag{723}
$$

The gradient term can be expanded as follows

$$
\begin{aligned}
\nabla\frac{e^{-jkR}}{4\pi R} &= \frac{1}{4\pi}\frac{(\overrightarrow{r}-\overrightarrow{r}')}{R}\frac{\partial}{\partial R}\frac{e^{-jkR}}{R} \\
&= \frac{1}{4\pi}\frac{(\overrightarrow{r}-\overrightarrow{r}')}{R}\frac{-jRke^{-jkR} - e^{-jkR}}{R^2} \\
&= -\frac{1}{4\pi}(\overrightarrow{r}-\overrightarrow{r}')\frac{1+jkR}{R^3}e^{-jkR}
\end{aligned}
$$

where

$$
(\overrightarrow{r}-\overrightarrow{r}') = \widehat{x}\left(\rho\cos\phi - \rho'\cos\phi'\right) + \widehat{y}\left(\rho\sin\phi - \rho'\sin\phi'\right) + \widehat{z}(z-z')
$$

The electric current on the wall of the cavity has two components $\widehat{t}$ and $\widehat{\phi}$

$$\begin{aligned}\overrightarrow{J}\left(\overrightarrow{r}'\right) &= \widehat{t}'J_t\left(s',\phi'\right)+\widehat{\phi}'J_\phi\left(s',\phi'\right)\\ &= \widehat{t}'\sum_{n,k}A_{nk}^{t'}\frac{T_k\left(s'\right)}{\rho\left(s'\right)}e^{+jn\phi'}-\widehat{\phi}'\sum_{n,k}A_{nk}^{\phi'}\frac{T_{k'}\left(s'\right)}{\rho\left(s'\right)}e^{+jn\phi'}\end{aligned}$$

where in Cartesian coordinates

$$\widehat{t}=\widehat{x}\sin\theta\cos\phi+\widehat{y}\sin\theta\sin\phi+\widehat{z}\cos\theta$$

and

$$\widehat{\phi}=-\widehat{x}\sin\phi+\widehat{y}\cos\phi$$

$A_{nk}^{t'}$ is the coefficient of the $\widehat{t'}$ directed basis, $A_{nk}^{\phi'}$ is the coefficient of the $\widehat{\phi}'$ directed basis, $T_k\left(s'\right)$ is a triangle basis function along the curve that generates the body of revolution, and $\rho\left(s'\right)$ is the radius of the body. In the $\widehat{\phi}'$ direction the basis is a Fourier mode $e^{jn\phi'}$. The negative sign in the second series comes from the definition in the original **CICERO** code.

Looking at just the n^{th} Fourier mode, the $\widehat{x}$ component of $\overrightarrow{H}$ is

$$\begin{aligned}H_x^s &= -\widehat{x}\cdot\int_{s'}\overrightarrow{J}\left(\overrightarrow{r}'\right)\times\nabla\frac{e^{-jkR}}{4\pi R}da'\\ &= H_x^{t'}+H_x^{\phi'}\\ &= -\int_{s'}\sum_k A_{nk}^{t'}\frac{T_{k'}\left(s'\right)}{\rho\left(s'\right)}e^{+jn\phi'}\widehat{x}\cdot\widehat{t}'\times\nabla\frac{e^{-jkR}}{4\pi R}da'+\int_{s'}\sum_k A_{nk}^{\phi'}\frac{T_{k'}\left(s'\right)}{\rho\left(s'\right)}e^{+jn\phi'}\widehat{x}\cdot\widehat{\phi}'\times\nabla\frac{e^{-jkR}}{4\pi R}da'\end{aligned}$$

Let us, for convenience, look at the two terms of the above equation separately. First for the $\widehat{t'}$ component term we obtain

$$H_x^{t'}=+\frac{1}{4\pi}\sum_k A_{nk}^{t'}\int_{\Delta_k'}\frac{T_{k'}\left(s'\right)}{\rho\left(s'\right)}\int_0^{2\pi}e^{+jn\phi'}\widehat{x}\cdot\left[\widehat{t'}\times\left(\overrightarrow{r}-\overrightarrow{r}'\right)\right]\frac{1+jkR}{R^3}e^{-jkR}\rho\left(s'\right)d\phi' ds'$$

Performing the cross product in Cartesian coordinates

$$\widehat{t'}\times\left(\overrightarrow{r}-\overrightarrow{r}'\right)=\left\|\begin{array}{lll}\widehat{x} & \widehat{y} & \widehat{z}\\ \sin\theta'\cos\phi' & \sin\theta'\sin\phi' & \cos\theta'\\ \left(\rho\cos\phi-\rho'\cos\phi'\right) & \left(\rho\sin\phi-\rho'\sin\phi'\right) & \left(z-z'\right)\end{array}\right\|$$

$$\begin{aligned}\widehat{t'}\times\left(\overrightarrow{r}-\overrightarrow{r}'\right) &= \widehat{x}\left[\sin\theta'\sin\phi'\left(z-z'\right)-\cos\theta'\left(\rho\sin\phi-\rho'\sin\phi'\right)\right]\\ &\quad+\widehat{y}\left[\cos\theta'\left(\rho\cos\phi-\rho'\cos\phi'\right)-\sin\theta'\cos\phi'\left(z-z'\right)\right]\\ &\quad+\widehat{z}\left[\sin\theta'\cos\phi'\left(\rho\sin\phi-\rho'\sin\phi'\right)-\sin\theta'\sin\phi'\left(\rho\cos\phi-\rho'\cos\phi'\right)\right]\end{aligned}\tag{724}$$

$$
\begin{aligned}
H_x^{t'} &= \frac{1}{4\pi}\sum_k A_{nk}^{t'} \int_{\Delta_k'} \frac{T_{k'}(s')}{\rho(s')} \int_0^{2\pi} \left[\cos n\phi' + j\sin n\phi'\right]\left[\sin\theta'\sin\phi'(z-z') - \cos\theta'\left(\rho\sin\phi - \rho'\sin\phi'\right)\right] \\
&\quad \frac{1+jkR}{R^3} e^{-jkR}\rho(s')\,d\phi' ds' \\
&= \frac{1}{4\pi}\sum_k A_{nk}^{t'} \int_{\Delta_k'} T_{k'}(s') \int_0^{2\pi} \left\{\left[\sin\theta'\cos n\phi'\sin\phi'(z-z') - \cos\theta'\cos n\phi'\left(\rho\sin\phi - \rho'\sin\phi'\right)\right]\right. \\
&\quad \left. + \left[\sin\theta' j\sin n\phi'\sin\phi'(z-z') - \cos\theta' j\sin n\phi'\left(\rho\sin\phi - \rho'\sin\phi'\right)\right]\right\} \frac{1+jkR}{R^3} e^{-jkR} d\phi' ds' \\
&= \frac{1}{4\pi}\sum_k A_{nk}^{t'} \int_{\Delta_k'} T_{k'}(s') \int_0^{2\pi} \left\{\sin\theta'(z-z')\cos n\phi'\sin\phi'\right. \\
&\quad -\rho\sin\phi\cos\theta'\cos n\phi' + \rho'\cos\theta'\sin\phi'\cos n\phi' + j\sin\theta'(z-z')\sin n\phi'\sin\phi' \\
&\quad \left. -j\rho\sin\phi\cos\theta'\sin n\phi' + j\cos\theta'\rho'\sin\phi'\sin n\phi'\right\} \frac{1+jkR}{R^3} e^{-jkR} d\phi' ds'
\end{aligned}
$$

R is even in ϕ' about the observation angle ϕ; the other quantities being integrated over ϕ' are even or odd about $\phi' = 0$. This way of thinking complicates the elimination of certain terms because we are looking at components along x, y, and z rather than components along t and ϕ. Let's specialize our observation plane to $\phi = 0$, then R is even around $\phi' = 0$ just like the other quantities. Some of the terms being integrated are odd around $\phi' = 0$ and don't contribute to the integral while the even terms can be integrated over half the support and multiplied by 2.

$$
H_x^{t'} = \frac{1}{2\pi}\sum_k A_{nk}^{t'} \int_{\Delta_k'} T_{k'}(s') \int_0^{\pi} j\left[\sin\theta'(z-z')\sin n\phi'\sin\phi' + \cos\theta'\left(\rho'\sin\phi'\sin n\phi'\right)\right] \frac{1+jkR}{R^3} e^{-jkR} d\phi' ds'
$$

$$
H_x^{t'} = \frac{j}{2\pi}\sum_k A_{nk}^{t'} \int_{\Delta_k'} T_{k'}(s')\left[\sin\theta'(z-z') + \rho'\cos\theta'\right] \int_0^{\pi} \sin\phi'\sin n\phi' \frac{1+jkR}{R^3} e^{-jkR} d\phi' ds' \tag{725}
$$

For the $\widehat{\phi}'$ component term we obtain

$$
H_x^{\phi'} = +\int_{s'} \sum_k A_{nk}^{\phi'} \frac{T_{k'}(s')}{\rho(s')} e^{+jn\phi'} \left[\widehat{x}\cdot\widehat{\phi}' \times (\overrightarrow{r} - \overrightarrow{r}')\right] \times \nabla \frac{e^{-jkR}}{4\pi R} da'
$$

Performing the cross product

$$
\widehat{\phi}' \times (\overrightarrow{r} - \overrightarrow{r}') = \left\|\begin{array}{lll} \widehat{x} & \widehat{y} & \widehat{z} \\ -\sin\phi' & \cos\phi' & 0 \\ \left(\rho\cos\phi - \rho'\cos\phi'\right) & \left(\rho\sin\phi - \rho'\sin\phi'\right) & (z-z') \end{array}\right\|
$$

$$
\begin{aligned}
\widehat{\phi}' \times (\overrightarrow{r} - \overrightarrow{r}') &= \widehat{x}\cos\phi'(z-z') \\
&\quad + \widehat{y}\sin\phi'(z-z') \\
&\quad + \widehat{z}\left[-\sin\phi'\left(\rho\sin\phi - \rho'\sin\phi'\right) - \cos\phi'\left(\rho\cos\phi - \rho'\cos\phi'\right)\right]
\end{aligned} \tag{726}
$$

$$
\begin{aligned}
H_x^{\phi'} &= -\frac{1}{4\pi}\sum_k A_{nk}^{\phi'} \int_{\Delta_k'} T_{k'}(s') \int_0^{2\pi} \left[\cos n\phi' + j\sin n\phi'\right]\left[\cos\phi'(z-z')\right] \frac{1+jkR}{R^3} e^{-jkR} d\phi' ds' \\
&= -\frac{1}{4\pi}\sum_k A_{nk}^{\phi'} \int_{\Delta_k'} T_{k'}(s')(z-z') \int_0^{2\pi} \left[\cos\phi'\cos n\phi' + j\cos\phi'\sin n\phi'\right] \frac{1+jkR}{R^3} e^{-jkR} d\phi' ds'
\end{aligned}
$$

Invoking even and odd symmetry and setting the observation point at $\phi = 0$ to eliminate terms we obtain

$$H_x^{\phi'} = -\frac{1}{2\pi} \sum_k A_{nk}^{\phi'} \int_{\Delta_k'} T_{k'}(s')(z - z') \int_0^{\pi} \cos\phi' \cos n\phi' \frac{1 + jkR}{R^3} e^{-jkR} d\phi' ds' \tag{727}$$

The $\widehat{z}$ component of $\overrightarrow{H}$ is

$$\begin{aligned} H_z^s &= -\widehat{z} \cdot \int_{s'} \overrightarrow{J}(\overrightarrow{r}') \times \nabla \frac{e^{-jkR}}{4\pi R} da' \\ &= H_z^{t'} + H_z^{\phi'} \\ &= -\int_{s'} \sum_k A_{nk}^{t'} \frac{T_{k'}(s')}{\rho(s')} e^{+jn\phi'} \widehat{z} \cdot \widehat{t}' \times \nabla \frac{e^{-jkR}}{4\pi R} da' + \int_{s'} \sum_k A_{nk}^{\phi'} \frac{T_{k'}(s')}{\rho(s')} e^{+jn\phi'} \widehat{z} \cdot \widehat{\phi}' \times \nabla \frac{e^{-jkR}}{4\pi R} da' \end{aligned}$$

Again doing the calculation term-by-term, for the $\widehat{t}'$ component term we obtain

$$\begin{aligned} H_z^{t'} &= +\frac{1}{4\pi} \sum_k A_{nk}^{t'} \int_{\Delta_k'} \frac{T_{k'}(s')}{\rho(s')} \int_0^{2\pi} e^{+jn\phi'} \widehat{z} \cdot \left[\widehat{t}' \times (\overrightarrow{r} - \overrightarrow{r}')\right] \frac{1 + jkR}{R^3} e^{-jkR} \rho(s') d\phi' ds' \\ &= +\frac{1}{4\pi} \sum_k A_{nk}^{t'} \int_{\Delta_k'} \frac{T_{k'}(s')}{\rho(s')} \int_0^{2\pi} \Big\{ \left[\cos n\phi' + j \sin n\phi'\right] \\ &\quad \left[\sin\theta' \cos\phi' \left(\rho \sin\phi - \rho' \sin\phi'\right) - \sin\theta' \sin\phi' \left(\rho \cos\phi - \rho' \cos\phi'\right)\right] \\ &\quad \frac{1 + jkR}{R^3} e^{-jkR} \rho(s') d\phi' \Big\} ds' \\ &= +\frac{1}{4\pi} \sum_k A_{nk}^{t'} \int_{\Delta_k'} T_{k'}(s') \int_0^{2\pi} \big\{ \rho \sin\phi \sin\theta' \cos\phi' \cos n\phi' - \rho' \sin\theta' \sin\phi' \cos\phi' \cos n\phi' \\ &\quad - \rho \cos\phi \sin\theta' \sin\phi' \cos n\phi' + \rho' \sin\theta' \cos\phi' \sin\phi' \cos n\phi' \\ &\quad + j\rho \sin\phi \sin\theta' \sin n\phi' \cos\phi' - j\rho' \sin\theta' \sin n\phi' \cos\phi' \sin\phi' \\ &\quad - j\rho \cos\phi \sin\theta' \sin n\phi' \sin\phi' + j\rho' \sin\theta' \sin n\phi' \sin\phi' \cos\phi' \big\} \frac{1 + jkR}{R^3} e^{-jkR} d\phi' ds' \end{aligned}$$

Invoking even and odd symmetry and setting the observation point at $\phi = 0$ to eliminate terms we obtain

$$\begin{aligned} H_z^{t'} &= -\frac{j}{2\pi} \sum_k A_{nk}^{t'} \int_{\Delta_k'} T_{k'}(s') \int_0^{\pi} \left[\rho' \sin\theta' \sin n\phi' \cos\phi' \sin\phi' + \rho \sin\theta' \sin n\phi' \sin\phi' \right. \\ &\quad \left. - \rho' \sin\theta' \sin n\phi' \sin\phi' \cos\phi'\right] \frac{1 + jkR}{R^3} e^{-jkR} d\phi' ds' \end{aligned}$$

$$H_z^{t'} = -\frac{j}{2\pi} \sum_k A_{nk}^{t'} \int_{\Delta_k'} T_{k'}(s') \rho \sin\theta' \int_0^{\pi} \sin\phi' \sin n\phi' \frac{1 + jkR}{R^3} e^{-jkR} d\phi' ds' \tag{728}$$

For the $\widehat{\phi}'$ component term we obtain

$$H_z^{\phi'} = \int_{s'} \sum_k A_{nk}^{\phi'} \frac{T_{k'}(s')}{\rho(s')} e^{+jn\phi'} \widehat{z} \cdot \widehat{\phi}' \times \nabla \frac{e^{-jkR}}{4\pi R} da'$$

$$
\begin{aligned}
H_z^{\phi'} &= -\frac{1}{4\pi}\sum_k A_{nk}^{\phi'}\int_{\Delta'_k}\frac{T_{k'}(s')}{\rho(s')}\int_0^{2\pi} e^{jn\phi}\left[\widehat{z}\cdot\widehat{\phi}'\times(\overrightarrow{r}-\overrightarrow{r}')\right]\frac{1+jkR}{R^3}e^{-jkR}\rho(s')\,d\phi' ds' \\
&= -\frac{1}{4\pi}\sum_k A_{nk}^{\phi'}\int_{\Delta'_k} T_{k'}(s')\int_0^{2\pi}\left[\cos n\phi' + j\sin n\phi'\right] \\
&\quad \left[-\sin\phi'\left(\rho\sin\phi-\rho'\sin\phi'\right)-\cos\phi'\left(\rho\cos\phi-\rho'\cos\phi'\right)\right] \\
&\quad \frac{1+jkR}{R^3}e^{-jkR}d\phi' ds' \\
&= -\frac{1}{4\pi}\sum_k A_{nk}^{\phi'}\int_{\Delta'_k} T_{k'}(s')\int_0^{2\pi}\left\{-\rho\sin\phi\cos n\phi'\sin\phi' + \rho'\cos n\phi'\sin\phi'\sin\phi'\right. \\
&\quad -\rho\cos\phi\cos n\phi'\cos\phi' + \rho'\cos n\phi'\cos\phi'\cos\phi' - j\rho\sin\phi\sin n\phi'\sin\phi' + j\rho'\sin n\phi'\sin\phi'\sin\phi' \\
&\quad \left.-j\rho\cos\phi\sin n\phi'\cos\phi' + j\rho'\sin n\phi'\cos\phi'\cos\phi'\right\}\frac{1+jkR}{R^3}e^{-jkR}d\phi' ds'
\end{aligned}
$$

Invoking even and odd symmetry and setting the observation point at $\phi = 0$ to eliminate terms we obtain

$$
\begin{aligned}
H_z^{\phi'} &= -\frac{1}{2\pi}\sum_k A_{nk}^{\phi'}\int_{\Delta'_k} T_{k'}(s')\int_0^{\pi}\left[\rho'\cos n\phi'\sin^2\phi' - \rho\cos n\phi'\cos\phi' + \rho'\cos n\phi'\cos^2\phi'\right] \\
&\quad \frac{1+jkR}{R^3}e^{-jkR}d\phi' ds'
\end{aligned}
$$

$$
H_z^{\phi'} = +\frac{1}{2\pi}\sum_k A_{nk}^{\phi'}\int_{\Delta'_k} T_{k'}(s')\left[\rho\int_0^{\pi}\cos\phi'\cos n\phi'\frac{1+jkR}{R^3}e^{-jkR}d\phi' - \rho'\int_0^{\pi}\cos n\phi'\frac{1+jkR}{R^3}e^{-jkR}d\phi'\right]ds' \tag{729}
$$

The $\widehat{y}$ component of $\overrightarrow{H}$ is

$$
\begin{aligned}
H_y &= -\widehat{y}\cdot\int_{s'}\overrightarrow{J}(\overrightarrow{r}')\times\nabla\frac{e^{-jkR}}{4\pi R}da' \\
&= H_y^{t'} + H_y^{\phi'} \\
&= -\int_{s'}\sum_k A_{nk}^{t'}\frac{T_{k'}(s')}{\rho(s')}e^{+jn\phi'}\widehat{y}\cdot\widehat{t}'\times\nabla\frac{e^{-jkR}}{4\pi R}da' + \int_{s'}\sum_k A_{nk}^{\phi'}\frac{T_{k'}(s')}{\rho(s')}e^{+jn\phi'}\widehat{y}\cdot\widehat{\phi}'\times\nabla\frac{e^{-jkR}}{4\pi R}da'
\end{aligned}
$$

Again doing the calculation term-by-term, for the $\widehat{t}'$ component term we obtain

$$\begin{aligned}
H_y^{t'} &= +\frac{1}{4\pi}\sum_k A_{nk}^{t'}\int_{\Delta_k'}\frac{T_{k'}(s')}{\rho(s')}\int_0^{2\pi} e^{+jn\phi'}\widehat{y}\cdot\left[\widehat{t}'\times\left(\overrightarrow{r}-\overrightarrow{r}'\right)\right]\frac{1+jkR}{R^3}e^{-jkR}\rho(s')\,d\phi' ds' \\
&= +\frac{1}{4\pi}\sum_k A_{nk}^{t'}\int_{\Delta_k'} T_{k'}(s')\int_0^{2\pi}\left[\cos n\phi' + j\sin n\phi'\right]\left[\cos\theta'\left(\rho\cos\phi-\rho'\cos\phi'\right)-\sin\theta'\cos\phi'(z-z')\right] \\
&\quad \frac{1+jkR}{R^3}e^{-jkR}d\phi' ds' \\
&= +\frac{1}{4\pi}\sum_k A_{nk}^{t'}\int_{\Delta_k'} T_{k'}(s')\int_0^{2\pi}\left\{\cos n\phi'\left[\cos\theta'\left(\rho\cos\phi-\rho'\cos\phi'\right)-\sin\theta'\cos\phi'(z-z')\right]\right. \\
&\quad \left.+j\sin n\phi'\left[\cos\theta'\left(\rho\cos\phi-\rho'\cos\phi'\right)-\sin\theta'\cos\phi'(z-z')\right]\right\}\frac{1+jkR}{R^3}e^{-jkR}d\phi' ds' \\
&= +\frac{1}{4\pi}\sum_k A_{nk}^{t'}\int_{\Delta_k'} T_{k'}(s')\int_0^{2\pi}\left\{\rho\cos\phi\cos\theta'\cos n\phi' - \rho'\cos\theta'\cos n\phi'\cos\phi'\right. \\
&\quad -\sin\theta'(z-z')\cos\phi'\cos n\phi' + j\rho\cos\phi\cos\theta'\sin n\phi' \\
&\quad \left.-j\rho'\cos\theta'\sin n\phi'\cos\phi' - j\sin\theta'(z-z')\cos\phi'\sin n\phi'\right\}\frac{1+jkR}{R^3}e^{-jkR}d\phi' ds'
\end{aligned}$$

Invoking even and odd symmetry and setting the observation point at $\phi=0$ to eliminate terms we obtain

$$\begin{aligned}
H_y^{t'} &= +\frac{1}{2\pi}\sum_k A_{nk}^{t'}\int_{\Delta_k'} T_{k'}(s')\int_0^{\pi}\left\{\rho\cos\theta'\cos n\phi' - \rho'\cos\theta'\cos n\phi'\cos\phi' - \sin\theta'(z-z')\cos\phi'\cos n\phi'\right\} \\
&\quad \frac{1+jkR}{R^3}e^{-jkR}d\phi' ds'
\end{aligned}$$

$$\begin{aligned}
H_y^{t'} &= -\frac{1}{2\pi}\sum_k A_{nk}^{t'}\int_{\Delta_k'} T_{k'}(s')\left[\left[\sin\theta'(z-z')+\rho'\cos\theta'\right]\int_0^{\pi}\cos\phi'\cos n\phi'\frac{1+jkR}{R^3}e^{-jkR}d\phi'\right. \\
&\quad \left.-\rho\cos\theta'\int_0^{\pi}\cos n\phi'\frac{1+jkR}{R^3}e^{-jkR}d\phi'\right]
\end{aligned} \tag{730}$$

For the $\widehat{\phi}'$ component we obtain

$$H_y^{\phi'} = \int_{s'}\sum_k A_{nk}^{\phi}\frac{T_{k'}(s')}{\rho(s')}e^{+jn\phi'}\widehat{y}\cdot\widehat{\phi}'\times\nabla\frac{e^{-jkR}}{4\pi R}da'$$

$$H_y^{\phi'} = -\frac{1}{4\pi}\sum_k A_{nk}^{\phi}\int_{\Delta_k'} T_{k'}(s')\int_0^{2\pi}\left[(z-z')\sin\phi'\cos n\phi' + j(z-z')\sin\phi'\sin n\phi'\right]\frac{1+jkR}{R^3}e^{-jkR}d\phi' ds'$$

Invoking even and odd symmetry and setting the observation point at $\phi=0$ to eliminate terms we obtain

$$H_y^{\phi'} = -\frac{j}{2\pi}\sum_k A_{nk}^{\phi}\int_{\Delta_k'} T_{k'}(s')\left[(z-z')\int_0^{\pi}\sin\phi'\sin n\phi'\frac{1+jkR}{R^3}e^{-jkR}d\phi' ds'\right] \tag{731}$$

Alternative Derivation of $\overrightarrow{H}^s(r)$ Since this procedure proved successful for E^s, we do the same thing with $\overline{\overline{K}}(\cdot)$ for H^s.

$$\overrightarrow{H}^s(\rho_p, 0, z_p) = -\frac{1}{2\pi\eta_0}\overline{\overline{K}}\left(\overrightarrow{J}\right) \tag{732}$$

where [21]

$$K_{kl}^{tt} = -j\eta_0 \left\{ \sin\nu_p \sin\nu_q (z_p - z_q) + \rho_q \sin\nu_p \cos\nu_q - \rho_p \cos\nu_p \sin\nu_q \right\} T_p T_q H_{sn}$$

$$K_{kl}^{\phi t} = \eta_0 \left\{ \left[\sin\nu_q (z_p - z_q) + \rho_q \cos\nu_q \right] H_{cn} - \rho_p \cos\nu_q H_n \right\} T_p T_q$$

$$K_{kl}^{t\phi} = -\eta_0 \left\{ \left[-\sin\nu_p (z_p - z_q) + \rho_p \cos\nu_p \right] H_{cn} - \rho_q \cos\nu_p H_n \right\} T_p T_q$$

$$K_{kl}^{\phi\phi} = j\eta_0 (z_p - z_q) T_p T_q H_{sn}$$

and where

$$H_n = \Delta t_q \int_0^{\pi} \cos n\phi \frac{(1+jkR)}{R^3} e^{-jkR}$$

$$H_{cn} = \Delta t_q \int_0^{\pi} \cos\phi \cos n\phi \frac{(1+jkR)}{R^3} e^{-jkR}$$

$$H_{sn} = \Delta t_q \int_0^{\pi} \sin\phi \sin n\phi \frac{(1+jkR)}{R^3} e^{-jkR}$$

Note the division by η_0 in Equation 732. This arises because both impedance matrix and right hand side are multiplied by η_0 in order for the magnetic field integral equation to be similar in scale to the electric field integral equation [21]

To look at the $\widehat{x}$ component we set $\nu_p = \pi/2$.

$$K_{kl}^{xt} = -j\eta_0 \left\{ \sin\nu_q (z_p - z_q) + \rho_q \cos\nu_q \right\} T_q^t H_{sn}$$

$$K_{kl}^{x\phi} = \eta_0 \left\{ \left[(z_p - z_q) \right] \right\} T_q^{\phi} H_{cn}$$

$$H_x^s = \frac{j}{2\pi} \left\{ \sin\nu_q (z_p - z_q) + \rho_q \cos\nu_q \right\} T_q^t H_{sn} - \frac{1}{2\pi} (z_p - z_q) T_q^{\phi} H_{cn}$$

The terms in this expression are identical to a combination of Equations 725 and 727.

To look at the $\widehat{z}$ component we set $\nu_p = 0$

$$K_{kl}^{zt} = j\eta_0 \rho_p \sin\nu_q T_q^t H_{sn}$$

$$K_{kl}^{z\phi} = -\eta_0 \left\{ \rho_p H_{cn} - \rho_q H_n \right\} T_q^{\phi}$$

$$H_z^s = -\frac{j}{2\pi} \rho_p \sin\nu_q T_q^t H_{sn} + \frac{1}{2\pi} \left\{ \rho_p H_{cn} - \rho_q H_n \right\} T_q^{\phi}$$

The terms in this expression are identical to a combination of Equations 728 and 729.

To look at the $\widehat{y}$ component we have contributions from $K^{\phi t}$ and $K^{\phi\phi}$.

$$K_{kl}^{\phi t} = \eta_0 \left\{ \left[\sin\nu_q (z_p - z_q) + \rho_q \cos\nu_q \right] H_{cn} - \rho_p \cos\nu_q H_n \right\} T_q^t$$

$$K_{kl}^{\phi\phi} = j\eta_0 (z_p - z_q) T_q^{\phi} H_{sn}$$

$$H_y^s = -\frac{1}{2\pi} \left\{ \left[\sin\nu_q (z_p - z_q) + \rho_q \cos\nu_q \right] H_{cn} - \rho_p \cos\nu_q H_n \right\} T_q^t - \frac{j}{2\pi} (z_p - z_q) T_q^{\phi} H_{sn}$$

The terms in this expression are identical to a combination of Equations 730 and 731.

The above equations for $\overrightarrow{H}^s$ are implemented in the **near_field_module** in the subroutine **kop_nf**. In the **near_field_module** the variable **field_scatt** is filled in the subroutine **nfield_fill** which sets the observation point r_{obs} and z_{obs} and calls **get_nearfield** which returns all the field components. In **get_nearfield** calls are made in sequence to **lop_nf**, which calculates the components contributing to $\overrightarrow{E}^s$ and to **assemble_nf**, which multiplies the components by the eigenvector, to **kop_nf** which calculates the components contributing to $\overrightarrow{H}^s$ and again to **assemble_nf**. The field is written in a standardized format for plotting by **write_near_field**.

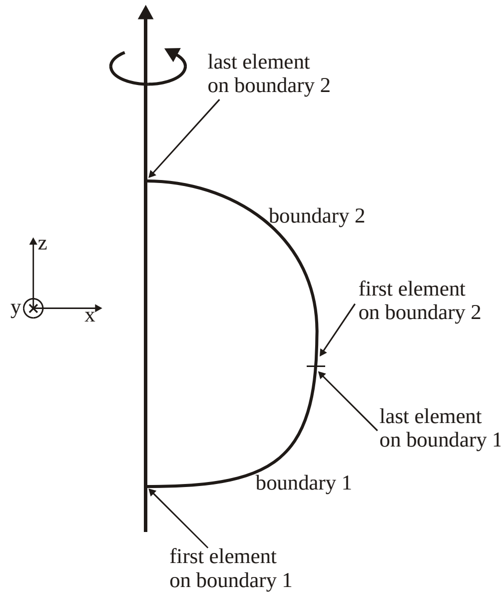

Figure 46. BOR geometry

7.4 Running CICERO_EV

In this section we describe the steps needed to run **CICERO_EV** and get out a set of files containing the electric fields in a cross section of the BOR so that we can get statistics on them. These steps are complicated in part because we have to generate an input deck for **CICERO**. Although we have software that does this from a mesh generated by **PATRAN**, we don't have software for a mesh generated by **I-deas**, so we have to kluge an input deck together.

Step 1. Use **I-deas** to generate a boundary of the BOR in the xz plane as shown in Figure 46. Other than the exception discussed in Step 3, **I-deas** orders the element and node ids on a boundary so that they increase monotonically as we progress along the boundary. This feature is a requirement for the **CICERO** input deck. Also, the number of points on the boundary must be odd.

Step 2. Convert the **I-deas *.unv** file to a ***.jfg** file using **i2j** [23].

Step 3. **I-deas** first writes out the end node ids of a boundary and then writes out the intervening ids. We use a text editor to edit the ***.jfg** file and put the nodes in the order needed by **CICERO**. Using the nodes on the two boundaries shown in Figure 47 as an example , we change the node order in the ***.jfg** file from column 1 to the order shown in column 2.

original order	modified order
id1	id1
id2	id3
id3	id4
id4	id2
id5	id5

We also change very small numbers $\left(x = 10^{-17}\right)$ to $(x = 0.0)$. Use the display of nodes in **I-deas** to help with editing the ***.jfg** file.

Step 4. Run **convert_jfg.f** which converts node data given in the **input.jfg** file to a format needed

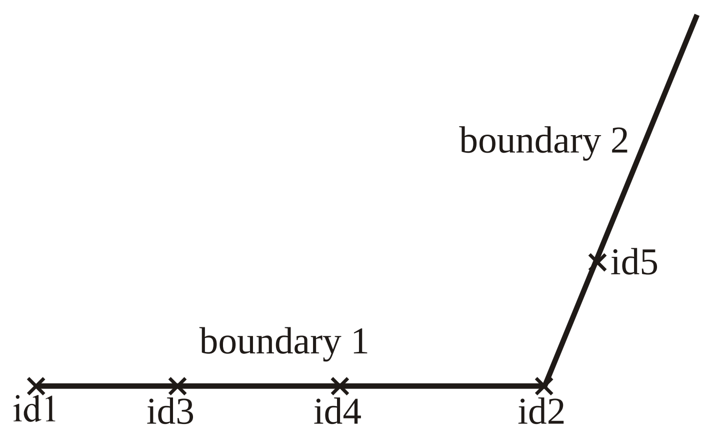

Figure 47. I-deas node numbering scheme

by the **CICERO_EV** input deck **input**. The formatted file is in **input.pnts**.

Step 5. Edit **input** by creating the first six lines in accordance with the description in Section 7.1 above and then inserting **input.pnts**.

Step 6. To run a frequency sweep to find the cavity resonances by the behavior of the condition number. Modify **CICERO_EV.f** by setting **calculate_nf = .FALSE**. Run **generate_frequency_lst.f**, which outputs the file **cicero_freq.lst**. Run **CICERO_EV** with **input** and **cicero_freq.lst** to obtain the output file **condition_number.out.**

Step 7. Run the code **general_wg.f** which takes the file **condition_number.out** and outputs the file **frequency.lst,** which gives the frequencies where a peak in the condition number occurs.

Step 8. Use **I-deas** to generate a visualization grid. Output the grid as a ***.unv** file. Convert the grid to a ***.jfg** file using **i2j**. Modify **cicero_ev.f** by setting **calculate_nf=.TRUE**. Edit **frequency.lst** to have the correct format (**frequency.lst** groups the frequencies in groups of twenties) and copy to **cicero_freq.lst**. Run **CICERO_EV** again to get individual files containing the electric field on the grid for each frequency in **cicero_freq.lst**. These files are labeled with a base name given by the user and an extension ***.nfld??**. Where ?? designates a number long enough to accommodate all the frequencies in **cicero_freq.lst**.

Step 9. Change the names of the electric field files using scripts. For example, with a base name of **axbt** so that the near field files generated are: **axbt.nfld00, axbt.nfld01, ... axbt.nfld99** , we change the file names to ones that have the frequency in the base name and an extension of ***.nfld0**. The previous example gets changed to **axbt_1.4GHz.nfld0, axbt_1.5GHz.nfld0,...axbt_2.0GHz.nfld0**. Make a list of the base name of all ***.nfl0** files that we are going to get statistics on in **nearfield.lst**.

Step 10. Run **abt_m0.f** Inputs are all the ***.nfld0** files and the **nearfield.lst** from Step 9 and the file **kp.lst**, which contains a list of the resonant wavenumbers for a bouncing ball mode. The codes **bowtie_kp_even.f** and **bowtie_kp_odd.f** generate **kp.lst**. Output of **abt_m0.f** is **bin_stat_m0.txt**, which provides data for a histogram plot and **ef_scatter_m0.txt**, which is the un-binned histogram data.

7.5 Analytical Problem

In this section we will analytically solve for the modal E and H fields inside a PEC cylindrical cavity having a radius $\rho = a$ and length $z = d$. We will also calculate the wall current $\overrightarrow{J}$ and a quantity related to the wall charge: $\nabla \cdot \overrightarrow{J}$ analytically. The numerical calculations using **CICERO_EV** for a cylindrical cavity will be compared to these analytic solutions to give us confidence that the numerical calculations are correct.

7.5.1 TM Modes

For modes TM to $\widehat{z}$ we have the magnetic potential [24]

$$\overrightarrow{A} = \widehat{z}\psi$$

For the cylindrical cavity ψ must have the form

$$\psi^{TM}_{npq} = J_n\left(\frac{x_{np}\rho}{a}\right)e^{jn\phi}\cos\left(\frac{q\pi}{d}z\right)$$

where $n = 0, 1, 2, ...; p = 1, 2, 3, ...;$ and $q = 0, 1, 2, ...;$ J_n is the Bessel function of the first kind, order n. x_{np}

is the p$^{\text{th}}$ zero of J_n. There is a degeneracy for $n \neq 0$. The E fields due to this potential are

$$\begin{aligned} E_\rho &= \frac{1}{j\omega\varepsilon}\frac{\partial^2 \psi_{npq}^{TM}}{\partial\rho\partial z} \\ &= -\frac{1}{j\omega\varepsilon}\frac{x_{np}}{a}\frac{q\pi}{d}J_n'\left(\frac{x_{np}\rho}{a}\right)e^{jn\phi}\sin\left(\frac{q\pi}{d}z\right) \end{aligned} \tag{733}$$

$$\begin{aligned} E_\phi &= \frac{1}{j\omega\varepsilon\rho}\frac{\partial^2 \psi_{npq}^{TM}}{\partial\phi\partial z} \\ &= \frac{jn}{j\omega\varepsilon\rho}\frac{q\pi}{d}J_n\left(\frac{x_{np}\rho}{a}\right)e^{jn\phi}\sin\left(\frac{q\pi}{d}z\right) \end{aligned} \tag{734}$$

$$\begin{aligned} E_z &= \frac{1}{j\omega\varepsilon}\left(\frac{\partial^2}{\partial z^2}+k^2\right)\psi_{npq}^{TM} \\ &= \frac{1}{j\omega\varepsilon}\left[k^2-\left(\frac{q\pi}{d}\right)^2\right]J_n\left(\frac{x_{np}\rho}{a}\right)e^{jn\phi}\cos\left(\frac{q\pi}{d}z\right) \end{aligned} \tag{735}$$

and the H fields are

$$\begin{aligned} H_\rho &= \frac{1}{\rho}\frac{\partial\psi_{npq}^{TM}}{\partial\phi} \\ &= \frac{jn}{\rho}J_n\left(\frac{x_{np}\rho}{a}\right)e^{jn\phi}\cos\left(\frac{q\pi}{d}z\right) \end{aligned}$$

$$\begin{aligned} H_\phi &= -\frac{\partial\psi_{npq}^{TM}}{\partial\rho} \\ &= -\frac{x_{np}}{a}J_n'\left(\frac{x_{np}\rho}{a}\right)e^{jn\phi}\cos\left(\frac{q\pi}{d}z\right) \end{aligned}$$

$$H_z = 0$$

In cylindrical coordinates

$$\nabla\cdot J = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho J_\rho)+\frac{1}{\rho}\frac{\partial J_\phi}{\partial\phi}+\frac{\partial J_z}{\partial z}$$

On the bottom of the cylinder ($z=0, \widehat{n}=\widehat{z}$)

$$\begin{aligned} \overrightarrow{J} &= \widehat{z}\times\widehat{\phi}H_\phi+\widehat{z}\times\widehat{\rho}H_\rho \\ &= -\widehat{\rho}\frac{x_{np}}{a}J_n'\left(\frac{x_{np}\rho}{a}\right)e^{jn\phi}+\widehat{\phi}\frac{jn}{\rho}J_n\left(\frac{x_{np}\rho}{a}\right)e^{jn\phi} \end{aligned}$$

$$\begin{aligned} \nabla\cdot J &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(-\rho\frac{x_{np}}{a}J_n'\left(\frac{x_{np}\rho}{a}\right)e^{jn\phi}\right)+\frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\frac{jn}{\rho}J_n\left(\frac{x_{np}\rho}{a}\right)e^{jn\phi}\right) \\ &= -\frac{1}{\rho}\frac{x_{np}}{a}e^{jn\phi}\frac{\partial}{\partial\rho}\left(\rho J_n'\left(\frac{x_{np}\rho}{a}\right)\right)+\frac{1}{\rho}\frac{jn}{\rho}J_n\left(\frac{x_{np}\rho}{a}\right)\frac{\partial}{\partial\phi}\left(e^{jn\phi}\right) \\ &= -\frac{1}{\rho}\frac{x_{np}}{a}e^{jn\phi}\left(\rho\frac{x_{np}}{a}J_n''\left(\frac{x_{np}\rho}{a}\right)+J_n'\left(\frac{x_{np}\rho}{a}\right)\right)+\frac{1}{\rho}\frac{jn}{\rho}J_n\left(\frac{x_{np}\rho}{a}\right)jne^{jn\phi} \\ &= -\frac{1}{\rho}\left[\frac{x_{np}}{a}\left(\rho\frac{x_{np}}{a}J_n''\left(\frac{x_{np}\rho}{a}\right)+J_n'\left(\frac{x_{np}\rho}{a}\right)\right)+\frac{n^2}{\rho}J_n\left(\frac{x_{np}\rho}{a}\right)\right]e^{jn\phi} \end{aligned}$$

On the side of the cylinder ($\rho=a, \widehat{n}=-\widehat{\rho}$)

$$\begin{aligned}
\overrightarrow{J} &= -\widehat{\rho}\times\widehat{\phi}H_\phi-\widehat{\rho}\times\widehat{z}H_z\\
&= -\widehat{z}\frac{x_{np}}{a}J_n'\left(x_{np}\right)e^{jn\phi}\cos\left(\frac{q\pi}{d}z\right)\\
\nabla\cdot J &= \frac{\partial}{\partial z}\left[-\frac{x_{np}}{a}J_n'\left(x_{np}\right)e^{jn\phi}\cos\left(\frac{q\pi}{d}z\right)\right]\\
&= -\frac{x_{np}}{a}J_n'\left(x_{np}\right)e^{jn\phi}\frac{\partial}{\partial z}\left[\cos\left(\frac{q\pi}{d}z\right)\right]\\
&= \frac{x_{np}}{a}J_n'\left(x_{np}\right)\frac{q\pi}{d}\sin\left(\frac{q\pi}{d}z\right)e^{jn\phi}
\end{aligned}$$

On the top of the cylinder $(z=d,\widehat{n}=-\widehat{z})$

$$\begin{aligned}
\overrightarrow{J} &= -\widehat{z}\times\widehat{\phi}H_\phi-\widehat{z}\times\widehat{\rho}H_\rho\\
&= +\widehat{\rho}\frac{x_{np}}{a}J_n'\left(\frac{x_{np}\rho}{a}\right)(-1)^q e^{jn\phi}-\widehat{\phi}\frac{jn}{\rho}J_n\left(\frac{x_{np}\rho}{a}\right)(-1)^q e^{jn\phi}
\end{aligned}$$

$$\begin{aligned}
\nabla\cdot J &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{x_{np}}{a}J_n'\left(\frac{x_{np}\rho}{a}\right)(-1)^q e^{jn\phi}\right)-\frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\frac{jn}{\rho}J_n\left(\frac{x_{np}\rho}{a}\right)(-1)^q e^{jn\phi}\right)\\
&= \frac{1}{\rho}(-1)^q e^{jn\phi}\frac{x_{np}}{a}\frac{\partial}{\partial\rho}\left(\rho J_n'\left(\frac{x_{np}\rho}{a}\right)\right)-\frac{1}{\rho}(-1)^q\frac{jn}{\rho}J_n\left(\frac{x_{np}\rho}{a}\right)\frac{\partial}{\partial\phi}\left(e^{jn\phi}\right)\\
&= \frac{1}{\rho}(-1)^q e^{jn\phi}\frac{x_{np}}{a}\frac{\partial}{\partial\rho}\left(\rho J_n'\left(\frac{x_{np}\rho}{a}\right)\right)-\frac{1}{\rho}(-1)^q\frac{jn}{\rho}J_n\left(\frac{x_{np}\rho}{a}\right)\frac{\partial}{\partial\phi}\left(e^{jn\phi}\right)\\
&= \frac{1}{\rho}(-1)^q e^{jn\phi}\frac{x_{np}}{a}\left(\rho\frac{x_{np}}{a}J_n''\left(\frac{x_{np}\rho}{a}\right)+J_n'\left(\frac{x_{np}\rho}{a}\right)\right)-\frac{1}{\rho}(-1)^q\frac{jn}{\rho}J_n\left(\frac{x_{np}\rho}{a}\right)jne^{jn\phi}\\
&= \frac{(-1)^q}{\rho}\left[\frac{x_{np}}{a}\left(\rho\frac{x_{np}}{a}J_n''\left(\frac{x_{np}\rho}{a}\right)+J_n'\left(\frac{x_{np}\rho}{a}\right)\right)+\frac{n^2}{\rho}J_n\left(\frac{x_{np}\rho}{a}\right)\right]e^{jn\phi}
\end{aligned}$$

The derivatives of the Bessel function are for the first derivative

$$\begin{aligned}
J_0'(x) &= -J_1(x)\\
J_n'(x) &= -J_{n+1}(x)+\frac{n}{x}J_n(x)
\end{aligned}$$

and for the second derivative

$$\begin{aligned}
J_n''(x) &= -J_{n+1}'(x)+\left(\frac{n}{x}J_n(x)\right)'\\
&= -\left(-J_{n+2}(x)+\frac{n+1}{x}J_{n+1}(x)\right)+\left(\frac{n}{x}J_n'(x)-\frac{n}{x^2}J_n(x)\right)\\
&= J_{n+2}(x)-\frac{n+1}{x}J_{n+1}(x)+\frac{n}{x}\left(-J_{n+1}(x)+\frac{n}{x}J_n(x)\right)-\frac{n}{x^2}J_n(x)\\
&= J_{n+2}(x)-\frac{n+1}{x}J_{n+1}(x)-\frac{n}{x}J_{n+1}(x)+\frac{n^2}{x^2}J_n(x)-\frac{n}{x^2}J_n(x)\\
&= J_{n+2}(x)-\left(\frac{2n+1}{x}\right)J_{n+1}(x)+\left(\frac{n^2-n}{x^2}\right)J_n(x)
\end{aligned}$$

The resonant frequencies of the TM_{npq} modes are

$$\left(f_r\right)_{npq}^{TM}=\frac{1}{2\pi a\sqrt{\varepsilon\mu}}\sqrt{x_{np}^2+\left(\frac{q\pi a}{d}\right)^2}$$

7.5.2 TE Modes

For modes TE to $\widehat{z}$ we have the electric potential [24]

$$\overrightarrow{F} = \widehat{z}\psi$$

For the cylindrical cavity ψ must have the form

$$\psi_{npq}^{TE} = J_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi}\sin\left(\frac{q\pi}{d}z\right)$$

where $n = 0, 1, 2, ...; p = 1, 2, 3, ...;$ and $q = 1, 2, ...;$ and x'_{np} is the p$^{\text{th}}$ zero of J'_n. There is a degeneracy for $n \neq 0$. The E fields due to this potential are

$$\begin{aligned} E_\rho &= -\frac{1}{\rho}\frac{\partial\psi_{npq}^{TE}}{\partial\phi} \\ &= -\frac{jn}{\rho}J_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi}\sin\left(\frac{q\pi}{d}z\right) \end{aligned} \tag{736}$$

$$\begin{aligned} E_\phi &= \frac{\partial\psi_{npq}^{TE}}{\partial\rho} \\ &= \frac{x'_{np}}{a}J'_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi}\sin\left(\frac{q\pi}{d}z\right) \end{aligned} \tag{737}$$

$$E_z = 0 \tag{738}$$

$$\begin{aligned} H_\rho &= \frac{1}{j\omega\mu}\frac{\partial^2\psi_{npq}^{TE}}{\partial\rho\partial z} \\ &= \frac{1}{j\omega\mu}\frac{x'_{np}}{a}\frac{q\pi}{d}J'_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi}\cos\left(\frac{q\pi}{d}z\right) \end{aligned}$$

$$\begin{aligned} H_\phi &= \frac{1}{j\omega\mu\rho}\frac{\partial^2\psi_{npq}^{TE}}{\partial\phi\partial z} \\ &= \frac{jn}{j\omega\mu\rho}\frac{q\pi}{d}J_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi}\cos\left(\frac{q\pi}{d}z\right) \end{aligned}$$

$$\begin{aligned} H_z &= \frac{1}{j\omega\mu}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\psi_{npq}^{TE} \\ &= \frac{1}{j\omega\mu}\left[k^2 - \left(\frac{q\pi}{d}\right)^2\right]J_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi}\sin\left(\frac{q\pi}{d}z\right) \end{aligned}$$

On the bottom of the cylinder $(z = 0, \widehat{n} = \widehat{z})$

$$\begin{aligned} \overrightarrow{J} &= \widehat{z}\times\widehat{\phi}H_\phi + \widehat{z}\times\widehat{\rho}H_\rho \\ &= -\widehat{\rho}\frac{jn}{j\omega\mu\rho}\frac{q\pi}{d}J_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi} + \widehat{\phi}\frac{1}{j\omega\mu}\frac{x'_{np}}{a}\frac{q\pi}{d}J'_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi} \\ &= -\widehat{\rho}\frac{nq\pi}{\omega\mu\rho d}J_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi} + \widehat{\phi}\frac{q\pi}{j\omega\mu d}\frac{x'_{np}}{a}J'_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi} \end{aligned}$$

$$
\begin{aligned}
\nabla \cdot J &= -\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{nq\pi}{\omega\mu\rho d}J_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi}\right)+\frac{1}{\rho}\frac{\partial}{\partial \phi}\left(\frac{q\pi}{j\omega\mu d}\frac{x'_{np}}{a}J'_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi}\right)\\
&= -\frac{1}{\rho}\frac{nq\pi}{\omega\mu d}\frac{x'_{np}}{a}J'_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi}+\frac{1}{\rho}\frac{jnq\pi}{j\omega\mu d}\frac{x'_{np}}{a}J'_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi}\\
&= -\frac{1}{\rho}\frac{nq\pi}{\omega\mu d}\frac{x'_{np}}{a}J'_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi}+\frac{1}{\rho}\frac{nq\pi}{\omega\mu d}\frac{x'_{np}}{a}J'_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi}\\
&= 0
\end{aligned}
$$

On the side of the cylinder ($\rho = a, \widehat{n} = -\widehat{\rho}$)

$$
\begin{aligned}
\overrightarrow{J} &= -\widehat{\rho}\times\widehat{\phi}H_\phi-\widehat{\rho}\times\widehat{z}H_z\\
&= -\widehat{z}\frac{jn}{j\omega\mu\rho}\frac{q\pi}{d}J_n\left(\frac{x'_{np}a}{a}\right)e^{jn\phi}\cos\left(\frac{q\pi}{d}z\right)+\widehat{\phi}\frac{1}{j\omega\mu}\left[k^2-\left(\frac{q\pi}{d}\right)^2\right]J_n\left(\frac{x'_{np}a}{a}\right)e^{jn\phi}\sin\left(\frac{q\pi}{d}z\right)\\
&= -\widehat{z}\frac{n}{\omega\mu\rho}\frac{q\pi}{d}J_n\left(x'_{np}\right)e^{jn\phi}\cos\left(\frac{q\pi}{d}z\right)+\widehat{\phi}\frac{1}{j\omega\mu}\left[k^2-\left(\frac{q\pi}{d}\right)^2\right]J_n\left(x'_{np}\right)e^{jn\phi}\sin\left(\frac{q\pi}{d}z\right)
\end{aligned}
$$

$$
\begin{aligned}
\nabla \cdot J &= \frac{1}{\rho}\frac{\partial J_\phi}{\partial \phi}+\frac{\partial J_z}{\partial z}\\
&= \frac{1}{a}\frac{\partial}{\partial\phi}\left[\frac{1}{j\omega\mu}\left[k^2-\left(\frac{q\pi}{d}\right)^2\right]J_n\left(x'_{np}\right)e^{jn\phi}\sin\left(\frac{q\pi}{d}z\right)\right]-\frac{\partial}{\partial z}\left[\frac{n}{\omega\mu a}\frac{q\pi}{d}J_n\left(x'_{np}\right)e^{jn\phi}\cos\left(\frac{q\pi}{d}z\right)\right]\\
&= \frac{n}{\omega\mu a}\left[k^2-\left(\frac{q\pi}{d}\right)^2\right]J_n\left(x'_{np}\right)e^{jn\phi}\sin\left(\frac{q\pi}{d}z\right)+\left[\left(\frac{q\pi}{d}\right)^2\frac{n}{\omega\mu a}J_n\left(x'_{np}\right)e^{jn\phi}\sin\left(\frac{q\pi}{d}z\right)\right]\\
&= \frac{nk^2}{\omega\mu a}J_n\left(x'_{np}\right)e^{jn\phi}\sin\left(\frac{q\pi}{d}z\right)\\
&= \frac{jnk^2}{j\omega\mu a}J_n\left(x'_{np}\right)e^{jn\phi}\sin\left(\frac{q\pi}{d}z\right)
\end{aligned}
$$

On the top of the cylinder ($z = d, \widehat{n} = -\widehat{z}$)

$$
\begin{aligned}
\overrightarrow{J} &= -\widehat{z}\times\widehat{\phi}H_\phi-\widehat{z}\times\widehat{\rho}H_\rho\\
&= \widehat{\rho}\frac{jn}{j\omega\mu\rho}\frac{q\pi}{d}J_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi}-\widehat{\phi}\frac{1}{j\omega\mu}\frac{x'_{np}}{a}\frac{q\pi}{d}J'_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi}\\
&= \widehat{\rho}\frac{nq\pi}{\omega\mu\rho d}J_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi}-\widehat{\phi}\frac{q\pi}{j\omega\mu d}\frac{x'_{np}}{a}J'_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi}
\end{aligned}
$$

$$
\begin{aligned}
\nabla \cdot J &= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{nq\pi}{\omega\mu\rho d}J_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi}\right)-\frac{1}{\rho}\frac{\partial}{\partial \phi}\left(\frac{q\pi}{j\omega\mu d}\frac{x'_{np}}{a}J'_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi}\right)\\
&= \frac{1}{\rho}\frac{nq\pi}{\omega\mu d}\frac{x'_{np}}{a}J'_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi}-\frac{1}{\rho}\frac{jnq\pi}{j\omega\mu d}\frac{x'_{np}}{a}J'_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi}\\
&= \frac{1}{\rho}\frac{nq\pi}{\omega\mu d}\frac{x'_{np}}{a}J'_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi}-\frac{1}{\rho}\frac{nq\pi}{\omega\mu d}\frac{x'_{np}}{a}J'_n\left(\frac{x'_{np}\rho}{a}\right)e^{jn\phi}\\
&= 0
\end{aligned}
$$

The resonant frequencies of the TE_{npq} modes are

$$(f_r)_{npq}^{TE} = \frac{1}{2\pi a\sqrt{\varepsilon\mu}}\sqrt{x_{np}'^2 + \left(\frac{q\pi a}{d}\right)^2}$$

7.6 Verification of n=0 Modes

If we numerically solve for the modes in PEC cylindrical cavity where the radius $a = 1$ m, and the length $d = 1$ m using **CICERO_EV** and compare to the analytic solution from the preceding section, we obtain the following results for the resonant frequencies when $n = 0$.

CICERO_EV (MHz)	Analytic (MHz)	Mode
115.4	114.75	TM_{010}
190.6	188.78	TM_{011}
238.7	236.43	TE_{011}
266.4	263.38	TM_{020}
306.8	303.04	TM_{021}
326.1	321.00	TM_{012}

CICERO_EV is a method-of-moments (MOM) code which has two parts. We first solve for the current on the wall of the cavity; then in a separate step we solve for the electric field throughout the interior of the cavity due to the wall currents. This naturally divides the verification procedure into two parts: 1) verification of the current calculation and 2) after demonstrating that the current is correct, verifying the field calculation.

7.6.1 TM_{010} Mode

This is an $n = 0$ mode that **CICERO_EV** finds at 115.4 MHz (114.75 MHz analytically). We will first do a comparison of the wall current calculated numerically and analytically. For this mode, only J_t exists on the wall of the cavity. This is compared in Figure 48 which plots J_t along the boundary s from $\rho = 0$, $z = -0.5$ to $\rho = 1$, $z = -0.5$ to $\rho = 1$, $z = 0.5$, to $\rho = 0$, $z = 0.5$. This boundary description will be used for all subsequent plots of $\overrightarrow{J}(s)$ in this report. Since we are dealing with comparisons of eigenvectors, which can differ from each other by a complex constant, we scaled all analytical quantities $(J, E$ and $H)$ associated with this mode by the factor $(-0.2672, 0.0)$. This factor was obtained by requiring the analytic and numerical current to match at $s = 1.5$ m. The comparison of current is good except for deviations adjacent to the axis, which comes from representing J/ρ as a triangular basis function in **CICERO_EV**, and at the corners ($s = 1$ and $s = 2$ m), which we believe is due to discretization.

Equations 733-735 show that for this mode $E_\rho = E_\phi = 0$ and

$$E_z = -j\omega\mu J_0(x_{01}\rho)$$

Figure 49 shows the E_ρ component of the field as calculated by **CICERO_EV**. The field is small throughout the space except at the ends of the cylinder on the axis where there is a hot spot that doesn't obey the boundary condition and is asymmetrical in z. This is obviously an error possibly caused by the afore-mentioned basis representation at the axis.

Figure 50 shows proper behavior for the E_z field component except near the axis ends where, again there is an un-physical hot spot – not as obvious as with E_ρ because it is being masked by the correct on-axis maximum of E_z. Figure 51 is a more detailed comparison of $E_z(\rho)$ at $z = 0$ as calculated by **CICERO_EV** to the scaled analytic result. The agreement is good even as the observation point approaches the boundary.

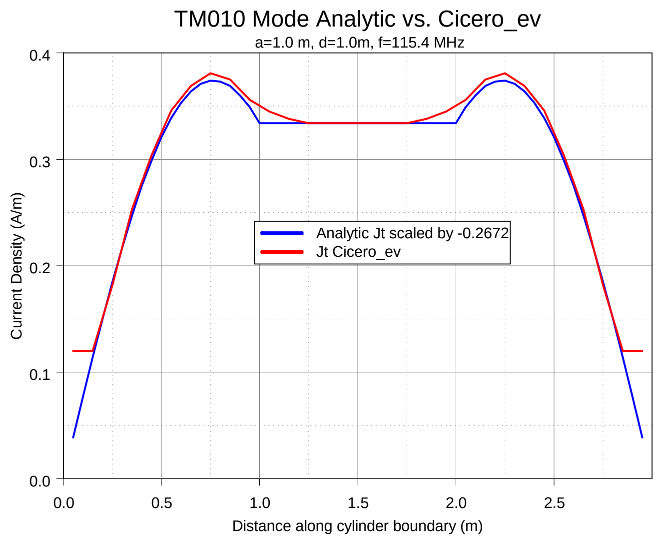

Figure 48. TM_{010} mode current on boundary. Only J_t is non-zero

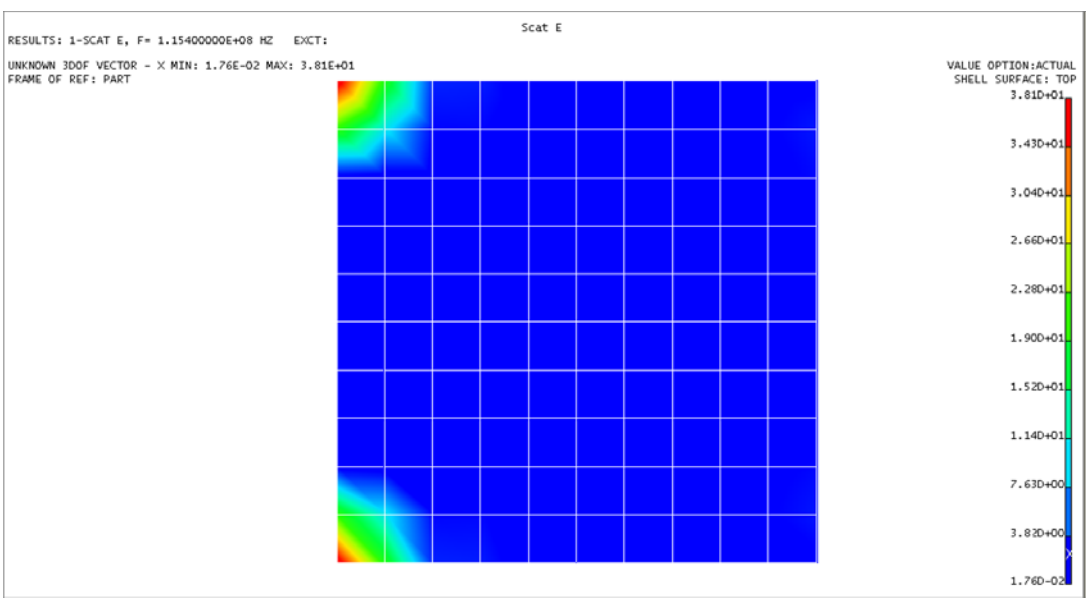

Figure 49. TM_{010} mode E_ρ

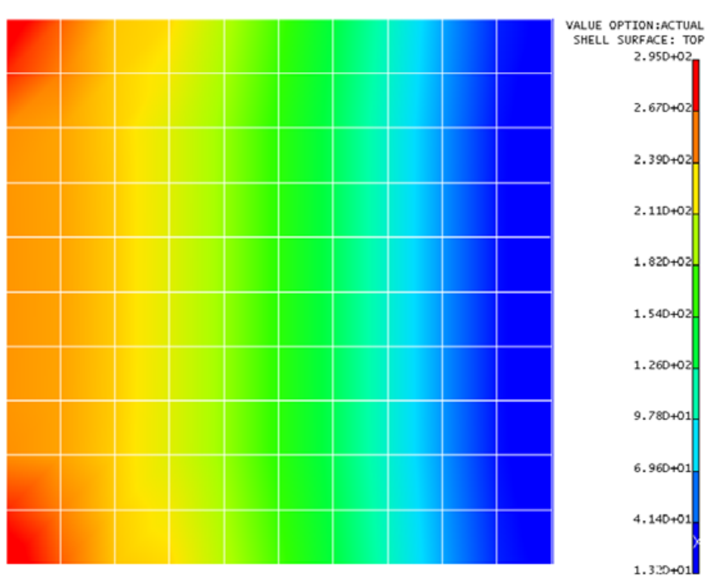

Figure 50. TM_{010} mode E_z

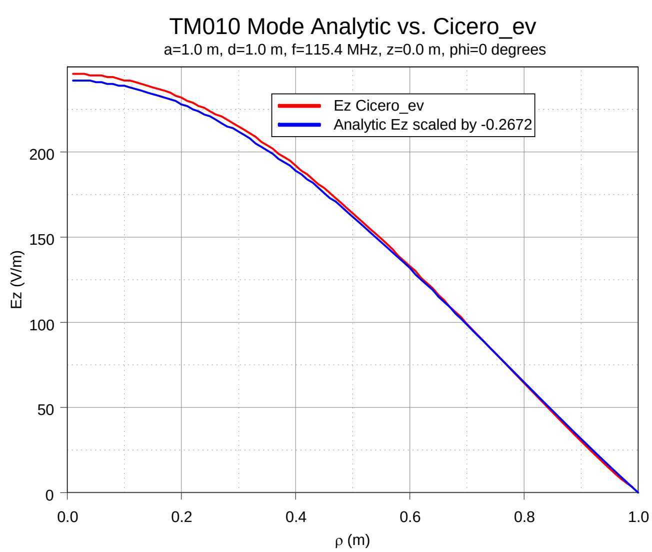

Figure 51. Comparison of $E_z\,(\rho)$ at $z = 0.0$ m, $\phi = 0^0$

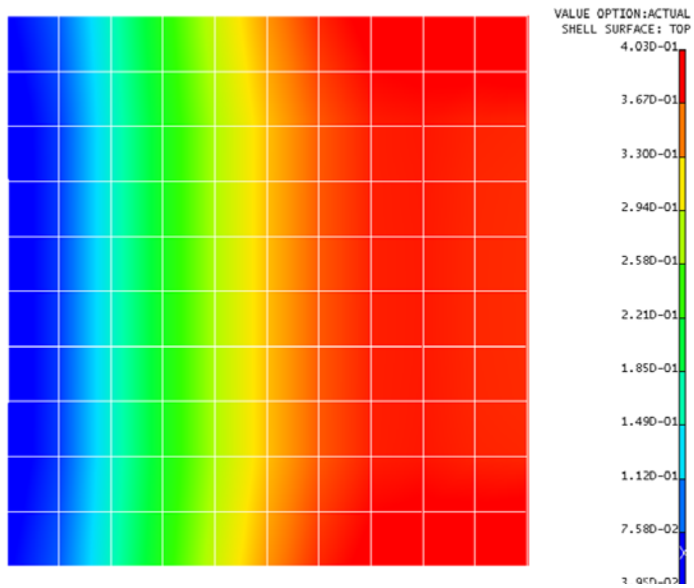

Figure 52. TM_{010} mode H_ϕ

The only component of H field in the cavity for this mode is H_ϕ which is plotted unscaled in Figure 52. Again, the form looks good except for a slight non-uniformity in z. Figure 53 compares $H_\phi(\rho)$ at $z = 0$ calculated by **CICERO_EV** to the scaled analytic result. The agreement is good until about one-half a cell away from the boundary (the boundary elements in this problem are 0.1 m long), then H_ϕ calculated by **CICERO_EV** deviates from the analytic results and doesn't match the wall current calculated by **CICERO_EV** $\left(\widehat{n} \times \overrightarrow{H} = \overrightarrow{J}\right)$ This behavior indicates that this discrepancy is due to the singularity contribution from the wall element that is being approached. This singularity, which is contained in the $\overline{\overline{\widetilde{K}}}(\cdot)$ operator for calculating the current, but is also specific to test locations on the wall, was ignored in calculating $\overrightarrow{H}$. To avoid this problem we have to maintain a one-half wall cell distance between the wall and the near field location when obtain the cavity statistics. This occurs automatically if the elements forming the near-field grid locations near the wall are the same size as the wall elements. This is the case in Figure 52.

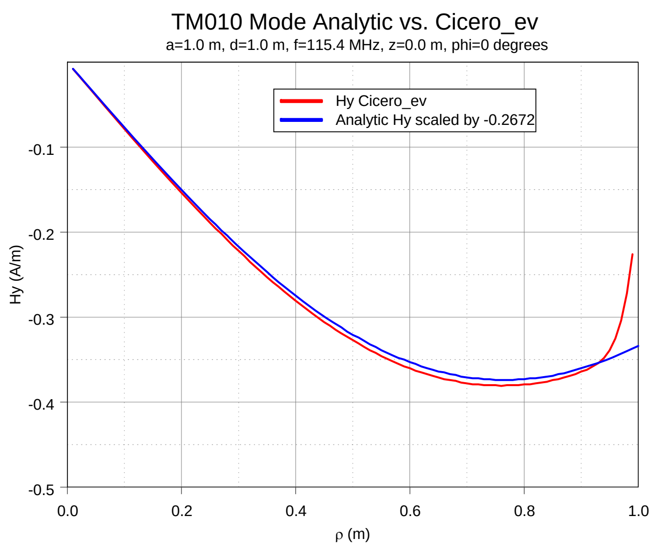

Figure 53. Comparison of $H_{\phi}\left(\rho\right)$ at $z = 0.0$ m, $\phi = 0^{0}$

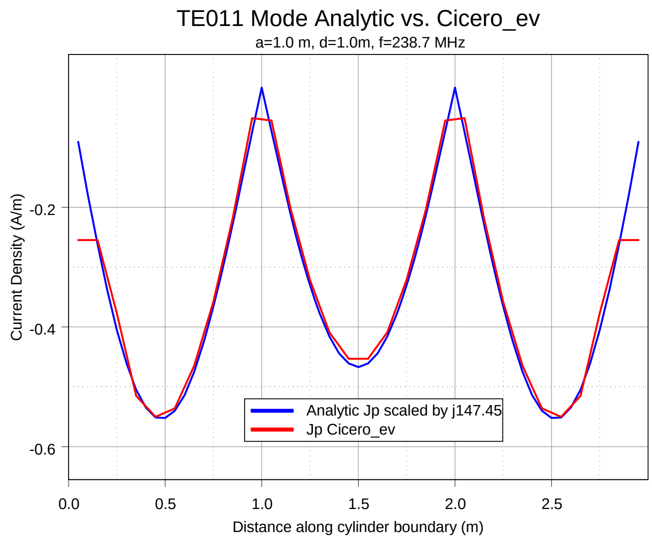

Figure 54. TE_{011} mode current on boundary. Only J_ϕ is non-zero

7.6.2 TE_{011} Mode

This is an $n = 0$ mode that **CICERO_EV** finds at 238.7 MHz (236.43 MHz analytically). Figure 54 shows a comparison of $J_\phi(s)$ which is the only component of the current for this mode. The comparison is good except for the axis and corners. For this mode we scaled the analytical quantities by the complex factor $(0.0, 147.45)$.

Equations 736 - 738 shows that TE_{011} mode has $E_\rho = E_z = 0$ and

$$E_\phi = x'_{np} J'_0 \left(x'_{np}\rho\right) \sin(\pi z)$$

E_ρ is plotted unscaled in Figure 55. The form of the field looks good as discussed in [25] and better than the above TM results. Figure 56 shows the comparison of $E_\phi(\rho)$ at $z = 0.0$ m. There are no discrepancies for the tangential E field as it approaches the wall because the singular behavior is included in the $\overline{\overline{L}}(\cdot)$ operator and thus in the tangential E field calculation.

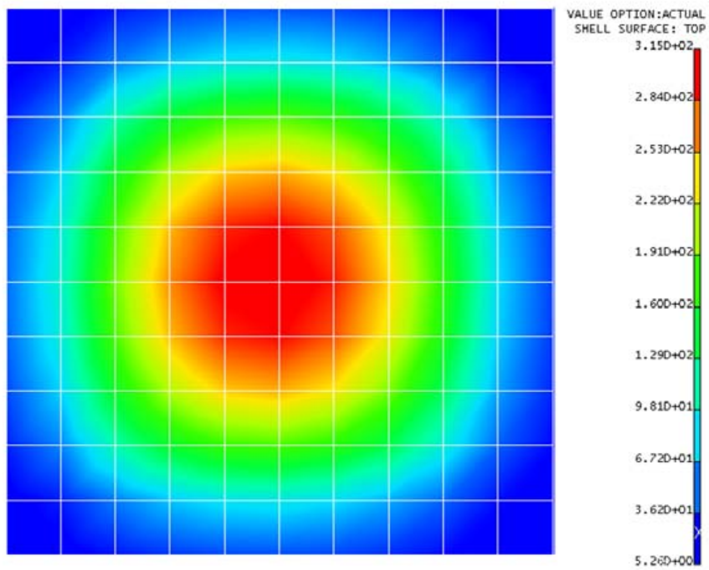

Figure 55. TE_{011} mode E_ϕ

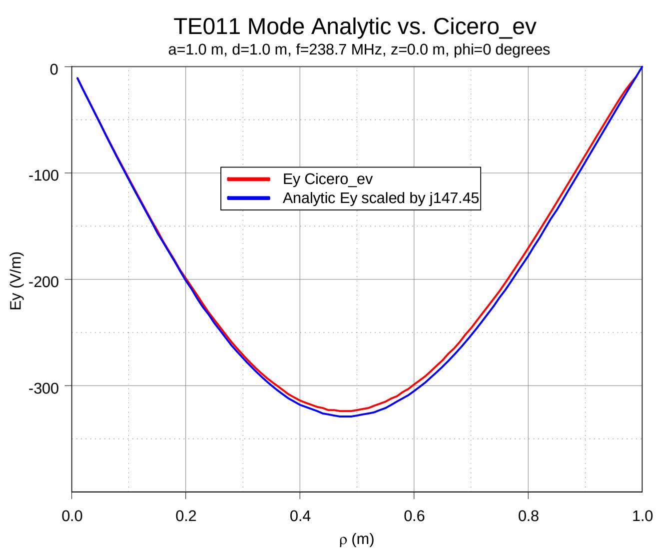

Figure 56. Comparison of $E_\phi\,(\rho)$ at $z = 0.0$ m, $\phi = 0^0$

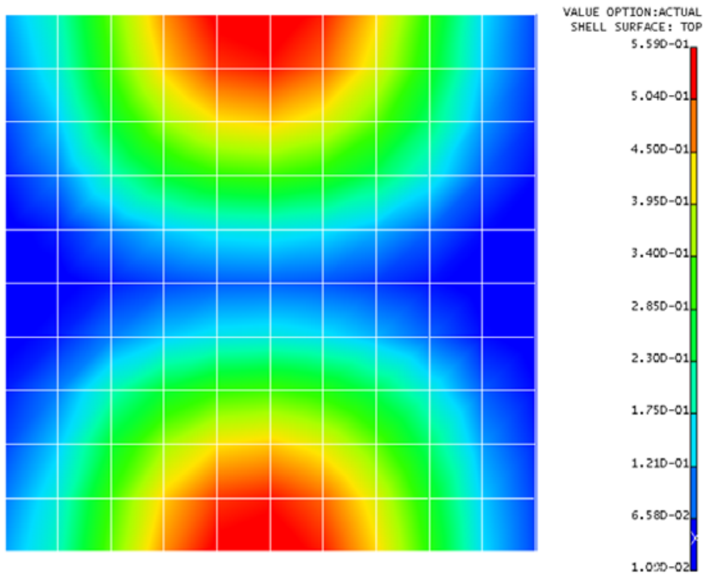

Figure 57. TE_{011} mode H_ρ

Figure 57 shows the unscaled H_ρ over the cavity cross-section. Figure 58 shows $H_\rho(z)$ at $\rho = 0.5$ m. The comparison is good except for near the wall for the reasons given previously.

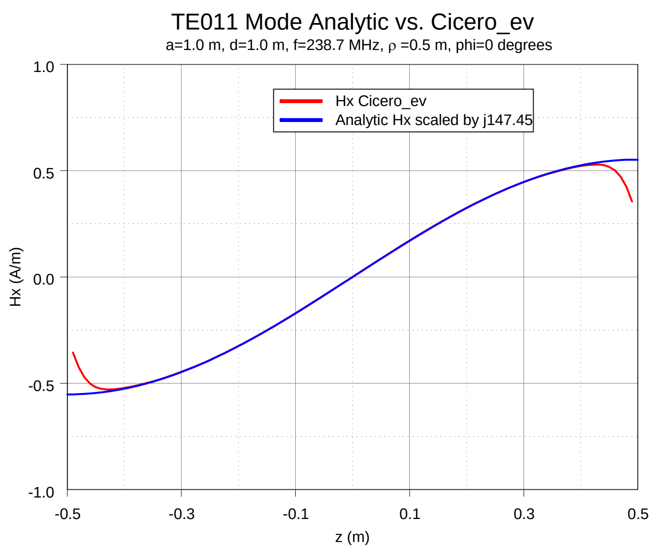

Figure 58. Comparison of $H_\rho(z)$ at $\rho = 0.5$ m, $\phi = 0^0$

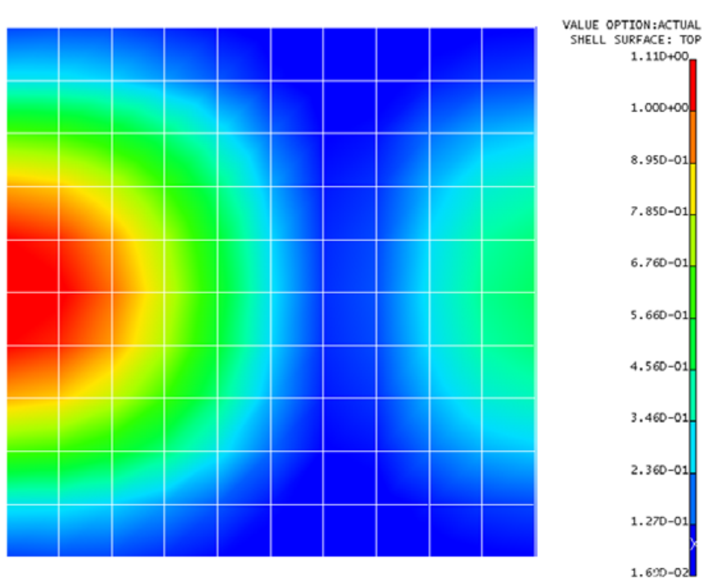

Figure 59. TE_{011} mode H_z

Figure 59 shows the unscaled H_z field over the cavity cross-section and Figure 60 shows $H_z(\rho)$ at $z = 0.0$ m.

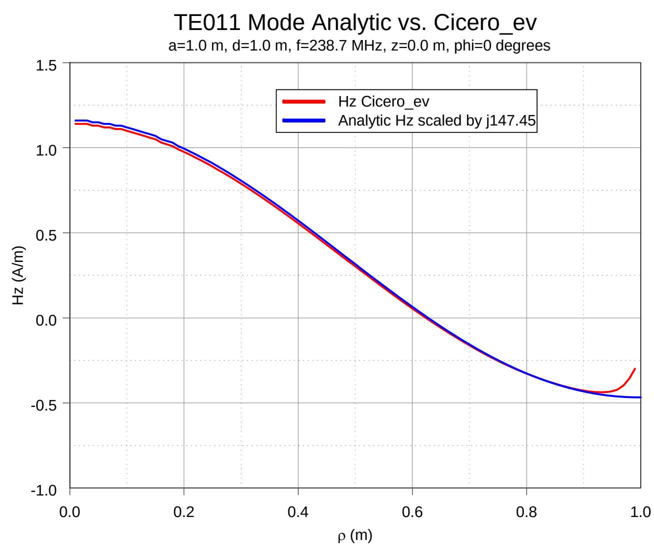

Figure 60. Comparison of $H_z(\rho)$ at $z = 0.0$ m, $\phi = 0^0$

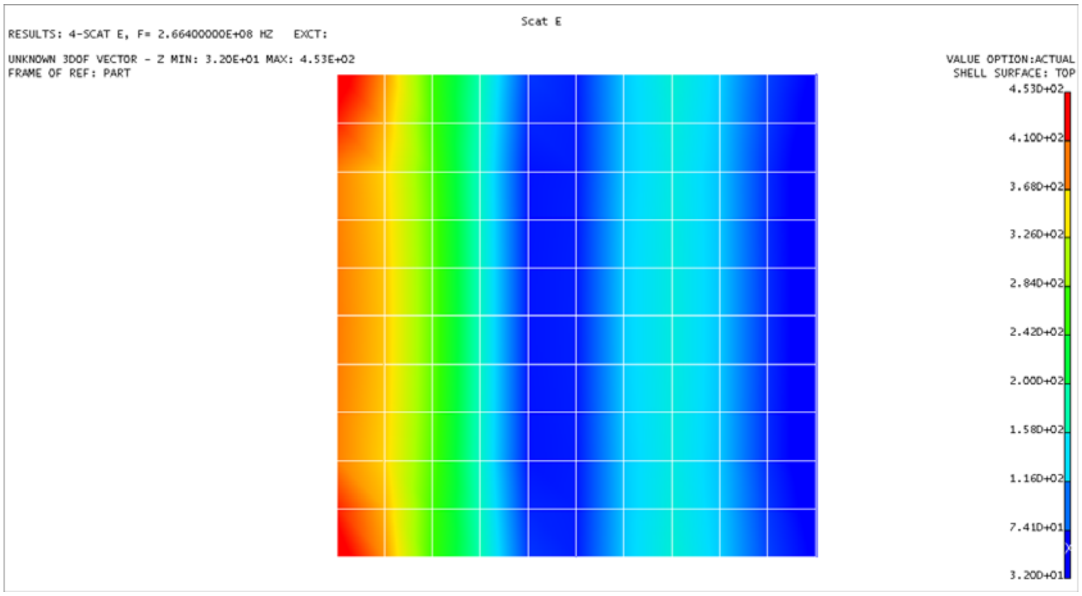

Figure 61. TM_{020} mode E_z

7.6.3 TM_{020} Mode

The TM_{020} mode has $E_\rho = E_\phi = 0$ and

$$E_z = -j\omega\mu J_0\left(x_{02}\rho\right)$$

This is the behavior shown by the **CICERO_EV** results in Figure 61 with the same problems near the ends of the axis discussed previously.

7.7 Verification of n=1 Modes

If we numerically solve for the modes in PEC cylindrical cavity where $a = 1$ m, and $d = 1$ m using **CICERO_EV** and compare to the analytic solution from the preceding section, we obtain the following results for the resonant frequencies when $n = 1$.

CICERO_EV (MHz)	Analytic (MHz)	Mode
175.9	173.7	TE_{111}
184.0	182.5	TM_{110}
238.1	236.4	TM_{111}

The resonant frequencies predicted by **CICERO_EV** have correspondence with the analytical modes as indicated in the table. We decided to use the TE_{111} and TM_{111} modes as our $n = 1$ test cases.

7.7.1 TE_{111} Mode

For the TE_{111} case, in order to understand the connection between current and field results, we examined in detail why E_z of this mode was small compared to the other components. We knew that E_z due to the vector potential from currents on the top and bottom of the cavity had to be identically zero because both J_t and J_ϕ on these two surfaces didn't have a z component. We checked the code and $E_z\left(A\right)$ was indeed zero. Therefore, $E_z\left(\phi\right)$ from the top and bottom surfaces also had to be zero. This meant that

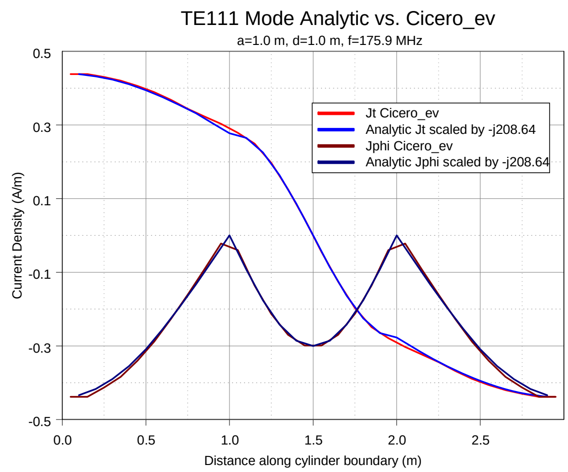

Figure 62. Comparison of analytical and numerical wall current for TE_{111}

ϕ had to be constant with respect to z variation, which could only happen if the charge on top and bottom of the cavity was zero and this would happen only if J_t and J_ϕ were in related in very specific ratios. We used the analytical calculation of the wall current and wall charge discussed above to understand the behavior of the current and charge on the walls and compare them to the values given by **CICERO_EV.**

The analytical expressions for the TE modes indicate that at each point along the bottom and top of the cylinder, the charge due to J_ρ and J_ϕ must cancel each other out. This gives the clue that the two components must be 90^0 out of phase with each other, i.e. if J_ρ is real and negative, then J_ϕ is purely imaginary and has a negative imaginary part. The two components must be in the correct ratio with respect to each other for this charge cancellation to occur. Since the charge has a sinusoidal variation along the side of the cylinder, there will be a z component of the field from the scalar potential which must be cancelled by the z component of the vector potential.

A complicating factor is that, as discussed previously, if an eigenvector is a solution to an eigenvalue problem, then a complex constant times the eigenvector is also a solution to the eigenvalue problem. If we find the complex constant that relate the analytical and numerical solution of one component of the current, then all quantities, such as charge and electric field should be related by the same complex constant. For the TE_{111} case of interest, we multiply the analytical solution by the factor $(-j208.64)$. A comparison of the numerical versus analytical current density is shown in Figure 62. This shows that **CICERO_EV** is calculating the current density correctly. There are slight discrepancies near the junctions between the bottom and the side surfaces (1 on the abscissa) and between the top and side surfaces (2 on the abscissa). Figure 63 compares the numerical versus the analytic $\nabla \cdot \overrightarrow{J}$ which is related to charge density. Here the numerical results oscillate around the analytical results, particularly on the top and bottom surface where the charge density is analytically zero. This oscillation is to be expected, since the current density is being differentiated and numerically we should expect the field from the charge to be small, but not identically zero.

In Figures 64-67 we compare various components (ρ and ϕ) of the analytical E field to the same components from **CICERO_EV** at various cuts in the $\phi = 0^0$ plane. The cuts are at $z = 0.5$ m (midway between the end planes of the cylinder) and at $\rho = 0.5$ (half-way to the outer radius). Most of the results show good agreement except for E_ρ along the $z = 0.5$ m cut near $\rho = a$ in Figure 64. Initially we thought this was due to the fact that we were approaching a junction between two elements on the outer surface, but when we moved the end point down to where it intersected the outer surface in the center of the element,

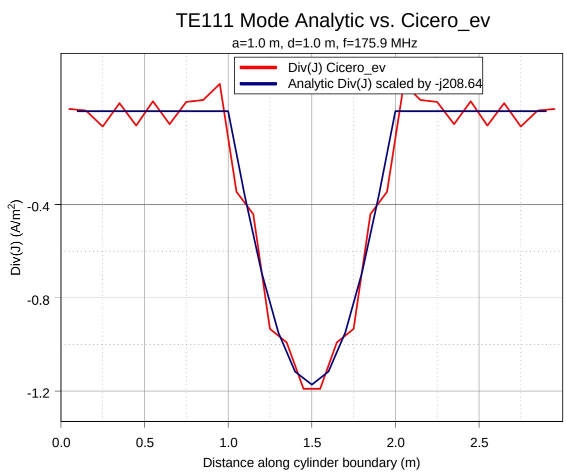

Figure 63. Comparison of analytical and numerical $\nabla \cdot \overrightarrow{J}$ for TE_{111}

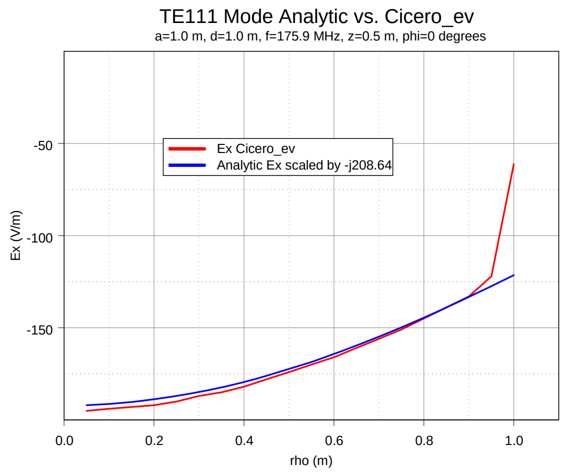

Figure 64. Comparison of $E_\rho\,(\rho)$ at $z = 0.5$ m , $\phi = 0^0$

the anomalous results remained. This bears further examination. Again because of the way we obtain our near fields when building cavity statistics this is not a problem.

For the TE_{111} modes the color plots of E_ρ, E_ϕ, and E_z shown in Figures 68 - 70 show good behavior of the results. Analytically E_z should be identically zero, which is not the case in Figure 70. If we look at the scale on the right side, however, the highest value of this component is the same order of magnitude as the lowest value of the other two components. This finite value of E_z is due to numerical error and should improve with a finer grid.

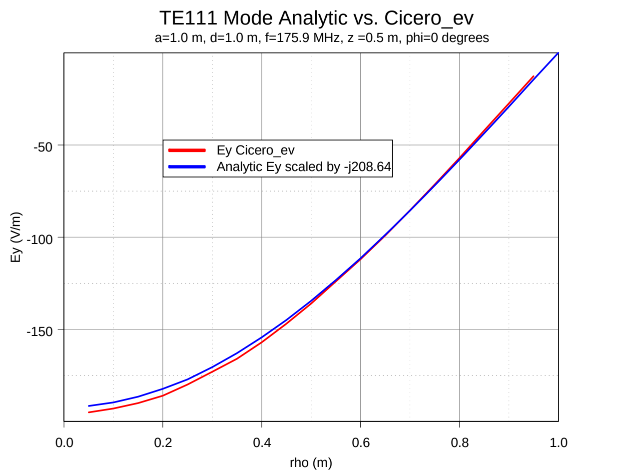

Figure 65. Comparison of $E_\phi(\rho)$ at $z = 0.5$ m , $\phi = 0^0$

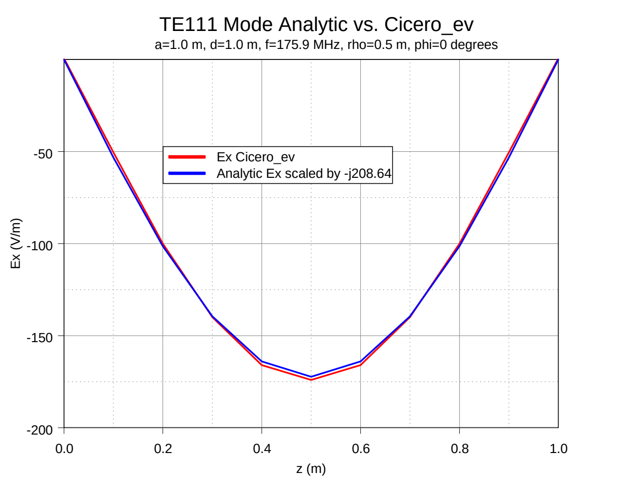

Figure 66. Comparison of $E_\rho(z)$ at $\rho = 0.5$ m , $\phi = 0^0$

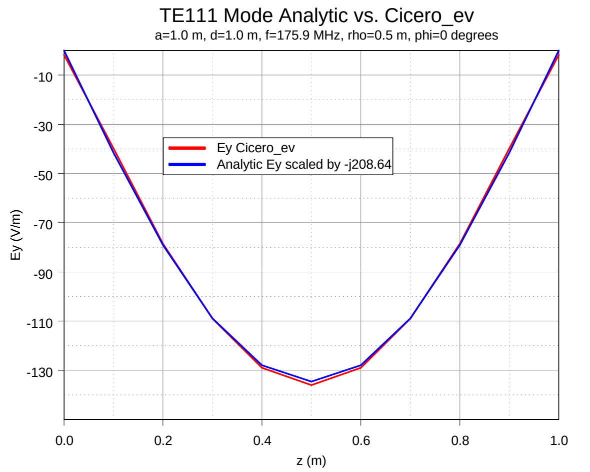

Figure 67. Comparison of $E_{\phi}(z)$ at $\rho = 0.5$ m, $\phi = 0^{0}$

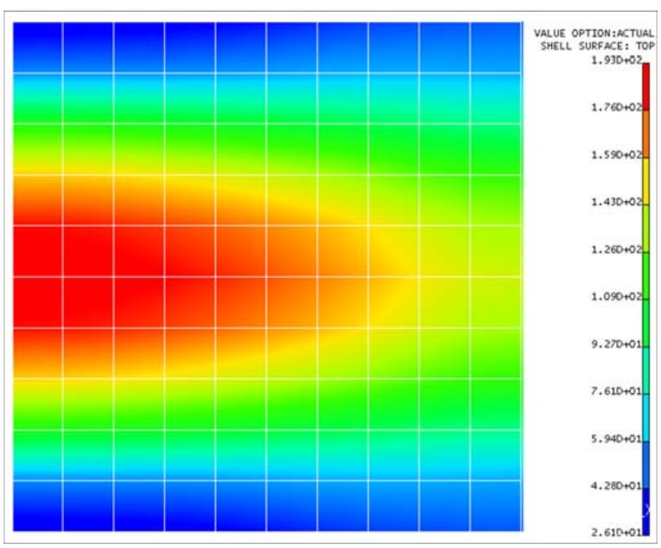

Figure 68. Correct E_{ρ} for TE_{111} mode

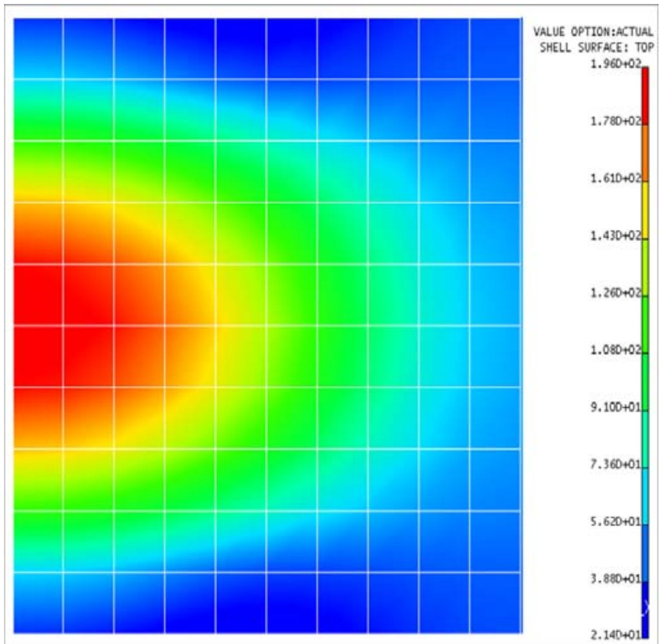

Figure 69. Correct E_ϕ for TE_{111} mode

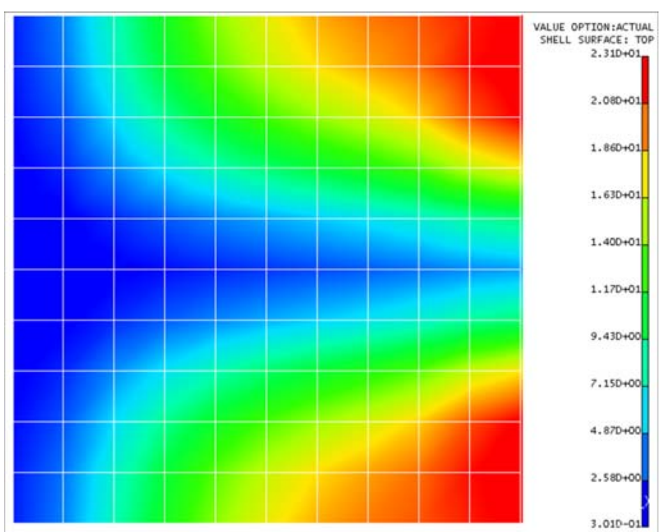

Figure 70. Correct E_z for TE_{111} mode

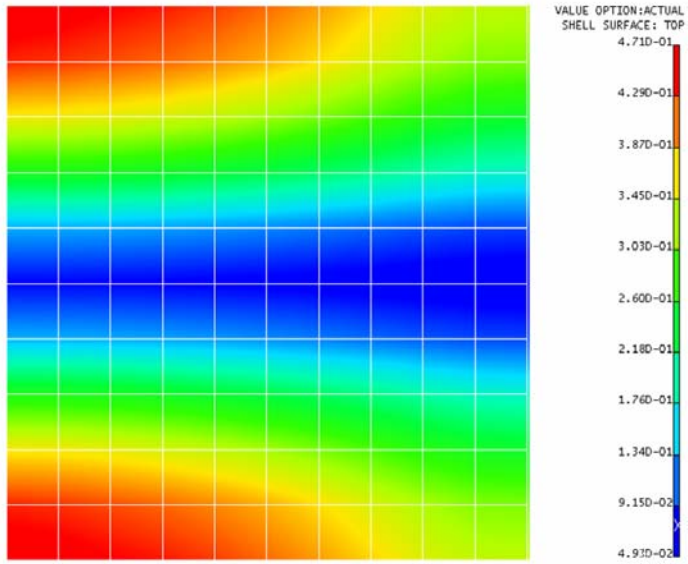

Figure 71. TE_{111} mode H_ρ

Next we will compare the H field to analytical results. Figures 71 through 76 shows H_ρ, H_ϕ and H_z over the cross-section of the cavity and comparisons with analytical results for various cuts across the cavity.

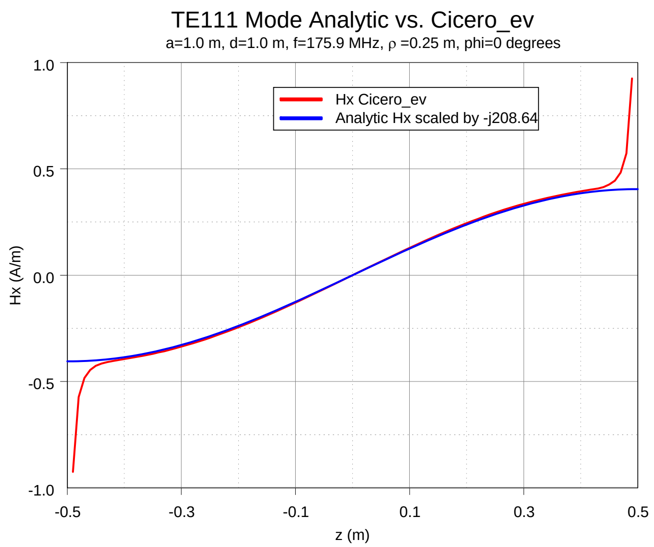

Figure 72. Comparison of $H_\rho(z)$ at $\rho = 0.25$ m, $\phi = 0^0$

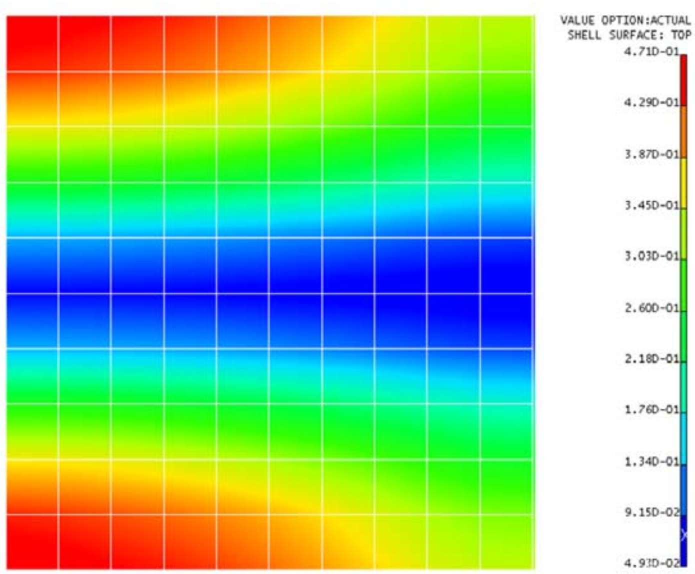

Figure 73. TE_{111} mode H_ϕ

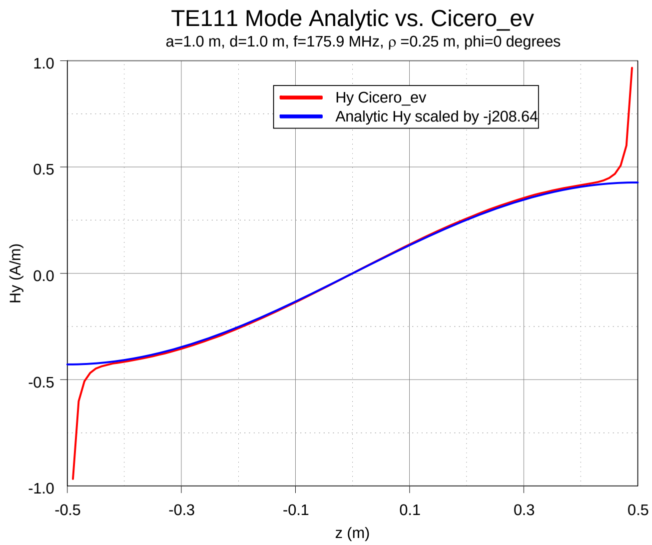

Figure 74. Comparison of $H_{\phi}(z)$ at $\rho = 0.25$ m, $\phi = 0^0$

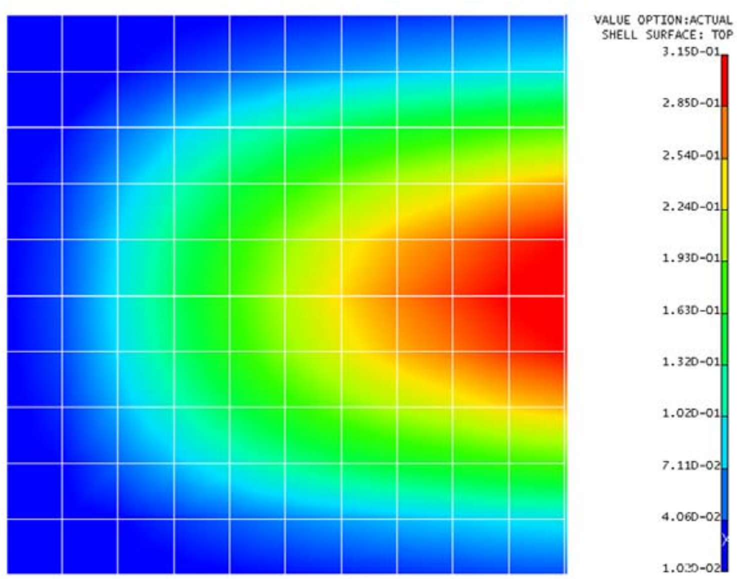

Figure 75. TE_{111} mode H_z

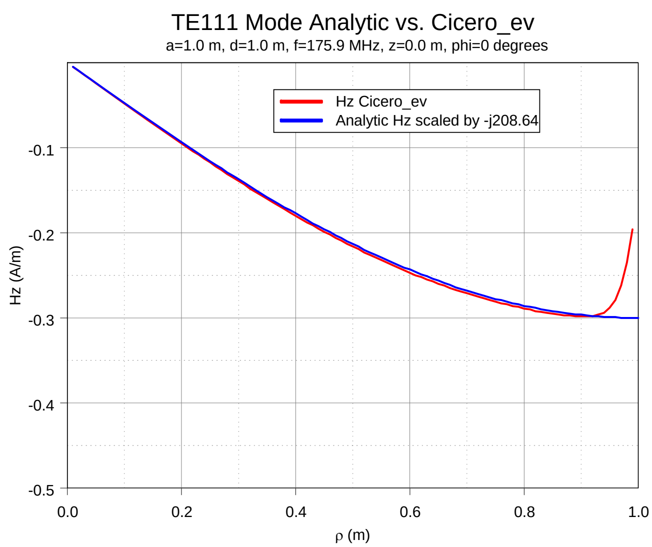

Figure 76. Comparison of $H_z(\rho)$ at $z = 0.0$ m, $\phi = 0^0$

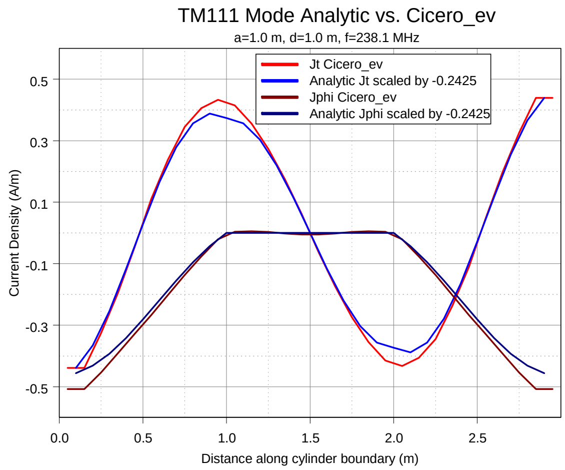

Figure 77. Comparison of analytical and numerical wall current for TM_{111}

7.7.2 TM_{111} Mode

In this section we will compare the TM_{111} mode at the resonant frequency of 238.1 MHz. Figure 77 shows good agreement between the analytical and numerical current density. There is error near the junctions between the side and the top and between the side and bottom and it is more error than we saw for the TE_{111} mode. In this case we have to scale the analytical results by the factor $(-0.2425, 0.0)$. Figure 78 shows comparison between analytical and numerical calculation of $\nabla \cdot \overrightarrow{J}$. For this case there is a significant deviation between analytical and numerical results near the axis of the cylinder. More disturbingly, there is a sign error between the two results – the scaling factor is $(+0.2425, 0.0)$. We will have to examine this further.

Figures 79 - 83 show comparisons between analytical and numerical results of E_ρ, E_ϕ and E_z in the $\phi = 0^0$ plane for cuts along ρ and z (the same as we did for the TE_{111} modes). Comparisons for E_z along ρ at $z = 0.5$ m are not plotted because both numerical and analytical calculations show $E_z = 0$. The results show the same trend as the TE_{111} results. E_ρ shows a sharp, anomalous discontinuity near $\rho = a$ in Figure 79. E_z exhibits the same behavior near its endpoints in Figure 83. E_ϕ shows a consistent sign error and $E_\rho(z)$ shows excessive error and non-conformance to the boundary condition (it should go to zero at the boundary) in Figure 81. The results overall are worse than the TE_{111} results.

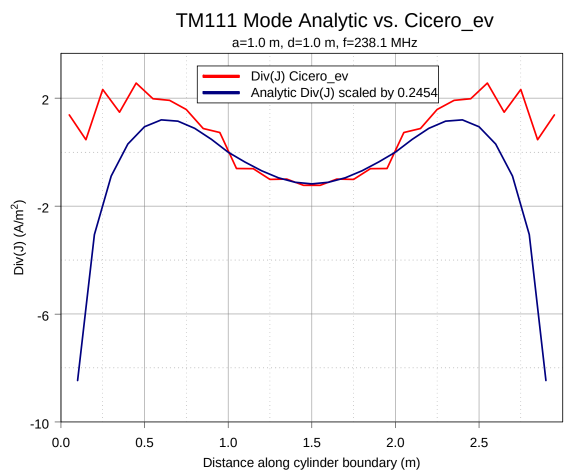

Figure 78. Comparison of analytical and numerical $\nabla \cdot \overrightarrow{J}$ for TM_{111}

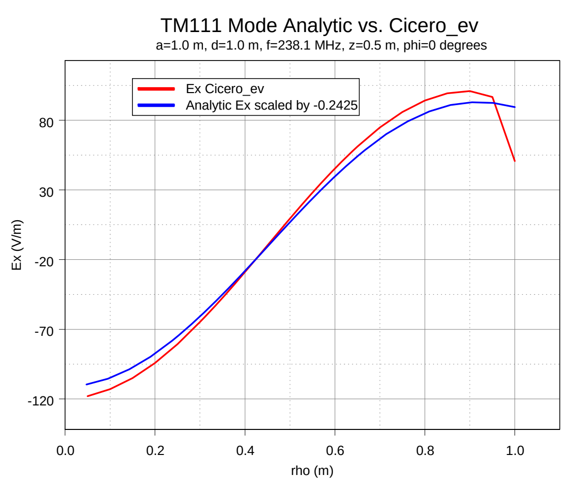

Figure 79. Comparison of $E_{\rho}(\rho)$ at $z = 0.5$ m , $\phi = 0^0$

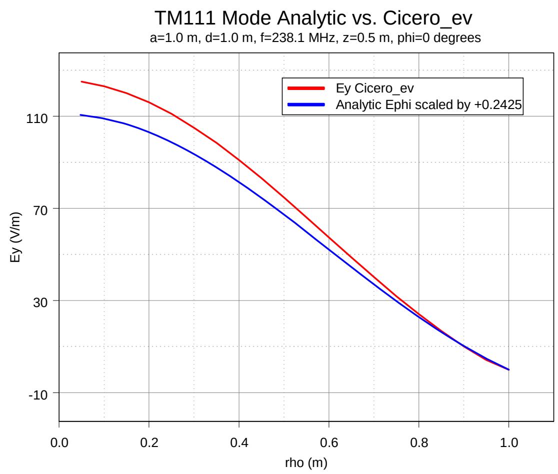

Figure 80. Comparison of $E_\phi(\rho)$ at $z = 0.5$ m, $\phi = 0^0$

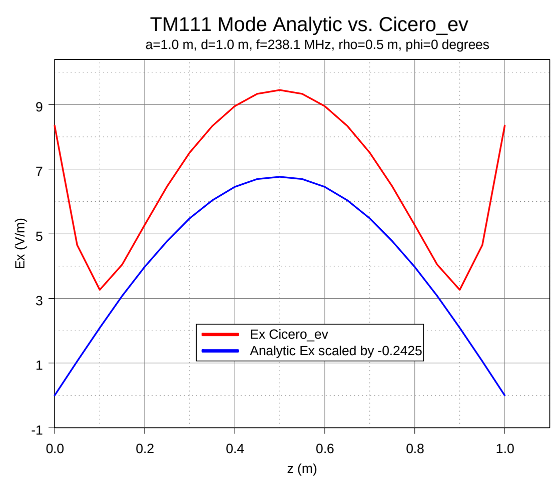

Figure 81. Comparison of $E_\rho(z)$ at $\rho = 0.5$ m, $\phi = 0^0$

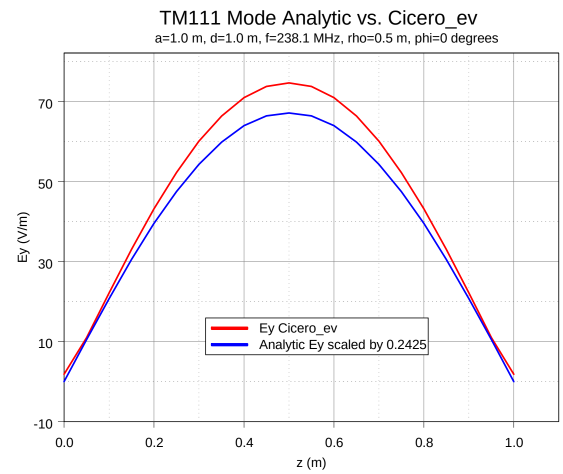

Figure 82. Comparison of $E_{\phi}\left(z\right)$ at $\rho = 0.5$ m , $\phi = 0^0$

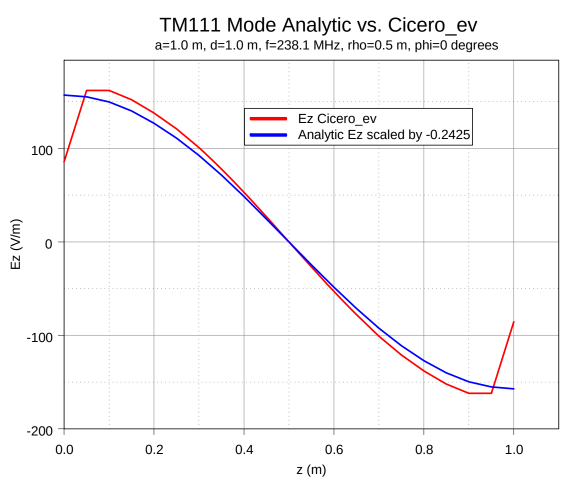

Figure 83. Comparison of $E_z\left(z\right)$ at $\rho = 0.5$ m , $\phi = 0^0$

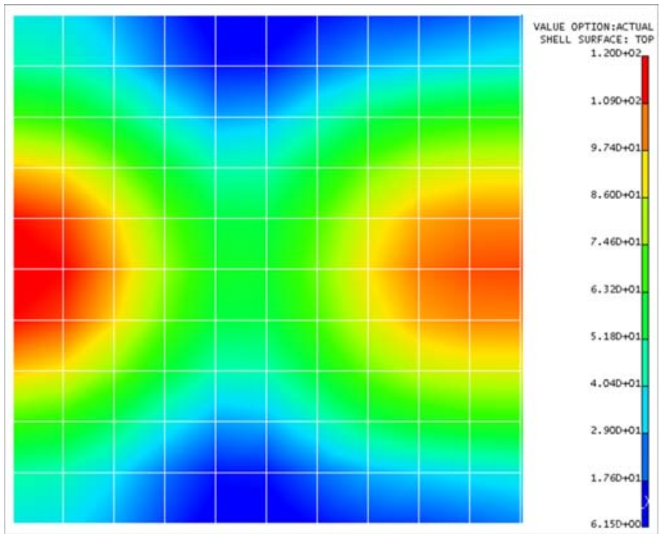

Figure 84. E_ρ for TM_{111} mode

Figures 84 - 86 show color plots of the TM_{111} mode. Overall the results look reasonable. The jumps in field near the boundary seen above are not seen in the color plots. The boundary condition for E_ρ and E_ϕ near the axis don't seem to be correct. This will have to be checked out in the future.

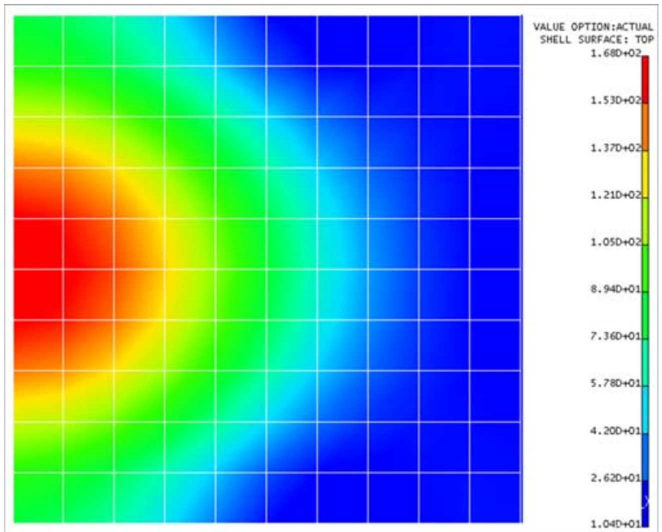

Figure 85. E_ϕ for TM_{111} mode

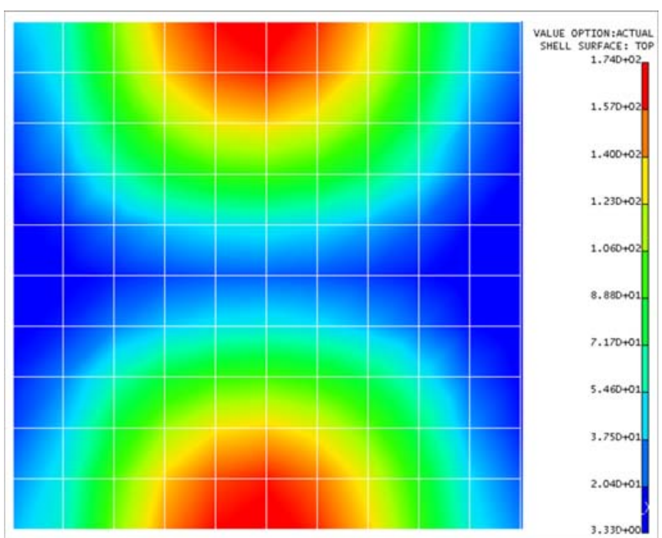

Figure 86. E_z for TM_{111} mode

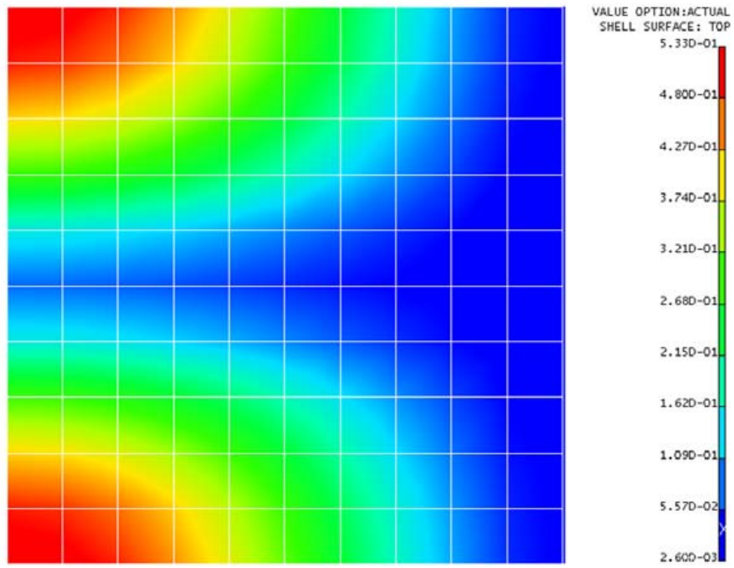

Figure 87. TM_{111} mode H_ρ

Figures 87 through 90 shows H_ρ, H_ϕ and H_z over the cross-section of the cavity and comparisons with analytical results for various cuts across the cavity.

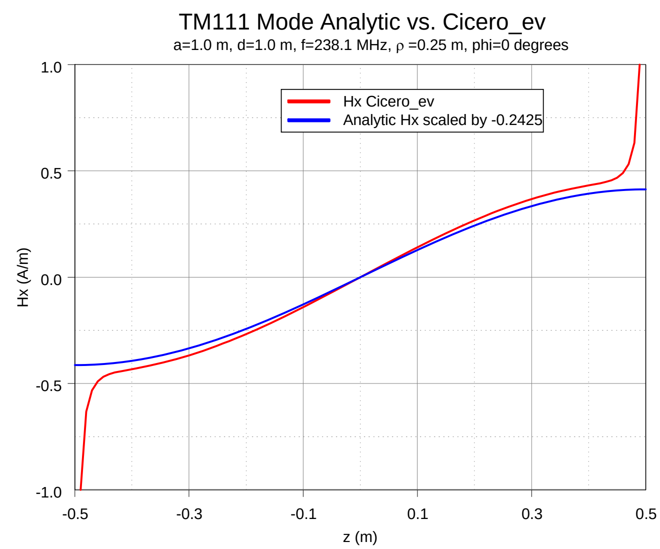

Figure 88. Comparison of $H_\rho\left(z\right)$ at $\rho = 0.25$ m, $\phi = 0^0$

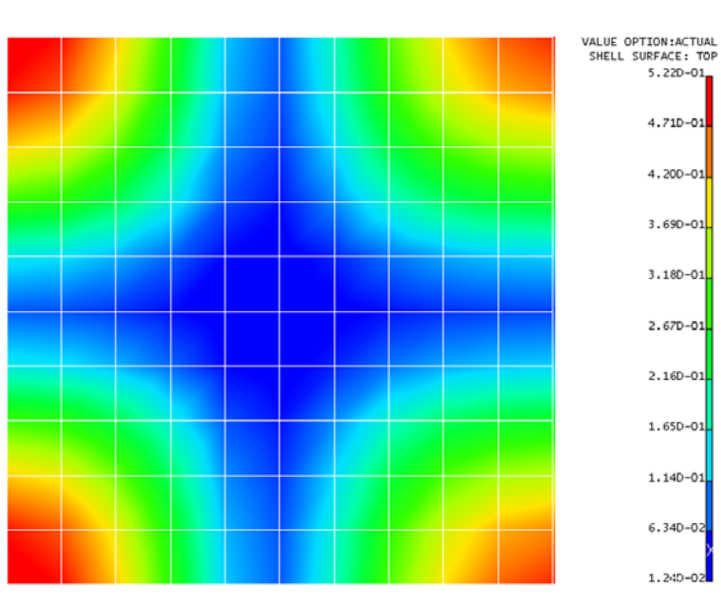

Figure 89. TM_{111} mode H_ϕ

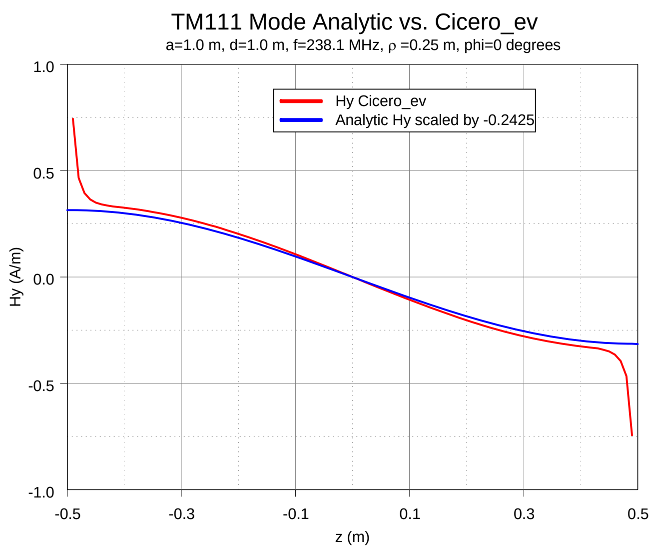

Figure 90. Comparison of $H_{\phi}(z)$ at $\rho = 0.25$ m, $\phi = 0^0$

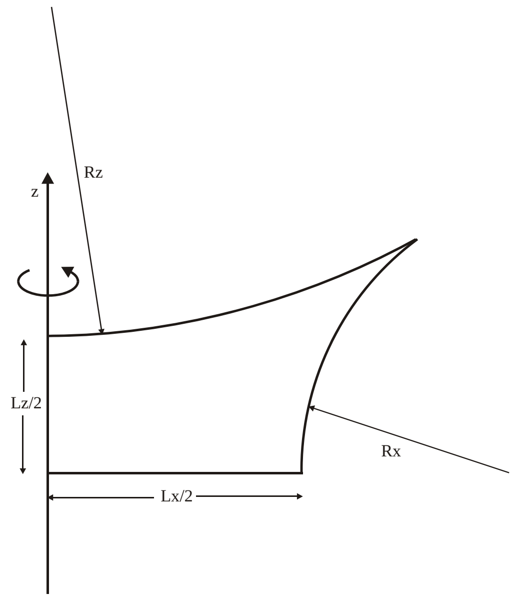

Figure 91. Quarter bowtie geometry

7.8 Statistical Results

The previous section demonstrated that **CICERO_EV** can find modes and predict E and H fields in a cross-section of the cavity for $n = 0$ and $n = 1$. In this section we will discuss obtaining statistics for the axisymmetric bowtie cavity. Figure 91 shows a quarter bowtie cross section that is revolved around the z axis to form a cavity. The body of revolution (BOR) has curved ends with a radius of curvature of R_z. The distance from end to end along the center line is L_z. The sides of the BOR are also curved with a radius of curvature of R_x. The radius of the BOR at point midway between its two ends is $L_x/2$. The cross section being considered was taken to be at $\phi = 0$, so instead of ρ we used x.

The particular geometry solved was where $L_x = L_z = 2.0$ m, $R_z = 10$ m and $R_x = 1.5$ m. We set $n = 0$ in **input** and swept over the range 1 GHz to 3 GHz. We found 472 modes, both TE and TM. We decided to concentrate only on the statistics of E_z, which is non-zero only for the TM modes. Of the 472 total modes only 166 were TM modes, the remaining 306 were TE modes. In obtaining the statistics we actually looped over all modes and skipped the results that had $E_z = 0$ over the entire cross section. A typical mode structure plotting E_z for the TM mode at 1.0164 GHz is shown in Figure 92. Originally we solved a complete cavity rather than the half cavity but found that there was a degeneracy for each mode that caused the field to be asymmetrical in the z direction. Adding the PEC plane through the center of the cavity at $z = 0$, removed the degeneracy and enforced symmetry.

Let $l = L_z/2$ and d be the focus location

$$d = l\sqrt{1 + R_z/l} \tag{739}$$

Then the scar frequency separation factor (s) is

$$s = \frac{(k_0 - k_p)\, L_z}{\ln\left[\frac{d+l}{d-l}\right]} \tag{740}$$

where $k_0 = 2\pi f_r/c$ (f_r is one of the 106 resonant frequencies calculated by **CICERO_EV** and c is the speed of light) and $k_p = p2\pi/L_z; p = 1, 2, 3, 4....$

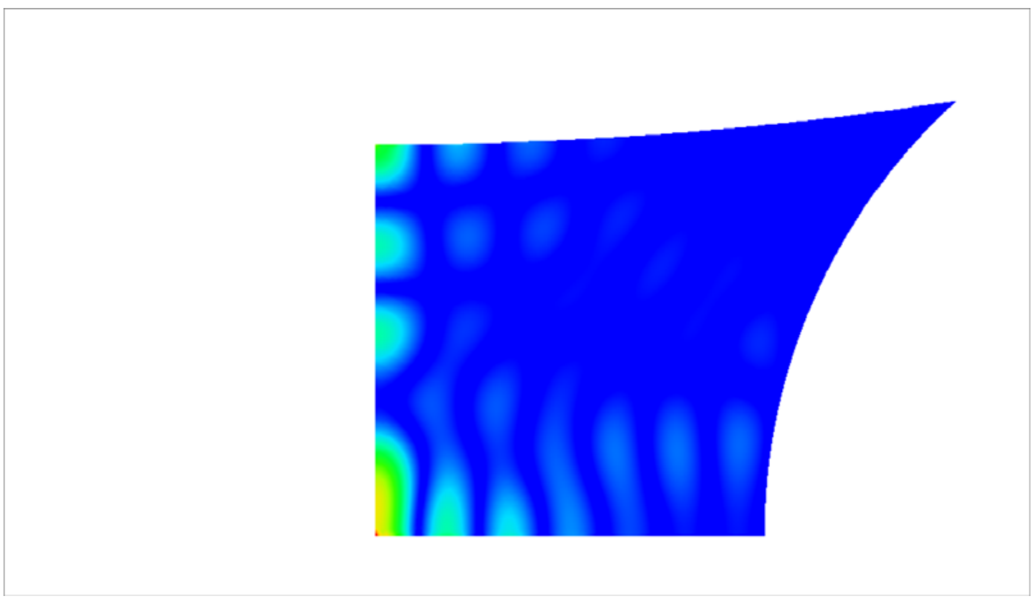

Figure 92. TM mode at 1.1064 GHz

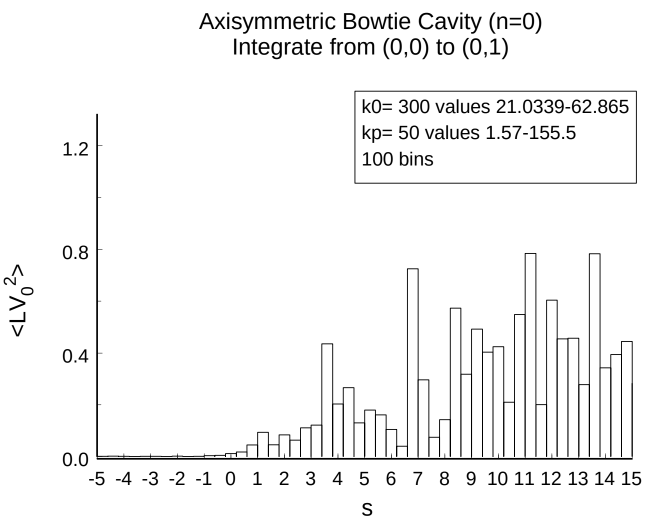

Figure 93. Histogram of quarter bowtie BOR

For each value of k_0 we first normalize each value of E_z contained in the ***.nfld0** file by the quantity

$$\iint E_z(x,z)\, E_z^*(x,z)\, dxdz$$

where the integral is taken over the area of the quarter-bowtie cross section. At resonance the fields should be all real to within a complex constant. We find the phase of the largest E_z field within the cavity and divide each value of E_z by this phase to remove the complex constant and make the fields real. For every value of k_0 we loop over all values of k_p, calculate s for each combination and calculate the quantity

$$V_0 = \int_0^{L_z/2} \widetilde{E}_z(0,z)\cos(k_p z)\, dz$$

by integrating numerically, where $\widetilde{E}_z(0,z)$ is the E_z field along the z axis of the BOR normalized as described above. The quantity $L_z V_0^2$ is calculated and added to one of 100 bins between the values of $s = [-15, 25]$. At the end of accounting for all k_0, k_p combinations the values of each bin are divided by the number of entries contributing to each bin. The result is shown in Figure 93.

Next we will discuss obtaining statistics for the axisymmetric bowtie cavity for the $n = 1$ mode. The particular geometry solved was where $L_x = L_z = 2.0$ m, $R_z = 10$ m and $R_x = 1.5$ m. The boundary elements and the elements used to get locations to calculate the near field both had edge lengths of around 1 cm. We set $n = 1$ in the file **input** and swept over the range 1 GHz to 3 GHz monitoring the condition number of the impedance matrix. The condition number as a function of frequency is shown in Figure 94. Each peak of the condition number is a frequency where a mode of this cavity occurs. Counting the peaks,we found 874 modes over this frequency range, that could be TE or TM. For each of these modal frequencies we solve for $\overrightarrow{E}$ and $\overrightarrow{H}$ at a series of points that span the cross-section of the bowtie. The fields for each modal frequency are stored in a separate ***.nfld0** file. A typical mode structure plotting E_y for the mode at 1.0164 GHz is shown in Figure 95.

Where d and s_p were defined above in Equations 739 and 740 and where $k_0 = 2\pi f_r/c$ (f_r is one of the 874 resonant frequencies given at the peaks in Figure 94 and c is the speed of light) and $k_p = p\pi/L_z$

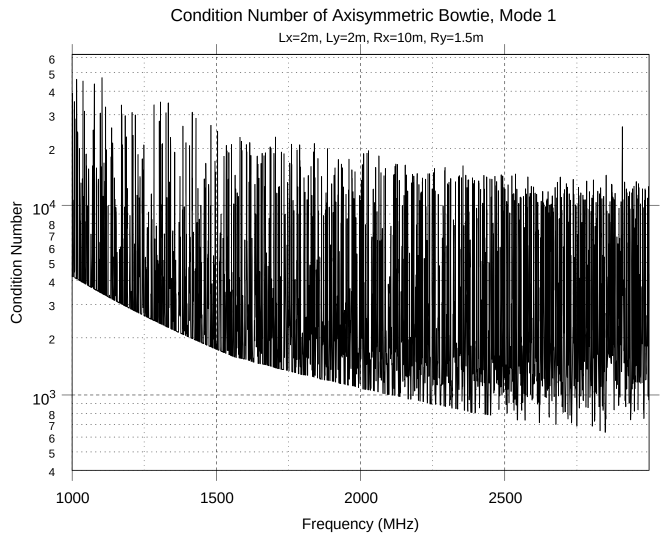

Figure 94. Condition number of the impedance matrix vs. frequency in the axisymmetric bowtie cavity for $n = 1$.

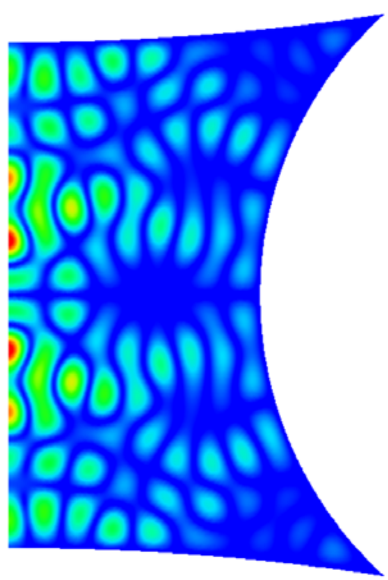

Figure 95. E_y at 1.3134 GHz

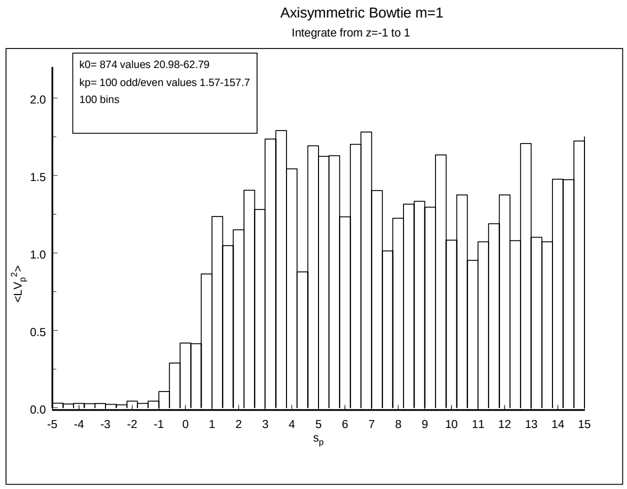

Figure 96. Histogram of $< L_x V_p^2 >$ as a function of separation parameter s_p

; $p = 1, 2, 3, 4....100$. To obtain the statistics we loop over all the different combinations of k_0 and k_p.

For each value of k_0 we first normalize each value of $E_x\left(0, z\right)$ (along the $\widehat{z}$ axis) by the quantity

$$\iint x\overrightarrow{E}\left(x, z\right) \cdot \overrightarrow{E}^*\left(x, z\right) dxdz$$

where the integral is taken over the area of the bowtie cross-section. At resonance, all the field components should be real to within a complex constant. We find the phase of the largest E_z field within the cavity and divide each value of $E_x\left(0, z\right)$ by this phase to make the quantity real. Then for each k_0, we loop over all values of k_p, calculate s_p for each combination and calculate the quantity

$$V_p = \begin{cases} \int_0^{L_z} D\widetilde{E}_z\left(0, z\right) \cos\left(k_p z + p_0\right) dz & \text{if } p \text{ is odd} \\ \int_0^{L_z} D\widetilde{E}_z\left(0, z\right) \sin\left(k_p z + p_0\right) dz & \text{if } p \text{ is even} \end{cases}$$

by integrating numerically, where $\widetilde{E}_x(0, z)$ is the E_x field along the z axis of the BOR normalized as described above,

$$p_0 = \left(k_0 - k_p\right) z - \frac{1}{2} s_p \ln\left[\frac{d+z}{d-z}\right]$$

and

$$D = \left[1 - \frac{z^2}{d^2}\right]^{-\frac{1}{2}}$$

The quantity $L_z V_p^2$ is calculated and added to one of 100 bins between the values of $s_p = [-15, 25]$ depending on the calculated value of s_p. After accounting for all k_0, k_p combinations, the values of each bin are divided by the number of entries contributing to each bin. The result is shown in Figure 96.

7.9 Numerical Conclusions

In this section we have discussed how we modified **CICERO** to obtain **CICERO_EV.** This allows us to find modes and calculate the fields inside a PEC axisymmetric cavity. We have carefully checked the results from **CICERO_EV** with analytical results on a circular cylinder for $n = 0$ and $n = 1$ and have found good agreement. We believe that the code will work for all values of n. We used the code to calculate field statistics in an axisymmetric bowtie cavity. These statistics are compared to the vector scar theory in [28].

During the course of obtaining the bowtie cavity statistics, we found that the scaled fields inside the cavity were not entirely real and this could indicate a modal degeneracy. This could be due to formulating the fields in terms of $e^{jn\phi}$ rather than a combination of $\sin(n\phi)$ and $\cos(n\phi)$. A second problem is the problem seen at the wall-axis junction for the TM010 mode discussed in [25]. This could be solved by using a finer grid. Finally, we could try to add the singular terms to the H field calculation so that it will give accurate results next to the cavity wall. This is the subject of future work.

8 CONCLUSIONS

This report explores the construction of high frequency electromagnetic scar enhancements along periodic orbits in axisymmetric convex (bowtie-like geometry) and concave (stadium-like geometry) walled cavities. In particular the method that Vaynshteyn [5] introduced to treat high frequency electromagnetic stable orbits is adapted to the unstable scar case. A random phase reflection is introduced, as Antonsen did in the two-dimensional bowtie cavity [1], to simply represent the outer chaotic region of the cavity.

The acoustic scalar problem is treated first. Next, emphasis is given to the vector nature of the electromagnetic problem; in particular, the quasi-rectangular coordinate system introduced by Vaynshteyn [5] is explored and the vector modal representation is constructed. The critical normalization condition introduced by Antonsen [1] is examined through the use of a source free electromagnetic energy theorem where the boundary of the scarred orbit connects the scar amplitude to the total energy in the cavity. Several approaches to this normalization are discussed, and the approach where the quasirectangular Hertz potential components are transformed to the prolate spheroidal system seems to be successful. The distribution plots are constructed for the axisymmetric scar along the center line of the cavity. We also consider the contribution from the magnetic Hertz potential (or alternatively the orthogonal polarization of the electric Hertz potential) to the polarization of the field.

The random plane wave construction is also carried out for the scalar and vector cases. The first azimuthal mode of the field is extracted and renormalized over the cavity volume; comparisons to this result confirms the proper asymptotic limits of the scar theory. We also discuss the enhancement of this first azimuthal mode on the central axis of the cavity due to the axisymmetric symmetry of the problem.

Comparisons of the theoretical scar distribution with the axisymmetric numerical simulations are discussed for the first azimuthal mode in both convex and concave walled cavities. These comparisons include both Fourier projections along the scarred orbit, as well as field point statistics along the scar orbit and near the interior foci (in the concave case). Finally, a comparison is made for a single realization of the scar mode in the concave walled stadium cavity, which confirms that the theoretical scar construction captures the focal region features of the high frequency electromagnetic solution.

9 REFERENCES

[1] T. M. Antonsen, Jr., E. Ott, A. Chen, and R. N. Oerter, "Statistics of wave-function scars, Vol. 51, No. 1, Physical Review E, January 1995, pp. 111-121.

[2] L. K. Warne, R. E. Jorgenson, J. D. Kotulski, and K. S. H. Lee, "Two-Dimensional Unstable Scar Statistics," Sandia Laboratories Report, SAND2006-7511, Dec. 2006.

[3] L. K. Warne, R. E. Jorgenson, J. D. Kotulski, and K. S. H. Lee, "Time Harmonic Scar Statistics in Two Dimensional Cavities," IEEE AP-S International Symposium, June 11-15, Honolulu, HI, 2007.

[4] A. Vagov, H. Schomerus, and V. V. Zalipaev, "Asymptotic-boundary-layer method for unstable trajectories: Semiclassical expansion for individual scar wave functions," Physical Review E, Vol. 80, p. 056202-1-8, 2009.

[5] L. A. Vaynshteyn, **Open Waveguides and Resonators**, Boulder, CO: Golem Press, 1969, Ch. 4.

[6] L. K. Warne, R. E. Jorgenson, J. D. Kotulski, K. S. H. Lee, "Time Harmonic Two-Dimensional Cavity Scar Statistics: Convex Mirrors and Bowtie," Electromagnetics, Vol. 31, No. 2, Feb. 2011, pp. 77-100.

[7] L. K. Warne, R. E. Jorgenson, J. D. Kotulski, K. S. H. Lee, "Time Harmonic Two-Dimensional Cavity Scar Statistics: Concave Mirrors and Stadium," Electromagnetics, Vol. 34, Feb. 2014, pp. 431-462.

[8] L. K. Warne, R. E. Jorgenson, J. D. Kotulski, and K. S. H. Lee, "Time Harmonic Scar Statistics in Convex and Concave Cavities," IEEE AP-S/URSI International Symposium, June 6-11, San Diego, CA, 2008.

[9] L. K. Warne and R. E. Jorgenson, "Scalar Formulation of Three Dimensional High Frequency Axisymmetric Cavity Scars," Sandia Internal Report, Sept. 5, 2010.

[10] M. R. Spiegel, **Vector Analysis**, Schaum's Outline Series, New York: McGraw-Hill Book Co., Ch. 7, 1959.

[11] M. Abramowitz and I. A. Stegun (editors), **Handbook of Mathematical Functions**, National Bureau of Standards, 1972.

[12] A. E. Siegman, **An Introduction to Lasers and Masers**, New York: McGraw-Hill, Inc., 1971, Section 8-1.

[13] C. H. Papas, **Theory of Electromagnetic Wave Propagation**, New York: McGraw-Hill Book Co., 1965, Ch. 1.

[14] P. M. Morse and K. U. Ingard, **Theoretical Acoustics**, New York: McGraw-Hill Book Co., 1968, Chapter 6.

[15] R. Courant and D. Hilbert, **Methods of Mathematical Physics**, Vol. I, New York: John Wiley & Sons, 1953, pp. 429-445.

[16] M. C. Gutzwiller, **Chaos in Classical and Quantum Mechanics**, New York: Springer-Verlag, Inc., 1990, Ch. 15, p. 259.

[17] F. E. Borgnis and C. H. Papas, **Handbuch Der Physik Volume XVI**: Electromagnetic Waveguides and Resonators, Berlin: Springer-Verlag, 1958, pp. 411-414.

[18] J. Van Bladel, **Electromagnetic Fields**, New York: Hemisphere Pub. Co., 1985, pp. 326-330.

[19] R. E. Jorgenson and L. K. Warne, "Vector Formulation of 3D High Frequency Axisymmetric Scars: Numerical Simulations," Sandia Internal Report, Sept. 12, 2012.

[20] I. S. Gradshteyn and I. M. Ryzhik, **Table Of Integrals Series And Products**, New York: Academic Press, 1965, pp. 402, 731.

[21] L. N. Medgyesi-Mitschang, P. L. Huddleston, and J. M. Putnam, "Radar scattering analysis: Mathematical formulations, Volume I, Final Report", *Report MDC Q1230, Contract No. DASG60-83-C-0008*, 22 November 1984.

[22] J. M. Putnam and L. N. Medgyesi-Mitschang, "Combined field integral equation formulation for axially inhomogeneous bodies of revolution (combined field formulation of CICERO), Volume I, Final

Technical Report", *Report MDCQA003, Contract No. 33-4257*, 1 December 1987.

[23] Eiger website at located at "https://ppscweb.sandia.gov/pulsed-power-sciences/ppt_tech/compphys/Projects/eiger/eiger.html"

[24] R. F. Harrington, **Time-Harmonic Electromagnetic Fields**, McGraw-Hill Co., 1961, p. 214.

[25] L. K. Warne and R. E. Jorgenson, "Scalar formulation of 3D high frequency axisymmetric cavity scars", memorandum to M. C. Caldwell, September 14, 2010.

[26] L. K. Warne and R. E. Jorgenson, "Vector formulation of three dimensional high frequency axisymmetric convex cavity scars", memorandum to M. C. Caldwell, September 21, 2011.

[27] R. E. Jorgenson and L. K. Warne, "Vector formulation of 3D high-frequency axisymmetric cavity scars: numerical solution", memorandum to M. C. Caldwell, September 14, 2011.

[28] L. K. Warne and R. E. Jorgenson, "Vector formulation of three dimensional high frequency axisymmetric convex cavity scars: scar and random plane wave statistics", memorandum to M. C. Caldwell, September 14, 2012.

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