LA-UR-14-27710

Approved for public release; distribution is unlimited.

Title: The Conforming Virtual Element Method for the convection-diffusion-reaction equation with variable coeffcients.

Author(s): Manzini, Gianmarco
Cangiani, Andrea
Sutton, Oliver

Intended for: Report

Issued: 2014-10-02

The Conforming Virtual Element Method for the convection-diffusion-reaction equation with variable coefficients.

A. Cangiani, [b] G. Manzini, [a] and O. Sutton [b]

[a] *Los Alamos National Laboratory, Theoretical Division, Group T-5, MS B284, Los Alamos, NM-87545, USA*

[b] *Department of Mathematics University of Leicester University Road - Leicester LE1 7RH United Kingdom*

[c] *Dipartimento di Matematica ed Applicazioni, Università degli Studi di Milano - Bicocca, via Cozzi 53, 20125 Milano, Italy*

Abstract

This document describes the *conforming* formulations for *virtual element approximation* of the convection-reaction-diffusion equation with variable coefficients. Emphasis is given to construction of the projection operators onto polynomial spaces of appropriate order. These projections make it possible the virtual formulation to achieve any order of accuracy. We present the construction of the *internal* and the *external* formulation. The difference between the two is in the way the projection operators act on the derivatives (laplacian, gradient) of the partial differential equation. For the diffusive regime we prove the well-posedness of the external formulation and we derive an estimate of the approximation error in the H^1-norm. For the convection-dominated case, the streamline diffusion stabilization (aka SUPG) is also discussed.

Key words: High-order method, unstructured polygonal mesh, virtual element method, diffusion, convection-dominated, reaction problem

1. The convection-diffusion-reaction equation

We consider the problem

$$-\nabla\cdot(\mathsf{K}(\boldsymbol{x})\nabla u)+\boldsymbol{\beta}(\boldsymbol{x})\cdot\nabla u+c(\boldsymbol{x})u=f(\boldsymbol{x})\qquad \text{in }\Omega\subset\mathbb{R}^d \tag{1a}$$

$$u=0\qquad \text{on }\partial\Omega, \tag{1b}$$

where $d=2,3$ is the spatial dimension, $\mathsf{K}\in(C^1(\Omega))^{d\times d}$ is the diffusion tensor, $\boldsymbol{\beta}\in(C(\Omega))^d$ is the convection field, $c\in C(\Omega)$ is the reaction field, and $f\in C(\Omega)$ is the right-hand side forcing function. The diffusion tensor K is a full symmetric $d\times d$-sized matrix and is strongly elliptic, i.e., there exist two strictly positive real constants ξ and η such that

$$\xi|\boldsymbol{v}|^2\leq\boldsymbol{v}\cdot\mathsf{K}(\boldsymbol{x})\boldsymbol{v}\leq\eta|\boldsymbol{v}|^2 \tag{2}$$

for almost every $\boldsymbol{x}\in\Omega$ and for any sufficiently smooth vector field $\boldsymbol{v}$ defined on Ω, where $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^d$. We also suppose that there exists a real constant $m_0>0$ such that

$$c(\boldsymbol{x})-\frac{1}{2}\nabla\cdot\boldsymbol{\beta}(\boldsymbol{x})\geq m_0>0 \tag{3}$$

for almost every $\boldsymbol{x}\in\Omega$.

The variational form of (1) is

$$A(u,v) := \int_\Omega \mathsf{K}\nabla u \cdot \nabla v \, dV + \int_\Omega (\boldsymbol{\beta} \cdot \nabla u)\, v \, dV + \int_\Omega c u \, v \, dV = \int_\Omega f \, v \, dV.$$

We define the following names for the components of the bilinear form $A(\cdot,\cdot)$:

$$a(u,v) := \int_\Omega \mathsf{K}\nabla u \cdot \nabla v \, dV, \qquad b(u,v) := \int_\Omega (\boldsymbol{\beta} \cdot \nabla u)\, v \, dV, \qquad c(u,v) := \int_\Omega c u \, v \, dV.$$

Each term is a bilinear form and can be decomposed in the sum of local contributions that for each element E read as

$$a^E(u,v) := \int_E \mathsf{K}\nabla u \cdot \nabla v \, dV, \qquad b^E(u,v) := \int_E (\boldsymbol{\beta} \cdot \nabla u)\, v \, dV, \qquad c^E(u,v) := \int_E c u \, v \, dV.$$

Hence, it holds that $a(u,v) = \sum_E a^E(u,v)$, etc. The coercivity of the bilinear form A with respect to the energy norm may be shown in the usual way, bounding

$$\begin{aligned} A(v,v) &= \int_\Omega \mathsf{K}\nabla v \cdot \nabla v \, dV + \int_\Omega (\boldsymbol{\beta} \cdot \nabla v) v \, dV + \int_\Omega c v \, v \, dV \\ &\geq \xi \|\nabla v\|_0^2 + \int_\Omega \Big(c - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \Big) v^2 \geq \xi \|\nabla v\|_0^2 + m_0 \|v\|_0^2 \geq \min\{\xi, m_0\} \|v\|_1^2. \end{aligned} \tag{4}$$

Combining inequality (4) with the continuity of A and the linearity of (f,v), it follows that there exists a unique solution to the variational form of problem (1). In practice, the approximate bilinear form is evaluated by using a quadrature rule and the evaluation is not exact when the coefficients are non-polynomial. Instead, by approximating the coefficients with polynomials and using a quadrature rule of sufficient degree, we can quantify and control the error produced. In light of this, the coefficients K, $\boldsymbol{\beta}$ and c are approximated by the polynomials $\widehat{\mathsf{K}}$, $\widehat{\boldsymbol{\beta}}$ and $\widehat{c}$ respectively.

It is required that $\widehat{\mathsf{K}}$ is elliptic on the polynomial space $\mathcal{P}_{k-1}(E)$ for each mesh element E, so that there exist two positive constants $\widehat{\xi}$ and $\widehat{\eta}$ such that

$$\widehat{\xi} |\boldsymbol{p}_{k-1}|^2 \leq \boldsymbol{p}_{k-1} \cdot \widehat{\mathsf{K}}(\boldsymbol{x}) \boldsymbol{p}_{k-1} \leq \widehat{\eta} |\boldsymbol{p}_{k-1}|^2 \tag{5}$$

for almost every $\boldsymbol{x} \in \Omega$ and for any $\boldsymbol{p}_{k-1} \in (\mathcal{P}_{k-1}(E))^d$ (recall that $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^d$). Similarly, we assume that there exists a constant $\widehat{m}_0 > 0$ such that

$$\widehat{c}(\boldsymbol{x}) \geq \widehat{m}_0 > 0 \tag{6}$$

for almost every $\boldsymbol{x} \in \Omega$. We also suppose that $\widehat{\mathsf{K}}$, $\widehat{\boldsymbol{\beta}}$ and $\widehat{c}$ are such that

$$\big\| \widehat{\mathsf{K}} - \mathsf{K} \big\|_{W^{r,\infty}(E)} \leq C h_E^{s-r} \|\mathsf{K}\|_{W^{s,\infty}(E)}, \tag{7a}$$

$$\big\| \widehat{\boldsymbol{\beta}} - \boldsymbol{\beta} \big\|_{W^{r,\infty}(E)} \leq C h_E^{s-r} \|\boldsymbol{\beta}\|_{W^{s,\infty}(E)}, \tag{7b}$$

$$\big\| \widehat{c} - c \big\|_{W^{r,\infty}(E)} \leq C h_E^{s-r} \|c\|_{W^{s,\infty}(E)}, \tag{7c}$$

for some pair of integer numbers s and r with $r < s$. Requirements (5) and (6) ensure the coercivity of the method, while (7a)-(7c) is necessary to maintain the accuracy. Conditions for choosing suitable polynomial degrees and quadrature rules to ensure these properties can be found in [4].

2. Virtual Element Method: the external formulation

Throughout the paper we will make use of the acronyms *VE(s)* for Virtual Element(s) and *VEM* for Virtual Element Method. For the description of the VEM, the choice of the degrees of freedom and the proof of there uni-solvency we mainly refer to [1, 2].

2.1. *The Virtual Element space*

Let E be a closed polygon with a finite number of vertices (and edges).

Definition 1 *For every $k \geq 1$, we define the finite dimensional space $V_h^k(E)$ as*

$$V_h^k(E) := \left\{ v \in H^1(E) \,:\, \Delta v \in \mathbb{P}_{k-2}(E),\, v \in C^0(\partial E), v_{|e} \in \mathbb{P}_k(e) \;\forall e \in \partial E \right\}$$

From this definition, it immediately follows that $\mathbb{P}_k(E) \subset V_h^k(E)$. The conforming VEM is formulated through the following projection operators, whose precise definition and properties will be the subject of the next section:

Elliptic Projection:
– $\Pi_k^{\nabla} \,:\, V_h^k(E) \to \mathbb{P}_k(E)$;
– $\widetilde{\Pi}_k^{\mathsf{K}\nabla} \,:\, V_h^k(E) \to \mathbb{P}_k(E)$;
– $\Pi_k^{\boldsymbol{\beta}} \,:\, V_h^k(E) \to \mathbb{P}_k(E)$;

L^2 Projection:
– $\Pi_k^0 \,:\, V_h^k(E) \to \mathbb{P}_k(E)$;
– $\Pi_{k-1}^0(\nabla\cdot) \,:\, \nabla(V_h^k(E)) \to \big(\mathbb{P}_k(E)\big)^d$;
– $\Pi_k^{\nabla} \,:\, V_h^k(E) \to \mathbb{P}_k(E)$;

2.2. *Approximation of the bilinear form $A(\cdot,\cdot)$*

The bilinear form $A(u,v)$ is approximated on the VE space V_h by the bilinear form

$$A_h(u_h,v_h) := a_h(u_h,v_h) + b_h(u_h,v_h) + c_h(u_h,v_h), \tag{8}$$

where each term in the right-hand side approximates the corresponding term of A. We assume that such terms can be decomposed into the sum of elemental terms, thus definining the approximate bilinear form

$$A_h^E(u_h,v_h) := a_h^E(u_h,v_h) + b_h^E(u_h,v_h) + c_h^E(u_h,v_h), \tag{9}$$

for each element E. We define the elemental contributions to A_h^E by

$$a_h^E(u_h,v_h) := \int_\Omega \widehat{\mathsf{K}}\, \Pi_{k-1}^0(\nabla u_h)\cdot \Pi_{k-1}^0(\nabla v_h)\, dV + S_a^E((\mathrm{I}-\Pi_k^0)u_h, (\mathrm{I}-\Pi_k^0)v_h), \tag{10a}$$

$$b_h^E(u_h,v_h) := \int_\Omega \widehat{\boldsymbol{\beta}}\, \Pi_{k-1}^0(\nabla u_h)\cdot \Pi_k^0(v_h)\, dV, \tag{10b}$$

$$c_h^E(u_h,v_h) := \int_\Omega \widehat{c}\, \Pi_k^0(u_h)\Pi_k^0(v_h)\, dV + S_c^E((\mathrm{I}-\Pi_k^0)u_h, (\mathrm{I}-\Pi_k^0)v_h), \tag{10c}$$

where S_a^E and S_c^E are the stabilising terms. These terms are symmetric and positive definite on the quotient space $V_h^E/\mathcal{P}_k(E)$ and satisfy the stability property:

$$\alpha_* a^E(v,v) \leq S_a^E(v,v) \leq \alpha^* a^E(v,v), \tag{11a}$$

$$\gamma_* c^E(v,v) \leq S_c^E(v,v) \leq \gamma^* c^E(v,v), \tag{11b}$$

for all $v_h \in V_h^E$ with $\Pi_k^0(v_h) = 0$. The first term of each bilinear form is responsible for ensuring the polynomial consistency property. The second term in a_h^E and c_h^E ensures that stability holds. We formally define the properties of *polynomial consistency* and *stability* on each element E of the mesh as follows.

Definition 2

(i) **Polynomial consistency***: Whenever either u_h or v_h or both are elements of the polynomial space $\mathcal{P}_k(E)$, the components of the approximate bilinear forms satisfy*

$$a_h^E(u_h, v_h) = \int_E \widehat{\mathsf{K}}\Pi^0_{k-1}(\nabla u_h) \cdot \Pi^0_{k-1}(\nabla v_h)\, dV,$$

$$b_h^E(u_h, v_h) = \int_E \big(\widehat{\boldsymbol{\beta}} \cdot \Pi^0_{k-1}(\nabla u_h)\big)\Pi^0_k(v_h)\, dV,$$

$$c_h^E(u_h, v_h) = \int_E \widehat{c}\Pi^0_k(u_h)\Pi^0_k(v_h)\, dV.$$

(ii) **Stability***: There exist two pairs of positive constants α_*, α^* and γ_*, γ^* that are independent of h and such that*

$$\alpha_* a^E(v_h, v_h) \leq a_h^E(v_h, v_h) \leq \alpha^* a^E(v_h, v_h) \tag{12a}$$

$$\gamma_* c^E(v_h, v_h) \leq c_h^E(v_h, v_h) \leq \gamma^* c^E(v_h, v_h) \tag{12b}$$

for all $v_h \in V_h^E$ and mesh elements E.

□

Remark 1 The polynomial consistency is an *exactness property.* For example, if K is a polynomial tensor, i.e., $\mathsf{K} = \widehat{\mathsf{K}}$, whenever one of the two functions u_h and v_h is a polynomial, it holds that $a_h^E(u_h, v_h) = a^E(u_h, v_h)$. Moreover, the degrees of freedom that due to the unisolvence property determine uniquely the other (possibly non-polynomial) functions are the minimum knowledge required to compute the value of the bilinear forms $a_h^E(\cdot,\cdot)$ and $a^E(\cdot,\cdot)$. The same is true for b_h^E and c_h^E. □

Remark 2 In the asymptotic diffusive regime, the stability of A_h is provided by the stabilising terms S_a^E and S_c^E. In facts, note here that we may take

$$b_h^E(u_h, v_h) := \int_E \widehat{\boldsymbol{\beta}} \cdot \Pi^0_{k-1}(\nabla u_h)\Pi^0_k(v_h)\, dV \qquad \forall u_h, v_h \in V_h^E,$$

to satisfy the above conditions. The precise form of the stabilising terms is left until Section 3. □

Method 1 (Virtual Element Approximation) *Let A_h be the bilinear form defined in* (8)*, whose construction is detailed above. Suppose that the right-hand side of the variational formulation is given by*

$$(f_h, v_h) = \int_\Omega f_h v_h\, dV \quad \textit{where} \quad f_h := \begin{cases} \Pi^0_{k-2}(f) & \textit{if } k \geq 2, \\ \Pi^0_0(f) & \textit{if } k = 1. \end{cases}$$

The Virtual Element Approximation of problem (1) *reads as:* Find $u_h \in V_h$ such that

$$A_h(u_h, v_h) = (f_h, v_h) \quad \forall v_h \in V_h. \tag{13}$$

□

The well-posedness of the method is discussed in Section 8. The convergence behavior is analyzed in Section 9.

3. The stabilising terms for the diffusive regime

In this section we discuss the construction of the VE stabilizing terms S_a^E and S_c^E. As mentioned above, the present framework is only applicable to the diffusion-dominated case when the Péclet number is sufficiently small, as described in Theorem 1. The convection-dominated case requires extra work to stabilize the method.

The structures of the bilinear forms a_h and c_h are designed to mimic those of

$$\mathcal{A}(u,v) := \int_\Omega \widehat{\mathsf{K}}\Pi^0_{k-1}(\nabla u)\cdot\Pi^0_{k-1}(\nabla v)\,dV + \int_\Omega \mathsf{K}\big(\mathrm{I}-\Pi^0_{k-1}\big)\nabla u\cdot\big(\mathrm{I}-\Pi^0_{k-1}\big)\nabla v\,dV,$$

$$\mathcal{C}(u,v) := \int_\Omega \widehat{c}\Pi^0_k(u)\,\Pi^0_k(v)\,dV + \int_\Omega c\big(\mathrm{I}-\Pi^0_k\big)u\,\big(\mathrm{I}-\Pi^0_k\big)v\,dV,$$

which we will show to satisfy the stability requirements. This result can then be used as a guide to constructing appropriate stabilizing terms. Consider

$$\mathcal{A}^E(u,u) = \int_E \widehat{\mathsf{K}}\Pi^0_{k-1}(\nabla u)\cdot\Pi^0_{k-1}(\nabla u)\,dV + \int_\Omega \mathsf{K}\big(\mathrm{I}-\Pi^0_{k-1}\big)\nabla u\cdot\big(\mathrm{I}-\Pi^0_{k-1}\big)\nabla u\,dV,$$

$$\mathcal{C}^E(u,u) = \int_E \widehat{c}\Pi^0_k(u)\,\Pi^0_k(u)\,dV + \int_\Omega c\big(\mathrm{I}-\Pi^0_k\big)u\,\big(\mathrm{I}-\Pi^0_k\big)u\,dV.$$

For $\mathcal{A}^E(u,u)$, the strong ellipticity of the diffusion tensors K and $\widehat{\mathsf{K}}$ and the definition of the L^2 projector imply that

$$\begin{aligned}
\mathcal{A}^E(u,u) &\geq \widehat{\xi}\big\|\Pi^0_{k-1}(\nabla u)\big\|^2_{0,E} + \xi\big\|\big(\mathrm{I}-\Pi^0_{k-1}\big)\,\nabla u\big\|^2_{0,E} \\
&\geq \min\{\widehat{\xi},\xi\}\left(2\big\|\Pi^0_{k-1}(\nabla u)\big\|^2_{0,E} + \|\nabla u\|^2_{0,E} - 2\int_E \nabla u\cdot\Pi^0_{k-1}(\nabla u)\,dV\right) \\
&= \min\{\widehat{\xi},\xi\}\left(2\big\|\Pi^0_{k-1}(\nabla u)\big\|^2_{0,E} + \|\nabla u\|^2_{0,E} - 2\int_E \big|\Pi^0_{k-1}(\nabla u)\big|^2\,dV\right) \\
&= \min\{\widehat{\xi},\xi\}\|\nabla u\|^2_{0,E}.
\end{aligned}$$

Similarly, for $\mathcal{C}^E(u,u)$ the lower bounds on $c(\boldsymbol{x})$ and $\widehat{c}(\boldsymbol{x})$ mean that

$$\begin{aligned}
\mathcal{C}^E(u,u) &\geq \widehat{m}_0\big\|\Pi^0_k(u)\big\|^2_{0,E} + m_0\big\|\big(\mathrm{I}-\Pi^0_k\big)\,u\big\|^2_{0,E} \\
&\geq \min\{\widehat{m}_0,m_0\}\left(2\big\|\Pi^0_k(u)\big\|^2_{0,E} + \|u\|^2_{0,E} - 2\int_E u\,\Pi^0_k(u)\,dV\right) \\
&= \min\{\widehat{m}_0,m_0\}\left(2\big\|\Pi^0_k(u)\big\|^2_{0,E} + \|u\|^2_{0,E} - 2\int_E \big|\Pi^0_k(u)\big|^2\,dV\right) \\
&= \min\{\widehat{m}_0,m_0\}\|u\|^2_{0,E},
\end{aligned}$$

Hence, $\mathcal{A}^E$ is coercive in the $H^1(E)$ semi-norm and $\mathcal{C}^E$ is coercive in the $L^2(E)$ norm. Since $a^E(v,v)$ and $c^E(v,v)$ are also *continuous* in these norms, the bilinear forms $\mathcal{A}^E$ and $\mathcal{C}^E$ must satisfy the lower part of the stability requirement:

$$a^E(u,u) \leq \zeta^*_a|u|^2_{1,E} \quad\Rightarrow\quad \mathcal{A}^E(u,u) \geq \min\{\widehat{\xi},\xi\}|u|^2_{0,E} \geq \frac{\min\{\widehat{\xi},\xi\}}{\zeta^*}a^E(u,u),$$

$$c^E(u,u) \leq \zeta^*_c\|u\|^2_{0,E} \quad\Rightarrow\quad \mathcal{C}^E(u,u) \geq \min\{\widehat{m}_0,m_0\}\|u\|^2_{0,E} \geq \frac{\min\{\widehat{m}_0,m_0\}}{\zeta^*_c}c^E(u,u).$$

To see that $\mathcal{A}^E(u,u)$ and $\mathcal{C}^E(u,u)$ also satisfy the upper part of the stability requirement, we proceed similarly. Focussing again on $\mathcal{A}^E(u,u)$, the ellipticity of the diffusion tensor and the definition of Π^0_{k-1} imply that

$$\begin{aligned}
\mathcal{A}^E(u,u) &\leq \widehat{\eta}\big\|\Pi^0_{k-1}(\nabla u)\big\|^2_{0,E} + \eta\big\|\big(\mathrm{I}-\Pi^0_{k-1}\big)\,\nabla u\big\|^2 \\
&\leq \max\{\widehat{\eta},\eta\}\left(2\big\|\Pi^0_{k-1}(\nabla u)\big\|_{0,E} + \|\nabla u\|^2_{0,E} - 2\left(\nabla u,\Pi^0_{k-1}(\nabla u)\right)\right) \\
&= \max\{\widehat{\eta},\eta\}\|\nabla u\|^2_{0,E},
\end{aligned}$$

while treating $\mathcal{C}^E(u,u)$ similarly provides

$$\begin{aligned}\mathcal{C}^E(u,u) &\leq \|\widehat{c}\|_\infty \left\|\Pi^0_k u\right\|^2_{0,E} + \|c\|_\infty \left\|\left(\mathrm{I}-\Pi^0_k\right)u\right\|^2 \\ &\leq \max\{\|\widehat{c}\|_\infty, \|c\|_\infty\}\left(2\left\|\Pi^0_k u\right\|_{0,E} + \|u\|^2_{0,E} - 2\left(u, \Pi^0_k u\right)\right) \\ &= \max\{\|\widehat{c}\|_\infty, \|c\|_\infty\}\|u\|^2_{0,E}.\end{aligned}$$

Since $a^E(v,v)$ and $c^E(v,v)$ are *coercive* in these norms, the bilinear forms $\mathcal{A}^E$ and $\mathcal{C}^E$ must satisfy the upper part of the stability requirement:

$$\begin{aligned}\zeta_{a,*}|u|^2_{1,E} \leq a^E(u,u) \quad &\Rightarrow \quad \frac{\max\{\widehat{\xi},\xi\}}{\zeta^*} a^E(u,u) \leq \max\{\widehat{\xi},\xi\}|u|^2_{0,E} \leq \mathcal{A}^E(u,u), \\ \zeta^*_c\|u\|^2_{0,E} \leq c^E(u,u) \quad &\Rightarrow \quad \frac{\max\{\widehat{m}_0, m_0\}}{\zeta^*_c} c^E(u,u) \leq \max\{\widehat{m}_0, m_0\}\|u\|^2_{0,E} \leq \mathcal{C}^E(u,u).\end{aligned}$$

Consequently, we may conclude that the bilinear form A^E_h with stabilising terms given by

$$\begin{aligned}T^E_a(u,v) &:= \int_\Omega \mathsf{K}\left(\mathrm{I}-\Pi^0_{k-1}\right)\nabla u \cdot \left(\mathrm{I}-\Pi^0_{k-1}\right)\nabla v \, dV, \\ T^E_c(u,v) &:= \int_\Omega c\left(\mathrm{I}-\Pi^0_k\right)u \left(\mathrm{I}-\Pi^0_k\right)u \, dV.\end{aligned}$$

instead of S^E_a and S^E_c satisfies the stability property.

For the VE form, however, T^E_a and T^E_c must be approximated by *computable* stabilising terms S^E_a and S^E_c. The next problem, then, is to construct such stabilising terms that maintain the coercivity of the method. First, note that both T^E_a and T^E_c are seminorms on the VE space V^E_h, zero on the polynomial space $\mathcal{P}_k(E)$ and norms on the (finite dimensional) quotient space $V^E_h/\mathcal{P}_k(E)$. Now, let S^E_a and S^E_c be any other bilinear forms which are also seminorms on V^E_h, zero on $\mathcal{P}_k(E)$ and norms on $V^E_h/\mathcal{P}_k(E)$. By the equivalence of norms on finite dimensional spaces, there must exist two pairs of constants (c_1,c_2) and (d_1,d_2) such that

$$\begin{aligned}c_1 T^E_a(v,v) \leq & S^E_a(v,v) \leq c_2 T^E_a(v,v), \\ d_1 T^E_c(v,v) \leq & S^E_c(v,v) \leq d_2 T^E_c(v,v)\end{aligned}$$

for all $v \in V^E_h$; so, the bilinear form

$$\begin{aligned}A_h(u,v) = &\int_\Omega \widehat{\mathsf{K}}\Pi^0_{k-1}(\nabla u)\cdot\Pi^0_{k-1}(\nabla v)\, dV + S_a(u,v) + \int_\Omega \widehat{\boldsymbol{\beta}}\cdot\Pi^0_{k-1}(\nabla u)\,\Pi^0_k(v)\, dV \\ &+ \int_\Omega \widehat{c}\Pi^0_k(u)\,\Pi^0_k(v)\, dV + S_c(u,v)\end{aligned}$$

is coercive and continuous in the H^1 norm.

However, these constants are required to be independent of the mesh parameter h. This can be satisfied by ensuring that the bilinear form S^E_a scales like T_a (i.e. like h^{d-2}) and that S^E_c scales like T_c (i.e. like h^d).

An example of bilinear forms satisfying these requirements is given by

$$\begin{aligned}S^E_a(u,v) &:= \overline{\mathsf{K}}_E h^{d-2}_E \sum_{r=1}^{n_E} \mathrm{dof}_r\left((\mathrm{I}-\Pi^0_k)u\right)\mathrm{dof}_r\left((\mathrm{I}-\Pi^0_k)v\right), \\ S^E_c(u,v) &:= \overline{c}_E h^d_E \sum_{r=1}^{n_E} \mathrm{dof}_r\left((\mathrm{I}-\Pi^0_k)u\right)\mathrm{dof}_r\left((\mathrm{I}-\Pi^0_k)v\right),\end{aligned}$$

where $\overline{\mathsf{K}}_E$ and $\overline{c}_E$ are some constant approximations of K_E and c_E, respectively, and $\mathrm{dof}_r(v_h)$ is the r-th degrees of freedom of v_h.

4. Projection operators

In this section, we review the construction and implementation of the projection operators used in the external and internal VEM. For exposition's sake, we use a compact matrix notation for the shape functions and the polynomials. Indexed notation can be easily recovered by substituting the symbol $\underline{\phi}$ with ϕ_i, $\underline{\phi}^T$ with ϕ_j, $\underline{m}$ with m_α, $\underline{m}^T$ with m_β (and the same for $\underline{\widehat{m}}$ and $\underline{\widehat{m}}^T$) and converting consistently all matrix-vector products into indexed summations. For example, the relation between the monomial basis and the shape functions involves the matrix D and is given by

$$\underline{m}^T = \underline{\phi}^T \mathsf{D} \qquad \text{which is equivalent to} \qquad m_\alpha = \sum_{i=1}^{n_d} \phi_i (\mathsf{D})_{i\alpha} \qquad \text{for } \alpha = 1, \dots, n_p.$$

The notation of matrices B, C, D, G, $\widetilde{\mathsf{G}}$, H is consistent with the notation already used in [3]. Instead, we denote the matrix representation of the "∇"-projection and the L^2-projection of the shape functions with respect of the monomials by $\mathbf{\Pi}_k^\nabla$ and $\mathbf{\Pi}_k^0$, respectively, and with respect to the shape functions by $\mathbf{\Pi}_k^{\nabla,\phi}$ and $\mathbf{\Pi}_k^{0,\phi}$, respectively (we do not use the "starred" notation of [3]). All these quantities are presented and discussed in the next subsections.

4.1. *Shape functions and polynomials*

Shape functions. We denote the shape functions of the local virtual element space $V_h^k(E)$ by ϕ_i for $i = 1, \dots, n_d$, where n_d is the cardinality of the basis set. The indices of the shape functions are in roman fonts, i.e., $i, j, k, \dots$. We use the compact notation:

$$\underline{\phi}^T = \big[\phi_1, \phi_2, \dots, \phi_{n_d}\big].$$

Polynomial basis. We denote the scaled monomials forming a basis of the local polynomials space $\mathbb{P}_k(E)$ by m_α for $\alpha = 1, \dots, n_p$, where n_p is the cardinality of the basis set. The indices of the monomials are in greek fonts, i.e., $\alpha, \beta, \gamma \dots$. We use the compact notation:

$$\underline{m}^T = \big[m_1, m_2, \dots, m_{n_p}\big].$$

Reduced polynomial basis. When we consider the linear space of the polynomials of degree up to $k-1$ (instead of k), we denote the cardinality of the basis by $\widehat{n}_p$ and use the compact notation:

$$\underline{\widehat{m}}^T = \big[m_1, m_2, \dots, m_{\hat{n}_p}\big].$$

Obviously, the monomials in $\underline{\widehat{m}}$ coincide with the first $\hat{n}_p$ monomials in $\underline{m}$.

Matrix D. We collect the degrees of freedom of the monomials m_α with respect to the shape functions ϕ_i in the α-th column of matrix D:

$$\mathsf{D}_{i\alpha} = \mathrm{dof}_i(m_\alpha), \qquad \text{i.e.,} \qquad \underline{m}^T = \underline{\phi}^T \mathsf{D}.$$

4.2. *Projector* $\Pi_k^\nabla : V_h^k(E) \to \mathbb{P}_k(E)$

The projection operator Π_k^∇ is defined through its action on the shape functions $\underline{\phi}$. We use the compact notation:

$$\Pi_k^\nabla(\underline{\phi}^T) = \big[\Pi_k^\nabla(\phi_1), \Pi_k^\nabla(\phi_2), \dots, \Pi_k^\nabla(\phi_{n_d})\big]$$

Formal definition. The projection operator Π_k^∇ is the solution of the elliptic projection problem:

$$P_0\big(\Pi_k^\nabla(\underline{\phi}^T)\big) = P_0\big(\underline{\phi}^T\big), \tag{14a}$$

$$\int_E \nabla\underline{m} \cdot \nabla\Pi_k^\nabla(\underline{\phi}^T)\, dV = \int_E \nabla\underline{m} \cdot \nabla\underline{\phi}^T\, dV, \tag{14b}$$

where P_0 is a suitable projector onto the constant functions defined on E (note, indeed, that second relation only involves the gradient of the shape functions). In matrix form, since $\Pi_k^\nabla(\underline{\phi}^T)$ are polynomials and functions in the virtual element space $V_h^k(E)$ we consider these expansions:

$$\Pi_k^\nabla(\underline{\phi}^T) = \underline{m}^T \boldsymbol{\Pi}_k^\nabla = \underline{\phi}^T \mathsf{D}\boldsymbol{\Pi}_k^\nabla = \underline{\phi}^T \boldsymbol{\Pi}_k^{\nabla,\phi}.$$

By comparison, it follows that:

$$\boldsymbol{\Pi}_k^{\nabla,\phi} = \mathsf{D}\boldsymbol{\Pi}_k^\nabla.$$

Matrices B, $\widetilde{\mathsf{B}}$. The right-hand side of the elliptic projection problem (14a)-(14b) is written as:

$$\widetilde{\mathsf{B}} = \left[\int_E \nabla\underline{m}\cdot\nabla\underline{\phi}^T\,dV\right], \qquad \mathsf{B} = \widetilde{\mathsf{B}} + \begin{bmatrix} P_0(\underline{\phi}^T) \\ 0 \end{bmatrix}.$$

Matrices G, $\widetilde{\mathsf{G}}$. The left-hand side of the elliptic projection problem (14a)-(14b) is written as

$$\widetilde{\mathsf{G}} = \left[\int_E \nabla\underline{m}\,\nabla\underline{m}^T\,dV\right], \qquad \mathsf{G} = \widetilde{\mathsf{G}} + \begin{bmatrix} P_0(\underline{m}^T) \\ 0 \end{bmatrix}.$$

By construction, matrix G is non-singular.

Elliptic projection problem in matrix form. The elliptic projection problem (14a)-(14b) takes the form of the matrix equation

$$\mathsf{G}\boldsymbol{\Pi}_k^\nabla = \mathsf{B} \qquad \text{which implies that} \qquad \boldsymbol{\Pi}_k^\nabla = \mathsf{G}^{-1}\mathsf{B}.$$

Computability issue. To prove that the elliptic projection problem is solvable, we need to prove that the integrals of the right-hand side matrix $\widetilde{\mathsf{B}}$ are computable using only the degrees of freedom of the shape functions. To this end, we integrate by parts:

$$\widetilde{\mathsf{B}} = \int_E \nabla\underline{m}\cdot\nabla\underline{\phi}^T\,dV = -\int_E \Delta\underline{m}\,\underline{\phi}^T\,dV + \sum_{e\in\partial E}\int_e (\mathbf{n}_e\cdot\nabla\underline{m})\,\underline{\phi}^T\,dS,$$

and we note that

$$\int_E \Delta\underline{m}\,\underline{\phi}^T\,dV \ \ \text{is computable using the polynomial moments of degree} \le k-2 \text{ of } \underline{\phi}^T;$$

$$\int_e (\mathbf{n}_e\cdot\nabla\underline{m})\,\underline{\phi}^T\,dS \ \ \text{is computable since the trace of } \underline{\phi}^T \text{ on } e \text{ is a polynomial.}$$

Lemma 1 (Consistency relation)

$$\mathsf{BD} = \mathsf{G}.$$

Proof: We use $\underline{m}^T = \underline{\phi}^T\mathsf{D}$ and the definition of $\widetilde{\mathsf{G}}$ and G:

$$\begin{aligned}\mathsf{BD} &= \left[\int_E \nabla\underline{m}\cdot\nabla\underline{\phi}^T\,dV\right]\mathsf{D} + \begin{bmatrix} P_0(\underline{\phi}^T) \\ 0 \end{bmatrix}\mathsf{D} = \left[\int_E \nabla\underline{m}\cdot\nabla\big(\underline{\phi}^T\mathsf{D}\big)\,dV\right] + \begin{bmatrix} P_0\big(\underline{\phi}^T\mathsf{D}\big) \\ 0 \end{bmatrix} \\ &= \left[\int_E \nabla\underline{m}\cdot\nabla\underline{m}^T\,dV\right] + \begin{bmatrix} P_0(\underline{m}^T) \\ 0 \end{bmatrix} = \widetilde{\mathsf{G}} + \begin{bmatrix} P_0(\underline{m}^T) \\ 0 \end{bmatrix} = \mathsf{G}\end{aligned}$$

□

4.3. *Projector* $\widetilde{\Pi}^{\mathsf{K}\nabla}_k : V_h^k(E) \rightarrow \mathbb{P}_k(E)$

The modified projection operator $\widetilde{\Pi}^{\mathsf{K}\nabla}$ is defined through its action on the shape functions $\underline{\phi}$. We use the compact notation:

$$\widetilde{\Pi}^{\mathsf{K}\nabla}(\underline{\phi}^T) = \big[\widetilde{\Pi}^{\mathsf{K}\nabla}(\phi_1), \widetilde{\Pi}^{\mathsf{K}\nabla}(\phi_2), \dots, \widetilde{\Pi}^{\mathsf{K}\nabla}(\phi_{n_d})\big].$$

Formal definition. The modified projection of the shape functions $\widetilde{\Pi}^{\mathsf{K}\nabla}(\underline{\phi}^T)$ is the solution of the elliptic projection problem defined by (14a) and

$$\int_E \nabla \underline{m} \cdot \nabla \widetilde{\Pi}^{\mathsf{K}\nabla}(\underline{\phi}^T)\, dV = \int_E \Pi^0_{k-1}(\mathsf{K}\nabla \underline{m}) \cdot \nabla \underline{\phi}\, dV. \tag{15}$$

The diffusion tensor K is incorporated in the definition of $\widetilde{\Pi}^{\mathsf{K}\nabla}(u_h)$. The matrix reformulation of (14a) and (15) is straightforward and not presented here.

Computability issue. To prove that the elliptic projection problem (14a) and (15) is solvable, we need to prove that the integrals of the right-hand side of (15) are computable using only the degrees of freedom of the shape functions. To this end, we integrate by part:

$$\int_E \Pi^0_{k-1}(\mathsf{K}\nabla \underline{m}) \cdot \nabla \underline{\phi}^T\, dV = -\int_E \mathrm{div}\big(\Pi^0_{k-1}(\mathsf{K}\nabla \underline{m})\big)\, \underline{\phi}^T\, dV + \sum_{e\in\partial E} \int_e \mathbf{n}_e \cdot \nabla\big(\Pi^0_{k-1}(\mathsf{K}\nabla \underline{m})\big)\, \underline{\phi}^T\, dS.$$

and we note that

$\int_E \mathrm{div}\big(\Pi^0_{k-1}(\mathsf{K}\nabla \underline{m})\big)\, \underline{\phi}^T\, dV$ is computable using the polynomial moments of degree $\leq k-2$ of $\underline{\phi}^T$ because $\mathrm{div}\big(\Pi^0_{k-1}(\mathsf{K}\nabla \underline{m})\big)$ is a polynomial of degree $k-2$;

$\int_e (\mathbf{n}_e \cdot \nabla \underline{m})\, \underline{\phi}^T\, dS$ is computable since the trace of $\underline{\phi}^T$ on e is a polynomial.

4.4. *Projector* $\Pi^{\boldsymbol{\beta}}_k : V_h^k(E) \rightarrow \mathbb{P}_k(E)$

The projection operator $\Pi^{\boldsymbol{\beta}}_k$ is defined through its action on the shape functions $\underline{\phi}$. We use the compact notation:

$$\Pi^{\boldsymbol{\beta}}_k(\underline{\phi}^T) = \big[\Pi^{\boldsymbol{\beta}}_k(\phi_1), \Pi^{\boldsymbol{\beta}}_k(\phi_2), \dots, \Pi^{\boldsymbol{\beta}}_k(\phi_{n_d})\big]$$

The operator $\Pi^{\boldsymbol{\beta}}_k$ is used as internal projector for the advection term $\boldsymbol{\beta} \cdot \nabla(\cdot)$.

Formal definition. The projection operator $\Pi^{\boldsymbol{\beta}}_k$ is the solution of the elliptic projection problem:

$$P_0\big(\Pi^{\boldsymbol{\beta}}_k(\underline{\phi}^T)\big) = P_0\big(\underline{\phi}^T\big), \tag{16}$$

$$\int_E \big[\boldsymbol{\beta} \cdot \nabla \Pi^{\boldsymbol{\beta}}_k \underline{m}\big]\, \big[\boldsymbol{\beta} \cdot \nabla \underline{\phi}^T\big]\, dV = \int_E \big[\boldsymbol{\beta} \cdot \nabla \underline{m}\big]\, \big[\boldsymbol{\beta} \cdot \nabla \underline{\phi}^T\big]\, dV \tag{17}$$

The matrix reformulation of (16)-(17) is straightforward and not presented here.

Computability issue. To prove that the elliptic projection problem is solvable, we need to prove that the integrals of the right-hand side of (17) are computable using only the degrees of freedom of the shape functions. As above, we integrate by parts

$$\int_E \big[\boldsymbol{\beta} \cdot \nabla \underline{m}\big]\, \big[\boldsymbol{\beta} \cdot \nabla \underline{\phi}^T\big]\, dV = -\int_E \big[\boldsymbol{\beta} \cdot \nabla\big]\, \big[\boldsymbol{\beta} \cdot \nabla \underline{m}\big]\, \underline{\phi}^T\, dV + \sum_{e\in\partial E} \int_e \big[\boldsymbol{\beta} \cdot \nabla \underline{m}\big]\, \boldsymbol{\beta} \cdot \mathbf{n}_e \underline{\phi}^T\, dS,$$

and we note that

$$\int_E \left[\boldsymbol{\beta}\cdot\nabla\right]\left[\boldsymbol{\beta}\cdot\nabla\underline{m}\right]\underline{\phi}^T\,dV \quad \text{is computable using the polynomial moments of degree} \leq k-2 \text{ of } \underline{\phi}^T \text{ because } (\boldsymbol{\beta}\cdot\nabla)(\boldsymbol{\beta}\cdot\nabla\underline{m}) \text{ is a polynomial of degree } k-2;$$

$$\int_e \left[\boldsymbol{\beta}\cdot\nabla\underline{m}\right]\boldsymbol{\beta}\cdot\mathbf{n}_e\underline{\phi}^T\,dS \quad \text{is computable since the trace of } \underline{\phi}^T \text{ on } e \text{ is a polynomial.}$$

For the implementation, we consider the expansion:

$$\boldsymbol{\beta}\cdot\nabla(\boldsymbol{\beta}\cdot\nabla\underline{m}) = \boldsymbol{\beta}^T\mathcal{H}(\underline{m})\boldsymbol{\beta} + \big(\boldsymbol{\beta}\cdot\nabla\beta_x\big)\frac{\partial\underline{m}}{\partial x} + \big(\boldsymbol{\beta}\cdot\nabla\beta_y\big)\frac{\partial\underline{m}}{\partial y}$$

where we introduced the Hessian of the polynomial basis $\mathcal{H}(\underline{m}) = \big[\mathcal{H}(m_1), \mathcal{H}(m_2), \ldots, \mathcal{H}(m_{n_p})\big]^T$:

$$\mathcal{H}(m_\alpha) = \begin{bmatrix} \dfrac{\partial^2 m_\alpha}{\partial x^2} & \dfrac{\partial^2 m_\alpha}{\partial x\partial y} \\ \dfrac{\partial^2 m_\alpha}{\partial y\partial x} & \dfrac{\partial^2 m_\alpha}{\partial y^2} \end{bmatrix} \qquad \text{so that} \qquad \boldsymbol{\beta}^T\mathcal{H}(\underline{m})\boldsymbol{\beta} = \beta_x^2\frac{\partial^2\underline{m}}{\partial x^2} + \beta_x\beta_y\frac{\partial^2\underline{m}}{\partial x\partial y} + \beta_y^2\frac{\partial^2\underline{m}}{\partial y^2},$$

(again the compact notation is such that $\frac{\partial^2\underline{m}}{\partial x^2} = \big[\frac{\partial^2 m_1}{\partial x^2}, \frac{\partial^2 m_2}{\partial x^2}, \ldots, \frac{\partial^2 m_{n_p}}{\partial x^2}\big]$, $\frac{\partial^2\underline{m}}{\partial x\partial y} = \ldots$, $\frac{\partial^2\underline{m}}{\partial y^2} = \ldots$).

4.5. *Projector* $\Pi^0_k : V^k_h(E) \to \mathbb{P}_k(E)$

The projection operator Π^0_k is defined through its action on the shape functions $\underline{\phi}$. We use the compact notation:

$$\Pi^0_k(\underline{\phi}^T) = \big[\Pi^0_k(\phi_1), \Pi^0_k(\phi_2), \ldots, \Pi^0_k(\phi_{n_d})\big]$$

Formal definition. The projection operator Π^0_k is the solution of the L^2 orthogonal projection problem:

$$\int_E \underline{m}\,\Pi^0_k(\underline{\phi}^T)\,dV = \int_E \underline{m}\,\underline{\phi}^T\,dV. \tag{18}$$

In matrix form, since $\Pi^0_k(\underline{\phi}^T)$ are polynomials and functions in the virtual element space $V^k_h(E)$ we consider these expansions:

$$\Pi^0_k(\underline{\phi}^T) = \underline{m}^T\boldsymbol{\Pi}^0_k = \underline{\phi}^T\mathsf{D}\boldsymbol{\Pi}^0_k = \underline{\phi}^T\boldsymbol{\Pi}^{0,\phi}_k.$$

By comparison, it follows that:

$$\boldsymbol{\Pi}^{0,\phi}_k = \mathsf{D}\boldsymbol{\Pi}^0_k.$$

Matrix C. The right-hand side of the L^2-orthogonal projection problem (18) is written as:

$$\mathsf{C} = \left[\int_E \underline{m}\underline{\phi}^T\,dV\right].$$

Matrix H. The left-hand side of the L^2-orthogonal projection problem (18) is written as:

$$\mathsf{H} = \left[\int_E \underline{m}\,\underline{m}^T\,dV\right].$$

L^2-orthogonal projection problem in matrix form. The L^2-orthogonal projection problem (18) takes the form of the matrix equation

$$\mathsf{H}\boldsymbol{\Pi}^0_k = \widetilde{\mathsf{C}} \qquad \text{which implies that} \qquad \boldsymbol{\Pi}^0_k = \mathsf{H}^{-1}\widetilde{\mathsf{C}}.$$

Computability issue. In the *conforming formulation*, matrix C is only partially computable using the degrees of freedom of $\underline{\phi}^T$. The full computability is ensured by the *enhancement* [1].

Matrix $\widetilde{\mathsf{C}}$ (enhancement). We define the last rows of C using Π^{∇}:

$$\mathrm{row}_\alpha(\widetilde{\mathsf{C}}) = \begin{cases} \displaystyle\int_E m_\alpha \underline{\phi}^T \, dV & \text{if } \mathsf{degree}(m_\alpha) \le k-2 \\ \displaystyle\int_E m_\alpha \Pi_k^{\nabla}(\underline{\phi}^T) \, dV & \text{if } \mathsf{degree}(m_\alpha) = k-1, k, \end{cases}$$

where $\mathsf{degree}(m_\alpha)$ returns the degree of m_α.

Matrix $\widetilde{\mathsf{C}}$ (alternative enhancement). We define the last rows of C using $\mathsf{H}(\mathsf{D}^T\mathsf{D})\mathsf{D}^T$

$$\mathrm{row}_\alpha(\widetilde{\mathsf{C}}) = \begin{cases} \displaystyle\int_E m_r \cdot \underline{\phi}^T \, dV & \text{if } \mathsf{degree}(m_\alpha) \le k-2 \\ \mathrm{row}_r\big(\mathsf{H}(\mathsf{D}^T\mathsf{D})\mathsf{D}^T\big) & \text{if } \mathsf{degree}(m_\alpha) = k-1, k, \end{cases}$$

where $\mathsf{degree}(m_\alpha)$ returns the degree of m_α.

Lemma 2 (Consistency relation)

$$\mathsf{CD} = \mathsf{H}$$

Proof: We use $\underline{m}^T = \underline{\phi}^T\mathsf{D}$ and the definition of matrix H:

$$\mathsf{CD} = \left[\int_E \underline{m}\,\underline{\phi}^T \, dV\right]\mathsf{D} = \left[\int_E \underline{m}\,(\underline{\phi}^T\mathsf{D})\, dV\right] = \left[\int_E \underline{m}\,\underline{m}^T \, dV\right] = \mathsf{H}$$

□

Remark 3 Since $\mathsf{D}^T\mathsf{D}$ is non-singular:

$$(\mathsf{D}^T\mathsf{D})^{-1}\mathsf{D}^T\,\mathsf{D} = \mathsf{I} \quad\Longrightarrow\quad \big[\mathsf{H}(\mathsf{D}^T\mathsf{D})^{-1}\mathsf{D}^T\big]\,\mathsf{D} = \mathsf{H}$$

By comparison with $\mathsf{CD} = \mathsf{H}$ it follows that

$$\mathsf{C} = \big[\mathsf{H}(\mathsf{D}^T\mathsf{D})^{-1}\mathsf{D}^T\big] + \mathsf{C}_0$$

where $\mathsf{C}_0^T \in \ker(\mathsf{D}^T)$, i.e., $\mathsf{C}_0\mathsf{D} = 0$. Matrix C_0 is normally unknown, and we can take $\mathsf{C}_0 = 0$ to compute the algebraic enhancement, but we cannot use this position for an alternative definition of the projections $\Pi_k^0(\underline{\phi}^T)$. □

Remark 4 *For $k = 1, 2$ the projector operators Π_k^{∇} and Π_k^0 coincide.* □

4.6. *Projector* $\Pi_{k-1}^0(\nabla\cdot) \,:\, \nabla(V_h^k(E)) \to \big(\mathbb{P}_k(E)\big)^d$

This projector operator is defined through its action on the gradients of the shape functions. For $d = 2$, we have

$$\nabla\underline{\phi}^T = \left[\frac{\partial\underline{\phi}^T}{\partial x}, \frac{\partial\underline{\phi}^T}{\partial y}\right]^T$$

and we use the compact notation

$$\Pi_{k-1}^0(\nabla\underline{\phi}^T) = \begin{bmatrix} \Pi_{k-1}^0\left(\dfrac{\partial\underline{\phi}^T}{\partial x}\right) \\ \Pi_{k-1}^0\left(\frac{\partial\underline{\phi}^T}{\partial y}\right) \end{bmatrix} = \begin{bmatrix} \Pi_{k-1}^0\left(\dfrac{\partial\phi_1}{\partial x}\right), \Pi_{k-1}^0\left(\dfrac{\partial\phi_2}{\partial x}\right), \ldots, \Pi_{k-1}^0\left(\dfrac{\partial\phi_{n_d}}{\partial x}\right) \\ \Pi_{k-1}^0\left(\dfrac{\partial\phi_1}{\partial y}\right), \Pi_{k-1}^0\left(\dfrac{\partial\phi_2}{\partial y}\right), \ldots, \Pi_{k-1}^0\left(\dfrac{\partial\phi_{n_d}}{\partial y}\right) \end{bmatrix}.$$

The extension to $d = 3$ is straightforward.

Formal definition. The L^2-orthogonal projection of $\nabla\underline{\phi}$ is defined onto the polynomial space $(\mathbb{P}_{k-1}(E))^d$ by

$$\int_E \underline{\widehat{m}}\,\Pi^0_{k-1}(\nabla\underline{\phi}^T)\,dV = \int_E \underline{\widehat{m}}\,\nabla\underline{\phi}^T\,dV$$

We use the reduced polynomial basis $\underline{\widehat{m}}$ because Π^0_{k-1} is projecting the components of $\nabla\underline{\phi}^T$ onto the polynomials of degree at most $k-1$.

Computability issue. The right-hand side is computable (without enhancement) after the integration by parts:

$$\int_E \underline{\widehat{m}}\,\nabla\underline{\phi}^T\,dV = -\int_E \nabla\underline{\widehat{m}}\,\underline{\phi}^T\,dV + \sum_{\mathsf{e}\in\partial E}\int_e \mathbf{n}_e\cdot\nabla\underline{\widehat{m}}\,\underline{\phi}^T\,dV.$$

Now,

$$\int_E \nabla\underline{\widehat{m}}\,\underline{\phi}^T\,dV \quad \text{is computable using the polynomial moments of degree} \leq k-2 \text{ of } \underline{\phi}^T;$$

$$\int_e (\mathbf{n}_e\cdot\nabla\underline{\widehat{m}})\,\underline{\phi}^T\,dS \quad \text{is computable since the trace of } \underline{\phi}^T \text{ on } e \text{ is a polynomial.}$$

5. Discretization of bilinear forms

5.1. *Virtual element decomposition of local bilinear forms*

Let $\mathbf{u} = [u_1, u_2, \ldots, u_{n_d}]$ and $\mathbf{v} = [v_1, v_2, \ldots, v_{n_d}]$ be the degrees of freedom of the fields u_h and v_h; hence, $u_h = \underline{\phi}^T\mathbf{u}$ and $v_h = \underline{\phi}^T\mathbf{v}$. Then,

$$\int_E \nabla u_h\cdot\nabla v_h\,dV + \int_E (\boldsymbol{\beta}\cdot\nabla u_h)\,v_h\,dV + \int_E c u_h v_h\,dV = \mathbf{u}^T\left[\int_E \nabla\underline{\phi}\cdot\nabla\underline{\phi}^T\,dV + \int_E (\boldsymbol{\beta}\cdot\nabla\underline{\phi})\,\underline{\phi}^T\,dV + \int_E c\underline{\phi}\underline{\phi}^T\,dV\right]\mathbf{v}$$

We decompose the shape functions, their gradients and directional derivatives along $\boldsymbol{\beta}$ by applying the projection operators of the previous section. Gradients and directional derivatives can have an *internal* or an *external* decomposition. In the former case, we take the derivatives of the projected shape functions; in the latter case, we apply the projection operator to the derivatives of the shape functions.

– **Shape functions:**

$$\underline{\phi} = \Pi^0_k(\underline{\phi}) + (I - \Pi^0_k)(\underline{\phi})$$

– **Gradient of the shape functions:**

$$\nabla\underline{\phi} = \Pi^0_{k-1}(\nabla\underline{\phi}) + (I - \Pi^0_{k-1})\nabla\underline{\phi} \qquad \text{(external)}$$
$$\nabla\underline{\phi} = \nabla\Pi^\nabla_k(\underline{\phi}) + \nabla(I - \Pi^\nabla_k)\underline{\phi} \qquad \text{(internal).}$$

– **Directional derivative of the shape functions:**

$$\boldsymbol{\beta}\cdot\nabla\underline{\phi} = \boldsymbol{\beta}\cdot\Pi^0_{k-1}(\nabla\underline{\phi}) + \boldsymbol{\beta}\cdot(I - \Pi^0_{k-1})\nabla\underline{\phi} \qquad \text{(external)}$$
$$\boldsymbol{\beta}\cdot\nabla\Pi^{\boldsymbol{\beta}}_k(\underline{\phi}) = \boldsymbol{\beta}\cdot\nabla\Pi^{\boldsymbol{\beta}}_k(\underline{\phi}) + \boldsymbol{\beta}\cdot\nabla(I - \Pi^{\boldsymbol{\beta}}_k)\underline{\phi} \qquad \text{(internal).}$$

In the last identity, we can consider the projectors Π^0_k and Π^∇_k instead of $\Pi^{\boldsymbol{\beta}}_k$.

The projection onto the polynomials is always computable, while the remaining part is not. We substitute these decompositions into the local bilinear forms and for each case we underline the computable and the non computable part of the decomposition. The "mixed" terms, e.g., $\Pi^0_k(\underline{\phi})\,(I-\Pi^0_k)\underline{\phi}^T$, are normally zero by definition of the projection operators if the coefficients are constant.

- **Diffusion bilinear form (using internal projections)**
  We distinguish between the case with constant coefficient, where for simplicity's sake we take $\mathsf{K}=\mathsf{I}$, and the case with variable coefficients, which uses the modified ∇-projector $\widetilde{\Pi}^{\mathsf{K}\nabla}_k$ defined in Section 4.3.

  - constant coefficients ($\mathsf{K}=\mathsf{I}$):

$$\int_E \nabla\underline{\phi}\cdot\nabla\underline{\phi}^T\,dV = \underbrace{\int_E \nabla\Pi^\nabla_k\underline{\phi}\cdot\nabla\Pi^\nabla_k\underline{\phi}^T\,dV}_{\text{computable}} + \underbrace{\int_E \nabla(I-\Pi^\nabla_k)\underline{\phi}\cdot\nabla(I-\Pi^\nabla_k)\underline{\phi}^T\,dV}_{\text{non computable}}.$$

  - variable coefficients:

$$\int_E \mathsf{K}\nabla\underline{\phi}\cdot\nabla\underline{\phi}^T\,dV = \underbrace{\int_E \nabla\widetilde{\Pi}^{\mathsf{K}\nabla}_k\underline{\phi}\cdot\nabla\Pi^\nabla_k\underline{\phi}^T\,dV}_{\text{computable}} + \underbrace{\int_E \nabla\widetilde{\Pi}^{\mathsf{K}\nabla}_k\underline{\phi}\cdot\nabla(I-\Pi^\nabla_k)\underline{\phi}^T\,dV}_{\text{non computable}}$$

$$+ \underbrace{\int_E \nabla(I-\widetilde{\Pi}^{\mathsf{K}\nabla}_k)\underline{\phi}\cdot\nabla\Pi^\nabla_k(\underline{\phi}^T)\,dV}_{\text{non computable}} + \underbrace{\int_E \nabla(I-\widetilde{\Pi}^{\mathsf{K}\nabla}_k)\underline{\phi}\cdot\nabla(I-\Pi^\nabla_k)\underline{\phi}^T\,dV}_{\text{non computable}}.$$

  Remark 5 *The "mixed" terms are always zero if K is constant; in such a case, the two definitions above coincide.* □

- **Diffusion bilinear form (using external projections)**

$$\int_E \mathsf{K}\nabla\underline{\phi}\cdot\nabla\underline{\phi}^T\,dV = \underbrace{\int_E \mathsf{K}\Pi^0_{k-1}(\nabla\underline{\phi})\cdot\Pi^0_{k-1}(\nabla\underline{\phi}^T)\,dV}_{\text{computable}} + \underbrace{\int_E \mathsf{K}\Pi^0_{k-1}(\nabla\underline{\phi})\cdot(I-\Pi^0_{k-1})\nabla\underline{\phi}^T\,dV}_{\text{non computable}}$$

$$+ \underbrace{\int_E \mathsf{K}(I-\Pi^0_{k-1})\nabla\underline{\phi}\cdot\Pi^0_{k-1}(\nabla\underline{\phi}^T)\,dV}_{\text{non computable}} + \underbrace{\int_E \mathsf{K}(I-\Pi^0_{k-1})\nabla\underline{\phi}\cdot(I-\Pi^0_{k-1})\nabla\underline{\phi}^T\,dV}_{\text{non computable}}.$$

  Remark 6 *The two "mixed" terms are zero if K is constants or if we redefine the orthogonal projections with respect to an inner product that is weighted by K.* □

- **Convection bilinear form (using internal projections)**

$$\int_E \underline{\phi}\boldsymbol{\beta}\cdot\nabla\underline{\phi}^T\,dV = \underbrace{\int_E \Pi^0_k(\underline{\phi})\,\boldsymbol{\beta}\cdot\nabla(\Pi^{\boldsymbol{\beta}}_k(\underline{\phi})^T)\,dV}_{\text{computable}} + \underbrace{\int_E \Pi^0_k(\underline{\phi})\,\boldsymbol{\beta}\cdot\nabla(I-\Pi^{\boldsymbol{\beta}}_k)\underline{\phi}^T\,dV}_{\text{non computable}}$$

$$+ \underbrace{\int_E (I-\Pi^0_k)\underline{\phi}\,\boldsymbol{\beta}\cdot\nabla\Pi^{\boldsymbol{\beta}}_k(\underline{\phi}^T)\,dV}_{\text{non computable}} + \underbrace{\int_E (I-\Pi^0_k)\underline{\phi}\,\boldsymbol{\beta}\cdot\nabla(I-\Pi^{\boldsymbol{\beta}}_k)\underline{\phi}^T\,dV}_{\text{non computable}}.$$

  Remark 7 *The second "mixed" term is zero if $\boldsymbol{\beta}$ is a constant vector field.* □

Remark 8 *Alternatively, in the directional derivative $\boldsymbol{\beta}\cdot\nabla$ we can consider one of the two projectors Π_k^{∇} and Π_k^0 instead of $\Pi_k^{\boldsymbol{\beta}}$.* □

- **Convection bilinear form (using external projections)**

$$\int_E \underline{\phi}\boldsymbol{\beta}\cdot\nabla\underline{\phi}^T\,dV = \underbrace{\int_E \Pi_k^0(\underline{\phi})\,\boldsymbol{\beta}\cdot\Pi_{k-1}^0(\nabla\underline{\phi}^T)\,dV}_{\text{computable}} + \underbrace{\int_E \Pi_k^0(\underline{\phi})\,\boldsymbol{\beta}\cdot(I-\Pi_{k-1}^0)\nabla\underline{\phi}^T\,dV}_{\text{non computable}}$$
$$+\underbrace{\int_E (I-\Pi_k^0)\underline{\phi}\,\boldsymbol{\beta}\cdot\Pi_{k-1}^0(\nabla\underline{\phi}^T)\,dV}_{\text{non computable}} + \underbrace{\int_E (I-\Pi_k^0)\underline{\phi}\,\boldsymbol{\beta}\cdot(I-\Pi_{k-1}^0)\nabla\underline{\phi}^T\,dV}_{\text{non computable}}\,.$$

Remark 9 *The second "mixed" term is zero if $\boldsymbol{\beta}$ is a constant vector field.* □

- **Reaction bilinear form**

$$\int_E c\,\underline{m}\,\underline{m}^T\,dV = \underbrace{\int_E c\,\Pi_k^0(\underline{m})\,\Pi_k^0(\underline{m}^T)\,dV}_{\text{computable}} + \underbrace{\int_E c\,\Pi_k^0(\underline{m})\,(I-\Pi_k^0)(\underline{m}^T)\,dV}_{\text{non computable}}$$
$$+\underbrace{\int_E c\,(I-\Pi_k^0)(\underline{m})\,\Pi_k^0(\underline{m}^T)\,dV}_{\text{non computable}} + \underbrace{\int_E c\,(I-\Pi_k^0)(\underline{m})\,(I-\Pi_k^0)(\underline{m}^T)\,dV}_{\text{non computable}}\,.$$

Remark 10 *The two "mixed" terms are zero if c is constant or if we redefine the orthogonal projections with respect to an inner product that is weighted by c.* □

5.2. *Virtual element discretization of local bilinear forms*

We defined the local bilinear forms as follows:

$$\begin{aligned}
a_h^E(\underline{\phi},\underline{\phi}^T) &:= [\mathsf{Consistency}] + [\mathsf{Stabilization}]\\
b_h^E(\underline{\phi},\underline{\phi}^T) &:= [\mathsf{Consistency}]\\
c_h^E(\underline{\phi},\underline{\phi}^T) &:= [\mathsf{Consistency}]\\
s_h^E(\underline{\phi},\underline{\phi}^T) &:= [\mathsf{Consistency}] + [\mathsf{Stabilization}]\\
(f_h,\underline{\phi}^T)_E &:= [\mathsf{Consistency}],
\end{aligned}$$

where [Consistency] is the *computable part* of the virtual element decompositions of the previous section, and [Stabilization] is a suitable "modeling" of the *non computable part.* To define the stabilization term, we do not consider the mixed terms. Both will be discussed in the next subsections. Note that only the diffusive term and the SUPG term include a stabilization term in their respective definition.

5.2.1. *Implementation of the diffusion term*

- The **internal VEM discretization**
  We distinguish between the case with constant coefficients, where for simplicity we take $\mathsf{K} = \mathsf{I}$, and the case with variable coefficients, which uses the modified ∇-projector $\widetilde{\Pi}_k^{\mathsf{K}\nabla}$ defined in Section 4.3.

- constant K:

$$\int_E \nabla\Pi^{\nabla}\underline{\phi}\cdot\nabla\Pi^{\nabla}\underline{\phi}^T\,dV = (\boldsymbol{\Pi}_k^{\nabla,\phi})^T\left[\int_E \nabla\underline{m}\cdot\nabla\underline{m}^T\right]\boldsymbol{\Pi}_k^{\nabla,\phi}\,dV = (\boldsymbol{\Pi}_k^{\nabla,\phi})^T\widetilde{\mathsf{G}}\boldsymbol{\Pi}_k^{\nabla,\phi}$$

- variable K:

$$\int_E \mathsf{K}\nabla\widetilde{\Pi}^{\mathsf{K}\nabla}\underline{\phi}\cdot\nabla\underline{\phi}^T\,dV = \int_E \Pi_{k-1}^0(\mathsf{K}\nabla\underline{\phi})\cdot\nabla\underline{\phi}^T\,dV = \left[\int_E \Pi_{k-1}^0(\mathsf{K}\nabla\underline{\phi})\cdot\nabla\underline{m}^T\,dV\right]\boldsymbol{\Pi}_k^{\nabla,\phi}$$
$$= (\widetilde{\boldsymbol{\Pi}}_{k-1}^{\nabla,\phi})^T\left[\int_E \nabla\underline{m}\cdot\nabla\underline{m}^T\,dV\right]\boldsymbol{\Pi}_k^{\nabla,\phi}.$$

where $\Pi_{k-1}^0(\mathsf{K}\nabla\underline{\phi}^T) = \underline{m}^T(\widetilde{\boldsymbol{\Pi}}_{k-1}^{\nabla,\phi})$ includes the diffusion tensor k.

- The **external VEM discretization**

$$\int_E \mathsf{K}\,\Pi_{k-1}^0(\nabla\underline{\phi})\cdot\Pi_{k-1}^0(\nabla\underline{\phi}^T)\,dV =$$
$$= \int_E \mathsf{K}_{xx}\,\Pi_{k-1}^0\left(\frac{\partial\underline{\phi}}{\partial x}\right)\Pi_{k-1}^0\left(\frac{\partial\underline{\phi}^T}{\partial x}\right)dV + \int_E \mathsf{K}_{xy}\,\Pi_{k-1}^0\left(\frac{\partial\underline{\phi}}{\partial x}\right)\Pi_{k-1}^0\left(\frac{\partial\underline{\phi}^T}{\partial y}\right)dV$$
$$+ \int_E \mathsf{K}_{yx}\,\Pi_{k-1}^0\left(\frac{\partial\underline{\phi}}{\partial y}\right)\Pi_{k-1}^0\left(\frac{\partial\underline{\phi}^T}{\partial x}\right)dV \int_E \mathsf{K}_{yy}\,\Pi_{k-1}^0\left(\frac{\partial\underline{\phi}}{\partial y}\right)\Pi_{k-1}^0\left(\frac{\partial\underline{\phi}^T}{\partial y}\right)dV.$$

Since the projections are polynomials of degree $\leq k$, it holds that

$$\Pi_{k-1}^0\left(\frac{\partial\underline{\phi}}{\partial x}\right) = \underline{\widehat{m}}^T\boldsymbol{\Pi}_{k-1}^{0,x}, \qquad \Pi_{k-1}^0\left(\frac{\partial\underline{\phi}}{\partial y}\right) = \underline{\widehat{m}}^T\boldsymbol{\Pi}_{k-1}^{0,y}. \tag{19}$$

Therefore,

$$\int_E \mathsf{K}_{xx}\,\Pi_{k-1}^0\left(\frac{\partial\underline{\phi}}{\partial x}\right)\Pi_{k-1}^0\left(\frac{\partial\underline{\phi}}{\partial x}\right)dV = \left(\boldsymbol{\Pi}_{k-1}^{0,x}\right)^T\left[\int_E \mathsf{K}_{xx}\,\underline{\widehat{m}}\,\underline{\widehat{m}}^T\,dV\right]\boldsymbol{\Pi}_{k-1}^{0,x} = \left(\boldsymbol{\Pi}_{k-1}^{0,x}\right)^T\mathsf{H}_{xx}^K\boldsymbol{\Pi}_{k-1}^{0,x}$$

$$\int_E \mathsf{K}_{xy}\,\Pi_{k-1}^0\left(\frac{\partial\underline{\phi}}{\partial x}\right)\Pi_{k-1}^0\left(\frac{\partial\underline{\phi}}{\partial y}\right)dV = \left(\boldsymbol{\Pi}_{k-1}^{0,x}\right)^T\left[\int_E \mathsf{K}_{xy}\,\underline{\widehat{m}}\,\underline{\widehat{m}}^T\,dV\right]\boldsymbol{\Pi}_{k-1}^{0,y} = \left(\boldsymbol{\Pi}_{k-1}^{0,x}\right)^T\mathsf{H}_{xy}^K\boldsymbol{\Pi}_{k-1}^{0,y}$$

$$\int_E \mathsf{K}_{yx}\,\Pi_{k-1}^0\left(\frac{\partial\underline{\phi}}{\partial y}\right)\Pi_{k-1}^0\left(\frac{\partial\underline{\phi}}{\partial x}\right)dV = \left(\boldsymbol{\Pi}_{k-1}^{0,y}\right)^T\left[\int_E \mathsf{K}_{yx}\,\underline{\widehat{m}}\,\underline{\widehat{m}}^T\,dV\right]\boldsymbol{\Pi}_{k-1}^{0,x} = \left(\boldsymbol{\Pi}_{k-1}^{0,y}\right)^T\mathsf{H}_{yx}^K\boldsymbol{\Pi}_{k-1}^{0,x}$$

$$\int_E \mathsf{K}_{yy}\,\Pi_{k-1}^0\left(\frac{\partial\underline{\phi}}{\partial y}\right)\Pi_{k-1}^0\left(\frac{\partial\underline{\phi}}{\partial y}\right)dV = \left(\boldsymbol{\Pi}_{k-1}^{0,y}\right)^T\left[\int_E \mathsf{K}_{yy}\,\underline{\widehat{m}}\,\underline{\widehat{m}}^T\,dV\right]\boldsymbol{\Pi}_{k-1}^{0,y} = \left(\boldsymbol{\Pi}_{k-1}^{0,y}\right)^T\mathsf{H}_{yy}^K\boldsymbol{\Pi}_{k-1}^{0,y}.$$

The last equalities in each formula below imply the obvious definitions:

$$\mathsf{H}_{xx}^K = \int_E \mathsf{K}_{xx}\,\underline{\widehat{m}}\,\underline{\widehat{m}}^T\,dV, \qquad \mathsf{H}_{xy}^K = \int_E \mathsf{K}_{xy}\,\underline{\widehat{m}}\,\underline{\widehat{m}}^T\,dV,$$

$$\mathsf{H}_{yx}^K = \int_E \mathsf{K}_{yx}\,\underline{\widehat{m}}\,\underline{\widehat{m}}^T\,dV, \qquad \mathsf{H}_{yy}^K = \int_E \mathsf{K}_{yy}\,\underline{\widehat{m}}\,\underline{\widehat{m}}^T\,dV,$$

5.2.2. *Implementation of the convection term*

- The **external VEM discretization**

Note that:

$$\boldsymbol{\beta}\cdot\Pi_{k-1}^0(\nabla\underline{\phi}^T) = \beta_x\Pi_{k-1}^0\left(\frac{\partial\underline{\phi}^T}{\partial x}\right) + \beta_y\Pi_{k-1}^0\left(\frac{\partial\underline{\phi}^T}{\partial y}\right). \tag{20}$$

Therefore, using (19) yields:

$$\int_E \boldsymbol{\beta}\Pi_k^0(\underline{\phi}) \cdot \Pi_{k-1}^0\big(\nabla\underline{\phi}^T\big)\, dV = \int_E \beta_x \Pi_k^0(\underline{\phi}) \cdot \Pi_{k-1}^0\left(\frac{\partial \underline{\phi}^T}{\partial x}\right) dV + \int_E \beta_y \Pi_k^0(\underline{\phi}) \cdot \Pi_{k-1}^0\left(\frac{\partial \underline{\phi}^T}{\partial y}\right) dV$$

$$= \big(\mathbf{\Pi}_k^0\big)^T \left[\int_E \beta_x\, \underline{m}\, \underline{\widehat{m}}^T\, dV\right] \mathbf{\Pi}_{k-1}^{0,x} + \big(\mathbf{\Pi}_k^0\big)^T \left[\int_E \beta_y\, \underline{m}\, \underline{\widehat{m}}^T\, dV\right] \mathbf{\Pi}_{k-1}^{0,y}$$

$$= \big(\mathbf{\Pi}_k^0\big)^T \mathsf{H}_x^\beta \mathbf{\Pi}_{k-1}^{0,x} + \big(\mathbf{\Pi}_k^0\big)^T \mathsf{H}_y^\beta \mathbf{\Pi}_{k-1}^{0,y}$$

with the obvious definitions:

$$\mathsf{H}_x^\beta = \int_E \beta_x \underline{m}\, \underline{\widehat{m}}^T\, dV, \qquad \mathsf{H}_y^\beta = \int_E \beta_y \underline{m}\, \underline{\widehat{m}}^T\, dV.$$

- The **internal VEM discretization**
Let $\mathbf{\Pi}_k^\beta$ be the matrix representation of $\Pi_k^{\boldsymbol{\beta}}$ on the polynomial basis $\underline{m}^T$, i.e., $\Pi_k^{\boldsymbol{\beta}}(\underline{\phi}^T) = \underline{m}^T \mathbf{\Pi}_k^\beta$. Then,

$$\int_E \Pi_k^0(\underline{\phi})\, \boldsymbol{\beta} \cdot \nabla\Pi(\underline{\phi}^T)\, dV = \big(\mathbf{\Pi}_k^0\big)^T \left[\int_E \underline{m}\, \boldsymbol{\beta} \cdot \nabla \underline{m}^T\, dV\right] \mathbf{\Pi} = \big(\mathbf{\Pi}_k^0\big)^T \mathbb{H}^\beta \mathbf{\Pi}$$

with the obvious definition:

$$\mathbb{H}^\beta = \int_E \underline{m}\, \boldsymbol{\beta} \cdot \nabla \underline{m}^T\, dV.$$

Alternatively, in the directional derivative we can consider one of the two projector operators Π_k^∇ or Π_k^0 instead of $\Pi_k^{\boldsymbol{\beta}}$.

5.3. *Implementation of the reaction term*

We use $\Pi_k^0(\underline{\phi}^T) = \underline{m}^T \mathbf{\Pi}_k^0$:

$$\int_E c\, \Pi_k^0(\underline{\phi}) \Pi_k^0(\underline{\phi}^T)\, dV = \big(\mathbf{\Pi}_k^0\big)^T \left[\int_E c\, \underline{m}\, \underline{m}^T\, dV\right] \mathbf{\Pi}_k^0 = \big(\mathbf{\Pi}_k^0\big)^T \mathbb{H}^c \mathbf{\Pi}_k^0 \qquad \text{where} \qquad \mathbb{H}^c = \int_E c\, \underline{m}\, \underline{m}^T\, dV.$$

6. Stabilization of bilinear forms

A stabilizing term is easily built by substituting $\phi \simeq 1$ and $\nabla\phi \simeq h_E^{-1}$, by substituting each coefficient with a constant estimate, and computing the integral of the remaining terms.

6.1. *Diffusion matrix*

Let $\mathsf{K} = \mathsf{I}$. We consider the virtual decomposition:

$$\int_E \nabla\underline{\phi} \cdot \nabla\underline{\phi}^T\, dV = \underbrace{\int_E \nabla\Pi_k^\nabla \underline{\phi} \cdot \nabla\Pi_k^\nabla \underline{\phi}^T\, dV}_{\text{computable}} + \underbrace{\int_E \nabla(I - \Pi_k^\nabla)\underline{\phi} \cdot \nabla(I - \Pi_k^\nabla)\underline{\phi}^T\, dV}_{\text{non computable}}.$$

Since $\Pi_k^\nabla(\underline{\phi})$ is a polynomial, we have

$$\nabla(I - \Pi_k^\nabla)\underline{\phi}^T = \underline{\phi}^T - \Pi_k^\nabla(\underline{\phi}^T) = \underline{\phi}^T - \underline{\phi}^T \mathbf{\Pi}_k^{\nabla,\phi} = \underline{\phi}^T(\mathsf{I} - \mathbf{\Pi}_k^{\nabla,\phi}).$$

We substitute this expression in the "non-computable" term and we find:

$$\int_E \nabla(I - \Pi_k^\nabla)\underline{\phi} \cdot \nabla(I - \Pi_k^\nabla)\underline{\phi}^T\, dV = (\mathsf{I} - \mathbf{\Pi}_k^{\nabla,\phi})^T \left[\int_E \nabla\underline{\phi} \cdot \nabla\underline{\phi}\, dV\right] (\mathsf{I} - \mathbf{\Pi}_k^{\nabla,\phi})$$

The stabilization term is provided by substituting

$$\int_E \nabla\underline{\phi}\cdot\nabla\underline{\phi}\,dV \approx |E|h_E^{-2}\mathsf{I}$$

into the integral above, which gives

$$\mathsf{stab}\big[a_h^E\big] := |E|h_E^{-2}(\mathsf{I}-\boldsymbol{\Pi}_k^{\nabla,\phi})^T(\mathsf{I}-\boldsymbol{\Pi}_k^{\nabla,\phi})$$

If $\mathsf{K}\neq\mathsf{I}$ and variable, we first approximate the diffusion tensor $\mathsf{K}\approx\mathsf{K}_E$, where K_E is constant on E, and then we consider

$$\mathsf{stab}\big[a_h^E\big] := \kappa_E|E|h_E^{-2}(\mathsf{I}-\boldsymbol{\Pi}_k^{\nabla,\phi})^T(\mathsf{I}-\boldsymbol{\Pi}_k^{\nabla,\phi})$$

where κ_E is the trace of K_E divided by d.

6.2. *Convection term*

We consider the virtual decomposition:

$$\int_E \underline{\phi}\boldsymbol{\beta}\cdot\nabla\underline{\phi}^T\,dV \approx \underbrace{\int_E \Pi_k^0(\underline{\phi})\,\boldsymbol{\beta}\cdot\nabla\Pi_k^{\nabla}(\underline{\phi}^T)\,dV}_{\text{computable}} + \underbrace{\int_E (I-\Pi_k^0)\underline{\phi}\,\boldsymbol{\beta}\cdot\nabla(I-\Pi_k^{\nabla})\underline{\phi}^T\,dV}_{\text{non computable}},$$

where the mixed terms are neglected. Since $\Pi_k^0(\underline{\phi})$ and $\Pi_{k-1}^0(\nabla\underline{\phi})$ are polynomials or vectors of polynomials, we have

$$\begin{aligned}
&\Pi_k^0(\underline{\phi}^T) = \underline{m}^T\boldsymbol{\Pi}_k^0 = \underline{\phi}^T\boldsymbol{\Pi}_k^{0,\phi}\\
&(I-\Pi_k^{\nabla})\underline{\phi}^T = \underline{\phi}^T - \Pi_k^{\nabla}(\underline{\phi}^T) = \underline{\phi}^T - \underline{\phi}^T\boldsymbol{\Pi}_k^{\nabla,\phi} = \underline{\phi}^T(\mathsf{I}-\boldsymbol{\Pi}_k^{\nabla,\phi})
\end{aligned}$$

We substitute these expressions in the "non-computable" term and we find:

$$\int_E (I-\Pi_k^0)\underline{\phi}\,\boldsymbol{\beta}\cdot\nabla(I-\Pi_k^{\nabla})\underline{\phi}^T\,dV = \boldsymbol{\Pi}_k^{0,\phi^T}\left[\int_E \underline{\phi}\,\boldsymbol{\beta}\cdot\nabla\underline{\phi}^T\,dV\right](\mathsf{I}-\boldsymbol{\Pi}_k^{\nabla,\phi}).$$

A stabilization term can be derived by substituting

$$\int_E \underline{\phi}\,\boldsymbol{\beta}\cdot\nabla\underline{\phi}^T\,dV \approx |\boldsymbol{\beta}||E|h_E^{-1}\mathsf{I}$$

into the integral above, which gives

$$\mathsf{stab}\big[b_h^E\big] := |\boldsymbol{\beta}||E|h_E^{-1}\boldsymbol{\Pi}_k^{0,\phi^T}(\mathsf{I}-\boldsymbol{\Pi}_k^{\nabla,\phi}).$$

6.3. *Reaction term*

We consider the virtual decomposition:

$$\int_E c\,\underline{m}\,\underline{m}^T\,dV \approx \underbrace{\int_E c\,\Pi_k^0(\underline{m})\,\Pi_k^0(\underline{m}^T)\,dV}_{\text{computable}} + \underbrace{\int_E c\,(I-\Pi_k^0)(\underline{m})\,(I-\Pi_k^0)(\underline{m}^T)\,dV}_{\text{non computable}},$$

where the mixed terms are neglected. Since $\Pi_k^0(\underline{\phi})$ are polynomials, we have

$$(I-\Pi_k^0)\underline{\phi}^T = \underline{\phi}^T - \underline{m}^T\boldsymbol{\Pi}_k^0 = \underline{\phi}^T(\mathsf{I}-\boldsymbol{\Pi}_k^{0,\phi}).$$

We substitute this expression in the "non-computable" term and we find:

$$\int_E c\,(I-\Pi_k^0)(\underline{m})\,(I-\Pi_k^0)(\underline{m}^T)\,dV = (\mathsf{I}-\boldsymbol{\Pi}_k^0)^T\left[\int_E c\,\underline{\phi}\,\underline{\phi}^T\,dV\right](\mathsf{I}-\boldsymbol{\Pi}_k^{0,\phi}).$$

We approximate $c \approx c_E$, this latter being a constant, as, for example, the cell-average or the value at the barycenter of E, and then we have the stabilization term by substituting

$$\int_E c\,\underline{\phi}\,\underline{\phi}^T\,dV \approx |c_E||E|\mathsf{I}$$

in the integral above and we obtain the stabilization term:

$$\mathsf{stab}\big[c_h^E\big] := |c_E||E|\,(\mathsf{I}-\mathbf{\Pi}_k^0)^T\,(\mathsf{I}-\mathbf{\Pi}_k^{0,\phi}).$$

6.4. *Streamline diffusion upwind stabilization (SUPG)*

We recall that $n_p = \#(\underline{m})$, $\widehat{n}_p = \#(\widehat{\underline{m}})$, $n_d = \#(\underline{\phi})$, where $\#(\xi)$ denotes the cardinality of set ξ.

6.4.1. *Implementation of the convection term*

- **External projection**; using (20):

$$\int_E \boldsymbol{\beta}\cdot\Pi_{k-1}^0(\nabla\underline{\phi})\,\boldsymbol{\beta}\cdot\Pi_{k-1}^0(\nabla\underline{\phi}^T)\,dV =$$

$$= \left(\mathbf{\Pi}_{k-1}^{0,x}\right)^T\left(\int_E \beta_x^2\widehat{\underline{m}}\,\widehat{\underline{m}}^T\,dV\right)\mathbf{\Pi}_{k-1}^{0,x} + \left(\mathbf{\Pi}_{k-1}^{0,x}\right)^T\left(\int_E \beta_x\beta_y\widehat{\underline{m}}\,\widehat{\underline{m}}^T\,dV\right)\mathbf{\Pi}_{k-1}^{0,y}$$

$$+ \left(\mathbf{\Pi}_{k-1}^{0,y}\right)^T\left(\int_E \beta_y\beta_x\widehat{\underline{m}}\,\widehat{\underline{m}}^T\,dV\right)\mathbf{\Pi}_{k-1}^{0,x} + \left(\mathbf{\Pi}_{k-1}^{0,y}\right)^T\left(\int_E \beta_y^2\widehat{\underline{m}}\,\widehat{\underline{m}}^T\,dV\right)\mathbf{\Pi}_{k-1}^{0,y}$$

$$= \left(\mathbf{\Pi}_{k-1}^{0,x}\right)^T\mathsf{H}_{xx}^{\beta}\mathbf{\Pi}_{k-1}^{0,x} + \left(\mathbf{\Pi}_{k-1}^{0,x}\right)^T\mathsf{H}_{xy}^{\beta}\mathbf{\Pi}_{k-1}^{0,y} + \left(\mathbf{\Pi}_{k-1}^{0,y}\right)^T\mathsf{H}_{yx}^{\beta}\mathbf{\Pi}_{k-1}^{0,x} + \left(\mathbf{\Pi}_{k-1}^{0,y}\right)^T\mathsf{H}_{yy}^{\beta}\mathbf{\Pi}_{k-1}^{0,y}$$

where

$$\mathsf{H}_{xx}^{\beta} = \int_E \beta_x^2\widehat{\underline{m}}\,\widehat{\underline{m}}^T\,dV \qquad \mathsf{H}_{xy}^{\beta} = \int_E \beta_x\beta_y\widehat{\underline{m}}\,\widehat{\underline{m}}^T\,dV$$

$$\mathsf{H}_{yx}^{\beta} = \int_E \beta_y\beta_x\widehat{\underline{m}}\,\widehat{\underline{m}}^T\,dV \qquad \mathsf{H}_{yy}^{\beta} = \int_E \beta_y^2\widehat{\underline{m}}\,\widehat{\underline{m}}^T\,dV$$

The sizes of the previous matrix operators are

$$\widehat{n}_p \times n_d = \mathsf{size}(\mathbf{\Pi}_{k-1}^{0,x}) = \mathsf{size}(\mathbf{\Pi}_{k-1}^{0,y})$$
$$\widehat{n}_p \times \widehat{n}_p = \mathsf{size}(\mathsf{H}_{xx}^{\beta}) = \mathsf{size}(\mathsf{H}_{xy}^{\beta}) = \mathsf{size}(\mathsf{H}_{yx}^{\beta}) = \mathsf{size}(\mathsf{H}_{yy}^{\beta}).$$

- **Internal projection**; since $\Pi_k^{\boldsymbol{\beta}}(\underline{\phi}^T)$ is a polynomial, we have $\Pi_k^{\boldsymbol{\beta}}(\underline{\phi}^T) = \underline{m}^T\mathbf{\Pi}_k^{\beta}$, where the matrix $\mathbf{\Pi}_k^{\beta}$ collects the coefficients of the polynomial expansion. Then, we have:

$$\int_E \big[\boldsymbol{\beta}\cdot\nabla\Pi_k^{\boldsymbol{\beta}}(\underline{\phi})\big]\,\big[\boldsymbol{\beta}\cdot\nabla\Pi_k^{\boldsymbol{\beta}}(\underline{\phi}^T)\big]\,dV = (\mathbf{\Pi}_k^{\beta})^T\left[\int_E \big[\boldsymbol{\beta}\cdot\nabla\underline{m}\big]\,\big[\boldsymbol{\beta}\cdot\nabla\underline{m}^T\big]\,dV\right]\mathbf{\Pi}_k^{\beta} = (\mathbf{\Pi}_k^{\beta})^T\mathsf{H}^{\beta\beta}\mathbf{\Pi}_k^{\beta}$$

with the obvious definition

$$\mathsf{H}^{\beta\beta} = \int_E \big[\boldsymbol{\beta}\cdot\nabla\underline{m}\big]\,\big[\boldsymbol{\beta}\cdot\nabla\underline{m}^T\big]\,dV = \int_E \nabla\underline{m}\;\boldsymbol{\beta}\boldsymbol{\beta}^T\,\nabla\underline{m}^T\,dV.$$

6.4.2. *Stabilization of the SUPG convection term ("stabilization of the stabilization")*

- **Internal projection**. We start from the decomposition

$$\int_E \left[\boldsymbol{\beta}\cdot\nabla\underline{\phi}\right]\left[\boldsymbol{\beta}\cdot\nabla\underline{\phi}^T\right] dV = \underbrace{\int_E \left[\boldsymbol{\beta}\cdot\nabla\Pi_k^\nabla(\underline{\phi})\right]\left[\boldsymbol{\beta}\cdot\nabla\Pi_k^\nabla(\underline{\phi}^T)\right] dV}_{\text{computable}}$$

$$+ \underbrace{\int_E \left[\boldsymbol{\beta}\cdot\nabla(I-\Pi_k^\nabla(\underline{\phi}))\right]\left[\boldsymbol{\beta}\cdot\nabla(I-\Pi_k^\nabla(\underline{\phi}^T))\right] dV}_{\text{non computable}}.$$

The second integral in the right-hand side is non-computable but it cannot be completely neglected because it stabilizes the full virtual bilinear form when the diffusive term is negligible. It is also called *the stabilization of the stabilization*. Note that

$$(I-\Pi_k^\nabla)\underline{\phi}^T = \underline{\phi}^T - \Pi_k^\nabla(\underline{\phi}^T) = \underline{\phi}^T - \underline{m}^T\boldsymbol{\Pi}_k^\nabla = \underline{\phi}^T - \underline{\phi}^T\mathsf{D}\boldsymbol{\Pi}_k^\nabla = \underline{\phi}^T(\mathsf{I}-\mathsf{D}\boldsymbol{\Pi}_k^\nabla). \tag{21}$$

Using this development, the stabilization of the stabilization becomes:

$$\int_E \left[\boldsymbol{\beta}\cdot\nabla(I-\Pi_k^\nabla(\underline{\phi}))\right]\left[\boldsymbol{\beta}\cdot\nabla(I-\Pi_k^\nabla(\underline{\phi}^T))\right] dV = \left(\mathsf{I}-\mathsf{D}\boldsymbol{\Pi}_k^\nabla\right)^T\left[\int_E \nabla\underline{\phi}\,\boldsymbol{\beta}\boldsymbol{\beta}^T\,\nabla\underline{\phi}^T\,dV\right]\left(\mathsf{I}-\mathsf{D}\boldsymbol{\Pi}_k^\nabla\right).$$

The integral in parenthesis is not computable unless we know $\nabla\underline{\phi}$. A possibility is to evaluate this term by a set of barycentric coordinates, e.g., the Weichspress shape functions $\{\underline{\phi}_{\mathrm{W}}\}$. Alternatively, we consider the approximation

$$\int_E \nabla\underline{\phi}\,\boldsymbol{\beta}\boldsymbol{\beta}^T\,\nabla\underline{\phi}^T\,dV \approx |\boldsymbol{\beta}|^2\,h_E^{d-2}\,\mathsf{I},$$

(I being the identity matrix). Since $\boldsymbol{\Pi}_k^{\nabla,\phi} = \mathsf{D}\boldsymbol{\Pi}_k^\nabla$, the resulting formula is

$$\begin{aligned}\int_E \boldsymbol{\beta}\cdot\nabla(I-\Pi_k^\nabla(\underline{\phi}))\,\boldsymbol{\beta}\cdot\nabla(I-\Pi_k^\nabla(\underline{\phi}^T))\,dV &\approx |\boldsymbol{\beta}|^2\,h_E^{d-2}\left(\mathsf{I}-\mathsf{D}\boldsymbol{\Pi}_k^\nabla\right)^T\left(\mathsf{I}-\mathsf{D}\boldsymbol{\Pi}_k^\nabla\right)\\ &= |\boldsymbol{\beta}|^2\,h_E^{d-2}\left(\mathsf{I}-\boldsymbol{\Pi}_k^{\nabla,\phi}\right)^T\left(\mathsf{I}-\boldsymbol{\Pi}_k^{\nabla,\phi}\right)\\ &=: \mathsf{stab}\left[s_h^E\right]\end{aligned}$$

which holds for $d=2,3$.

- **External projection**. We start from the decomposition

$$\int_E \left[\boldsymbol{\beta}\cdot\nabla\underline{\phi}\right]\left[\boldsymbol{\beta}\cdot\nabla\underline{\phi}^T\right] dV = \underbrace{\int_E \left[\boldsymbol{\beta}\cdot\Pi_{k-1}^0(\nabla\underline{\phi})\right]\left[\boldsymbol{\beta}\cdot\Pi_{k-1}^0(\nabla\underline{\phi}^T)\right] dV}_{\text{computable}}$$

$$+ \underbrace{\int_E \left[\boldsymbol{\beta}\cdot(I-\Pi_{k-1}^0)(\nabla\underline{\phi})\right]\left[\boldsymbol{\beta}\cdot(I-\Pi_{k-1}^0)(\nabla\underline{\phi}^T)\right] dV}_{\text{non computable}}.$$

Again,the second integral in the right-hand side is the so-called *stabilization of the stabilization*. This term is not computable, and we simply substitute is with the previous internal approximation.

6.4.3. *Implementation of the reaction term*

The reaction term is approximated through the following development:

$$
\begin{aligned}
\int_E c\,\Pi^0_k(\underline{\phi})\,\boldsymbol{\beta}\cdot\Pi^0_{k-1}(\nabla\underline{\phi}^T)\,dV &= \int_E c\,\Pi^0_k(\underline{\phi})\beta_x\Pi^0_{k-1}\left(\frac{\partial\underline{\phi}^T}{\partial x}\right)\,dV + \int_E c\,\Pi^0_k(\underline{\phi})\beta_y\Pi^0_{k-1}\left(\frac{\partial\underline{\phi}^T}{\partial y}\right)\,dV\\
&= \left(\boldsymbol{\Pi}^0_k\right)^T\left(\int_E c\beta_x\,\underline{m}\,\underline{\widehat{m}}^T\,dV\right)\boldsymbol{\Pi}^{0,x}_{k-1} + \left(\boldsymbol{\Pi}^0_k\right)^T\left(\int_E c\beta_y\,\underline{m}\,\underline{\widehat{m}}^T\,dV\right)\boldsymbol{\Pi}^{0,y}_{k-1}\\
&= \left(\boldsymbol{\Pi}^0_k\right)^T\mathsf{H}^{c\beta}_x\boldsymbol{\Pi}^{0,x}_{k-1} + \left(\boldsymbol{\Pi}^0_k\right)^T\mathsf{H}^{c\beta}_y\boldsymbol{\Pi}^{0,y}_{k-1}
\end{aligned}
$$

with the obvious definitions:

$$
\mathsf{H}^{c\beta}_x = \int_E c\beta_x\,\underline{m}\,\underline{\widehat{m}}^T\,dV, \qquad \mathsf{H}^{c\beta}_y = \int_E c\beta_y\,\underline{m}\,\underline{\widehat{m}}^T\,dV.
$$

The size of the previous matrix operator is

$$
\begin{aligned}
&n_p\times n_d = \mathsf{size}(\boldsymbol{\Pi}^0_k)\\
&\widehat{n}_p\times n_d = \mathsf{size}(\boldsymbol{\Pi}^{0,x}_{k-1}) = \mathsf{size}(\boldsymbol{\Pi}^{0,y}_{k-1})\\
&n_p\times\widehat{n}_p = \mathsf{size}(\mathsf{H}^{c\beta}_x) = \mathsf{size}(\mathsf{H}^{c\beta}_y).
\end{aligned}
$$

6.4.4. *Implementation of the forcing term*

The forcing term is approximated through the following development:

$$
\begin{aligned}
\int_E f\,\boldsymbol{\beta}\cdot\Pi^0_{k-1}(\nabla\underline{\phi}^T)\,dV &= \int_E f\,\beta_x\Pi^0_{k-1}\left(\frac{\partial\underline{\phi}^T}{\partial x}\right)\,dV + \int_E f\,\beta_y\Pi^0_{k-1}\left(\frac{\partial\underline{\phi}^T}{\partial y}\right)\,dV\\
&= \left(\int_E f\beta_x\,\underline{\widehat{m}}^T\,dV\right)\boldsymbol{\Pi}^{0,x}_{k-1} + \left(\int_E f\beta_y\,\underline{\widehat{m}}^T\,dV\right)\boldsymbol{\Pi}^{0,y}_{k-1}\\
&= \mathbf{f}^\beta_x\boldsymbol{\Pi}^{0,x}_{k-1} + \mathbf{f}^\beta_y\boldsymbol{\Pi}^{0,y}_{k-1}.
\end{aligned}
$$

with the obvious definitions:

$$
\mathbf{f}^\beta_x = \int_E f\beta_x\,\underline{\widehat{m}}^T\,dV, \qquad \mathbf{f}^\beta_y = \int_E f\beta_y\,\underline{\widehat{m}}^T\,dV
$$

7. Virtual formulations for variable coefficients

The stabilization term $s^E_h(\underline{\phi},\underline{\phi}^T)$ is present only when the scheme is working in the convection-dominated regime. We consider the following four possible formulations. Variants can be designed by combining differently the projection operators.

(**i**) The **external formulation**; the local bilinear forms and the right-hand side functional are given by:

$$
A^E_h(\underline{\phi},\underline{\phi}^T) := a^E_h(\underline{\phi},\underline{\phi}^T) + b^E_h(\underline{\phi},\underline{\phi}^T) + c^E_h(\underline{\phi},\underline{\phi}^T) + s^E_h(\underline{\phi},\underline{\phi}^T),
$$

with

$$
\begin{aligned}
a_h^E(\underline{\phi},\underline{\phi}^T) &:= \int_E \mathsf{K}\Pi^0_{k-1}(\nabla\underline{\phi})\cdot\Pi^0_{k-1}(\nabla\underline{\phi}^T)\,dV + \mathsf{stab}\big[a_h^E\big],\\
\mathsf{stab}\big[a_h^E\big] &:= \mathrm{Trace}(\mathsf{K})h_E^{d-2}\,(\mathsf{I}-\mathbf{\Pi}_k^{\nabla,\phi})^T(\mathsf{I}-\mathbf{\Pi}_k^{\nabla,\phi}),\\
b_h^E(\underline{\phi},\underline{\phi}^T) &:= \int_E \boldsymbol{\beta}\cdot\Pi^0_{k-1}(\nabla\underline{\phi})\,\Pi^0_k(\underline{\phi}^T)\,dV,\\
c_h^E(\underline{\phi},\underline{\phi}^T) &:= \int_E \Pi^0_k(\underline{\phi})\,\Pi^0_k(\underline{\phi}^T)\,dV,\\
s_h^E(\underline{\phi},\underline{\phi}^T) &:= \tau_E\int_E \big[\boldsymbol{\beta}\cdot\Pi^0_{k-1}(\nabla\underline{\phi})\big]\,\big[\boldsymbol{\beta}\cdot\Pi^0_{k-1}(\nabla\underline{\phi}^T)\big]\,dV + \tau_E\mathsf{stab}\big[s_h^E\big],\\
\mathsf{stab}\big[s_h^E\big] &:= |\boldsymbol{\beta}|^2\,h_E^{d-2}\,(\mathsf{I}-\mathbf{\Pi}_k^{\nabla,\phi})^T(\mathsf{I}-\mathbf{\Pi}_k^{\nabla,\phi}),
\end{aligned}
$$

and

$$
(f_h,\underline{\phi}^T)_E := \int_E f_h\,\big(\underline{\phi}^T+\boldsymbol{\beta}\cdot\Pi^0_k(\underline{\phi}^T)\big)\,dV
$$

Remark 11 *A possible choice for the stabilization parameter is given by $\tau_E = c_E h_E/|\boldsymbol{\beta}|^2$, where $c_E \in [0.1, 1]$.* □

Remark 12 *This formulation is suitable to both constant and variable coefficients.* □

(ii) The **internal formulation using** Π_k^∇; local bilinear forms and right-hand side functional are given by:

$$
A_h^E(\underline{\phi},\underline{\phi}^T) := a_h^E(\underline{\phi},\underline{\phi}^T) + b_h^E(\underline{\phi},\underline{\phi}^T) + c_h^E(\underline{\phi},\underline{\phi}^T) + s_h^E(\underline{\phi},\underline{\phi}^T),
$$

with

$$
\begin{aligned}
a_h^E(\underline{\phi},\underline{\phi}^T) &:= a_h^{E,0}(\underline{\phi},\underline{\phi}^T) + \mathsf{stab}\big[a_h^E\big]\\
a_h^{E,0}(\underline{\phi},\underline{\phi}^T) &:= \begin{cases} \displaystyle\int_E \mathsf{K}\nabla\Pi_k^\nabla(\underline{\phi})\cdot\nabla\Pi_k^\nabla(\underline{\phi}^T)\,dV & \big[\text{constant } \mathsf{K}\big]\\ \displaystyle\int_E \nabla\widetilde{\Pi}^{\mathsf{K}\nabla}(\underline{\phi})\cdot\nabla\Pi_k^\nabla(\underline{\phi}^T)\,dV & \big[\text{variable } \mathsf{K}\big]\end{cases}\\
\mathsf{stab}\big[a_h^E\big] &:= \mathrm{Trace}(\mathsf{K})h_E^{d-2}\,(\mathsf{I}-\mathbf{\Pi}_k^{\nabla,\phi})^T(\mathsf{I}-\mathbf{\Pi}_k^{\nabla,\phi}),\\
b_h^E(\underline{\phi},\underline{\phi}^T) &:= \int_E \big[\boldsymbol{\beta}\cdot\nabla\Pi_k^\nabla(\underline{\phi})\big]\,\Pi^0_k(\underline{\phi}^T)\,dV,\\
c_h^E(\underline{\phi},\underline{\phi}^T) &:= \int_E \Pi^0_k(\underline{\phi})\,\Pi^0_k(\underline{\phi}^T)\,dV,\\
s_h^E(\underline{\phi},\underline{\phi}^T) &:= \tau_E\int_E \big[\boldsymbol{\beta}\cdot\nabla\Pi_k^\nabla(\underline{\phi})\big]\,\big[\boldsymbol{\beta}\cdot\nabla\Pi_k^\nabla(\underline{\phi}^T)\big]\,dV + \tau_E\mathsf{stab}\big[s_h^E\big],\\
\mathsf{stab}\big[s_h^E\big] &:= |\boldsymbol{\beta}|^2\,h_E^{d-2}\,(\mathsf{I}-\mathbf{\Pi}_k^{\nabla,\phi})^T(\mathsf{I}-\mathbf{\Pi}_k^{\nabla,\phi}),
\end{aligned}
$$

and

$$
(f_h,\underline{\phi}^T)_E := \int_E f_h\,\big(\underline{\phi}^T+\boldsymbol{\beta}\cdot\Pi^0_k(\underline{\phi}^T)\big)\,dV.
$$

Remark 13 For a variable K the bilinear form a_h^E must use the modified projector $\widetilde{\Pi}^{\mathsf{K}\nabla}(\underline{\phi})$, which incorporate the diffusion tensor. Furthermore, note that $\Pi_k^\nabla = \Pi_k^0$ when $k = 1, 2$; so, for the lowest-order cases there is no difference between the internal formulation using Π_k^∇ and Π_k^0. □

(**iii**) The **internal formulation using** $\Pi_k^{\boldsymbol{\beta}}$; local bilinear forms and right-hand side functional are given by:

$$A_h^E(\underline{\phi}, \underline{\phi}^T) := a_h^E(\underline{\phi}, \underline{\phi}^T) + b_h^E(\underline{\phi}, \underline{\phi}^T) + c_h^E(\underline{\phi}, \underline{\phi}^T) + s_h^E(\underline{\phi}, \underline{\phi}^T)$$

with

$$\begin{aligned}
a_h^E(\underline{\phi}, \underline{\phi}^T) &:= a_h^{E,0}(\underline{\phi}, \underline{\phi}^T) + \mathsf{stab}\big[a_h^E\big] \\
a_h^{E,0}(\underline{\phi}, \underline{\phi}^T) &:= \begin{cases} \displaystyle\int_E \mathsf{K}\nabla\Pi_k^\nabla(\underline{\phi}) \cdot \nabla\Pi_k^\nabla(\underline{\phi}^T)\, dV & \big[\text{constant } \mathsf{K}\big] \\ \displaystyle\int_E \nabla\widetilde{\Pi}^{\mathsf{K}\nabla}(\underline{\phi}) \cdot \nabla\Pi_k^\nabla(\underline{\phi}^T)\, dV & \big[\text{variable } \mathsf{K}\big] \end{cases} \\
\mathsf{stab}\big[a_h^E\big] &:= \text{Trace}(\mathsf{K}) h_E^{d-2}\, (\mathsf{I} - \boldsymbol{\Pi}_k^{\nabla,\phi})^T(\mathsf{I} - \boldsymbol{\Pi}_k^{\nabla,\phi}), \\
b_h^E(\underline{\phi}, \underline{\phi}^T) &:= \int_E \big[\boldsymbol{\beta} \cdot \nabla\Pi_k^{\boldsymbol{\beta}}(\underline{\phi})\big]\, \Pi_k^0(\underline{\phi}^T)\, dV, \\
c_h^E(\underline{\phi}, \underline{\phi}^T) &:= \int_E \Pi_k^0(\underline{\phi})\, \Pi_k^0(\underline{\phi}^T)\, dV, \\
s_h^E(\underline{\phi}, \underline{\phi}^T) &:= \tau_E \int_E \big[\boldsymbol{\beta} \cdot \nabla\Pi_k^{\boldsymbol{\beta}}(\underline{\phi})\big]\, \big[\boldsymbol{\beta} \cdot \nabla\Pi_k^{\boldsymbol{\beta}}(\underline{\phi}^T)\big]\, dV + \tau_E \mathsf{stab}\big[s_h^E\big], \\
\mathsf{stab}\big[s_h^E\big] &:= |\boldsymbol{\beta}|^2\, h_E^{d-2}\, (\mathsf{I} - \boldsymbol{\Pi}_k^{\nabla,\phi})^T(\mathsf{I} - \boldsymbol{\Pi}_k^{\nabla,\phi}),
\end{aligned}$$

and

$$(f_h, \underline{\phi}^T)_E := \int_E f_h\, \big(\underline{\phi}^T + \boldsymbol{\beta} \cdot \Pi_k^0(\underline{\phi}^T)\big)\, dV$$

Remark 14 This formulation differs from the previous one because we use $\Pi_k^{\boldsymbol{\beta}}$ instead of Π_k^∇ as internal projector. Again, the bilinear form a_h^E has a different definition for constant and variable diffusion tensors K. □

(**iv**) The **internal-external formulation**; local bilinear forms and right-hand side functional are given by:

$$A_h^E(\underline{\phi}, \underline{\phi}^T) := a_h^E(\underline{\phi}, \underline{\phi}^T) + b_h^E(\underline{\phi}, \underline{\phi}^T) + c_h^E(\underline{\phi}, \underline{\phi}^T) + s_h^E(\underline{\phi}, \underline{\phi}^T)$$

with

$$a_h^E(\underline{\phi}, \underline{\phi}^T) := a_h^{E,0}(\underline{\phi}, \underline{\phi}^T) + \mathsf{stab}\big[a_h^E\big]$$

$$a_h^{E,0}(\underline{\phi}, \underline{\phi}^T) := \begin{cases} \displaystyle\int_E \mathsf{K}\nabla\Pi_k^\nabla(\underline{\phi}) \cdot \nabla\Pi_k^\nabla(\underline{\phi}^T)\, dV & \big[\text{constant } \mathsf{K}\big] \\ \displaystyle\int_E \nabla\widetilde{\Pi}^{\mathsf{K}\nabla}(\underline{\phi}) \cdot \nabla\Pi_k^\nabla(\underline{\phi}^T)\, dV & \big[\text{variable } \mathsf{K}\big] \end{cases}$$

$$\mathsf{stab}\big[a_h^E\big] := \mathrm{Trace}(\mathsf{K}) h_E^{d-2}\, (\mathsf{I} - \boldsymbol{\Pi}_k^{\nabla,\phi})^T (\mathsf{I} - \boldsymbol{\Pi}_k^{\nabla,\phi}),$$

$$b_h^E(\underline{\phi}, \underline{\phi}^T) := \int_E \big[\boldsymbol{\beta} \cdot \Pi_{k-1}^0(\nabla\underline{\phi})\big]\, \Pi_k^0(\underline{\phi}^T)\, dV,$$

$$c_h^E(\underline{\phi}, \underline{\phi}^T) := \int_E \Pi_k^0(\underline{\phi})\, \Pi_k^0(\underline{\phi}^T)\, dV,$$

$$s_h^E(\underline{\phi}, \underline{\phi}^T) := \tau_E \int_E \big[\boldsymbol{\beta} \cdot \nabla\Pi_k^{\boldsymbol{\beta}}(\underline{\phi})\big]\, \big[\boldsymbol{\beta} \cdot \nabla\Pi_k^{\boldsymbol{\beta}}(\underline{\phi}^T)\big]\, dV + \tau_E \mathsf{stab}\big[s_h^E\big],$$

$$\mathsf{stab}\big[s_h^E\big] := |\boldsymbol{\beta}|^2\, h_E^{d-2}\, (\mathsf{I} - \boldsymbol{\Pi}_k^{\nabla,\phi})^T (\mathsf{I} - \boldsymbol{\Pi}_k^{\nabla,\phi}),$$

and

$$(f_h, \underline{\phi}^T)_E := \int_E f_h\, \big(\underline{\phi}^T + \boldsymbol{\beta} \cdot \Pi_k^0(\underline{\phi}^T)\big)\, dV$$

Remark 15 This formulation differs from the previous ones because we use the internal (elliptic) projection in a_h^E and the external projection for the convection term. Again, the bilinear form a_h^E has a different definition for constant and variable diffusion tensors K. □

(v) Stabilization based on Weichspress shape functions. In convection-dominated problems, a variant of the internal and external formulations presented above is obtained by considering the stabilization:

$$\mathsf{stab}_W\big[s_h^E\big] := \big(\mathsf{I} - \boldsymbol{\Pi}_k^{\nabla,\phi}\big)^T \left[\int_E \nabla\underline{\phi}_W\, \boldsymbol{\beta}\boldsymbol{\beta}^T\, \nabla\underline{\phi}_W^T\, dV\right] \big(\mathsf{I} - \boldsymbol{\Pi}_k^{\nabla,\phi}\big),$$

where direct evaluation of the integral is done by using the Weichspress barycentric coordinates $\big\{\underline{\phi}_W\big\}$.

8. Well-posedness of the external VEM

In this section, we discuss the well-posedness of the external VEM presented in Section 2, i.e., the existence and uniqueness of the virtual element approximation. This result is a consequence of the coercivity of the VEM bilinear form.

Definition 3 (Péclet number) *For each element E we define the* local mesh Péclet number *as*

$$Pe_E := \frac{h_E\left(\|\boldsymbol{\beta}\|_{\infty,E} + \|\boldsymbol{\beta}\|_{W^{1,\infty}(E)}\right)}{\alpha_*\xi},$$

where α_* and ξ are the stability constant of (11a) *and the ellipticity constant of* (2)*. Considering all the local Péclet numbers, we define the* global mesh Péclet number *as $Pe := \max_E(Pe_E)$.* □

Lemma 3 (Coercivity of A_h) *Under the assumptions of polynomial consistency and stability along with those on the coefficients K, $\boldsymbol{\beta}$ and c and their approximations, the bilinear form A_h is coercive with respect to the energy norm if the mesh Péclet number Pe is sufficiently small. Thus, there exists a (conveniently defined) positive constant α independent of h such that*

$$A_h(v_h, v_h) \geq \alpha\|v_h\|_1^2 \qquad \forall v_h \in V_h.$$

Proof. From (9), the stability requirements (11a)-(11b) and the ellipticity of the diffusion tensor it can be seen that

$$\begin{aligned} A_h^E(v_h,v_h) &\geq \alpha_* a^E(v_h,v_h) + b_h^E(v_h,v_h) + \gamma_* c^E(v_h,v_h) \\ &\geq \alpha_*\xi|v_h|_{1,E}^2 + b^E(v_h,v_h) + \gamma_* c^E(v_h,v_h) + \Big[b_h^E(v_h,v_h) - b^E(v_h,v_h)\Big] \\ &\geq \alpha_*\xi|v_h|_{1,E}^2 + \min(1,\gamma_*)\big(b^E(v_h,v_h) + c^E(v_h,v_h)\big) + \Big[b_h^E(v_h,v_h) - b^E(v_h,v_h)\Big] \\ &\geq \alpha_*\xi|v_h|_{1,E}^2 + \min(1,\gamma_*)m_0\|v_h\|_{0,E}^2 - \big|b_h^E(v_h,v_h) - b^E(v_h,v_h)\big|. \qquad \forall v_h \in V_h^E. \end{aligned} \tag{22}$$

The convection term $b_h^E(v_h,v_h) - b^E(v_h,v_h)$ can be expanded as

$$\begin{aligned} b_h^E(v_h,v_h) - b^E(v_h,v_h) &= \int_E (\widehat{\boldsymbol\beta}\cdot\Pi_{k-1}^0(\nabla v_h))\,\Pi_k^0(v_h)\,dV - \int_E (\boldsymbol\beta\cdot\nabla v_h)\,v_h\,dV \\ &= \int_E \big((\widehat{\boldsymbol\beta}-\boldsymbol\beta)\cdot\nabla v_h\big)\,v_h\,dV + \int_E \big(\widehat{\boldsymbol\beta}(\Pi_k^0-\mathrm{I})v_h\big)\cdot\nabla v_h\,dV \\ &\quad + \int_E \widehat{\boldsymbol\beta}\Pi_k^0(v_h)\cdot(\Pi_{k-1}^0-\mathrm{I})(\nabla v_h)\,dV, \end{aligned}$$

from which

$$\begin{aligned} \big|b_h^E(v_h,v_h) - b^E(v_h,v_h)\big| &\leq \left|\int_E \big((\widehat{\boldsymbol\beta}-\boldsymbol\beta)\cdot\nabla v_h\big)\,v_h\,dV\right| + \left|\int_E \big(\widehat{\boldsymbol\beta}(\Pi_k^0-\mathrm{I})v_h\big)\cdot\nabla v_h\,dV\right| \\ &\quad + \left|\int_E \widehat{\boldsymbol\beta}\Pi_k^0(v_h)\cdot(\Pi_{k-1}^0-\mathrm{I})(\nabla v_h)\,dV\right|. \end{aligned}$$

To bound the first term, use must be made of the requirement (7b) and the Cauchy-Schwarz inequality, so

$$\left|\int_E \big((\widehat{\boldsymbol\beta}-\boldsymbol\beta)\cdot\nabla v_h\big)\,v_h\,dV\right| \leq \|\widehat{\boldsymbol\beta}-\boldsymbol\beta\|_\infty\,\|\nabla v_h\|_{0,E}\,\|v_h\|_{0,E} \leq Ch_E\|\boldsymbol\beta\|_{W^{1,\infty}(E)}\,|v_h|_{1,E}\,\|v_h\|_{0,E}.$$

The penultimate term is bounded by

$$\begin{aligned} \left|\int_E (\widehat{\boldsymbol\beta}(\Pi_k^0-\mathrm{I})v_h)\cdot\nabla v_h\,dV\right| &\leq \|\widehat{\boldsymbol\beta}\|_{\infty,E}\,\big\|(\Pi_k^0-\mathrm{I})v_h\big\|_{0,E}\,\|\nabla v_h\|_{0,E} \leq Ch_E\|\widehat{\boldsymbol\beta}\|_{\infty,E}\,|v_h|_{1,E}^2 \\ &\leq Ch_E\|\boldsymbol\beta\|_{\infty,E}\,|v_h|_{1,E}^2, \end{aligned}$$

since the triangular inequality and approximation property (7b) imply that

$$\|\widehat{\boldsymbol\beta}\|_{\infty,E} \leq \|\boldsymbol\beta\|_{\infty,E} + \|\widehat{\boldsymbol\beta}-\boldsymbol\beta\|_{\infty,E} \leq \|\boldsymbol\beta\|_{\infty,E} + Ch_E\|\boldsymbol\beta\|_{W^{1,\infty}(E)} \approx \|\boldsymbol\beta\|_{\infty,E} \tag{23}$$

for sufficiently small h_E. Since $\widehat{\boldsymbol\beta} = (\widehat{\boldsymbol\beta}_x,\widehat{\boldsymbol\beta}_y)^T$, we can split the third integral as

$$\int_E \widehat{\boldsymbol\beta}\Pi_k^0(v_h)\cdot(\Pi_{k-1}^0-\mathrm{I})\nabla v_h\,dV = \int_E \widehat{\boldsymbol\beta}_x\Pi_k^0(v_h)(\Pi_{k-1}^0-\mathrm{I})\left(\frac{\partial v_h}{\partial x}\right) + \int_E \widehat{\boldsymbol\beta}_y\Pi_k^0(v_h)(\Pi_{k-1}^0-\mathrm{I})\left(\frac{\partial v_h}{\partial y}\right).$$

The final term is bounded by [1]

[1] We use the fact that Π_{k-1}^0 is the orthogonal projection onto the polynomials of degree $k-1$:
$\int_E \Pi_k^0(v_h)\cdot\Pi_{k-1}^0\left(\frac{\partial v_h}{\partial x}\right)\,dV = \int_E \Pi_{k-1}^0\big(\Pi_k^0(v_h)\big)\Pi_{k-1}^0\left(\frac{\partial v_h}{\partial x}\right)\,dV = \int_E \Pi_{k-1}^0\big(\Pi_k^0(v_h)\big)\frac{\partial v_h}{\partial x}\,dV.$
A similar relation holds for the integral containing $\partial v_h/\partial y$.

$$
\begin{aligned}
&\left|\int_E \widehat{\boldsymbol{\beta}}\Pi_k^0(v_h)\cdot(\Pi_{k-1}^0-\mathrm{I})\nabla v_h\,dV\right| \\
&\quad\le \|\widehat{\boldsymbol{\beta}}_x\|_{\infty,E}\left|\int_E \Pi_k^0(v_h)(\Pi_{k-1}^0-\mathrm{I})\left(\frac{\partial v_h}{\partial x}\right)\right| + \|\widehat{\boldsymbol{\beta}}_y\|_{\infty,E}\left|\int_E \Pi_k^0(v_h)(\Pi_{k-1}^0-\mathrm{I})\left(\frac{\partial v_h}{\partial y}\right)\right| \\
&\quad= \|\widehat{\boldsymbol{\beta}}_x\|_{\infty,E}\left|\int_E (\Pi_{k-1}^0-\mathrm{I})\Pi_k^0(v_h)\,\frac{\partial v_h}{\partial x}\right| + \|\widehat{\boldsymbol{\beta}}_y\|_{\infty,E}\left|\int_E (\Pi_{k-1}^0-\mathrm{I})\Pi_k^0(v_h)\,\frac{\partial v_h}{\partial y}\right| \\
&\quad\le \|\widehat{\boldsymbol{\beta}}\|_{\infty,E}\|(\Pi_{k-1}^0-\mathrm{I})\,\Pi_k^0(v_h)\|_{0,E}\|\nabla v_h\|_{0,E} \\
&\quad\le Ch_E\|\widehat{\boldsymbol{\beta}}\|_{\infty,E}|\Pi_k^0(v_h)|_{1,E}\,\|\nabla v_h\|_{0,E} \\
&\quad\le Ch_E\|\widehat{\boldsymbol{\beta}}\|_{\infty,E}|v_h|_{1,E}^2 \\
&\quad\le Ch_E\|\boldsymbol{\beta}\|_{\infty,E}|v_h|_{1,E}^2.
\end{aligned}
$$

where we used again (23) in the last step. Thus,

$$
\begin{aligned}
-|b_h^E(v_h,v_h)-b^E(v_h,v_h)| &= -\left|\int_E \widehat{\boldsymbol{\beta}}\cdot\Pi_{k-1}^0(\nabla v_h)\,\Pi_k^0(v_h)\,dV - \int_E(\boldsymbol{\beta}\cdot\nabla v_h)\,v_h\,dV\right| \\
&\ge -Ch_E\big(\|\boldsymbol{\beta}\|_{\infty,E}|v_h|_{1,E}+\|\boldsymbol{\beta}\|_{W^{1,\infty}(E)}\|v_h\|_{0,E}\big)\,|v_h|_{1,E}. \qquad (24)
\end{aligned}
$$

Using (24) into (22) and noting that $\|v_h\|_{0,E}\le\|v_h\|_{1,E}$, $|v_h|_{1,E}\le\|v_h\|_{1,E}$ it follows that

$$
\begin{aligned}
A_h(v_h,v_h) &\ge \sum_E\left[\alpha_*\xi - Ch_E\left(\|\boldsymbol{\beta}\|_{\infty,E}+\|\boldsymbol{\beta}\|_{W^{1,\infty}(E)}\right)\right]\|v_h\|_{1,E}^2 + \min(1,\gamma_*)m_0\sum_E\|v_h\|_{0,E}^2 \\
&\ge \min_E\left\{\alpha_*\xi - Ch_E\left(\|\boldsymbol{\beta}\|_{\infty,E}+\|\boldsymbol{\beta}\|_{W^{1,\infty}(E)}\right)\right\}\|v_h\|_1^2 + \min(1,\gamma_*)m_0\|v_h\|_0^2 \\
&\ge \alpha\|v_h\|_1^2, \qquad (25)
\end{aligned}
$$

where

$$
\alpha := \min\left\{\min_E\left\{\alpha_*\xi - Ch_E\left(\|\boldsymbol{\beta}\|_{\infty,E}+\|\boldsymbol{\beta}\|_{W^{1,\infty}(E)}\right)\right\},\min\{\gamma_*,1\}m_0\right\}.
$$

From $\alpha>0$ and the fact that $\gamma_*, m_0>0$, it may be seen that the VE bilinear form is coercive when

$$
\frac{h_E\left(\|\boldsymbol{\beta}\|_{\infty,E}+\|\boldsymbol{\beta}\|_{W^{1,\infty}(E)}\right)}{\alpha_*\xi} < \frac{1}{C}.
$$

for all mesh elements E. This can be interpreted as showing that the method is coercive when the mesh Péclet number is sufficiently small. □

Lemma 4 (Continuity of A_h) *Under the assumptions of polynomial consistency and stability along with those on the coefficients* K*,* $\boldsymbol{\beta}$ *and* c *and their approximations, the bilinear form* A_h *is continuous.*

Proof. From the stability property for the diffusion and reaction terms, and using their coercivity and linearity, both a_h^E and c_h^E can be viewed as an inner product on the VE space V_h^E over each element E. Consequently,

$$
\begin{aligned}
a_h^E(u_h,v_h) &\le (a_h^E(u_h,u_h))^{\frac12}(a_h^E(v_h,v_h))^{\frac12} \le \alpha^*(a^E(u_h,u_h))^{\frac12}(a^E(v_h,v_h))^{\frac12} \\
&\le \alpha^*\|\mathsf{K}\|_\infty\|\nabla u_h\|_{0,E}\|\nabla v_h\|_{0,E},
\end{aligned}
$$

and, similarly, $c_h^E(u_h,v_h)\le\gamma^*\|c\|_\infty\|u_h\|_{0,E}\|v_h\|_{0,E}$. For the convection term, simply using the stability of the L^2 projector and the boundedness of the coefficient provides

$$
b_h^E(u_h,v_h)\le\|\widehat{\boldsymbol{\beta}}\|_\infty\|\nabla u_h\|_{0,E}\|v_h\|_{0,E}.
$$

Thus,

$$\begin{aligned}A_h^E(u_h, v_h) &\le C(\|\nabla u_h\|_{0,E}\|\nabla v_h\|_{0,E} + \|\nabla u_h\|_{0,E}\|v_h\|_{0,E} + \|u_h\|_{0,E}\|v_h\|_{0,E}) \\ &\le C\|u_h\|_{1,E}\|v_h\|_{1,E}\end{aligned}$$

where we take $C = \max\left\{\alpha^*\|\mathsf{K}\|_\infty,\ \|\widehat{\boldsymbol{\beta}}\|_\infty,\ \gamma^*\|c\|_\infty\right\}$ and so the bilinear form is continuous. □

Theorem 1 (Existence and Uniqueness of Solutions) *Under the assumptions of polynomial consistency and stability along with those on the coefficients* K*,* $\boldsymbol{\beta}$ *and* c *and their approximations, the problem: find* $u_h \in V_h$ *such that*

$$A_h(u_h, v_h) := a_h(u_h, v_h) + b_h(u_h, v_h) + c_h(u_h, v_h) = (f_h, v_h) \quad \forall v_h \in V_h,$$

where

$$(f_h, v_h) = \int_\Omega f_h v_h \, dV \tag{26}$$

possesses a unique solution provided that the mesh Péclet number is sufficiently small.

Proof. The previous lemmas and an appeal to the Lax-Milgram Lemma show that there exists a unique solution to the problem. □

9. Convergence theory in H^1 norm

Let u_{I} and u_π denote approximations of u with $u_{\mathrm{I}} \in V_h^k$ and u_π piecewise in $\mathbb{P}_k$ on the mesh partitioning of Ω. Assuming that the element E has a convex shape, standard approximation estimates yield immediately that

$$\|u - u_{\mathrm{I}}\|_{1,E} + \|u - u_\pi\|_{1,E} \le C h_E^k \|u\|_{k+1,E}. \tag{27}$$

Since $u_\pi \in \mathcal{P}_k(E)$, the polynomial consistency property implies that

$$A_h^E(u_\pi, \delta) = \int_E \widehat{\mathsf{K}}\nabla u_\pi \cdot \Pi_{k-1}^0(\nabla\delta)\, dV + \int_E \widehat{\boldsymbol{\beta}} \cdot \nabla u_\pi\, \Pi_k^0(\delta)\, dV + \int_E \widehat{c} u_\pi\, \Pi_k^0(\delta)\, dV. \tag{28}$$

Theorem 2 (H^1-norm error estimate) *Let* u *be the exact solution of problem* (1)*, with coefficients* $\mathsf{K}, \boldsymbol{\beta}, c \in W^{k,\infty}(\Omega)$ *satisfying conditions* (2) *and* (3)*. Let* u_h *be the solution to the virtual element approximation* (13)*, with polynomial coefficients* $\widehat{\mathsf{K}}$*,* $\widehat{\boldsymbol{\beta}}$ *and* $\widehat{c}$ *satisfying* (6) *and* (7a)-(7c)*. Then, if* $f \in H^{k-1}(\Omega)$ *and* $u \in H^{k+1}(\Omega)$*, the approximation error can be bounded as*

$$\|u - u_h\|_1 \le C h^k |u|_{k+1}.$$

Proof. Define $\delta := u_h - u_{\mathrm{I}}$. The coercivity of the VE bilinear form, c.f. (25), implies that

$$\alpha\|\delta\|_1^2 \le A_h(\delta, \delta) = A_h(u_h, \delta) - A_h(u_{\mathrm{I}}, \delta) = \int_\Omega f_h \delta \, dV - \sum_E A_h^E(u_{\mathrm{I}}, \delta). \tag{29}$$

We manipulate the summation's argument as follows; add and subtract $A_h^E(u_\pi, \delta)$:

$$\begin{aligned}A_h^E(u_{\mathrm{I}}, \delta) &= A_h^E(u_{\mathrm{I}} - u_\pi, \delta) + A_h^E(u_\pi, \delta) && \left[\text{add and subtract } A^E(u_\pi, \delta)\right] \\ &= \left(\mathsf{T}_1^E\right) + A_h^E(u_\pi, \delta) - A^E(u_\pi, \delta) + A^E(u_\pi, \delta) && \left[\text{add and subtract } A^E(u, \delta)\ \right] \\ &= \left(\mathsf{T}_1^E\right) + \left(\mathsf{T}_2^E\right) + A^E(u_\pi - u, \delta) + A^E(u, \delta) && \left[\text{substitute } A^E(u, \delta) = \int_E f v_h \, dV\ \right] \\ &= \left(\mathsf{T}_1^E\right) + \left(\mathsf{T}_2^E\right) + \left(\mathsf{T}_3^E\right) + \int_E f\, v_h \, dV.\end{aligned}$$

Substituting the last expression into (29) gives

$$\alpha\|\delta\|_1^2 \le \mathsf{T}^f - \sum_E \left(\mathsf{T}_1^E + \mathsf{T}_2^E + \mathsf{T}_3^E\right), \tag{30}$$

where the error has been split up into four parts:

$$\mathsf{T}^f := \int_\Omega f_h \delta \, dV - \int_\Omega f \delta \, dV, \qquad \mathsf{T}_1^E := A_h^E(u_{\mathrm{I}} - u_\pi, \delta),$$
$$\mathsf{T}_2^E := A_h^E(u_\pi, \delta) - A^E(u_\pi, \delta), \qquad \mathsf{T}_3^E := A^E(u_\pi - u, \delta).$$

Since $f_h = \Pi_{k-2}^0(f)$, the term T^f can be bounded as

$$\left|\mathsf{T}^f\right| \le Ch^k |u|_{k+1} \|\delta\|_1. \tag{31}$$

From the continuity of the bilinear form, the terms T_1^E and T_3^E may be bounded as

$$\left|\mathsf{T}_1^E\right| = \left|A_h^E(u_{\mathrm{I}} - u_\pi, \delta)\right| \le C\|u_{\mathrm{I}} - u_\pi\|_{1,E}\|\delta\|_{1,E},$$
$$\left|\mathsf{T}_3^E\right| = \left|A^E(u_\pi - u, \delta)\right| \le C\|u_\pi - u\|_{1,E}\|\delta\|_{1,E}.$$

From these inequalities and bounds (27) it follows that

$$\left|\mathsf{T}_1^E\right| + \left|T_3^E\right| \le C\left(\|u_{\mathrm{I}} - u_\pi\|_{1,E} + \|u_\pi - u\|_{1,E}\right)\|\delta\|_{1,E} \le Ch^k|u|_{k+1}\|\delta\|_{1,E}. \tag{32}$$

Now, we are left to estimate term T_2^E, which can be bounded as

$$\begin{aligned}\left|\mathsf{T}_2^E\right| &\le |a_h^E(u_\pi,\delta) - a^E(u_\pi,\delta)| + |b_h^E(u_\pi,\delta) - b^E(u_\pi,\delta)| + |c_h^E(u_\pi,\delta) - c^E(u_\pi,\delta)| \\ &= \left|\mathsf{T}_{21}^E\right| + \left|\mathsf{T}_{22}^E\right| + \left|\mathsf{T}_{23}^E\right|. \end{aligned} \tag{33}$$

We will estimate the terms T_{21}^E, T_{22}^E, T_{23}^E separately. To estimate T_{21}^E, we add and substract $\mathsf{K}\nabla u_\pi \cdot \Pi_{k-1}^0(\nabla\delta)$ inside the integral arguments and we use the definition of the projection operator Π_{k-1}^0 to obtain:

$$\begin{aligned} a_h^E(u_\pi,\delta) - a^E(u_\pi,\delta) &= \int_E \widehat{\mathsf{K}}\nabla u_\pi \cdot \Pi_{k-1}^0(\nabla\delta)\, dV - \int_E \mathsf{K}\nabla u_\pi \cdot \nabla\delta\, dV \\ &= \int_E (\widehat{\mathsf{K}} - \mathsf{K})\nabla u_\pi \cdot \Pi_{k-1}^0(\nabla\delta)\, dV + \int_E \mathsf{K}\nabla u_\pi \cdot (\Pi_{k-1}^0 - \mathrm{I})\nabla\delta\, dV \\ &= \int_E \Pi_{k-1}^0\big((\widehat{\mathsf{K}} - \mathsf{K})\nabla u_\pi\big) \cdot \nabla\delta\, dV + \int_E (\Pi_{k-1}^0 - \mathrm{I})(\mathsf{K}\nabla u_\pi) \cdot \nabla\delta\, dV. \end{aligned} \tag{34}$$

Now,

$$\begin{aligned} \left|a_h^E(u_\pi,\delta) - a^E(u_\pi,\delta)\right| &\le \left|\int_E \Pi_{k-1}^0\big((\widehat{\mathsf{K}} - \mathsf{K})\nabla u_\pi\big) \cdot \nabla\delta\, dV\right| + \left|\int_E (\Pi_{k-1}^0 - \mathrm{I})(\mathsf{K}\nabla u_\pi) \cdot \nabla\delta\, dV\right| \\ &\le C\|\delta\|_{1,E}\left(\left\|\widehat{\mathsf{K}} - \mathsf{K}\right\|_{\infty,E}\|\nabla u\|_{0,E} + \left\|\left(\Pi_{k-1}^0 - \mathrm{I}\right)(\mathsf{K}\nabla u_\pi)\right\|_{0,E}\right). \end{aligned} \tag{35}$$

We easily bound the first norm in (35) by using (7a). To bound the second norm, we first manipulate its argument by adding and subtracting $\Pi_{k-1}^0\big(\mathsf{K}\nabla u\big)$ and $\mathsf{K}\nabla u$ and rearranging the summation we obtain:

$$\begin{aligned} \big(\Pi_{k-1}^0 - \mathrm{I}\big)\big(\mathsf{K}\nabla u_\pi\big) &= \Pi_{k-1}^0\big(\mathsf{K}\nabla u_\pi\big) - \mathsf{K}\nabla u_\pi = \Pi_{k-1}^0\big(\mathsf{K}\nabla(u_\pi - u)\big) + \Pi_{k-1}^0\big(\mathsf{K}\nabla u\big) - \mathsf{K}\nabla(u_\pi - u) - \mathsf{K}\nabla u \\ &= \Pi_{k-1}^0\big(\mathsf{K}\nabla(u_\pi - u)\big) - \mathsf{K}\nabla(u_\pi - u) + (\Pi_{k-1}^0 - \mathrm{I})(\mathsf{K}\nabla u). \end{aligned} \tag{36}$$

Using the last expression of (36) as the norm's argument, applying the triangle inequality twice and using the stability of the projection operator Π_{k-1}^0 yield:

$$\begin{aligned} \left\|\left(\Pi_{k-1}^0 - \mathrm{I}\right)(\mathsf{K}\nabla u_\pi)\right\|_{0,E} &\le \left\|\Pi_{k-1}^0(\mathsf{K}\nabla(u_\pi - u))\right\|_{0,E} + \|\mathsf{K}\nabla(u_\pi - u)\|_{0,E} + \left\|\left(\Pi_{k-1}^0 - \mathrm{I}\right)(\mathsf{K}\nabla u)\right\|_{0,E} \\ &\le C\left(\|\mathsf{K}\nabla(u_\pi - u)\|_{0,E} + \left\|\left(\Pi_{k-1}^0 - \mathrm{I}\right)(\mathsf{K}\nabla u)\right\|_{0,E}\right). \end{aligned}$$

The last two terms above can be bounded using (27) and the fact that the coefficients are in L^∞:

$$\|\mathsf{K}\nabla(u_\pi - u)\|_{0,E} \le \|\mathsf{K}\|_\infty |u_\pi - u|_{1,E} \le C\|\mathsf{K}\|_\infty h^k |u|_{k+1}$$

$$\left\|\left(\Pi^0_{k-1} - \mathrm{I}\right)(\mathsf{K}\nabla u)\right\|_{0,E} \le C\|\mathsf{K}\|_\infty h^k |u|_{k+1}.$$

Combining all such inequalities for all the mesh elements E yields the global bound:

$$\left|\mathsf{T}^E_{21}\right| \le C\|\mathsf{K}\|_\infty h^k |u|_{k+1}\|\delta\|_1. \tag{37}$$

To estimate T^E_{22}, we add and substract $\boldsymbol{\beta}\cdot\nabla u_\pi\,\Pi^0_k(\delta)$ inside the integral argument and we use the definition of the projection operator Π^0_k to obtain:

$$\begin{aligned} b^E_h(u_\pi,\delta) - b^E(u_\pi,\delta) &= \int_E \widehat{\boldsymbol{\beta}}\cdot\nabla u_\pi\,\Pi^0_k(\delta)\,dV - \int_E \boldsymbol{\beta}\cdot\nabla u_\pi\,\delta\,dV \\ &= \int_E (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})\cdot\nabla u_\pi\,\Pi^0_k(\delta)\,dV + \int_E \boldsymbol{\beta}\cdot\nabla u_\pi\,(\Pi^0_k - \mathrm{I})(\delta)\,dV \\ &= \int_E \Pi^0_k\big((\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})\cdot\nabla u_\pi\big)\,\delta\,dV + \int_E (\Pi^0_k - \mathrm{I})(\boldsymbol{\beta}\cdot\nabla u_\pi)\,\delta\,dV. \end{aligned} \tag{38}$$

Now,

$$\begin{aligned} \left|b^E_h(u_\pi,\delta) - b^E(u_\pi,\delta)\right| &\le \left|\int_E \Pi^0_k\big((\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})\cdot\nabla u_\pi\big)\,\delta\,dV\right| + \left|\int_E (\Pi^0_k - \mathrm{I})(\boldsymbol{\beta}\cdot\nabla u_\pi)\,\delta\,dV\right| \\ &\le C\|\delta\|_{1,E}\left(\left\|\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right\|_{\infty,E}\|\nabla u_\pi\|_{0,E} + \left\|\left(\Pi^0_{k-1} - \mathrm{I}\right)(\boldsymbol{\beta}\cdot\nabla u_\pi)\right\|_{0,E}\right). \end{aligned} \tag{39}$$

Since for $h_E \to 0$

$$\|\nabla u_\pi\|_{0,E} \le \|\nabla(u_\pi - u)\|_{0,E} + \|\nabla u\|_{0,E} \le C(h_E + 1)\|\nabla u\|_{0,E} \approx C\|\nabla u\|_{0,E},$$

estimate (39) becomes

$$\left|b^E_h(u_\pi,\delta) - b^E(u_\pi,\delta)\right| \le C\|\delta\|_{1,E}\left(\left\|\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right\|_{\infty,E}\|\nabla u\|_{0,E} + \left\|\left(\Pi^0_{k-1} - \mathrm{I}\right)(\boldsymbol{\beta}\cdot\nabla u_\pi)\right\|_{0,E}\right) \tag{40}$$

We easily bound the first norm in (40) by using (7b), and we manipulate the second norm as we have done for the estimate of T^E_{21} to obtain

$$\begin{aligned} \left\|\left(\Pi^0_{k-1} - \mathrm{I}\right)(\boldsymbol{\beta}\cdot\nabla u_\pi)\right\|_{0,E} &\le \left\|\Pi^0_{k-1}(\boldsymbol{\beta}\cdot\nabla(u_\pi - u))\right\|_{0,E} + \|\boldsymbol{\beta}\cdot\nabla(u_\pi - u)\|_{0,E} + \left\|\left(\Pi^0_{k-1} - \mathrm{I}\right)(\boldsymbol{\beta}\cdot\nabla u)\right\|_{0,E} \\ &\le C\left(\|\boldsymbol{\beta}\cdot\nabla(u_\pi - u)\|_{0,E} + \left\|\left(\Pi^0_{k-1} - \mathrm{I}\right)(\boldsymbol{\beta}\cdot\nabla u)\right\|_{0,E}\right). \end{aligned}$$

The last two terms above can be bounded using (27) and the fact that the coefficients are in L^∞:

$$\|\boldsymbol{\beta}\cdot\nabla(u_\pi - u)\|_{0,E} \le \|\boldsymbol{\beta}\|_{\infty,E}\|\nabla(u_\pi - u)\|_{0,E} \le C\|\boldsymbol{\beta}\|_{\infty,E} h^k |u|_{k+1,E},$$

$$\left\|\left(\Pi^0_{k-1} - \mathrm{I}\right)(\boldsymbol{\beta}\cdot\nabla u)\right\|_{0,E} \le C\|\boldsymbol{\beta}\|_{\infty,E} h^k |u|_{k+1,E}.$$

Combining all such inequalities for all the mesh elements E yields the global bound:

$$\left|\mathsf{T}^E_{22}\right| \le C\|\boldsymbol{\beta}\|_\infty h^k |u|_{k+1}\|\delta\|_1. \tag{41}$$

To estimate T^E_{23}, we add and substract $cu_\pi\,\Pi^0_k(\delta)$ inside the integral arguments and we use the definition of the projection operator Π^0_k to obtain:

$$\begin{aligned} c^E_h(u_\pi,\delta) - b^E(u_\pi,\delta) &= \int_E \widehat{c}u_\pi\,\Pi^0_k(\delta)\,dV - \int_E cu_\pi\,\delta\,dV \\ &= \int_E (\widehat{c} - c)u_\pi\,\Pi^0_k(\delta)\,dV + \int_E cu_\pi\,(\Pi^0_k - \mathrm{I})(\delta)\,dV \\ &= \int_E \Pi^0_k\big((\widehat{c} - c)u_\pi\big)\,\delta\,dV + \int_E (\Pi^0_k - \mathrm{I})(cu_\pi)\,\delta\,dV. \end{aligned} \tag{42}$$

Now,

$$\left|c_h^E(u_\pi,\delta) - b^E(u_\pi,\delta)\right| \leq \left|\int_E \Pi_k^0\big((\widehat{c}-c)u_\pi\big)\,\delta\,dV\right| + \left|\int_E (\Pi_k^0 - \mathrm{I})(cu_\pi)\,\delta\,dV\right|$$
$$\leq C\|\delta\|_{1,E}\left(\|\widehat{c}-c\|_{\infty,E}\|u\|_{0,E} + \left\|\left(\Pi_k^0 - \mathrm{I}\right)(u_\pi)\right\|_{0,E}\right). \tag{43}$$

We easily bound the first norm in (43) by using (7c), and we manipulate the second norm as we have done for the estimate of T_{21}^E and T_{22}^E to obtain

$$\left\|\left(\Pi_k^0 - \mathrm{I}\right)(u_\pi)\right\|_{0,E} \leq \left\|\Pi_k^0(c(u_\pi - u))\right\|_{0,E} + \|c(u_\pi - u)\|_{0,E} + \left\|\left(\Pi_{k-1}^0 - \mathrm{I}\right)(cu)\right\|_{0,E}$$
$$\leq C\left(\|c(u_\pi - u)\|_{0,E} + \left\|\left(\Pi_{k-1}^0 - \mathrm{I}\right)(cu)\right\|_{0,E}\right).$$

The last two terms above can be bounded using (27) and the fact that the coefficients are in L^∞:

$$\|c(u_\pi - u)\|_{0,E} \leq \|c\|_{\infty,E}\|u_\pi - u\|_{0,E} \leq C\|c\|_{\infty,E}h^k|u|_{k+1,E},$$
$$\left\|\left(\Pi_{k-1}^0 - \mathrm{I}\right)(cu)\right\|_{0,E} \leq C\|c\|_{\infty,E}h^k|u|_{k+1,E}.$$

Combining all such inequalities for all the mesh elements E yields the global bound:

$$\left|\mathsf{T}_{23}^E\right| \leq C\|c\|_\infty h^k|u|_{k+1}\|\delta\|_1. \tag{44}$$

Inequalities (37), (41), and (44) gives the bound for T_2^E:

$$\left|\mathsf{T}_2^E\right| \leq Ch^k|u|_{k+1}\|\delta\|_{1,E}, \tag{45}$$

where constant C absorbs the coefficients $\|\mathsf{K}\|_\infty$, $\|\mathsf{K}\|_{\boldsymbol{\beta}}$, $\|c\|_\infty$.

Finally, we combine inequalities (31), (32), and (45) to form an $\mathcal{O}(h^k)$ bound for the approximation error:

$$\|\delta\|_1^2 \leq C(h^k + h^{k+1})|u|_{k+1}\|\delta\|_1.$$

The result then follows upon dividing through by $\|\delta\|_1^2$ and applying the triangle inequality and bounds (27) to

$$\|u - u_h\|_1 \leq \|u - u_{\mathrm{I}}\|_1 + \|u_{\mathrm{I}} - u_h\|_1.$$

□

10. Conclusions

In this work, we summarize some advances in the development of the Conforming Virtual Element methodology for the convection-diffusion-reaction problems with constant and variable coefficients. The Conforming VEM is based on a suitable virtual space definitions as well as some consistency property on a subset of polynomial functions that allows us to approximate all the bilinear form of the weak formulation without the explicit construction of the shape functions. In this work, we present several possible combinations of differential operators and projections. For one of them, we also prove the well-posedness and give an estimate of the approximation error in the H^1 norm.

Acknowledgements

The second author work was supported by the LDRD-ER Project ”From the Finite Element Method to the Virtual Element Method”.

References

[1] B. Ahmad, A. Alsaedi, F. Brezzi, L. D. Marini, and A. Russo. Equivalent projectors for virtual element methods. *Comput. Math. Appl.*, 66(3):376–391, 2013.

[2] L. Beirão da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L. D. Marini, and A. Russo. Basic principles of virtual element methods. *Math. Models Methods Appl. Sci.*, 23(1):199–214, 2013.

[3] L. Beirão da Veiga, F. Brezzi, L. D. Marini, and A. Russo. The hitchhiker's guide to the virtual element method. *Mathematical Models and Methods in Applied Sciences*, 24(08):1541–1573, 2014.

[4] Philippe G. Ciarlet. *The finite element method for elliptic problems.* North-Holland Publishing Co., Amsterdam, 1978. Studies in Mathematics and its Applications, Vol. 4.

Appendix A. Relation between the modified projection operator and the mimetic method

The mimetic method is based on the integration by parts:

$$\int_E \mathsf{K}\nabla u \cdot \nabla q \, dV = -\int_E \mathrm{div}(\mathsf{K}\nabla q)\, u \, dV + \sum_{e\in\partial E}\int_e \big(\mathbf{n}_e\cdot \mathsf{K}\nabla q\big)\, u\, dS$$

where we assume that q belongs to $\mathbb{P}_k(E)$. Note that the coefficient $\mathsf{K}(x,y)$ is always with ∇q.

Now we have two special cases:

- $k = 1$**,** q **is a linear polynomial,** ∇q **is a constant vector**. We can approximate $\mathsf{K}(x,y)$ by a constant K_E and note that
  - the integral on E in the right-hand side is zero because $\mathrm{div}(\mathsf{K}_E\nabla q)$ is zero;
  - $\mathbf{n}_e \cdot \nabla q$ is constant, we can take it out of the integral, and we are left with $\int_e u\, dS$ which can be approximated by using the trapezoidal rule if we know the values of u at the vertices of e. In the VEM, this is not even an approximation because the trace of u on e is a linear function and is determined by its values at the vertices.

- K **is constant**. Then, $\mathsf{K}\nabla q$ is a vector of polynomials of degree up to $k-1$ if q is in $\mathbb{P}_k(E)$. In this case, the integral of $\int_E \mathsf{K}\nabla u\cdot\nabla q\, dV$ is still computable if we know the cell moments of u against polynomials of degree up to $k-2$ and we approximate the trace of u on each edge e by the polynomial interpolant of degree k. To compute such interpolant, we need to define suitable degrees of freedom of u on e. In the conforming VEM this not an approximation because the trace of u is a polynomial of degree k on each edge.

In the general case for $k > 1$ and $\mathsf{K}(x,y)$ not constant on E it happens that $\mathsf{K}\nabla q$ is not anymore a vector of polynomials when q is a polynomial. So, the idea is to project $\mathsf{K}\nabla q$ onto the polynomials of degree $k-1$ and absorb the non-constant K inside this projection:

$$\begin{aligned}\int_E \mathsf{K}\nabla u\cdot\nabla q\, dV &= \int_E \nabla u\cdot \mathsf{K}\nabla q\, dV \approx \int_E \nabla u\cdot \Pi^0_{k-1}(\mathsf{K}\nabla q)\, dV\\ &= -\int_E \mathrm{div}(\Pi^0_{k-1}(K\nabla q))\, u\, dV + \sum_{e\in\partial E}\int_e \Big(\mathbf{n}_e\cdot\Pi^0_{k-1}(\mathsf{K}\nabla q)\Big)\, u\, dS\end{aligned}$$

Now, the two terms on the right are computable:

- $\mathrm{div}(\Pi^0_{k-1}(K\nabla q))$ is a polynomial of degree $k-2$ and $\int_E \mathrm{div}(\Pi^0_{k-1}(\mathsf{K}\nabla q))\, u\, dV$ is computable through the moments of degree up to $k-2$ of u;
- $\mathbf{n}_e\cdot\Pi^0_{k-1}(\mathsf{K}\nabla q)$ is a polynomial of degree $k-1$ and $\int_e \Big(\mathbf{n}_e\cdot\Pi^0_{k-1}(\mathsf{K}\nabla q)\Big)\, u\, dS$ by interpolating u on polynomials of degree k.

In VEM we do the same using the **modified projection oprrator**, but the interpretation is different. Let the VEM diffusive bilinear form be given by:

$$a_h^E(u_h, v_h) = \int_E \nabla\widetilde{\Pi}^{\mathsf{K}\nabla}(u_h)\cdot\nabla\Pi^{\nabla}_k(v_h)\, dV + \mathsf{stab}\big[a_h^E\big],$$

where the modified projector is defined by:

$$\int_E \nabla\widetilde{\Pi}^{\mathsf{K}\nabla}(u_h)\cdot\nabla q\, dV = \int_E \nabla u_h\cdot\Pi^0_{k-1}(\mathsf{K}\nabla q)\, dV \quad \forall q\in\mathbb{P}_k(E),$$

(plus the projection on constants as in the usual case of Π^{∇}_k). Using such definition, **the diffusion tensor** K **is incorporated in the definition of** $\widetilde{\Pi}^{\mathsf{K}\nabla}(u_h)$**.**

$\widetilde{\Pi}^{\mathsf{K}\nabla}(u_h)$ is computed explicitly by running q on the basis of monomials m_α:

$$\int_E \nabla\widetilde{\Pi}^{\mathsf{K}\nabla}(u_h)\cdot\nabla m_\alpha\,dV = \int_E \nabla u_h\cdot\Pi^0_{k-1}(\mathsf{K}\nabla m_\alpha)\,dV \quad \forall\alpha \text{ such that } \mathbb{P}_k(E) = \text{span}(\{m_\alpha\})$$

Note that the second integral is computable knowing only the degrees of freedom of u_h after integration by parts

$$\int_E \nabla u_h\cdot\Pi^0_{k-1}(\mathsf{K}\nabla m_\alpha)\,dV = -\int_E u_h\cdot\text{div}\big(\Pi^0_{k-1}(\mathsf{K}\nabla m_\alpha)\big)\,dV + \sum_{e\in\partial E}\int_e u_h\mathbf{n}_e\cdot\Pi^0_{k-1}(\mathsf{K}\nabla m_\alpha)\,dS,$$

since again $\text{div}\big(\Pi^0_{k-1}(\mathsf{K}\nabla m_\alpha)\big)$ is a polynomial of degree $k-2$ and the trace of u_h on e is a polynomial of degree k (use the same arguments as above).

So, the VEM with the modified projector works exactly like the normal VEM (without K), where instead of

$$\mathsf{M}_{ij} = \int_E \nabla\Pi^\nabla_k\phi_i\cdot\nabla\Pi^\nabla_k\phi_j + \mathsf{stab}\big[\mathsf{M}\big]$$

we use the (modified) stiffness matrix

$$\widetilde{\mathsf{M}}_{ij} = \int_E \nabla\widetilde{\Pi}^{\mathsf{K}\nabla}\phi_i\cdot\nabla\Pi^\nabla_k\phi_j\,dV + \mathsf{stab}\big[\mathsf{M}\big]$$

The stabilization term remains the same for both (at least this is what we do in MFD).

The approach using the modified projector requires two "nabla" projectors: Π^∇_k (which is the usual one) and $\widetilde{\Pi}^{\mathsf{K}\nabla}$ (defined as above), and the action of these operators on the shape functions must be computed explicitly.