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Title: Extension of Finite Element methods to general polygonal (and polyhedral) meshes

Author(s): Gyrya, Vitaliy
Lipnikov, Konstantin Nikolayevich
Manzini, Gianmarco
McGregor, Duncan Alisdair Odum

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Extension of Finite Element methods to general polygonal (and polyhedral) meshes

Vitaliy Gyrya
K. Lipnikov, G. Manzini, D. McGregor

Los Alamos National Laboratory
T-5, Applied Mathematics and Plasma Physics

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1. General view on generalization of FE to polygonal and polyhedral meshes

- Classical FEM construction on simplicial meshes: approximation space $\mathcal{V} = \mathcal{P} \rightarrow$d.o.f. (values at the vertices, average fluxes, etc.)
- Extension of FEM to polygonal/polyhedral meshes, while retaining the same d.o.f., $\Rightarrow$ number of d.o.f. grows.
- Dimension of the discr. approx. space grows beyond polynomial space $\mathcal{P}$. Need to extend the approximation space.
- Adding new (polynomial) functions raises *unisolvency* questions.

Main tools:

- Consistency: "correct" behaviour when polynomials are involved. Appropriate choice of d.o.f.
- Stability: "roughly correct" behavior for non-polynomials.

2. Focus

We will focus on *reaction-diffusion* equation:

$$\triangle u + u = 0$$

and the *wave equation*

$$u_{tt} = -\triangle u.$$

Need:

- specify d.o.f. and the approximation spaces.
- build stiffness matrix: $U^T \mathbb{A} V \approx \int_E \nabla u \cdot \nabla v$.
- build mass matrix: $U^T \mathbb{M} V \approx \int_E uv$.

3. Approximation space

The approximation space $\mathcal{V}$ is built in orthogonal form:

$$\mathcal{V} = \mathcal{P}_k \oplus_* \mathcal{B}_*, \qquad * = \mathbb{A}, \mathbb{M}.$$

- The polynomial space $\mathcal{P}_k$ is selected according to the desired accuracy of the scheme.
- The extension space $\mathcal{B}$ (not constructed explicitly) is added to satisfy the unisolvency between the continuous and the discrete approximation space.

Orthogonal projection operators simplify the construction and the analysis:

- π_∇ for the stiffness matrix, $\int_E \nabla u \cdot \nabla p = \int_E \nabla(\pi_\nabla u) \cdot \nabla p$, and
- π_0 for the mass matrix, $\int_E up = \int_E (\pi_0) p$.

4. Consistency and d.o.f.

Consistency condition: for any $P \leftrightarrow p \in \mathcal{P}_k$ and for any $V \leftrightarrow v \in \mathcal{V}$

$$P^T \mathbb{A}_E V = \int_E \nabla p \cdot \nabla v = -\int_E (\triangle p) v + \sum_{f_i \in \partial E} \int_{f_i} (\nabla p \cdot \mathbf{n}) v.$$

E – element, f_i – face of the element in 3D, edge in 2D.

- Choosing d.o.f. to be the moments allows to compute $P^T \mathbb{A} V = R_P^T V$ without knowing the exact shape of v inside the element:

  Boundary moments: $\int_{f_i} (\nabla p \cdot \mathbf{n}) u,$

  Internal moments: $\int_E (\triangle p) u.$

- Alternatively, if all functions in $\mathcal{V}$ satisfy appropriate quadrature then one can use other d.o.f. (e.g. values at the vertices).

5. Consistency conditions using projection operators

Construction based on the projection operator
(L. Beiro da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L.D. Marini, A. Russo: *Basic principles of Virtual Element Methods*, Math. Models Methods Appl. Sci. 23)

- For all monomials $m_\alpha \in \mathcal{P}_k \to D_\alpha$ – d.o.f. representation.
- For each monomial build the coefficients vector B_α

$$D_\alpha^T \mathbb{A}_E V = R_\alpha^T V \qquad \Leftrightarrow \qquad \mathbb{D}^T \mathbb{A}_E = \mathbb{B}^T.$$

  Possible due to the choice of the d.o.f., $\mathbb{R} = [R_\alpha]$, $\mathbb{D} = [D_\alpha]$.
- Build the orthogonal projector $\pi_\nabla = \mathbb{D}(\tilde{\mathbb{R}}^T \mathbb{D})^{-1} \tilde{\mathbb{R}}^T$.
  $\tilde{\mathbb{R}} \approx \mathbb{R}$ with the modified 1st column.
- Write the stiffness matrix as consistency and the stability terms

$$\mathbb{A}_E = \mathbb{D}(\mathbb{D}^T \mathbb{D})^{-1} \mathbb{R}^T + (I - \pi_\nabla)^T \text{extra}(I - \pi_\nabla).$$

6.a Alternative construction/view

- Complete the basis ($\mathbb{D}_p$ is the old $\mathbb{D}$, new $\mathbb{D}$ is invertible):

$$\mathbb{D} = \left[\ \mathbb{D}_p \ \middle|\ \mathbb{D}_c\ \right], \qquad \mathbb{D}_p = [D_\alpha], \qquad \mathbb{D}_c = [D_\beta]$$

D_α – d.o.f. for monomials m_α; m_α – basis in $\mathcal{P}_k$;
D_β – d.o.f. for basis r_α; r_β – basis in $\mathcal{B}$.

- Ex. Stiffness matrix:

$$\mathbb{A}_E = \mathbb{D}^{-T}\widetilde{\mathbb{A}}_E\mathbb{D}^{-1}, \qquad \widetilde{\mathbb{A}} = \begin{bmatrix} \widetilde{\mathbb{A}}_{11} & 0 \\ 0 & \widetilde{\mathbb{A}}_{22} \end{bmatrix}$$

$\widetilde{\mathbb{A}}_{11}$ – products of monomials with monomials (computable);
$\widetilde{\mathbb{A}}_{22}$ – products within $\mathcal{B}$ (not computable).

6.b Alternative construction/view

$$\mathbb{A}_E = \mathbb{D}^{-T} \widetilde{\mathbb{A}}_E \mathbb{D}^{-1}, \qquad \widetilde{\mathbb{A}} = \begin{bmatrix} \widetilde{\mathbb{A}}_{11} & 0 \\ 0 & \widetilde{\mathbb{A}}_{22} \end{bmatrix}, \qquad \mathbb{D} = \left[\; \mathbb{D}_p \; \middle| \; \mathbb{D}_c \; \right].$$

- Consistency condition is satisfied through a proper choice of $\mathbb{D}_c$ (depends on the operator $\mathbb{A}$ or $\mathbb{M}$: $\mathbb{D}_{c,\mathbb{A}}, \mathbb{D}_{c,\mathbb{B}}$):

$$0 = \mathbb{D}_p^T \mathbb{A} \mathbb{D}_c = \mathbb{R}^T \mathbb{D}_c \qquad \Leftrightarrow \qquad \text{span}\{\mathbb{D}_c\} \in \ker\{\mathbb{R}^T\}.$$

- $\text{span}\{\mathbb{D}_c\}/D_1$ is defined uniquely (D_1 – d.o.f. for a constant).
- Choice of $\mathbb{D}_{c,\mathbb{A}}$ is directly related to the projector π_∇. If columns of $\mathbb{D}_{c,\mathbb{A}}$ are ortho-normal, then

$$\pi_\nabla = I - \mathbb{D}_{c,\mathbb{A}} \mathbb{D}_{c,\mathbb{A}}^T.$$

7. Mass matrix

- Consistency condition can be enforced only for lower degree polynomials (internal moments only):

$$P^T \mathbb{M}_E V = \int_E pv, \qquad P \leftrightarrow p \in \mathcal{P}_{k-2}.$$

- Completion of the basis

$$\mathbb{D} = \left[\; \mathbb{D}_{\mathcal{P}_k} \; \middle| \; \mathbb{D}_{\mathcal{B}_A} \; \right] = \left[\; \mathbb{D}_{\mathcal{P}_{k-2}} \quad \mathbb{D}_{\tilde{\mathcal{P}}_{k-1},\tilde{\mathcal{P}}_k} \; \middle| \; \mathbb{D}_{\mathcal{B}_{\mathbb{M}}} \; \right],$$

$\tilde{\mathcal{P}}_k$ – homogeneous polynomials of degree k.

- Mass matrix:

$$\mathbb{M}_E = \mathbb{D}^{-T} \widetilde{\mathbb{M}}_E \mathbb{D}^{-1}, \qquad \widetilde{\mathbb{M}}_E = \left[\begin{array}{cc|c} \widetilde{\mathbb{M}}_{11} & \widetilde{\mathbb{M}}_{12} & 0 \\ \widetilde{\mathbb{M}}_{21} & \widetilde{\mathbb{M}}_{22} & \widetilde{\mathbb{M}}_{23} \\ \hline 0 & \widetilde{\mathbb{M}}_{32} & \widetilde{\mathbb{M}}_{33} \end{array} \right]$$

$\widetilde{\mathbb{M}}_{11}$, $\widetilde{\mathbb{M}}_{12} = \widetilde{\mathbb{M}}_{21}{}^T$ and $\widetilde{\mathbb{M}}_{22}$ – computable,
$\widetilde{\mathbb{M}}_{23} = \widetilde{\mathbb{M}}_{32}{}^T$ and $\widetilde{\mathbb{M}}_{33}$ – not computable.

8. Mass matrix (continued)

Mass matrix: $\mathbb{M}_E = \mathbb{D}^{-T}\widetilde{\mathbb{M}}_E\mathbb{D}^{-1}$,

$$\widetilde{\mathbb{M}}_E = \left[\begin{array}{cc|c} \widetilde{\mathbb{M}}_{11} & \widetilde{\mathbb{M}}_{12} & 0 \\ \widetilde{\mathbb{M}}_{21} & \widetilde{\mathbb{M}}_{22} & \widetilde{\mathbb{M}}_{23} \\ \hline 0 & \widetilde{\mathbb{M}}_{32} & \widetilde{\mathbb{M}}_{33} \end{array}\right], \qquad \mathbb{D} = \left[\ \mathbb{D}_{\mathcal{P}_k} \ \middle|\ \mathbb{D}_{\mathcal{B}_A}\ \right]$$

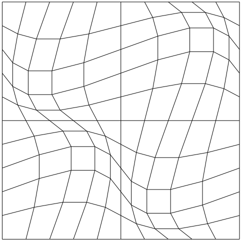
- Consistency condition:

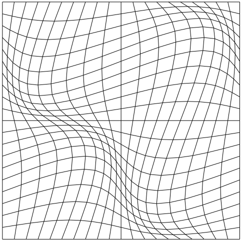
$$0 = \mathbb{D}^T_{\mathcal{P}_{k-2}}\mathbb{M}\mathbb{D}_{\mathcal{B}_{\mathbb{M}}} = \mathbb{R}^T\mathbb{D}_{\mathcal{B}_{\mathbb{M}}}.$$

  $\text{span}\{\mathbb{D}_{\mathcal{B}_{\mathbb{M}}}\}/\,\text{span}\{\mathbb{D}_{\tilde{\mathcal{P}}_{k-1},\tilde{\mathcal{P}}_k}\}$ is defined uniquely.
- Unisolvency condition: $\mathbb{D}$ is invertible.
  Can be enforced by making $\mathbb{D}_{\mathcal{B}_{\mathbb{M}}} \perp \mathbb{D}_{\tilde{\mathcal{P}}_{k-1},\tilde{\mathcal{P}}_k}$.

9.a Numerical experiment

$\triangle u + u = 0$, Dirichlet b.c., solution: $u(x,y) = e^{-2\pi xy}\sin(2\pi xy)$.

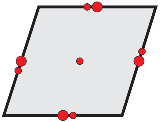




$\mathcal{V} = \mathcal{P}_2 \oplus \mathcal{B}$

$\dim\{\mathcal{P}_2\} = 6$

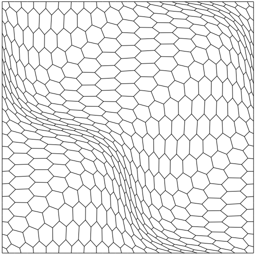
$\dim\{\mathcal{V}\} = 9$



N_E	N_{dof}	h_{max}	$\mathcal{E}_{H^1}$	$\mathcal{E}_{L^2}$
400	1681	1.1×10^{-1}	1.3×10^{-2}	8.7×10^{-4}
1500	6561	5.7×10^{-2}	3.2×10^{-3}	1.1×10^{-4}
6400	25921	2.9×10^{-2}	8.0×10^{-4}	1.6×10^{-5}
25600	103041	1.5×10^{-2}	2.0×10^{-4}	2.7×10^{-6}
102400	410881	7.2×10^{-3}	5.0×10^{-5}	5.7×10^{-7}

9.b Numerical experiment

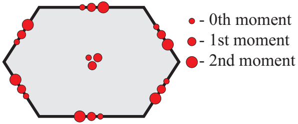
$$\triangle u + u = 0, \qquad \text{Dirichlet b.c., solution:} \qquad u(x,y) = e^{-2\pi xy}\sin(2\pi xy).$$



$$\mathcal{V} = \mathcal{P}_3 \oplus \mathcal{B}$$

$$\dim\{\mathcal{P}_3\} = 10$$

$$\dim\{\mathcal{V}\} = 21$$



N_E	N_{dof}	$h_{\max}$	$\mathcal{E}_{H^1}$	$\mathcal{E}_{L^2}$
441	5083	1.1×10^{-1}	6.4×10^{-4}	3.3×10^{-5}
1681	18963	5.4×10^{-2}	9.0×10^{-5}	2.4×10^{-6}
6561	73123	2.7×10^{-2}	1.2×10^{-5}	1.6×10^{-7}
25921	287043	1.4×10^{-2}	1.6×10^{-6}	1.1×10^{-8}

10. What can we do with the family?

On structured meshes for wave equation:

- match the speed of particular waves or asymptotically (long waves).
- increase the order of the discretization – super-convergence.

Super-convergence for second-order vertex-based discretizations of acoustic wave equation (Gyrya, Lipnikov):

- on **rectangular meshes** (one parameter in $\mathbb{A}_E$, 6 parameters in $\mathbb{M}_E$) – **4th order** one parameter family of schemes
- on **square meshes** – **4th order** scheme that is anisotropic up to **6th order**.
- on **cuboid meshes** (10 parameters in $\mathbb{A}_E$, 28 parameters in $\mathbb{M}_E$) **4th order** schemes.

For second-order edge-based discretization of Maxwell equation (Gyrya, McGregor, Bokil, Gibson):

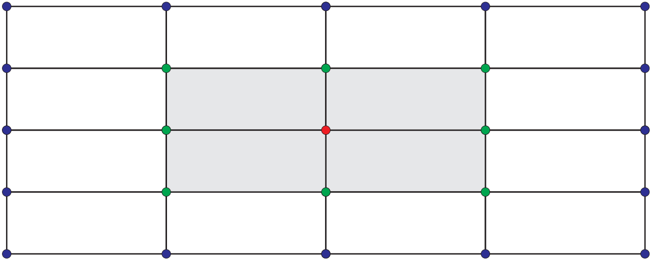
- on **square meshes** (no parameters in $\mathbb{A}_E$, 3 parameters in $\mathbb{M}_E$) – **4th order** scheme.

11. Tools

On structured meshes:

- Von-Neumann dispersion relation with reduction to local problem.
- Error cancelation for polynomial waves: spatial with temporal discretizations .

12. Von Neumann based analysis

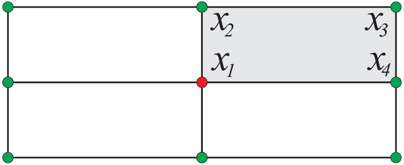


Von Neumann-type analysis: $u = e^{i(\boldsymbol{\kappa}\cdot\mathbf{x} - \kappa c_h t)}$, $\kappa = |\boldsymbol{\kappa}|$.

$$\frac{U^{k+1} - 2U^k + U^{k-1}}{\triangle t^2} = c^2 \mathbb{W}\mathbb{A}U^k, \qquad \mathbb{W} = \mathbb{D}^{-1}\mathbb{M}\mathbb{D}^{-1} \approx \mathbb{M}^{-1}.$$

- Need reduced-form dispersion relation (in terms of the elemental matrices) $\mathbb{M}_E$ and $\mathbb{A}_E$.

13. Reduced-form dispersion relation



$$\frac{2(1-\cos(c_h\kappa\triangle t))}{\triangle t^2} = c^2(U^*\mathbb{W}_E U)(U^*\mathbb{A}_E U),$$

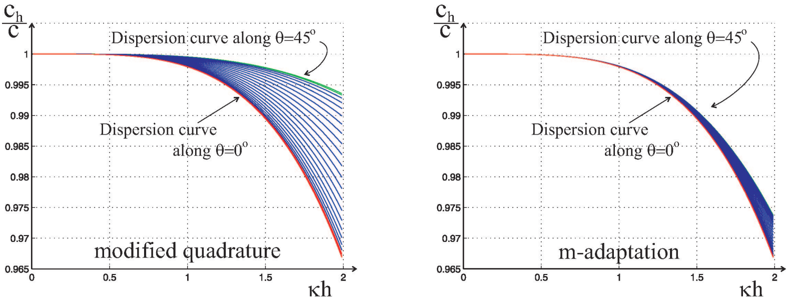
where

$$U = \begin{bmatrix} u(\mathbf{x}_1) \\ u(\mathbf{x}_2) \\ u(\mathbf{x}_3) \\ u(\mathbf{x}_4) \end{bmatrix} = \begin{bmatrix} e^{i\boldsymbol{\kappa}\cdot\mathbf{x}_1} \\ e^{i\boldsymbol{\kappa}\cdot\mathbf{x}_2} \\ e^{i\boldsymbol{\kappa}\cdot\mathbf{x}_3} \\ e^{i\boldsymbol{\kappa}\cdot\mathbf{x}_4} \end{bmatrix}$$

Asymptotic approach ($\triangle x, \triangle y \sim h$):

$$c_h = c + O(\kappa h)^d, \qquad \text{maximize } d.$$

14. Numerical experiment



- The dispersion curves for various angles θ between the planar wave and the mesh axis for the Courant number $\nu = 0.75$.

15. Conclusions & possible extensions

- Construction of the MFD/VE family of discretizations for stiffness and mass matrices based on completion of basis.
- The completion of basis respects the orthogonality condition.
- The completion of basis is different for mass and stiffness matrices and is not unique.
- The freedoms in the construction of the discretizations can be used to improve scheme properties (at least on structured meshes).
- The improved schemes are highly efficient: fourth order accuracy with second order complexity. No solution of global system is required.

Thank you!