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A practical method for reliability of dynamic systems under limited information

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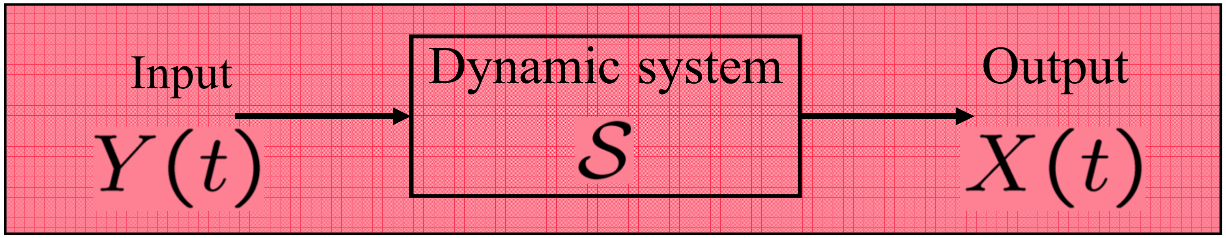
Reliability of dynamic systems



- $Y(t)$ = stochastic input (excitation)
  - Examples: launch environments for satellites, re-entry of RBs
  - Can be Gaussian or non-Gaussian; stationary or non-stationary
- S = dynamic system
  - Can be real hardware or model (e.g., Salinas FE model)
  - Can be linear or nonlinear
- $X(t)$ = stochastic output (response)
  - Examples: stress/accel in critical components
- Objective: calculate (time-dependent) probability that system output remains in safe set D during lifetime τ

Reliability: $p_S(\tau) = P(X(t) \in D,\ 0 \le t \le \tau)$

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Current methods for reliability analysis

Input, Y \ **Dynamic system, S**	*Linear*	*Nonlinear*
Gaussian	**1**	**2**
Non-Gaussian	**3**	**4**

Most methods require information typically not available for problems of practical interest

- Case 1 (classic linear random vibration)
  - Full probability law of output X is available
  - Reliability can be calculated directly
- Cases 2, 3, and 4
  - Reliability must be estimated
  - Techniques include path integral method, Fokker-Plank equation, perturbation, stochastic averaging, equivalent linearization, moment closure

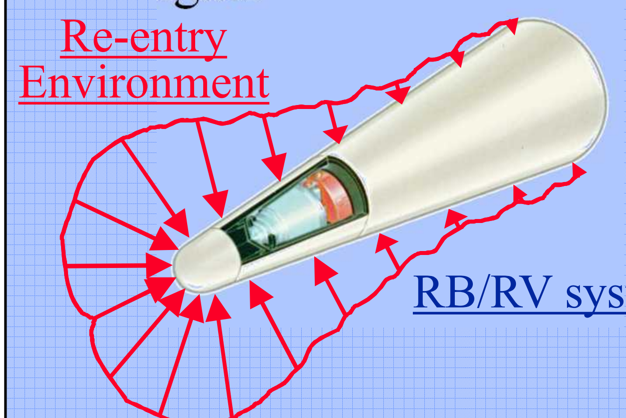
Available information for practical problems

1) One or more samples of output *X* (required)
2) Knowledge of some properties of system *S* (optional)

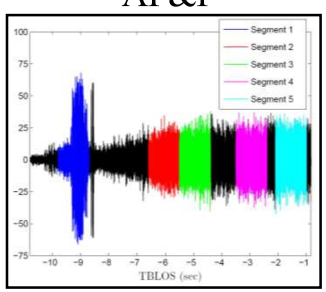
Example: re-entry random vibration

Experimentalist

Measurements at a few internal locations for a few flights

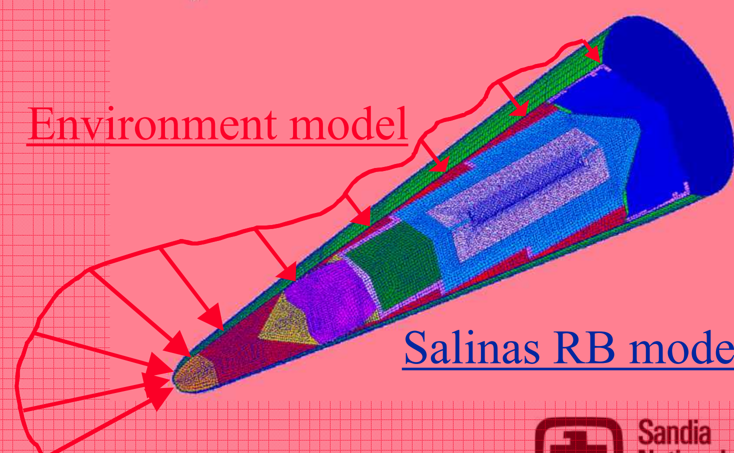


Response at base of AF&F



Analyst

Small number of runs of complex FE model



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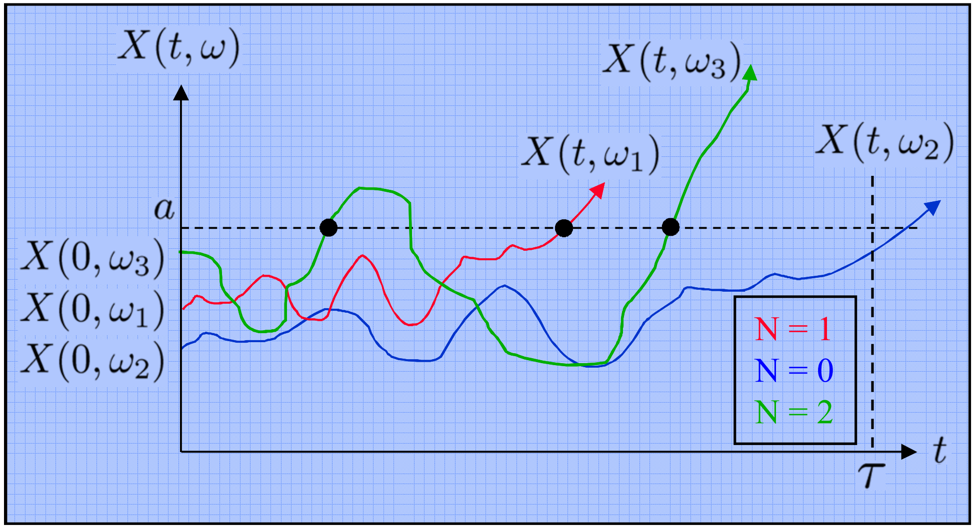
Outline

- Our approach is consistent with the types of information available for problems of practical interest
  - Response data and, if available, additional knowledge on system properties
- Two methods are used for analysis
  - Method 1: output statistics
  - Method 2: non-Gaussian translation processes
    - Additional knowledge can only be used by Method 2
  - Both make use of crossing theory of stochastic processes
  - Coefficient of variation provides measure of accuracy
- Applications
  - Simple dynamic systems with known solutions

Crossing theory of stochastic processes

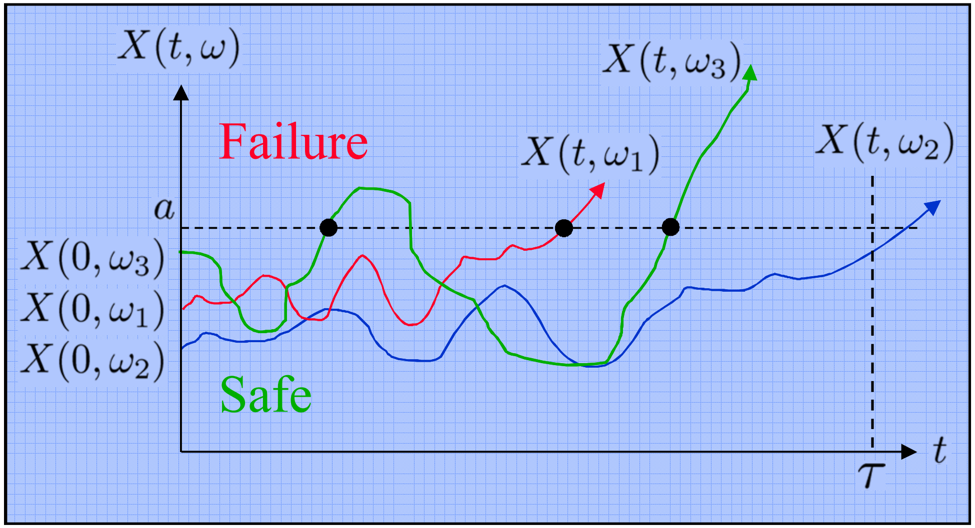


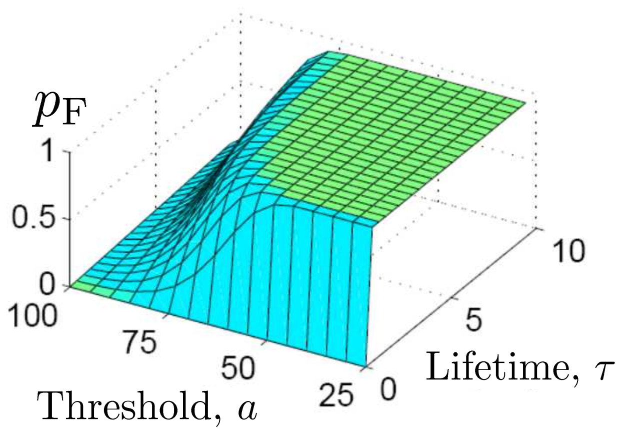
- $X(t)$ = stationary, scalar stochastic process (can be generalized)
- $N_a(\tau)$ = random # of times X "upcrosses" a during $[0,\tau]$
- $\nu(a) = \frac{1}{\tau}\,\mathrm{E}\left[N_a(\tau)\right]$ is mean upcrossing rate of X

- Example: X = Gaussian process with mean μ and variance σ^2

$$\nu(a) = \frac{\mathrm{Std}[\dot{X}(t)]}{2\,\pi\,\sigma}\,\exp\left[-\frac{1}{2}\left(\frac{a-\mu}{\sigma}\right)^2\right]$$

Reliability estimates by crossing theory





- Safe set: $D = (-\infty, a)$
- Assumptions
  - System is safe at $t = 0$
  - Failure events are rare
- Reliability

$$p_S(\tau) = P(X(t) \in D,\ 0 \le t \le \tau)$$
$$\approx e^{-\nu(a)\,\tau}$$

- Probability of failure

$$p_F(\tau) = 1 - p_S(\tau)$$

Accurate estimates of p_F require accurate estimates of ν

Method 1: Output statistics

- Available information: one sample of system response

$$X_1 = X(t_1), X_2 = X(t_2), \ldots, X_n = X(t_n)$$

  - Assume X is stationary/ergodic
  - Assume $\Delta t = t_k - t_{k\text{-}1}$ is constant and sufficiently small
- Statistical estimator for $\nu(a)$, the mean rate at which X upcrosses level a

$$V_n(a) = \frac{1}{n\,\Delta t} \sum_{i=1}^{n} 1\left(X_i \leq a,\; X_{i+1} > a\right)$$

- Accuracy
  - Depends on a, n, and correlation length of X
  - Quantified by estimates of C.O.V.$[V_n(a)]$

Method 2: Translation model (1 of 2)

- Available information
  - One sample of system response

$$X_1 = X(t_1), X_2 = X(t_2), \ldots, X_n = X(t_n)$$

  - Knowledge of system properties (optional)
    - Examples: (i) X takes values on bounded interval $[a, b]$; (ii) the distribution of X is symmetric about zero
- Assume response can be represented by a non-Gaussian translation process:

$$X_{\mathrm{T}}(t) = F_n^{-1} \circ \Phi[G(t)] = h_n[G(t)]$$

  - G is a zero-mean, unit-variance, stationary Gaussian process
  - Φ is the CDF of a N(0,1) random variable
  - X_{T} is strictly stationary with marginal CDF F_n

Method 2: Translation model (2 of 2)

- Choose F_n based on available sample and any additional information on the properties of X
  - Examples: (i) if X is bounded, use beta distribution; (ii) if X is unbounded and symmetric, use student-t distribution
  - Calibrate parameters of CDF F_n using available data
  - Without any additional information, use:

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} 1(X_i \leq x)$$

- Estimator for $\nu(a)$

$$V_{\mathrm{T},n}(a) = \frac{\mathrm{Std}[\dot{G}(t)]}{2\,\pi}\, e^{-1/2\,\left[h_n^{-1}(a)\right]^2}$$

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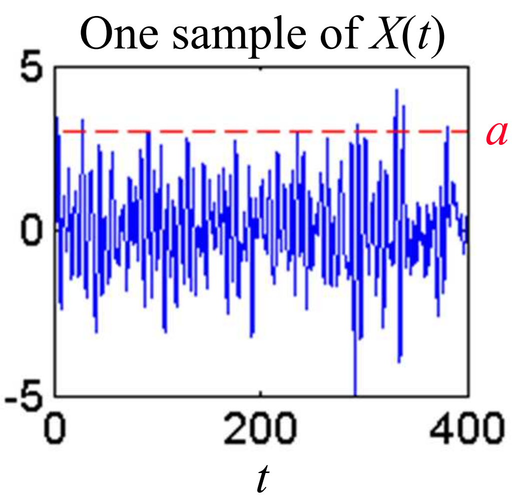
Application: linear oscillator

$$\ddot{X}(t) + 2\,\zeta\,\omega_0\,\dot{X}(t) + \omega_0^2\,X(t) = W(t),\;\; t \geq 0$$



- ω_0, ζ = natural frequency and damping ratio
- W = stationary, zero-mean, Gaussian white noise with one-sided PSD $1/\pi$
- Initial conditions:

  $X(0) \sim N(0, \sigma)$ and $\dot{X}(0) \sim N(0, \omega_0\,\sigma)$, where $\sigma^2 = 1/(4\,\zeta\,\omega_0^3)$
- X = stationary, zero-mean, Gaussian process with variance σ^2
- Exact solution for mean a–upcrossing rate of X:

$$\nu(a) = \frac{\omega_0}{2\pi}\,\exp\left(-\frac{a^2}{\sigma^2}\right)$$

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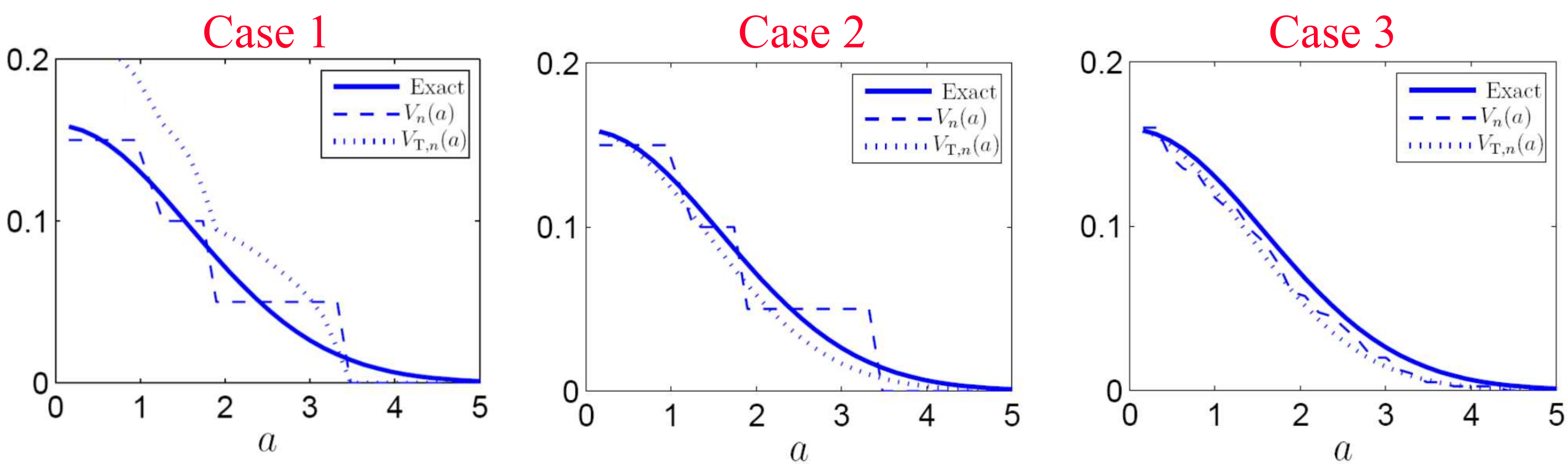
Estimates of ν(*a*) for linear oscillator



- Case 1: short sample, no additional information
  - Both methods perform poorly; no information provided for $a > 3.5$
- Case 2: short sample, X is known to have a symmetric distribution
  - Method 2 improves and provides information for any *a*
- Case 3: long sample, X is known to have a symmetric distribution
  - Both methods are adequate when sample is long

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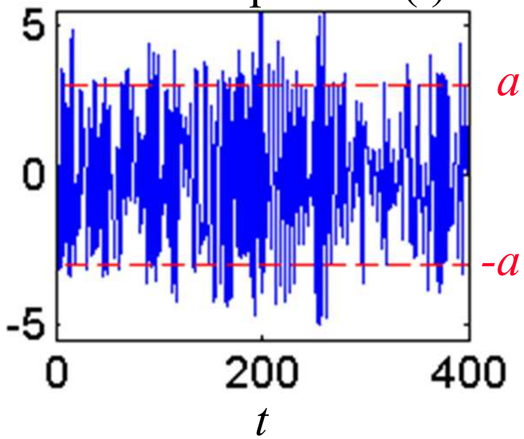
Application: nonlinear (Duffing) oscillator

$$\ddot{X}(t) + c\,\dot{X}(t) + \omega_0^2\,X(t)\left[1 + \epsilon\,X(t)^2\right] = W(t),\ \ t \geq 0$$

One sample of $X(t)$



- $\omega_0, c =$ initial frequency and damping coefficient
- $\epsilon =$ degree of nonlinearity
- $W =$ stationary, zero-mean, Gaussian white noise with one-sided PSD $1/\pi$
- $X =$ non-Gaussian process with stationary marginal PDF:

$$f(x) = \frac{\sqrt{2\,\epsilon}\,\exp\left(-\frac{c\,\omega_0^2}{4\,\epsilon}\right)}{K_{1/4}\left(\frac{c\,\omega_0^2}{4\,\epsilon}\right)}\,\exp\left[-c\,\omega_0^2\left(x^2 + \frac{\epsilon}{2}\,x^4\right)\right]$$

- Exact solution for mean $(-a, a)$–outcrossing rate of X:

$$2\,\nu(a) = \frac{1}{\sqrt{c\,\pi}}\,f(a)$$

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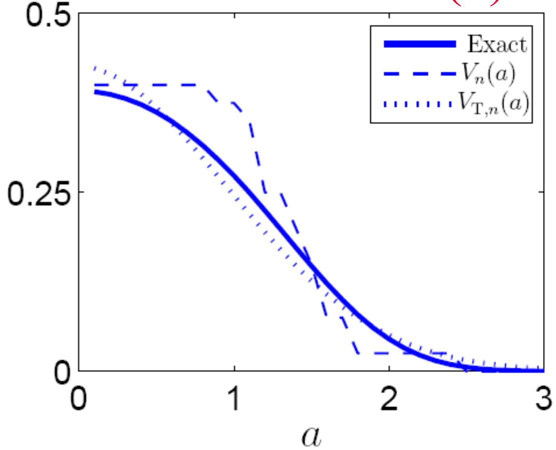
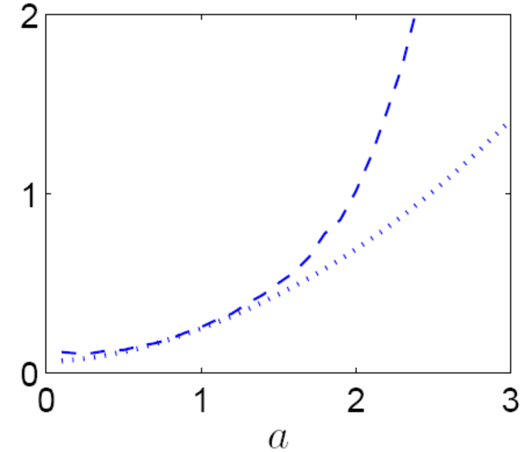
Results for Duffing oscillator

Estimates of $2\nu(a)$



C.O.V. of each estimate



- X is known to have a symmetric distribution
- Method 1
  - Poor performance; no information for *a* > 2.5
- Method 2
  - Adequate performance for all *a*
  - C.O.V. estimates demonstrate Method 2 is less sensitive to particular sample used

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Summary

- Developed method to assess reliability of dynamic systems under limited information
- Available information is consistent with practical problems
  - One sample of system output (required)
  - Knowledge of system properties (optional)
- Features
  - No requirement that system be linear
  - Output can be experimental data or from mathematical model
  - Special class of non-stationary output is considered
- Simple dynamic systems
  - Linear and nonlinear Duffing oscillators; MEMS dynamics
- Complex dynamic systems
  - RB re-entry random vibration