SAND2007-4412C

V&V Example Problem

Thomas L. Paez
Don Simons
Hans Mair
Bill Oberkampf
Len Schwer
Ben Thacker

Sandia National Laboratories

Goals

- Demonstrate a comprehensive example that addresses the most important elements in the ASME Guide to V&V
- Secondary elements are mentioned at the appropriate places, but references are given for details

Elements of V&V Essential to Include (Guide References in Brackets)

1. Conceptual model including intended use [Section 3.1]
2. Mathematical model [Section 3.2]
3. Computational model [Sect 3.3]
4. Code verification – convergence using analytical solution [Section 4.1]
5. Calculation verification – model convergence [Section 4.2]
6. Model parameter calibration [Section 3.4.1]
7. Validation experiments [Section 5.1, 5.2]
8. Comparison of model outputs to experimental measurements [Section 5.3]
9. Uncertainty quantification and propagation [Section 3.6]
10. Decision of model adequacy [Section 5.3.2]

Elements of V&V to be Avoided

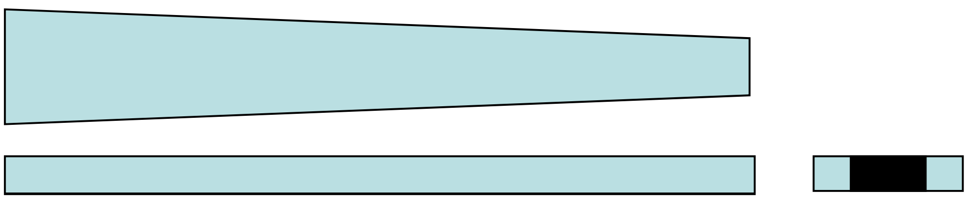
1. Complex physics or models
2. Numerical algorithms
3. Manufactured solutions
4. Consistency tests
5. Software quality engineering
6. Updates to model form
7. Sensitivity analysis
8. Experimental planning and design
9. Experimental measurement selection and sources of error
10. Extrapolation
11. Documentation of V&V and UQ

Ultimate Objective

- Ultimate objective: Generate validated solid mechanics model to accurately simulate static behavior of aircraft wing under distributed load
- To confirm (via validation comparisons) our capability to generate such a model, we might model and test other systems, for example:
  - Actual aircraft wing including non-structural elements and with distributed static load
  - Actual aircraft wing excluding non-structural elements and with distributed static load
  - Actual aircraft wing … with concentrated static load
  - …
  - Tapered beam
  - Prismatic beam

Physical System

- Consider system that is a tapered, cantilever beam



$$L = 72\ in$$
$$d = 2\ in$$
$$w_0 = 8\ in$$
$$w_L = 4\ in$$

  - Hollow cross-section
  - Deterministic physical dimensions (Precisely known and replicated)
  - Material modulus of elasticity - Random (but treated as deterministic, using mean)
  - Boundary constraint - nonlinear torsional spring w/ random char (Model 1 – Approximate, linear & deterministic, Model 2 – Approximate linear model with random parameter)
  - Deterministic load applied in vertical direction on right half of beam

Sandia National Laboratories

Conceptual Model

- *Conceptual model* of the physical system:
  - Cantilever beam with deterministic geometry and concentrated load
  - Material modulus of elasticity varies randomly along length
  - Supported at boundary via nonlinear spring with unknown form and random parameters

- The *intended use* of the model: to predict the tip deflection of a beam tested in the laboratory (Partial evidence that we can credibly model physics of ultimate reality of interest - aircraft wing)

Sandia National Laboratories

Conceptual Model

- *Model adequacy – Validation criteria* (2 criteria):
  - 1a. Deterministic: Absolute values of relative errors of predicted deflections

$$(rel\ error) = \frac{\Delta_{pred} - Avg[\Delta_{meas}]}{Avg[\Delta_{meas}]}$$

    must be equal to or less than ten percent at all locations where experimental deflections are measured
  - 1b. Deterministic model, statistical criterion: Predicted deflections must lie within the interval

$$[Avg(\Delta_{meas}) - 3 \times Std(\Delta_{meas}), Avg(\Delta_{meas}) + 3 \times Std(\Delta_{meas})]$$

Conceptual Model

- *Model adequacy – Validation criteria* (2 criteria):
  - 2. Probabilistic: Probability that random, model-predicted beam displacements differ from random, experimental validation displacements by a pre-established amount, or less, is bounded by a pre-established probability (Details presented later)

$$P\left(\left|\Delta_{pred} - \Delta_{meas}\right| \leq k\sigma_{\Delta_{meas}}\right) \geq P_0$$

Mathematical Model

- Model beam using the following assumptions:
  - Material linear, deterministic
  - Modulus of elasticity spatially constant, equal to “handbook” mean
  - Small deflections
  - Bernoulli-Euler beam theory
  - Perfect symmetry (deflection in a plane)
  - Boundary constraint spring is linear
    - Deterministic model
    - Probabilistic model

Computational Model

- Beam model implemented in the finite element framework
  - Elements – Bernoulli-Euler beam elements
  - Boundary conditions – fixed end used in computations, then end rotation associated with constraint spring added
  - Material linear - Modulus of elasticity constant, equal to “handbook” mean

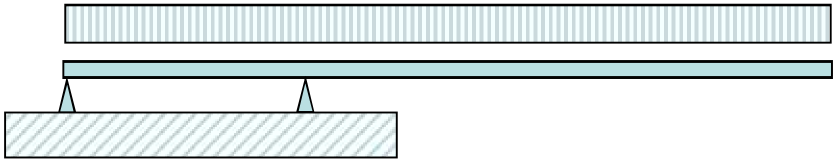
- Constraint spring model

$$M_0 = k_R \theta_0$$

  - k_R inferred from experiments in which θ_0 inferred
  - Model number 1 - k_R equals average of inferred values
  - Model number 2 - k_R modeled as normal random variable

Code Verification

- *Code verification* - perform via assessment of convergence of finite element code computations to analytical solution
  - Example – Compare code predictions to analytical solution for deflection of prismatic beam



$$L = 10\ in$$

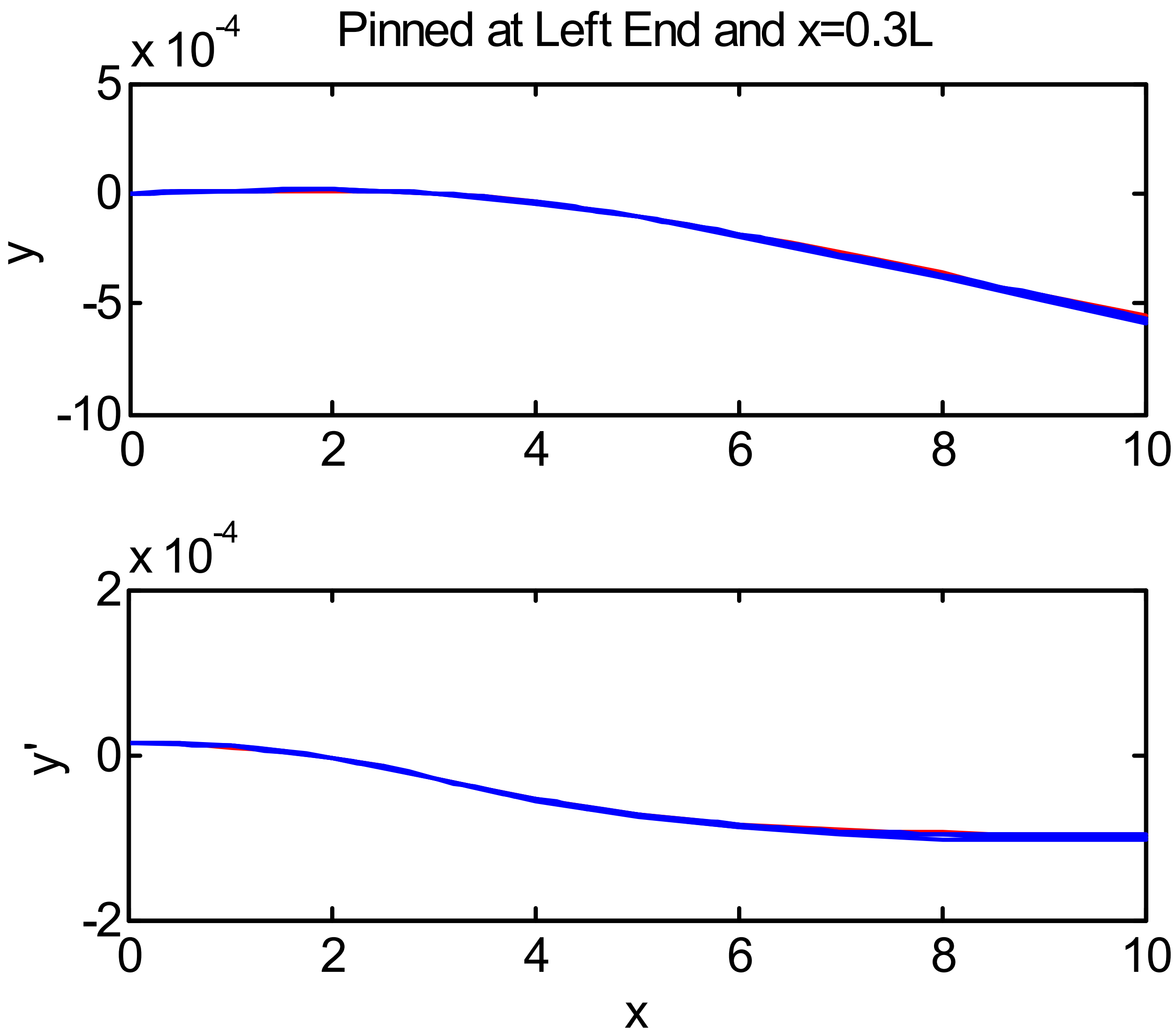
$$I = 1/12\ in^4$$

$$E = 10 \times 10^6\ psi$$

$$load = 1\ lb/in$$

  - Consider free end deflection
  - (Other cases also considered)

Code Verification

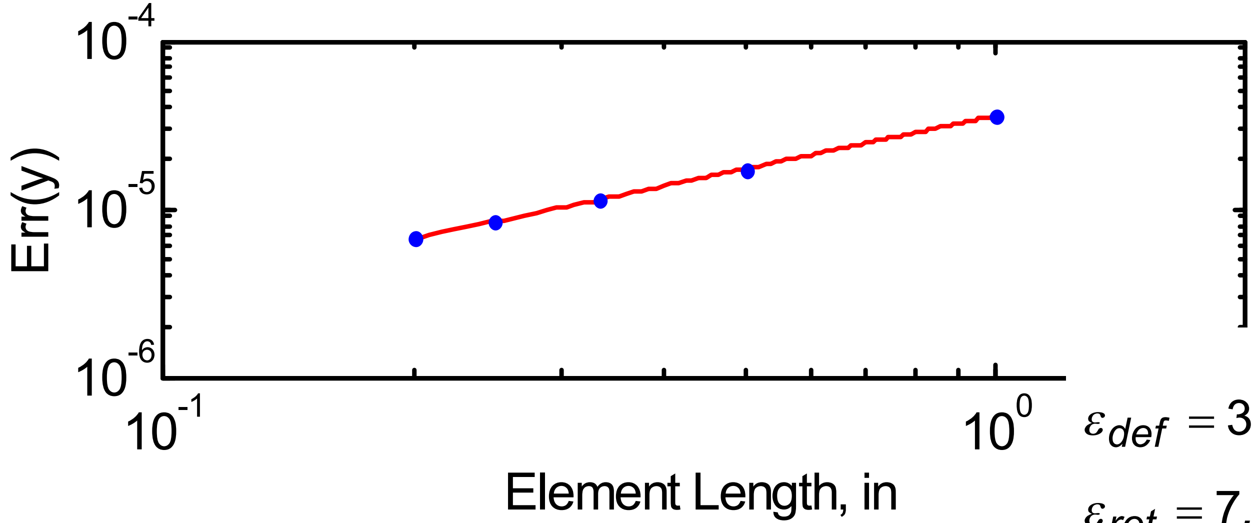


Deflections and rotations

Analytical solutions – red

FE code computations w/ 10, 20, 30, 40, 50 elements - blue

Code Verification

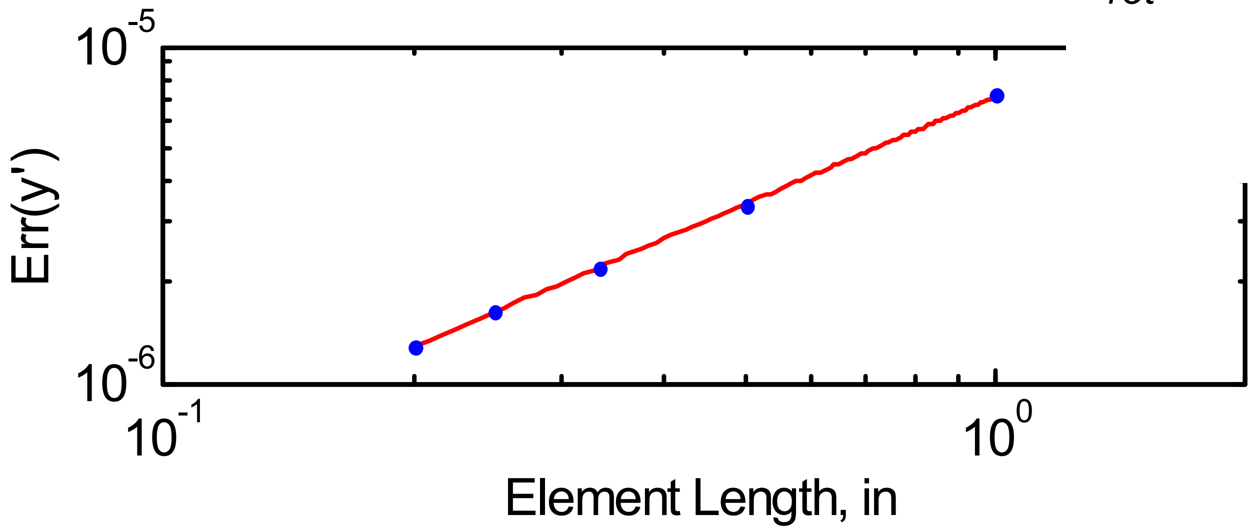


Convergence

Deflections and rotations

$$\varepsilon_{def} = 3.59 \times 10^{-5} (\Delta x)^{1.05} \qquad \Delta x > 0$$

$$\varepsilon_{rot} = 7.11 \times 10^{-6} (\Delta x)^{1.06} \qquad \Delta x > 0$$

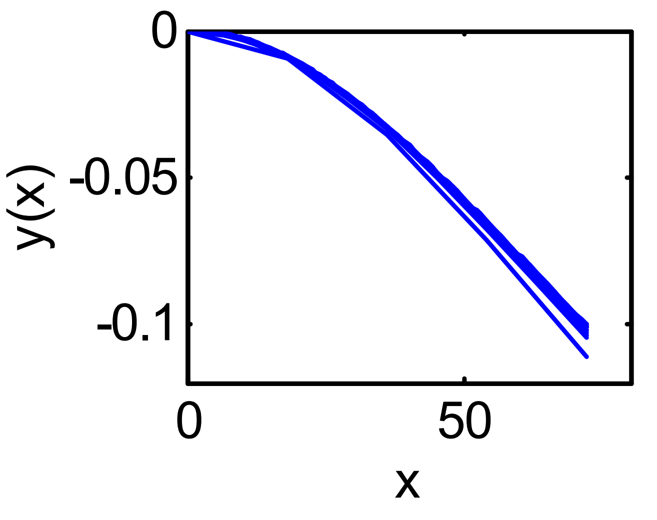


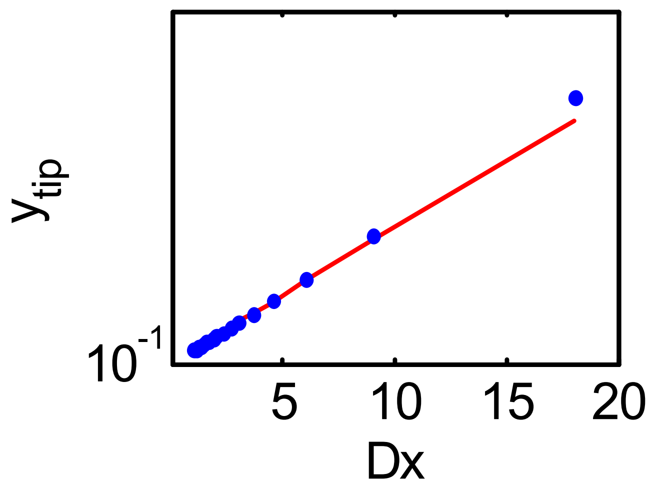
Sandia National Laboratories

Calculation Verification

- *Calculation verification* - perform via assessment of convergence of tapered beam finite element model to well-defined limit
  - Model is FE model of tapered beam
  - Required mesh refinement is related to required accuracy
  - Accuracy relates to experimental transducer error
  - Transducer error is ~ 1%
  - Required accuracy is 0.5%

Calculation Verification





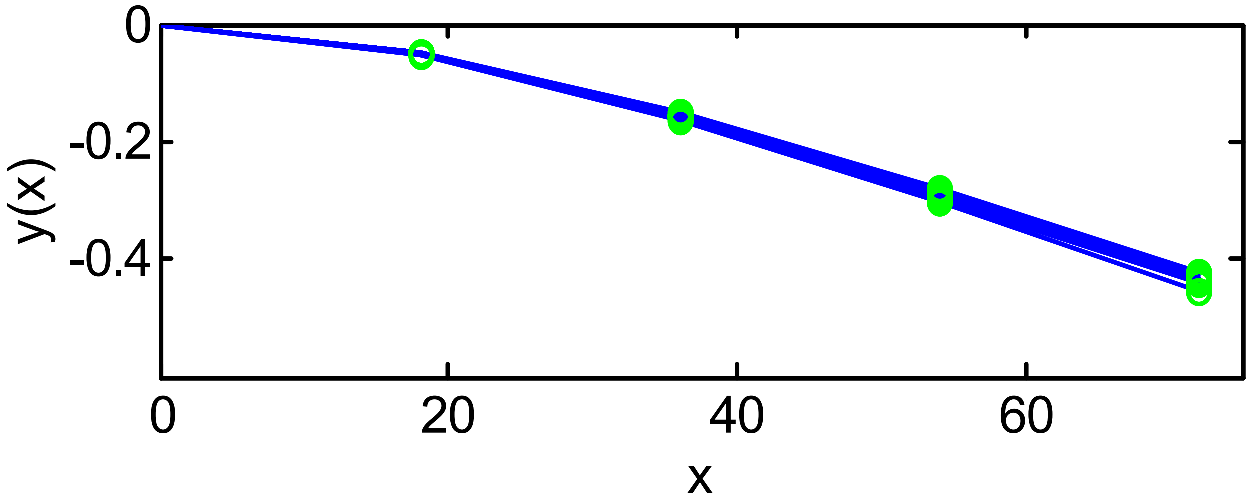
- Approx deflection at Dx=0 is 0.1001 in
- Model that yields deflection of 0.1006 in is satisfactory
- That model has 68 elements
- Element length is 72/68 in

Model Parameter Calibration

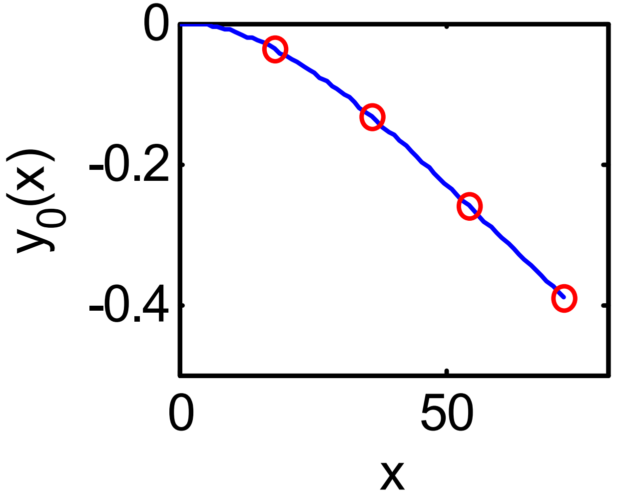
- *Calibration* experiments:
  - Construct 20 nominally identical prismatic beams w/ 72 in length and 2x6 in cross-section
  - Constrain each beam as physical system is constrained
  - Perform static experiments - Load each beam as beams will be loaded in validation experiment – 3.33 lb/in load on right half of beam
  - Measure deflections of beams from their equilibrium positions (deflections under their own weights) at four locations using displacement transducers following application of static load
  - Retain calibration data for use in identification of end constraint model

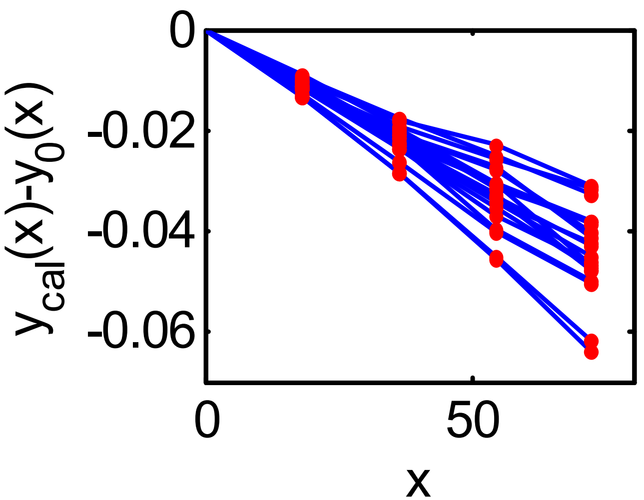
Model Parameter Calibration

- Results of *calibration* experiments (Recall that measurements contain noise) Twenty experiments. Measurements at four locations (green circles) joined by straight lines



Model Parameter Calibration

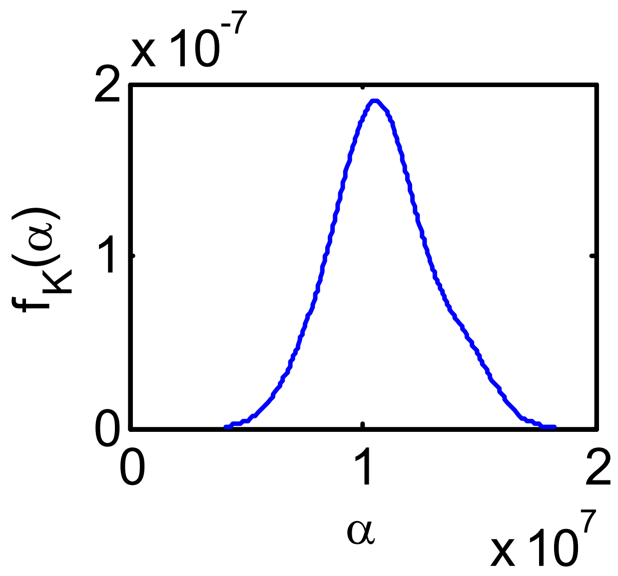




- Identification of model parameter
  - Modulus of elasticity assumed constant – equal to handbook mean – here, and in validation predictions
  - Use FE model to predict deflection of calibration beam with left end fixed (computed deflection shown at left, top)
  - Subtract this deflection from measured calibration deflections (results shown at left, bottom)
  - Estimate slope, θ_0, of each line
  - Infer constraint stiffness

$$k_R = \frac{M_0}{\theta_0}$$

Model Parameter Calibration



- Identification of constraint stiffness model parameter
  - One constraint stiffness for each experiment
  - Approximate PDF of estimated stiffnesses, k_R, shown at left
  - Sample mean and standard deviation of k_R

$$\overline{k_R} = 1.09 \times 10^7 \ in - lb / rad$$

$$s_k = 1.84 \times 10^6 \ in - lb / rad$$

  - (Stiffness will be assumed normal random variable in probabilistic model)

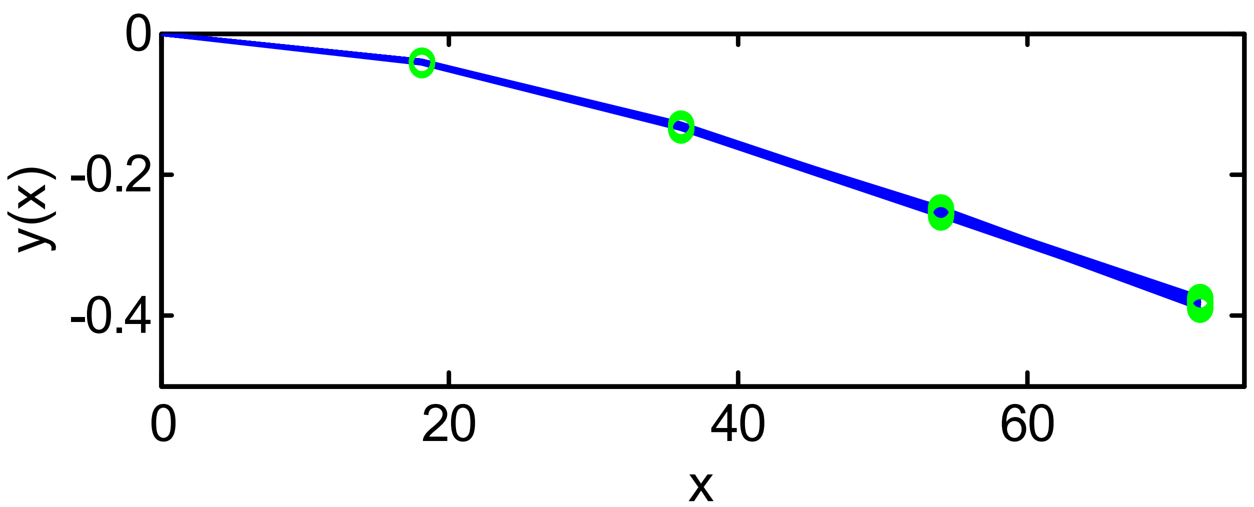
Sandia National Laboratories

Validation Experiments

- *Validation* Experiments
  - Construct 10 tapered beams described in “Physical System”
  - Constrain each beam as described in “Physical System”
  - Perform static experiments - Load each beam with 3.33 lb/in load on right half of beam
  - Measure deflections of beams from their equilibrium positions (deflections under their own weights) at four locations using displacement transducers following application of static load
  - Retain deflection data for use in validation comparisons

Validation Experiments

- Results of *validation* experiments (Recall that measurements contain noise) Ten experiments. Measurements at four locations (green circles) joined by straight lines



Comparison of Model to Experiments - Deterministic

- *Measure of response*
  - Beam deflection at four locations

- *Validation metric*
  - The absolute values of the relative errors of model-predicted deflections (deflections obtained using mean stiffness in rotational constraint)

$$(rel\ error) = \frac{\Delta_{pred} - Avg[\Delta_{meas}]}{Avg[\Delta_{meas}]}$$

must be equal to or less than ten percent at all locations where experimental deflections are measured

Comparison of Model to Experiments - Deterministic

- Example 1a
  - Ten validation experiments performed
  - Each experimental tapered beam loaded with distributed load over right half of beam
  - Deflections measured at four locations $x_i = 18i, 1 = 1,\ldots,4$; Experimental deflections are $y_{ij}, i = 1,\ldots,4,\ j = 1,\ldots,10$
  - Sample means of deflections are

$$\bar{y}_i = \frac{1}{10}\sum_{j=1}^{10} y_{ij} \qquad i = 1,\ldots,4$$

  - Model predictions of deflection are $y_i^{(mod)}, i = 1,\ldots,4$
  - Validation metrics are

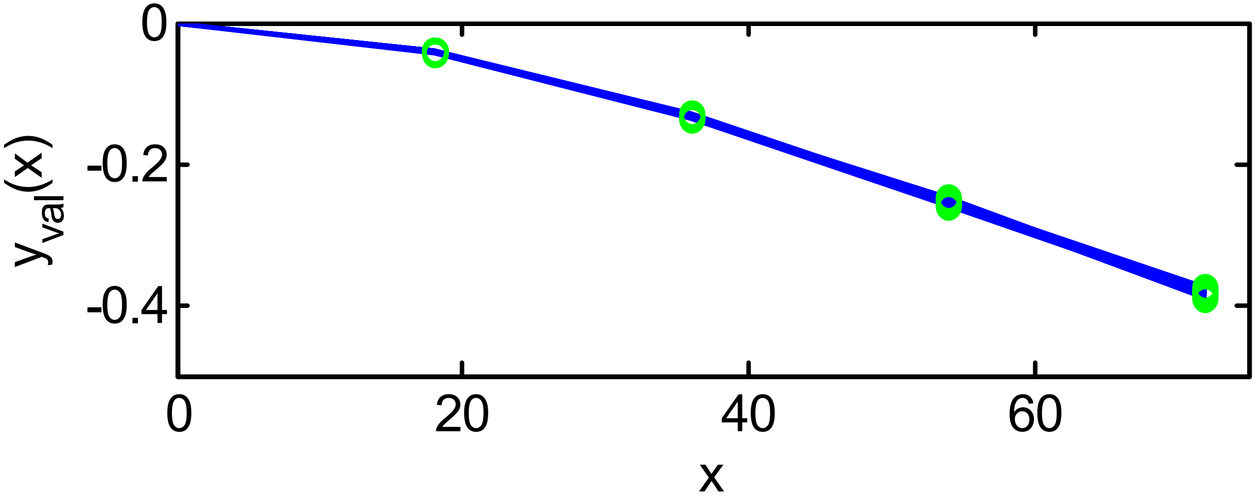
$$M_i = \left|\frac{\bar{y}_i - y_i^{(mod)}}{\bar{y}_i}\right| \qquad i = 1,\ldots,4$$

  - Adequacy (accuracy) criterion is

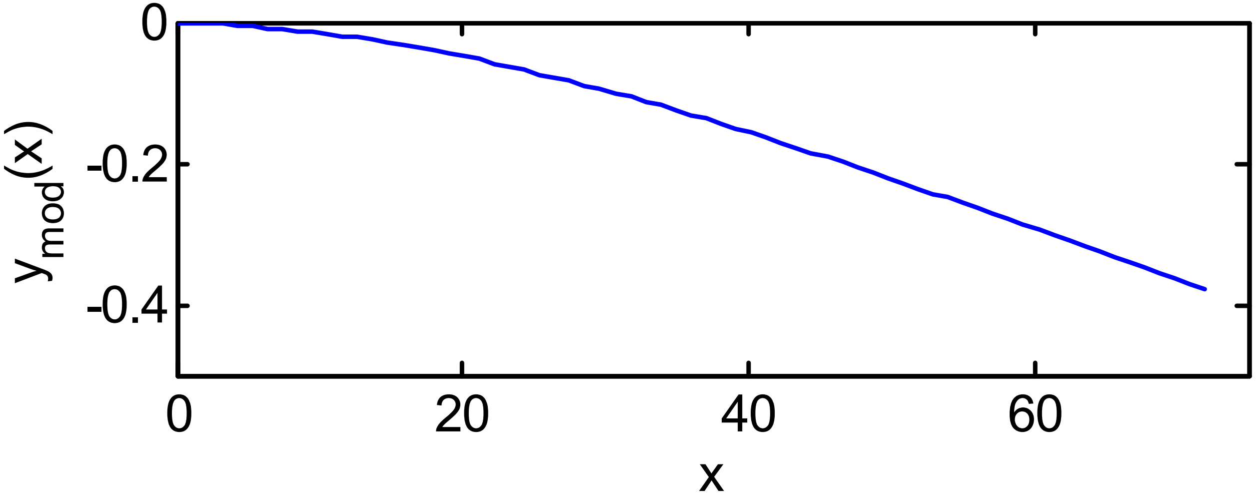
$$M_i \le 0.10 \qquad i = 1,\ldots,4$$

Sandia National Laboratories

Comparison of Model to Experiments - Deterministic

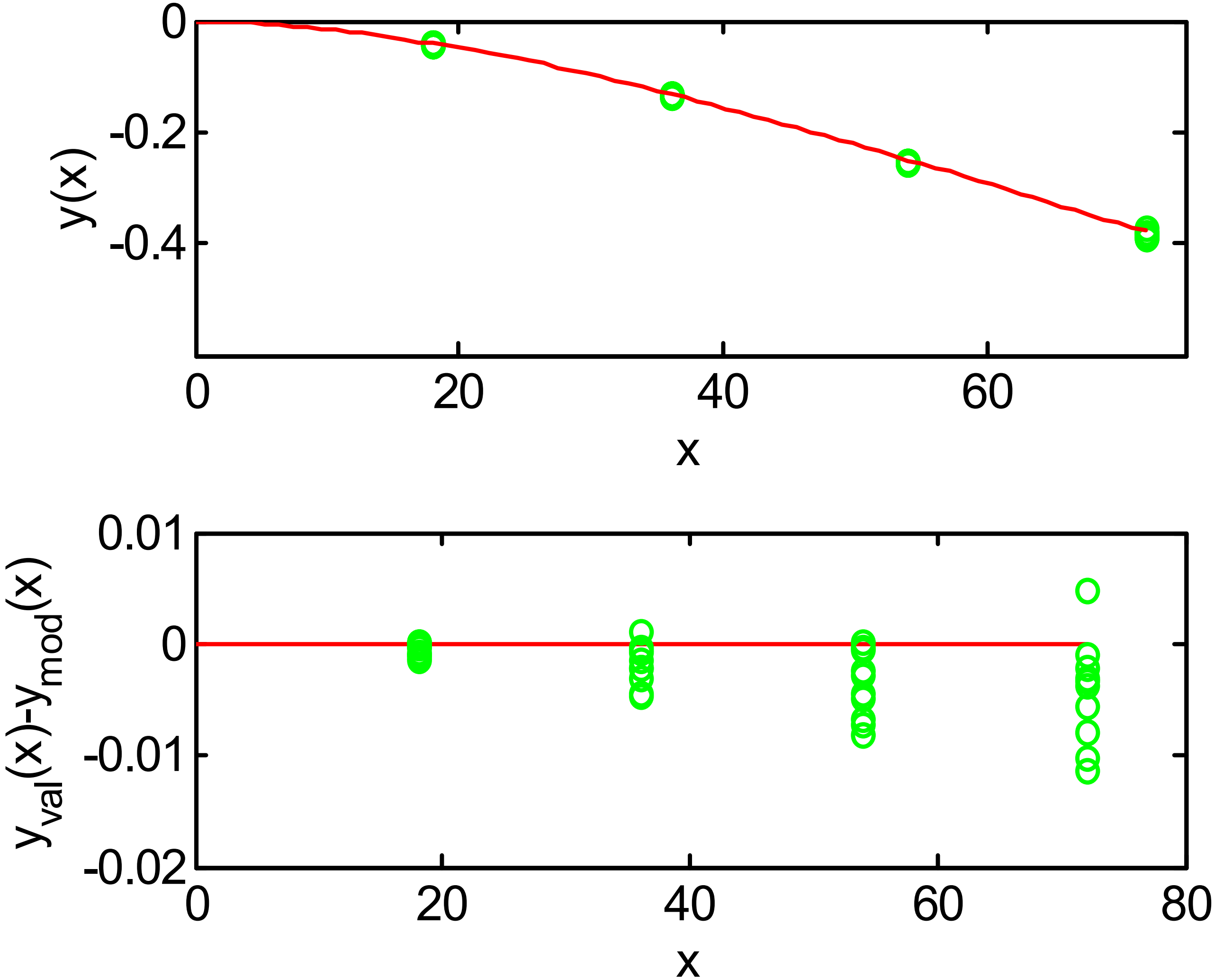


- *Validation* results (repeated)



- *Model prediction*

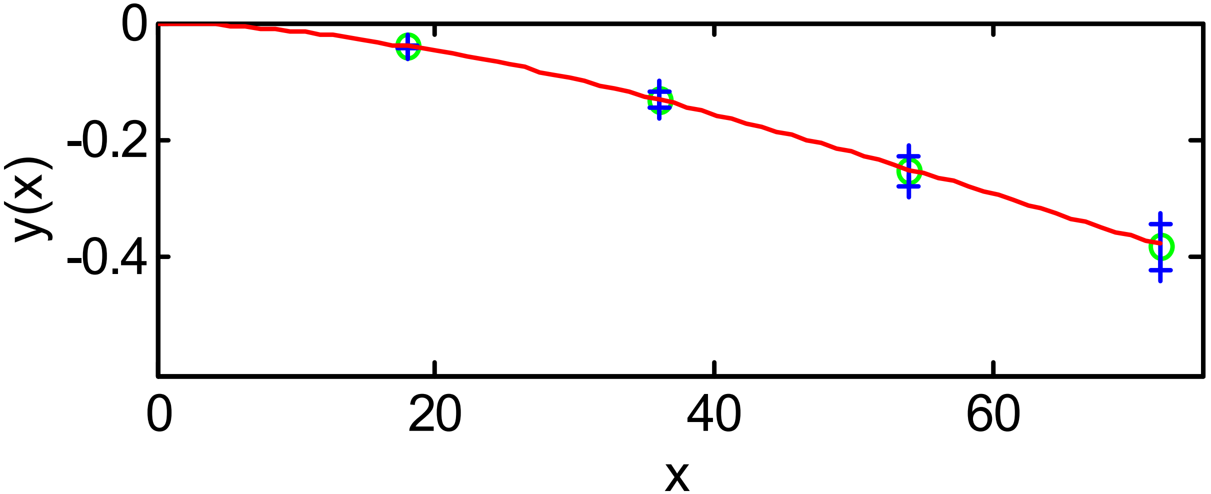
Comparison of Model to Experiments - Deterministic



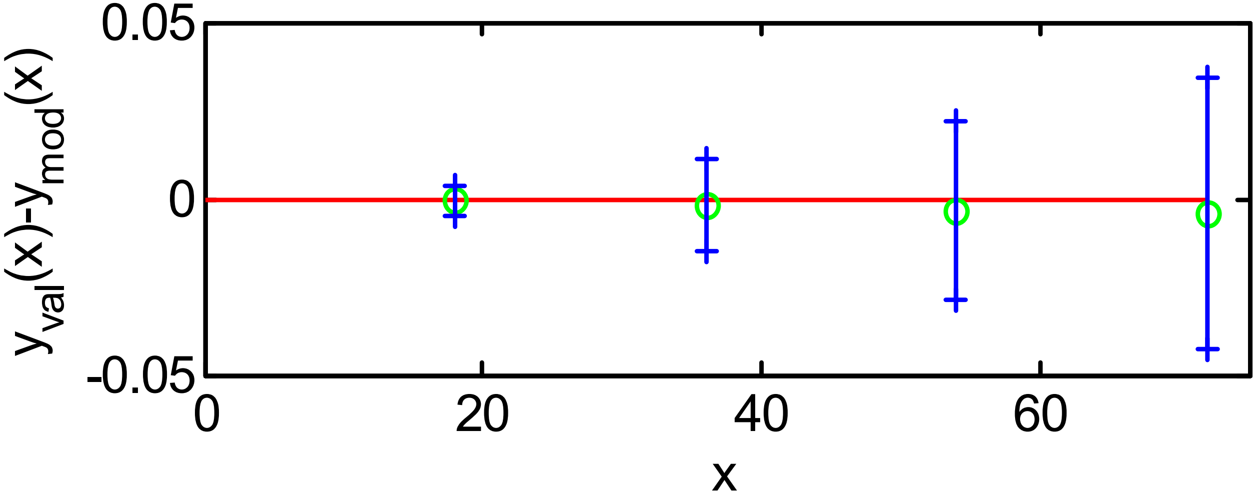
- *Validation* results (green circles) and *model prediction* (red line)
- *Validation* results minus *model prediction* (green circles)

Sandia National Laboratories

Comparison of Model to Experiments - Deterministic



- Mean of *validation* results (green), +/- 10 % limits (blue), model prediction (red)



- *Validation* results minus *model prediction* *(green)*, +/- 10 % limits (blue)

Sandia National Laboratories

Decision of Model Adequacy - Deterministic

- Validation criterion (relative model error less than 10%) satisfied in all four comparisons
- Model *valid* – adequate (accurate) – based on current criterion

Comparison of Model to Experiments - Statistical

- *Measure of response*
  - Beam deflection at four locations
- *Validation metric*
  - Predicted deflections must lie within the interval

$$\left[Avg\left(\Delta_{meas}\right)-3\times Std\left(\Delta_{meas}\right), Avg\left(\Delta_{meas}\right)+3\times Std\left(\Delta_{meas}\right)\right]$$

all locations where experimental deflections are measured

Comparison of Model to Experiments - Statistical

- Example 1b
  - Ten validation experiments performed
  - Each experimental tapered beam loaded with distributed load over right half of beam
  - Deflections measured at four locations $x_i = 18i, 1 = 1,\ldots,4$; Experimental deflections are $y_{ij}, i = 1,\ldots,4,\ j = 1,\ldots,10$
  - Sample means and standard deviations of deflections are

$$\bar{y}_i = \frac{1}{10}\sum_{j=1}^{10} y_{ij} \qquad s_i = \left[\frac{1}{9}\sum_{j=1}^{10}\left(y_{ij} - \bar{y}_i\right)^2\right]^{1/2} \qquad i = 1,\ldots,4$$

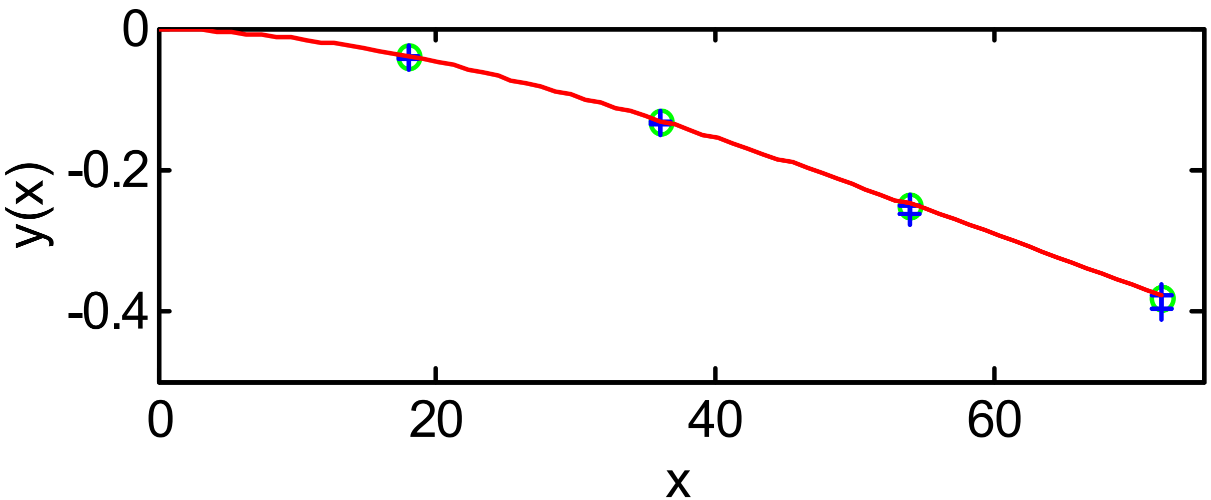
  - Model predictions of deflection are $y_i^{(mod)},\ i = 1,\ldots,4$
  - Adequacy (accuracy) criterion is

$$\bar{y}_i - 3s_i \le y_i^{(mod)} \le \bar{y}_i + 3s_i \qquad i = 1,\ldots,4$$

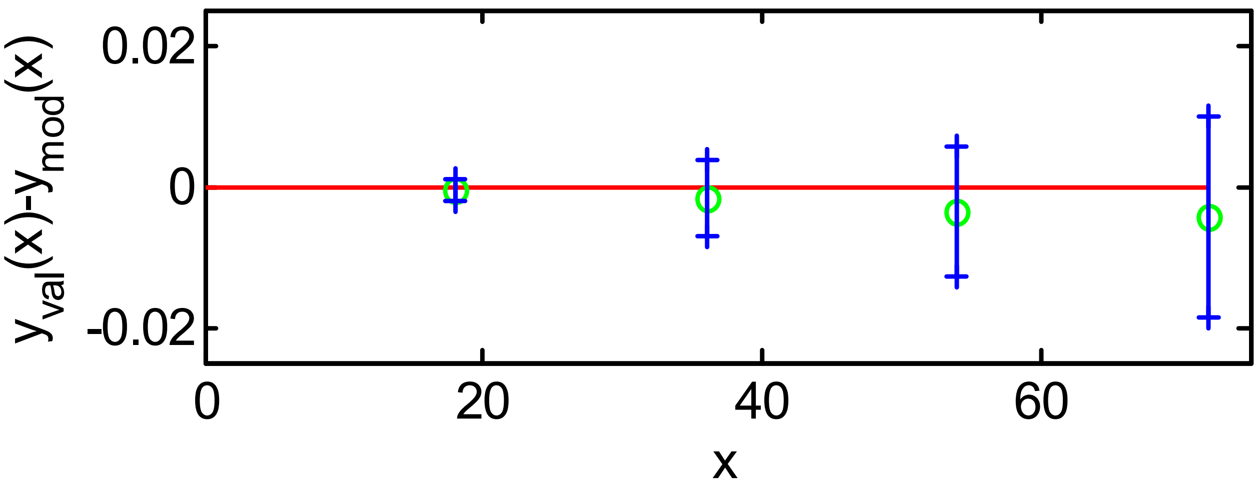
must be satisfied in four comparisons

Sandia National Laboratories

Comparison of Model to Experiments - Deterministic



- Mean of *validation* results (green), +/- 3σ limits (blue), model prediction (red)



- *Validation* results minus *model prediction* (green), +/- 3σ limits (blue)

Sandia National Laboratories

Decision of Model Adequacy - Statistical

- Validation criterion (model prediction within sample mean +/- three standard deviation interval) satisfied in all four comparisons
- Model *valid* – adequate (accurate) – based on current criterion

Comparison of Model to Experiments - Probabilistic

- *Measure of response*
  - Beam deflection at four locations

- *Validation metric*
  - Probability that random, model-predicted beam displacements differ from random, experimental validation displacements by a pre-established amount, or less, is bounded by a pre-established probability
  - Validation experiments yield ten measured results, with statistics given on a slide page 30

$$\bar{y}_i = \frac{1}{10}\sum_{j=1}^{10} y_{ij} \qquad s_i = \left[\frac{1}{9}\sum_{j=1}^{10}\left(y_{ij} - \bar{y}_i\right)^2\right]^{1/2} \qquad i = 1,\ldots,4$$

Comparison of Model to Experiments - Probabilistic

- *Validation metric*
  - Use probabilistic model (given on slide 20) for spring constraint
  - Constraint governed by $M_0 = k_R \theta_0$
  - k_R is a random variable with mean and standard deviation

$$\overline{k_R} = 1.09 \times 10^7 \ in - lb / rad$$

$$s_k = 1.84 \times 10^6 \ in - lb / rad$$

  - Generate 20 realizations of coefficient k_R
  - Compute 20 corresponding realizations of model-predicted response $y_{ij}^{(mod)}, i = 1,\ldots,4, \ j = 1,\ldots,20$
  - Estimate mean and standard deviation using formulas analogous to those on previous slide

$$\bar{y}_i^{(mod)} = \frac{1}{20} \sum_{j=1}^{20} y_{ij}^{(mod)} \qquad s_i^{(mod)} = \left[\frac{1}{19} \sum_{j=1}^{20} \left(y_{ij}^{(mod)} - \bar{y}_i^{(mod)} \right)^2 \right]^{1/2} \qquad i = 1,\ldots,4$$

Sandia National Laboratories

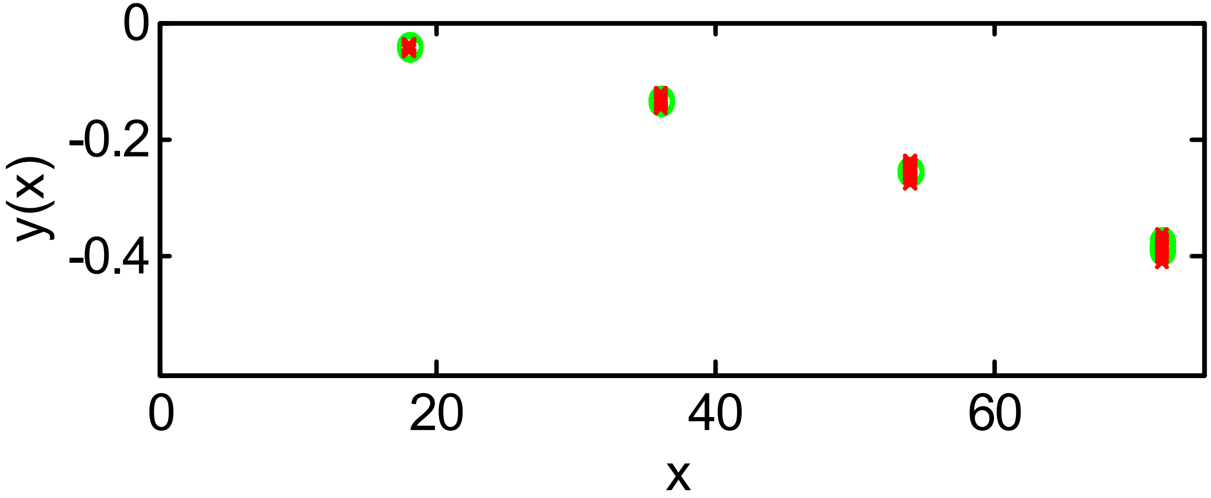
Comparison of Model to Experiments - Probabilistic

- *Validation metric*
  - If the measured validation experiment and model-predicted deflections come from the same random source (i.e., they are probabilistically indistinguishable) then, on average

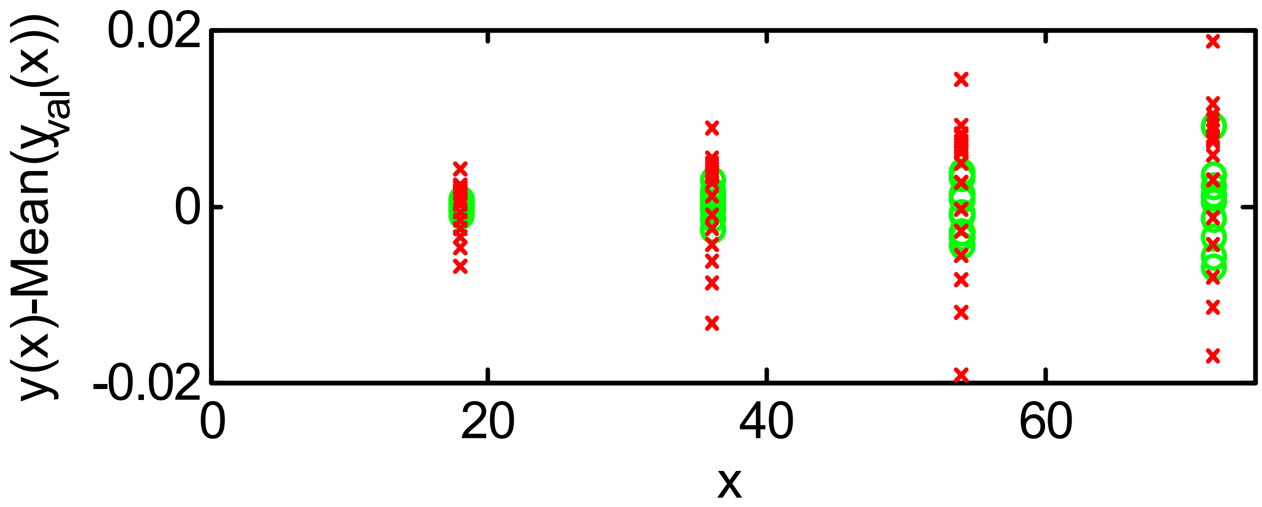
$$P\left(\left|Y_{exp}(x_{meas}) - Y_{val}(x_{meas})\right| \le 1.96\sqrt{2}\sigma_{Y_{meas}}\right) = 0.95$$

  - When the statistics are based on finite data (10 experiments, 20 model simulations) then the estimate of the probability on the left will be equal to or greater than 0.88 in 95% of trials
  - *Validation criterion* – The probability estimate given above, based on measured data and the normal assumption must be greater than 0.88 at two or more of the four measurement locations.

Comparison of Model to Experiments - Probabilistic



- *Validation* results (green circles) and *model prediction* (red x's)

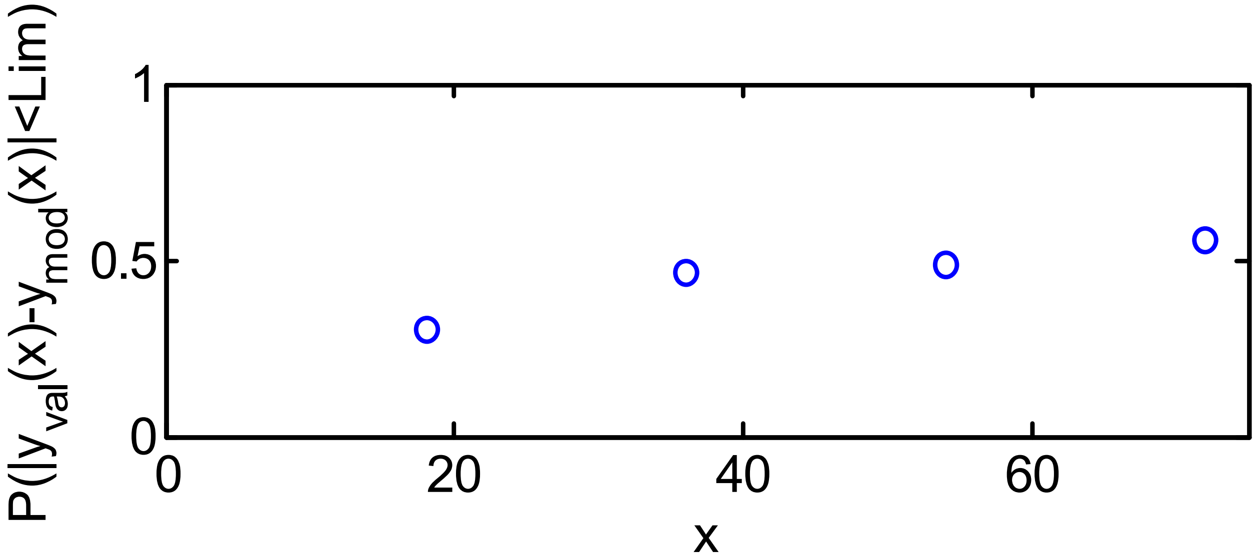


- *Validation* results minus mean validation results (green circles) and *model predictions* minus mean validation results (red x's)

Sandia National Laboratories

Comparison of Model to Experiments - Probabilistic

$$P\left(\left|Y_{exp}(x_{meas}) - Y_{val}(x_{meas})\right| \le 1.96\sqrt{2}\sigma_{Y_{meas}}\right)$$



Decision of Model Adequacy - Probabilistic

- Validation criterion (probability that experiment/model difference is bounded by multiple of experimental deflection standard deviation be equal to or greater than 0.88 in at least two of four cases) not satisfied in any case
- Model *not valid* – adequate (accurate) – based on current criterion

Uncertainty Quantification and Propagation

- Uncertainty quantification and propagation can be performed on physical system model
- Sources of *uncertainty*
  - Structure geometry
  - Modulus of elasticity of the beam material
  - Boundary conditions
  - Load – magnitude, location, direction
- *Quantification of uncertainty* - Form
  - Mean and variance of scalar random sources
  - Probability distribution (PDF, CDF, etc.) of scalar random sources
  - Joint probability distribution of multiple sources
  - UQ of epistemic sources

Uncertainty Quantification and Propagation

- Specification of *adequacy criterion* in case where experimental and model UQ are considered
  - Must involve specification of requirement on probabilistic (or other UQ framework) measure of random quantity
- Decision of model *adequacy* same as deterministic case, except that validation metrics stated in terms of probabilistic measure of random quantity