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An Anisotropic Sparse Grid Stochastic Collocation Method for PDEs with Random Input Data

Applications to Nonlinear Problems

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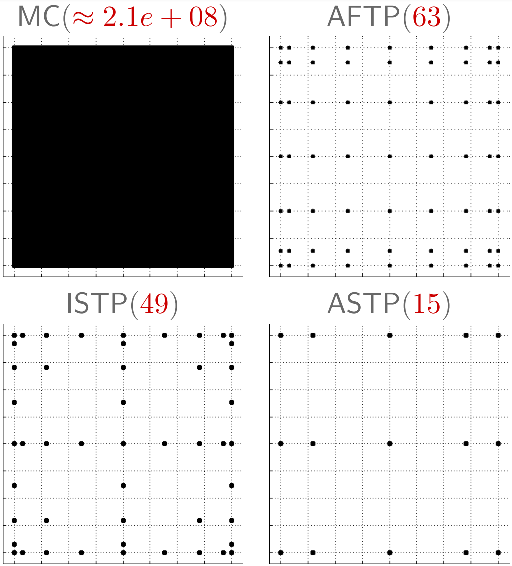
Motivation

Number of ensembles to reduce $\|\mathbb{E}[u - u^N]\|_{L^2(D)}$ by 10^4



Outline

1. Problem setting
2. Applications to linear and nonlinear elliptic SPDEs
3. Stochastic Collocation FEM
4. Anisotropic sparse grid Stochastic Collocation FEM
5. Error Analysis
6. Numerical examples
7. Concluding remarks

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Stochastic formulation of uncertainty

Consider an elliptic operator $\mathcal{L}$, linear or nonlinear, on a domain $D \subset \mathbb{R}^d$, which depends on some coefficients $a(\omega, x)$ with $x \in D$, $\omega \in \Omega$ and $(\Omega, \mathcal{F}, P)$ a complete probability space. The forcing $f = f(\omega, x)$ and the solution $u = u(\omega, x)$ are random functions.

$$\mathcal{L}(a)(u) = f \quad \text{in } D \tag{1}$$

equipped with suitable boundary conditions.

A_1. the solution to (1) has realizations in the Banach space $W(D)$, i.e. $u(\cdot, \omega) \in W(D)$ almost surely

$$\|u(\cdot, \omega)\|_{W(D)} \leq C \|f(\cdot, \omega)\|_{W^*(D)}$$

A_2. the forcing term $f \in L^2_P(\Omega; W^*(D))$ is such that the solution u is unique and bounded in $L^2_P(\Omega; W(D))$.

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Examples

Linear and Nonlinear Elliptic SPDEs

Example: The linear elliptic problem

$$\begin{cases} -\nabla \cdot (a(\omega,\cdot)\nabla u(\omega,\cdot)) = f(\omega,\cdot) & \text{in } \Omega \times D, \\ u(\omega,\cdot) = 0 & \text{on } \Omega \times \partial D, \end{cases}$$

with $a(\omega,\cdot)$ uniformly bounded and coercive and $f(\omega,\cdot)$ square integrable with respect to P, satisfies assumptions A_1 and A_2 with $W(D) = H_0^1(D)$.

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Example: The nonlinear elliptic problem

Similarly, for $k \in \mathbb{N}^+$,

$$\begin{cases} -\nabla \cdot (a(\omega,\cdot)\nabla u(\omega,\cdot)) + u(\omega,\cdot)|u(\omega,\cdot)|^k = f(\omega,\cdot) & \text{in } \Omega \times D, \\ u(\omega,\cdot) = 0 & \text{on } \Omega \times \partial D, \end{cases}$$

satisfies assumptions A_1 and A_2 with $W(D) = H_0^1(D) \cap L^{k+2}(D)$

On finite dimensional noise

Nonlinear Coefficients

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N terms Karhunen-Loève expansion of $z = \log(a - a_{min})$:

$$\log(a_N - a_{min}) = b_0(x) + \sum_{n=1}^{N} \sqrt{\lambda_n}\, b_n(x) Y_n(\omega)$$

- $(\lambda_n, b_n(x))$ eigenpairs of $T_{Cov[z]}$ and the random variables Y_n satisfy $\mathbb{E}[Y_n] = 0$, $Cov[Y_n, Y_m] = \delta_{nm}$.
- $\Gamma_n \equiv Y_n(\Omega)$ and WLOG we assume $Y_n(\omega)$ to be bounded. (i.e. $\Gamma_n = [-1, 1]$), $\Gamma^N = \prod_{n=1}^N \Gamma_n$ where $N \sim \mathcal{O}(10)$.
- $[Y_1, Y_2, \ldots, Y_N]$ have a joint p.d.f. $\rho : \Gamma^N \to \mathbb{R}_+$, with $\rho \in L^\infty(\Gamma^N)$, i.e. for $y \in \Gamma^N$

$$\mathbb{P}\left(Z \in \gamma \subset \Gamma^N\right) = \int_\gamma \rho(y)\, dy$$

GOAL: approximate a deterministic bounded functional $\mathbb{E}[\mathcal{J}(u_N)]$ or

$$\mathbb{E}[u_N](x) = \int_{\Gamma^N} u_N(y, x) \rho(y) dy, \quad y \in \Gamma^N, x \in \overline{D}$$

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Regularity

Let $\Gamma_n^* = \prod_{\substack{j=1 \\ j \neq n}}^{N} \Gamma_j$, and let $\mathbf{y}_n^*$ denote an arbitrary element of Γ_n^*.

Assumption: Regularity

For each $y_n \in \Gamma_n$, there exists $\tau_n > 0$ such that the function $u_N(y_n, \mathbf{y}_n^*, x)$ as a function of y_n, $u_N : \Gamma_n \to C^0(\Gamma_n^*; W(D))$ admits an analytic extension $u(z, \mathbf{y}_n^*, x)$, $z \in \mathbb{C}$, in the region of the complex plane

$$\Sigma(\Gamma_n; \tau_n) \equiv \{z \in \mathbb{C}, \ \ \mathrm{dist}(z, \Gamma_n) \leq \tau_n\}.$$

Moreover, $\forall z \in \Sigma(\Gamma_n; \tau_n)$,

$$\|u_N(z)\|_{C^0(\Gamma_n^*; W(D))} \leq \lambda$$

with λ a constant independent of n.

Applications to linear elliptic SPDEs

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By the **Lax-Milgram** theorem $\exists!\ u \in H_P = L^2_P(\Omega; H^1_0(D))$ s.t.

$$\int_D \mathbb{E}[a \nabla u \cdot \nabla v]\, dx = \int_D \mathbb{E}[fv]\, dx \quad \forall\, v \in H_P$$

$$\|u\|_{H_P} \leq \frac{C_P}{a_{min}} \left(\int_D \mathbb{E}[f^2]\, dx \right)^{1/2}$$

"**Deterministic**" equivalent:

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"**Deterministic**" equivalent: $a \leftrightarrow a_N,\ f \leftrightarrow f_N$

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"**Deterministic**" equivalent: find $u_N \in H_\rho = L^2_\rho(\Gamma^N; H^1_0(D))$ s.t.

$$\int_{\Gamma^N} \rho\, (a_N \nabla u_N, \nabla v)_{L^2(D)}\, dy = \int_{\Gamma^N} \rho\, (f_N, v)_{L^2(D)}\, dy, \quad \forall\, v \in H_\rho.$$

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Consider $u_N : \Gamma^N \to H^1_0(D)$

$$\int_D a_N(y) \nabla u_N(y) \cdot \nabla \phi\,dx = \int_D f_N(y)\phi\,dx, \quad \forall \phi \in H^1_0(D), \quad \rho\text{-a.e. in } \Gamma^N.$$

Applications to nonlinear SPDEs

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Well-posedness of $u \in \widetilde{H}_P \equiv L^2_P(\Omega; H^1_0(D) \cap L^{k+2}(D))$ s.t.

$$\mathscr{F}(u) = \frac{1}{2}\int_D \mathbb{E}\left[a|\nabla u|^2\right] dx + \frac{1}{k+2}\int_D \mathbb{E}\left[|u|^{k+2}\right] dx - \int_D \mathbb{E}\left[f\, u\right] dx.$$

1. Coercivity Condition, i.e. there exists $\epsilon, \delta > 0$ s.t.

$$\mathscr{F}(u) \geq \epsilon \|u\|^2_{\widetilde{H}_P} - \delta.$$

2. $\mathscr{F}$ is weakly lower semicontinuous on $\widetilde{H}_P$, i.e.

$$\mathscr{F}(u) \leq \liminf_{k\to\infty} \mathscr{F}(u_k) \quad \text{whenever} \quad u_k \rightharpoonup u \text{ weakly in } \widetilde{H}_P$$

3. Existence and Uniqueness of a minimizer, i.e. $\exists ! u \in \widetilde{H}_P$ s.t.

$$\mathscr{F}(u) = \min_{w \in \widetilde{H}_P} \mathscr{F}(w)$$

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Applications to nonlinear SPDEs (contd)

Theorem (Well-posedness)

$u \in \widetilde{H}_P$ is a minimizer of $\mathscr{F}(u)$ **iff** it is a weak solution, i.e.

$$\int_D \mathbb{E}\left[a\nabla u \cdot \nabla v\right] dx + \int_D \mathbb{E}\left[u\,|u|^k v\right] dx - \int_D \mathbb{E}\left[fv\right] dx = 0,\ \forall v \in \widetilde{H}_P$$

and the following *a priori* estimates hold:

$$\|u\|^2_{H_P} \leq \frac{4C_P}{a^2_{min}}\mathbb{E}\left[\|f\|^2_{L^2(D)}\right],\ \|u\|^2_{\widetilde{H}_P} \leq \frac{2}{C(k)}\left(\frac{C_P}{a_{min}}\mathbb{E}\left[\|f\|^2_{L^2(D)}\right]\right)$$

Consider $u_N : \Gamma^N \to L^{k+2}(D) \cap H^1_0(D)$.

$$\int_D a_N(y)\nabla u_N(y)\cdot\nabla\phi\, dx + \int_D u_N(y)\,|u_N(y)|^k \phi\, dx = \int_D f_N(y)\phi\, dx,$$

for all $\phi \in L^{k+2}(D) \cap H^1_0(D)$, ρ-a.e. in Γ^N.

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Region of Analyticity

Assume a_N is a $\log$-truncated Karhunen-Loève expansion and f_N deterministic then the analyticity region $\Sigma(\Gamma_n; \tau_n)$:

$$\tau_n = \frac{1}{\delta \sqrt{\lambda_n} \|b_n\|_{L^\infty(D)}}$$

$\delta = 4$ (linear), $\delta = 12$ (nonlinear, $k = 1$)

$$\sqrt{\lambda_n} \|b_n\|_{L^\infty(D)} \to 0 \text{ as } n \to \infty$$

anisotropic behavior with respect to the “direction” n

Stochastic Collocation FEM

- decouples computations as Monte Carlo does,
- treats efficiently the case of non independent random variables introducing an auxiliary density

$$\boxed{\hat{\rho}(y) = \prod_{n=1}^{N} \hat{\rho}_n(y_n), \quad \forall y \in \Gamma^N, \qquad \text{and s.t.} \quad \left\| \frac{\rho}{\hat{\rho}} \right\|_{L^\infty(\Gamma^N)} < \infty,}$$

- deals with unbounded random variables in the input data,
- essentially preserves the convergence speed to the stochastic Galerkin FEM.
- effectively handle problems that depend on random input data described by a moderately large number of random variables with the use of sparse collocation tensor product techniques

$$\mathcal{L}(a_N)(u_N) = f_N \quad \text{in } D. \tag{2}$$

Approximating spaces: Let $\mathcal{T}_h$ be a triangulation of D and

$\mathbf{p} = (p_1, \ldots, p_N)$ a multi-index.

- $W_h(D) \subset W(D)$ contains continuous piecewise polynomials defined in $\mathcal{T}_h$.
- $\mathcal{P}_{\mathbf{p}}(\Gamma^N) \subset L^2_\rho(\Gamma^N)$ is the span of tensor product polynomials with $deg \leq \mathbf{p}$.

Stochastic collocation $\Longleftrightarrow u_h^N(y_k) := \pi_h u_N(y_k) \in W_h(D)$, $y_k \in \Gamma^N$

Example: The linear SPDE

Let the semi-discrete approximation $u_h^N : \Gamma^N \to W_h(D) \subset H_0^1(D)$, satisfy, for a.e $y \in \Gamma^N$,

$$\int_D a_N(y) \nabla u_h^N(y) \cdot \nabla \phi_h \, dx = \int_D f_N(y) \phi_h \, dx, \quad \forall \phi_h \in W_h(D).$$

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Fully discrete approximation $u_{h,\mathbf{p}}^N \in C^0(\Gamma^N; W_h(D))$ is given by

$$u_{h,\mathbf{p}}^N(y, \cdot) = \sum_k u_h^N(y_k, \cdot) l_k^{\mathbf{p}}(y).$$

The anisotropic sparse grid SCFEM

An anisotropic (dimension weighted) Smolyak method I

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Let $i \in \mathbb{N}_+$ and for $u \in C^0(\Gamma^1; W(D))$ define $\mathscr{U}^0 = 0$,

$$\mathscr{U}^i(u)(y) = \sum_{j=1}^{m_i} u(y_j^i) \cdot l_j^i(y) \quad \text{and} \quad \Delta^i = \mathscr{U}^i - \mathscr{U}^{i-1}$$

- $l_j^i \in \mathcal{P}_{p_i}(\Gamma^1)$ are Lagrange polynomials of degree $p_i = m_i - 1$.

Basic idea: Consider the general class of simplices $\mathbf{i} \cdot \boldsymbol{\alpha} \le q$ where

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{R}_+^N \text{ with } \underline{\alpha} := \min_{1 \le n \le N} \alpha_n$$

is a N-dimensional weight vector for each stochastic direction.

For $w \in \mathbb{N}$, the **anisotropic** Smolyak algorithm is given by:

$$\mathscr{A}_{\boldsymbol{\alpha}}(w, N) = \sum_{\mathbf{i} \in X_{\boldsymbol{\alpha}}(w,N)} \left(\Delta^{i_1} \otimes \cdots \otimes \Delta^{i_N} \right)$$

$$X_{\boldsymbol{\alpha}}(w, N) = \left\{ \mathbf{i} \in \mathbb{N}_+^N, \mathbf{i} \ge \mathbf{1} : \sum_{n=1}^{N} (i_n - 1)\alpha_n \le w\underline{\alpha} \right\}.$$

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$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{R}_+^N \text{ with } \underline{\alpha} := \min_{1 \leq n \leq N} \alpha_n$$

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For $w \in \mathbb{N}$, the **anisotropic** Smolyak algorithm is given by:

$$\mathscr{A}_{\boldsymbol{\alpha}}(w, N) = \sum_{\mathbf{i} \in X_{\boldsymbol{\alpha}}(w,N)} \left(\Delta^{i_1} \otimes \cdots \otimes \Delta^{i_N} \right)$$

$$X_{\boldsymbol{\alpha}}(w, N) = \left\{ \mathbf{i} \in \mathbb{N}_+^N, \mathbf{i} \geq \mathbf{1} \,:\, \sum_{n=1}^{N} (i_n - 1)\alpha_n \leq w\underline{\alpha} \right\}.$$

The anisotropic sparse grid SCFEM

An anisotropic (dimension weighted) Smolyak method I

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Let $i \in \mathbb{N}_+$ and for $u \in C^0(\Gamma^1; W(D))$ define $\mathscr{U}^0 = 0$,

$$\mathscr{U}^i(u)(y) = \sum_{j=1}^{m_i} u(y_j^i) \cdot l_j^i(y) \quad \text{and} \quad \Delta^i = \mathscr{U}^i - \mathscr{U}^{i-1}$$

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The anisotropic sparse grid SCFEM

Anisotropic Smolyak for non-nested abscissas

Equivalently, for $w \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathbb{R}_+^N$:

$$\mathscr{A}_{\boldsymbol{\alpha}}(w, N) = \sum_{\mathbf{i} \in Y_{\boldsymbol{\alpha}}(w,N)} c_{\boldsymbol{\alpha}}(\mathbf{i}) \left(\mathscr{U}^{i_1} \otimes \cdots \otimes \mathscr{U}^{i_N} \right)$$

with

$$c_{\boldsymbol{\alpha}}(\mathbf{i}) := \sum_{\substack{\mathbf{j} \in \{0,1\}^N \\ \mathbf{i}+\mathbf{j} \in X_{\boldsymbol{\alpha}}(w,N)}} (-1)^{|\mathbf{j}|}$$

and

$$Y_{\boldsymbol{\alpha}}(w, N) := X_{\boldsymbol{\alpha}}(w, N) \setminus X_{\boldsymbol{\alpha}} \left(w - \frac{|\boldsymbol{\alpha}|}{\underline{\alpha}}, N \right).$$

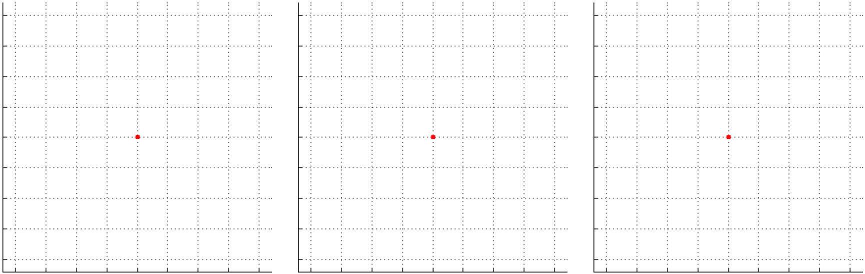
To compute $\mathscr{A}_{\boldsymbol{\alpha}}(w, N)(u)$ sample the "sparse grid"

$$\mathscr{H}_{\boldsymbol{\alpha}}(w, N) = \bigcup_{\mathbf{i} \in Y_{\boldsymbol{\alpha}}(w,N)} \left(\vartheta^{i_1} \times \cdots \times \vartheta^{i_N} \right).$$

Generated anisotropic sparse grids

Clenshaw-Curtis abscissas, $N = 2$

$N = 2$ anisotropic sparse grid: $\mathscr{A}_{\boldsymbol{\alpha}}(w, 2)$



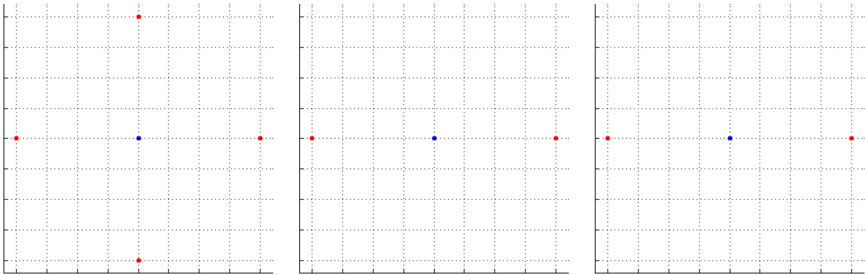
$\alpha_2/\alpha_1 = 1$ $\qquad$ $\alpha_2/\alpha_1 = 1.5$ $\qquad$ $\alpha_2/\alpha_1 = 2$

$w = 0$

Generated anisotropic sparse grids

Clenshaw-Curtis abscissas, $N = 2$

$N = 2$ anisotropic sparse grid: $\mathscr{A}_{\boldsymbol{\alpha}}(w, 2)$



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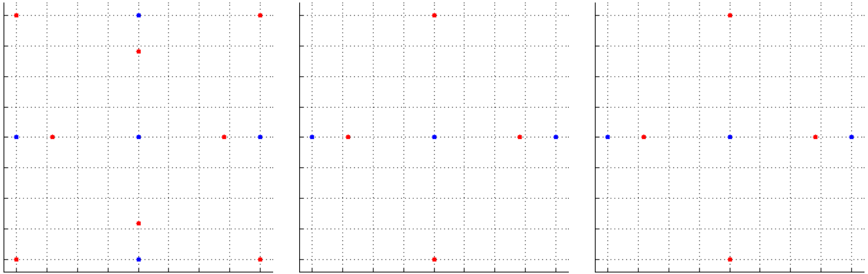

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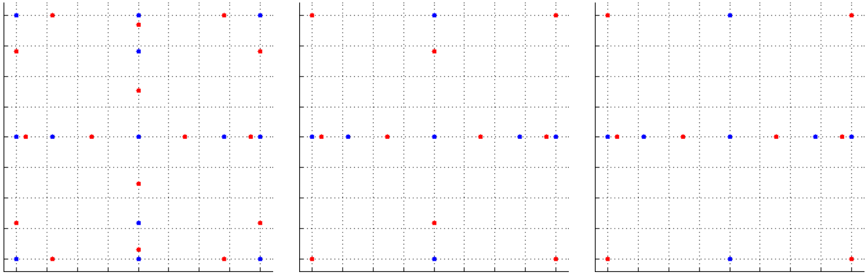


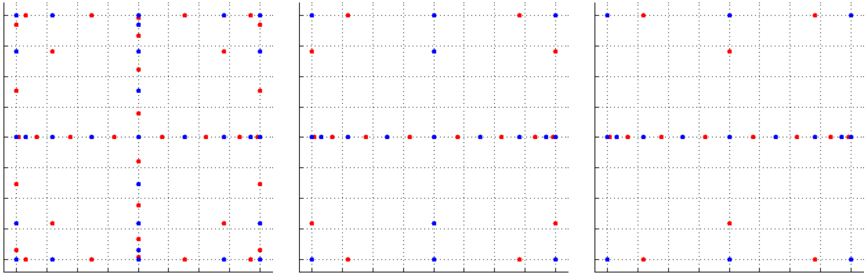
$\alpha_2/\alpha_1 = 1$ $\alpha_2/\alpha_1 = 1.5$ $\alpha_2/\alpha_1 = 2$

$w = 3$

Generated anisotropic sparse grids

Clenshaw-Curtis abscissas, $N = 2$

$N = 2$ anisotropic sparse grid: $\mathscr{A}_{\boldsymbol{\alpha}}(w, 2)$



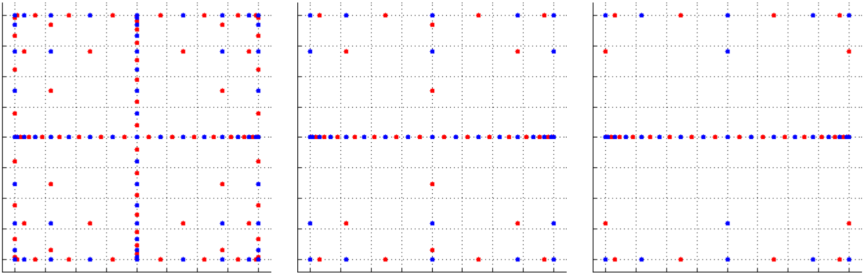
$\alpha_2/\alpha_1 = 1$ $\alpha_2/\alpha_1 = 1.5$ $\alpha_2/\alpha_1 = 2$

$w = 4$

Generated anisotropic sparse grids

Clenshaw-Curtis abscissas, $N = 2$

$N = 2$ anisotropic sparse grid: $\mathscr{A}_{\boldsymbol{\alpha}}(w, 2)$



$\alpha_2/\alpha_1 = 1$ $\alpha_2/\alpha_1 = 1.5$ $\alpha_2/\alpha_1 = 2$

$w = 5$

Generated anisotropic sparse grids

Clenshaw-Curtis abscissas, $N = 2$

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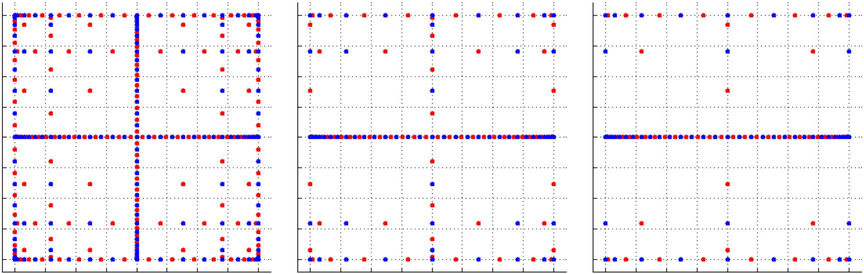
$N = 2$ anisotropic sparse grid: $\mathscr{A}_{\boldsymbol{\alpha}}(w, 2)$



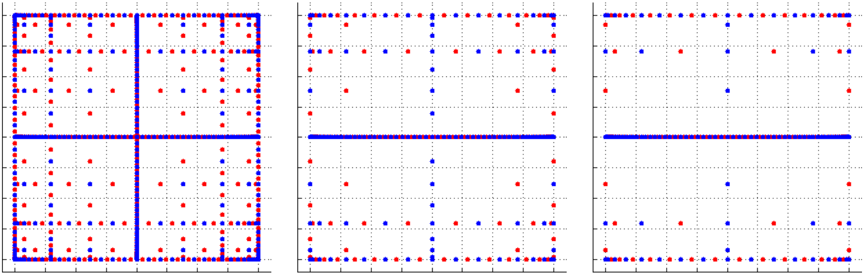
$\alpha_2/\alpha_1 = 1$ $\alpha_2/\alpha_1 = 1.5$ $\alpha_2/\alpha_1 = 2$

$w = 6$

Generated anisotropic sparse grids

Clenshaw-Curtis abscissas, $N = 2$

$N = 2$ anisotropic sparse grid: $\mathscr{A}_{\boldsymbol{\alpha}}(w, 2)$

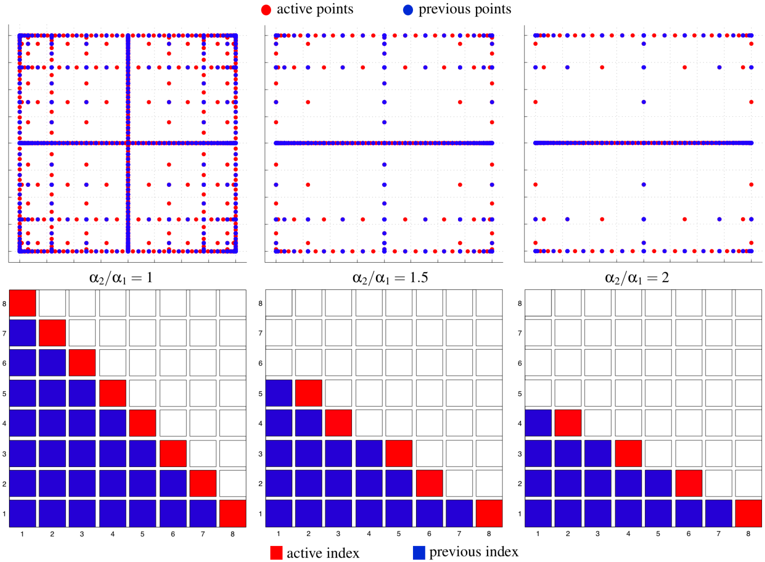


$\alpha_2/\alpha_1 = 1$ $\qquad$ $\alpha_2/\alpha_1 = 1.5$ $\qquad$ $\alpha_2/\alpha_1 = 2$

$w = 7$

Generated C-C anisotropic sparse grids

Correspondng indices $(i_1, i_2) \in X_{\alpha}(7, 2)$



Constructing general simplices

A priori, A posteriori α-weights

The rationale behind our anisotropic sparse grid approach is based on an examination of the total error

$$\varepsilon = \|u_N - \mathcal{I}_{\mathbf{p}}^N u_N\|_{L^2_\rho(\Gamma^N; W(D))},$$

produced by anisotropic full tensor product polynomial interpolation on Gaussian abscissas.

- When $\varepsilon \approx \varepsilon_1 + \cdots + \varepsilon_N$ is divided equally among the random variables $\Rightarrow$ **isotropic** Smolyak algorithm
- When $\varepsilon \approx \varepsilon_1 + \cdots + \varepsilon_N$ is dominated by certain directions $\Rightarrow$ **anisotropic** Smolyak algorithm

Basic Idea: for $n = 1, 2, \ldots, N$ link α_n-weights with the *rate of exponential convergence* in the corresponding stochastic direction.

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Basic Idea: for $n = 1, 2, \ldots, N$ link α_n-weights with the *rate of exponential convergence* in the corresponding stochastic direction.

A priori or *A posteriori*

Case $N = 1$:

Lemma: Best approximation error

Given a function $v \in C^0(\Gamma^1; W(D))$ which admits an analytic extension in the region of the complex plane $\Sigma(\Gamma^1; \tau) = \{z \in \mathbb{C},\ \text{dist}(z, \Gamma^1) \leq \tau\}$ for some $\tau > 0$, there holds

$$E_{m_i} \equiv \min_{w \in V_{m_i}} \|v - w\|_{C^0(\Gamma^1; W(D))} \leq C \varrho^{-m_i},$$

$1 < \varrho = \dfrac{2\tau}{|\Gamma^1|} + \sqrt{1 + \dfrac{4\tau^2}{|\Gamma^1|^2}}$ and C is a constant dependent on τ.

Case $N > 1$: the size of the analyticity region will depend, in general, on the direction n and it will be denoted by τ_n:

Assume: $\varrho_n \geq e^{g(n)}, g(n) > 0$ and **Define:** $\alpha_n = g(n)$

Notation: $\underline{\alpha} = \underline{g} = \min_{1 \leq n \leq N} \{g(n)\}$ and $\mathscr{G}(N) = \sum_{n=1}^{N} g(n)$.

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A priori or *A posteriori*

A priori knowledge: (Linear SPDE)

- First estimate the size of the analyticity region τ_n
- For $n = 1, 2, \ldots, N$ define the weight $\boldsymbol{\alpha} = \mathbf{g} \in \mathbb{R}_+^N$ as follows:

$$g(n) = \log\left(\frac{2\tau_n}{|\Gamma_n|} + \sqrt{1 + \frac{4{\tau_n}^2}{|\Gamma_n|^2}}\right)$$

A posteriori information: (Nonlinear SPDE)

- From the previous **Lemma** we expect an error decay of the form:

$$\varepsilon_n \approx d_n {\varrho_n}^{-p_n}, \quad \text{for all } n = 1, 2, \ldots, N,$$

  p_n is the number of collocation points in the direction n.
- To compute the weight vector $\mathbf{g} = \boldsymbol{\alpha} \in \mathbb{R}_+^N$, $\mathbf{g} \approx \log(\boldsymbol{\varrho})$:

$$\log_{10}(\varepsilon_n) \approx \log_{10}(d_n) - p_n \log_{10}(\varrho_n) \approx \log_{10}(d_n) - p_n \log_{10}(e) g(n).$$

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A posteriori selection: $N = 11$

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$$\begin{cases} -\nabla \cdot (a(\omega,\cdot)\nabla u(\omega,\cdot)) + (u(\omega,\cdot))^2 & = f(\omega,\cdot) \quad \text{in } \Omega \times D, \\ u(\omega,\cdot) & = 0 \quad \text{on } \Omega \times \partial D \end{cases}$$

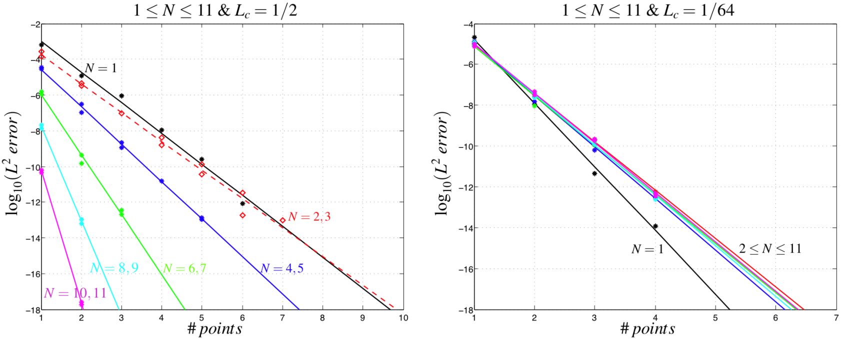

Figure: A linear least square approximation to fit $\log_{10}(\|E[\varepsilon_n]\|_{L^2(D)})$ versus p_n. For $n = 1, 2, \ldots, N = 11$ we plot: on the left, the highly anisotropic case $L_c = 1/2$ and on the right, the isotropic case $L_c = 1/64$.

Error analysis

Analysis of the interpolation error

Our aim is to give a priori estimates for the total error:

$$\epsilon = u - u^N_{h,\mathbf{p}} = u - \mathscr{A}_{\boldsymbol{\alpha}}(w,N)\pi_h u_N$$

- $\mathscr{A}_{\boldsymbol{\alpha}}(w,N)$ is the anisotropic Smolyak sparse interpolant
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$$\left\|\mathbb{E}[u - u^N_{h,\mathbf{p}}]\right\|_{W(D)} \le \mathbb{E}\left[\left\|u - u^N_{h,\mathbf{p}}\right\|_{W(D)}\right] \le \left\|u - u^N_{h,\mathbf{p}}\right\|_{L^2_P(\Omega;W(D))}.$$

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Clenshaw-Curtis interpolation estimates

Convergence wrt the level w

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Theorem [NTW,07]

For functions $u \in L^2_\rho(\Gamma^N; W(D))$ satisfying the **Regularity Assumption**, the anisotropic Smolyak method with the choice $\alpha_n = g(n)$ of the weights satisfies:

$$\|(I_N - \mathscr{A}_{\boldsymbol{\alpha}}(w, N))\,(u)\|_{L^2_\rho(\Gamma^N; W(D))} \leq \hat{C}(\mathbf{g}, N) e^{w - \lambda(w,N)}$$

$$\lambda(w, N) := \begin{cases} w \dfrac{\underline{g} \log(2) e}{2}, & \text{if } 0 \leq w \leq \dfrac{\mathscr{G}(N)}{\underline{g} \log(2)}, \\[2ex] \dfrac{\mathscr{G}(N)}{2} 2^{w \frac{\underline{g}}{\mathscr{G}(N)}}, & \text{otherwise} \end{cases},$$

and the function $\hat{C}(\mathbf{g}, N)$ does not depend on w.

Theorem [NTW,07]

Let $\eta = \eta(w, N) = \#\mathscr{H}_{\boldsymbol{\alpha}}(w, N)$, then for $u \in L^2_\rho(\Gamma^N; W(D))$ satisfying the assumptions of the previous Theorem, the anisotropic Smolyak method with the choice $\alpha_n = g(n)$ of the weights satisfies:

- *Algebraic convergence* $\left(0 \leq w \leq \frac{\mathscr{G}(N)}{\underline{g}\log(2)}\right)$, s.t. $\underline{g} \geq 1/(e\log(2))$

$$\|(I_N - \mathscr{A}_{\boldsymbol{\alpha}}(w, N))(u)\|_{\rho,N} \leq \hat{C}(\mathbf{g}, N)\eta^{-\boldsymbol{\mu}_1},$$

$$\text{with} \qquad \boldsymbol{\mu}_1 = \frac{\underline{g}\log(2)e - 1}{\log(2) + \sum_{n=1}^{N} \underline{g}/g(n)}$$

and constant $\hat{C}(\mathbf{g}, N)$ independent of η.

Theorem [NTW,07]

Let $\eta = \eta(w, N) = \#\mathscr{H}_{\boldsymbol{\alpha}}(w, N)$, then for $u \in L^2_\rho(\Gamma^N; W(D))$ satisfying the assumptions of the previous Theorem, the anisotropic Smolyak method with the choice $\alpha_n = g(n)$ of the weights satisfies:

- *Sub-exponential convergence* $\left(w > \frac{\mathscr{G}(N)}{\underline{g}\log(2)}\right)$

$$\|(I_N - \mathscr{A}_{\boldsymbol{\alpha}}(w, N))(u)\|_{\rho,N} \leq \hat{C}(\mathbf{g}, N)(2\eta)^{1/\log(2)} e^{-\frac{\mathscr{G}(N)}{2}\eta^{\mu_2}},$$

$$\text{with} \qquad \mu_2 = \frac{\underline{g}\log(2)}{\mathscr{G}(N)\left(\log(2) + \sum_{n=1}^{N} \underline{g}/g(n)\right)}$$

and constant $\hat{C}(\mathbf{g}, N)$ independent of η.

Numerical examples

The problems we consider are:

$$\begin{cases} -\nabla \cdot (a_N(\omega,\cdot)\nabla u_N(\omega,\cdot)) = f_N(\omega,\cdot) & \text{in } \Omega \times D \\ u_N(\omega,\cdot) = 0 & \text{on } \Omega \times \partial D \end{cases}$$

$$\begin{cases} -\nabla \cdot (a_N(\omega,\cdot)\nabla u_N(\omega,\cdot)) + (u_N(\omega,\cdot))^2 = f_N(\omega,\cdot) & \text{in } \Omega \times D \\ u_N(\omega,\cdot) = 0 & \text{on } \Omega \times \partial D \end{cases}$$

with $D = \{\mathbf{x} = (x,z) \in \mathbb{R}^2 \,:\, 0 \le x, z \le 1\}$ and a deterministic load $f_N(\omega,x,z) = \cos(x)\sin(z)$

$$\log(a_N(\omega,x) - 0.5) = 1 + Y_1(\omega)\left(\frac{\sqrt{\pi}L}{2}\right)^{1/2} + \sum_{n=2}^{N} \beta_n\, \varphi_n(x)\, Y_n(\omega).$$

$$\beta_n := (\sqrt{\pi}L)^{1/2} \exp\left(\frac{-(\lfloor\frac{n}{2}\rfloor\pi L)^2}{8}\right), \quad \text{if } n > 1$$

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Calculating the weighting parameters

- $\mathbb{E}[Y_n] = 0$ and $\mathbb{E}[Y_n Y_m] = \delta_{nm}$ for $n, m \in \mathbb{N}_+$, and are taken uniform in the interval $[-\sqrt{3}, \sqrt{3}]$.
- To resolve the noise $W_h(D) \equiv$ span of continuous piecewise quadratic functions over a uniform triangulation of D with maximum mesh size $h = 1/3N$. (Smallest Period $T = 4/N$) i.e. for $N = 11$ we have 4225 FE unknowns.

$$\varrho_n = \frac{2\tau_n}{|\Gamma_n|} + \sqrt{1 + \frac{4{\tau_n}^2}{|\Gamma_n|^2}} \geq e^{g(n)}$$

Where the weight vector $\mathbf{g}$ becomes:

$$g(n) = \begin{cases} \log\left(1 + \sqrt{\frac{1}{24\sqrt{\pi}L}}\right), & \text{for } n = 1 \\ \log\left(1 + \sqrt{\frac{1}{48\sqrt{\pi}L}} \exp\left(\frac{\lfloor \frac{n}{2} \rfloor^2 \pi^2 L^2}{8}\right)\right), & \text{for } n > 1 \end{cases}$$

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	$g(1)$	$g(2,3)$	$g(4,5)$	$g(6,7)$	$g(8,9)$	$g(10,11)$
$L = 1/2$	0.20	0.19	0.42	1.24	3.1	5.8
$L = 1/64$	0.79	0.62	0.62	0.62	0.62	0.62

Table: The $N = 11$ values of the function $g(n)$ constructed from *a priori* information. The components of $\boldsymbol{\alpha} = \mathrm{g}$ are used as the input information for the anisotropic Smolyak algorithm with correlation lengths $L = 1/2$ and $L = 1/64$.

To study the convergence of the anisotropic algorithm we consider a fixed dimension $N = 5, 11$ and estimate the error:

$$\|\mathbb{E}[\epsilon]\|_{L^2(D)} \approx \|\mathbb{E}[\mathscr{A}_{\boldsymbol{\alpha}}(w, N)\pi_h u_N - \mathscr{A}_{\widehat{\boldsymbol{\alpha}}}(\overline{w} + 1, N)\pi_h u_N]\|_{L^2(D)}$$

- $w = 0, 1, \ldots, \overline{w}$ and $\mathscr{A}_{\widehat{\boldsymbol{\alpha}}}(\overline{w} + 1, N)\pi_h u_N$ is an enriched solution that is an approximation to the exact solution.

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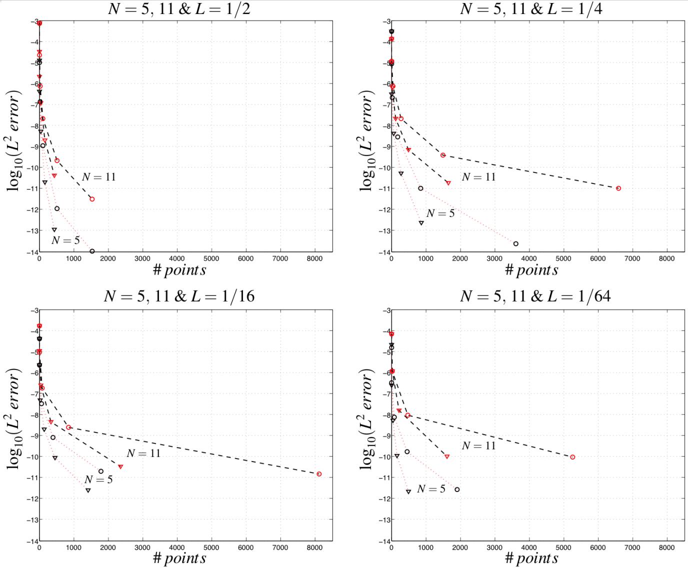
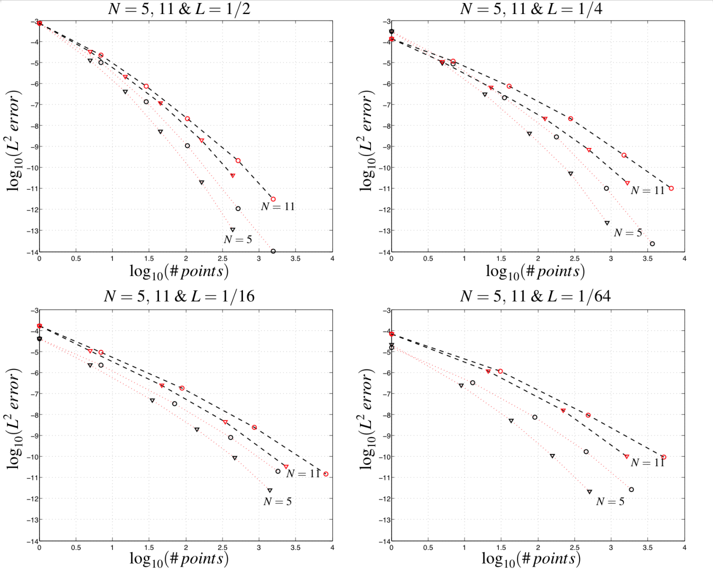
Anisotropic Smoyak rate of convergence

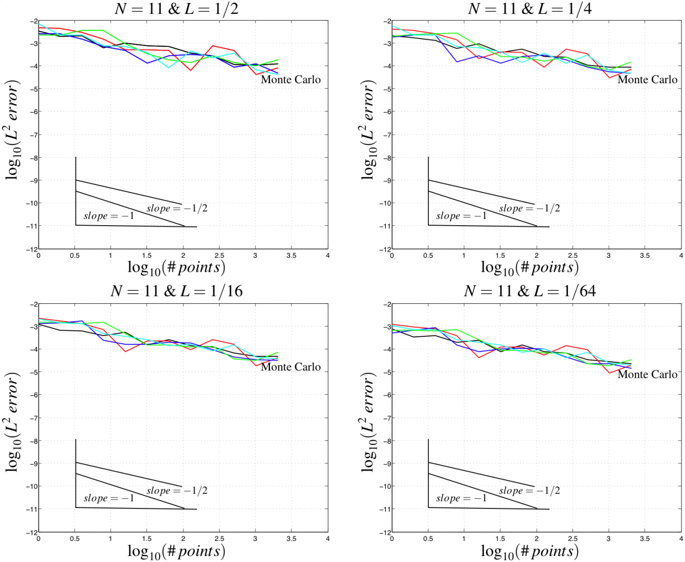


Anisotropic Smoyak convergence



Convergence Comparisons I

$N = 11$ random variables



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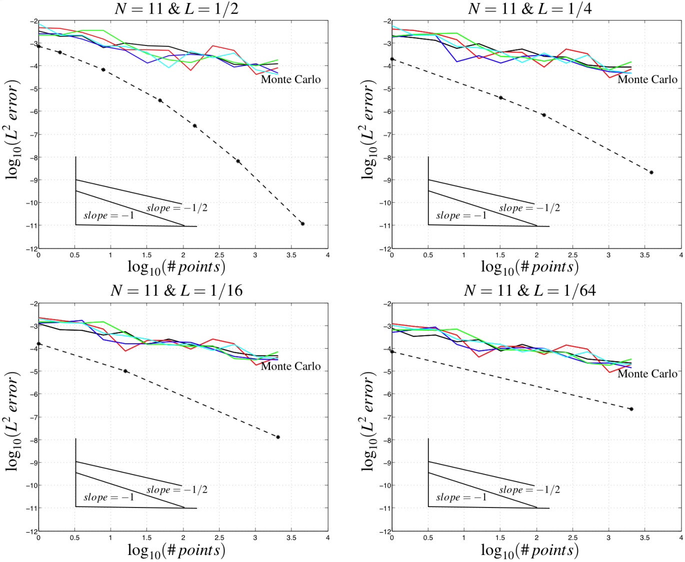

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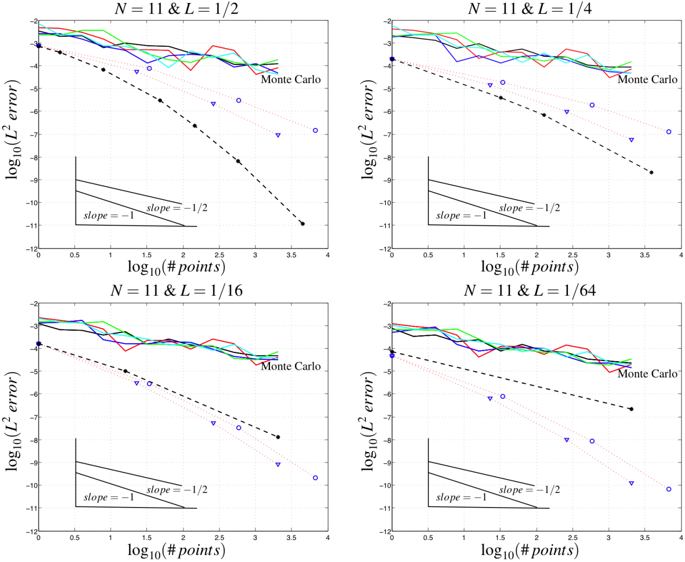

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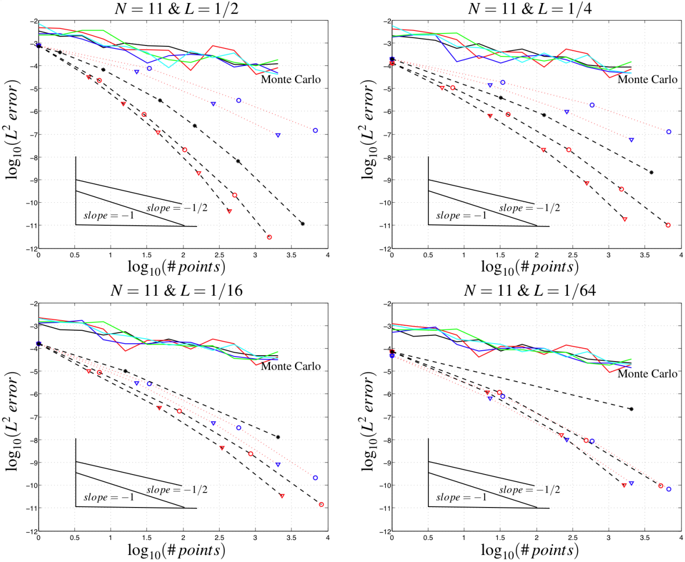


Convergence Comparisons II

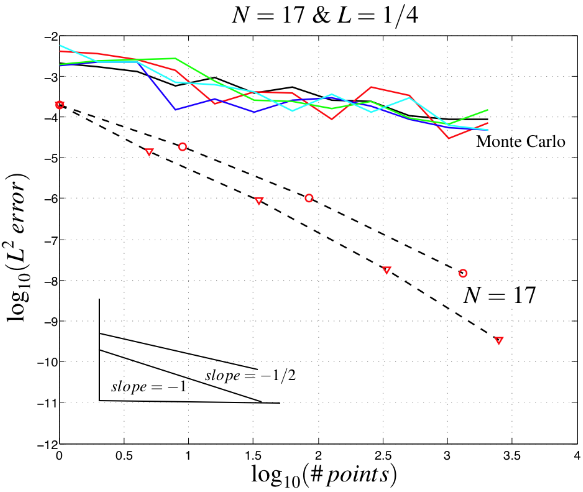
$N = 11$ random variables

L	AS	AF	IS	MC
$1/2$	50	252	2512	$5.0e+09$
$1/4$	158	1259	3981	$2.0e+09$
$1/16$	199	1958	501	$1.6e+09$
$1/64$	316	199530	360	$1.3e+09$

Table: For Γ^N, with $N = 11$, we compare the number of deterministic solutions required by the Anisotropic Smolyak (AS) using Clenshaw-Curtis abscissas, Anisotropic Full Tensor product method (AF) using Gaussian abscissas, Isotropic Smolyak (IS) using Clenshaw-Curtis abscissas and the Monte Carlo (MC) method using random abscissas, to reduce the original error by a factor of 10^4.

Convergence Comparisons: Nonlinear SPDE

$N = 17$ random variables



- -○· Anisotropic Smolyak with Gaussian abscissas ($N = 17$)
- -▽· Anisotropic Smolyak with Clenshaw-Curtis abscissas ($N = 17$)

Concluding remarks

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- Extended [Nobile-Tempone-Webster, 2006] using an anisotropic sparse grid stochastic collocation method for solving linear and nonlinear elliptic SPDEs whose coefficients and forcing terms depend on a moderately large number of random variables.
- Sparse grid Stochastic Collocation FEM yields:
  - uncoupled deterministic problems (fully parallelizable)
  - reduces considerably the *curse of dimensionality.*
  - for the problems under study the displayed convergence is faster than standard collocation techniques built upon full tensor product spaces, the isotropic sparse grid methods and Monte Carlo.
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