Isocontouring Higher Order Finite Elements: One Numerical Solution to Hilbert's 16^{th} Problem

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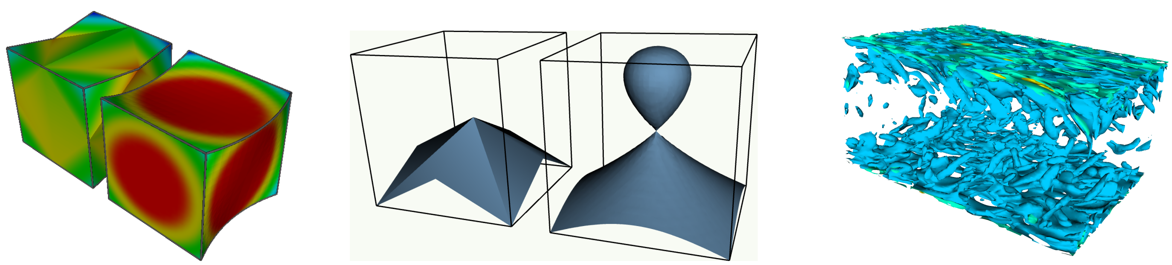
Finite element (FE) analysis is an important, widely used numerical technique for solving partial differential equations. Traditionally, the polynomial order of the FE basis functions used in practice has been low – typically linear or quadratic. In recent years, so-called p-adaptive methods that can elevate the order of the basis have been developed, in an effort to converge to a solution faster than previously possible, or to provide more accurate approximations than traditional FE simulation within the same amount of computational time. Moreover, FE solvers that incorporate hp-adaptivity are becoming popular since they often converge to a solution with fewer total degrees of freedom than h (hierarchical) or p-adaptivity alone. For instance, at Sandia, tridecadic hp-adaptive FE solvers are under development.

However, the resulting increased order of the basis functions poses a significant challenge to visualization systems. Currently, there is very limited support for quadratic elements: although a few visualization systems offer some tools for quadratic finite elements, they do not always do so correctly, as illustrated in Fig. 1(a). Furthermore, there is no way to visualize the solutions of simulations with cubic or higher order elements using existing visualization packages, and attempting to do so can yield aberrant results, as shown in Fig. 1(b).



(a) (b) (c)

Figure 1: (a) Surface rendering of a scalar field interpolated over a Lagrange Q2, incorrectly using ParaView (left) and correctly after our methodology has been implemented (right). (b) Isocontouring of a cubic function interpolated over a single Lagrange Q3 element, using a linear isocontouring technique (left), and using our approach (right). (c) Triquadratic isocontouring of the vorticity magnitude field in a channel flow computed with a 3-dimensional finite difference scheme (data set courtesy of A. Gruber (SINTEF)).

Higher order surface rendering is probably the most used visualization technique, which we have already addressed with a brute-force technique that scales with the size of the model. Arguably, the next most useful visualization tool is isocontouring. For higher order finite element solution fields, this pertains to a branch of mathematics completely different from that used to compute them. In fact, it involves solving a problem proposed by D. Hilbert as an

[...] investigation as to the number, form, and position of the sheets of an algebraic surface in space[1].

This problem has stymied mathematicians for more than a hundred years, and still does. Without a solution, we cannot even know the range of values that a field takes on over an element, much less the number, form, and position in space of an isosurface.

The goal of this presentation is to describe our numerical solution to this problem, borrowing tools from analysis, geometry, and algebra. We will also discuss why and how this approach has led us to develop an embarrassingly parallel tetrahedral mesh refiner[2] – the fastest existing one, in our knowledge. Finally, we will see that, when the problem of isocontouring is not well-defined, as is the case with finite difference solution fields (as in Fig. 1(c)), our technique can be employed as well, with different orders to gain additional information about how the simulation is behaving.

[1] D. Hilbert's 16^{th} Problem, International Congress of Mathematicians, Paris, 1900.

[2] The code is available in VTK as the `vtkStreamingTessellator` class.