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An R-adaptive Mesh Optimization Algorithm Based on Interpolation Error Bounds

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Anisotropic Adaptivity

- Engineering applications generate solutions with different scales of variation along different directions or solutions that are not smooth.

Uniform refinement (Coudière et al. - 2002):

For the linear interpolation of a discontinuous function on a 3D-mesh with N nodes, the error satisfies at best

$$\|u - \Pi(u)\|_2 \leq C \cdot N^{-1/6}$$

Isotropic refinement (Coudière et al. - 2002):

For an isotropic adaptive mesh method, the error satisfies at best

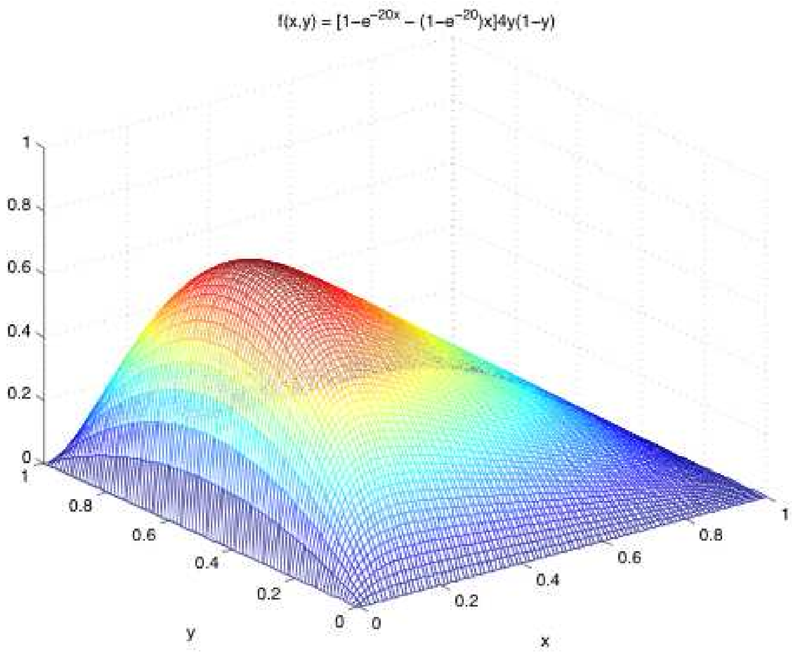
$$\|u - \Pi(u)\|_2 \leq C \cdot N^{-1/4}$$

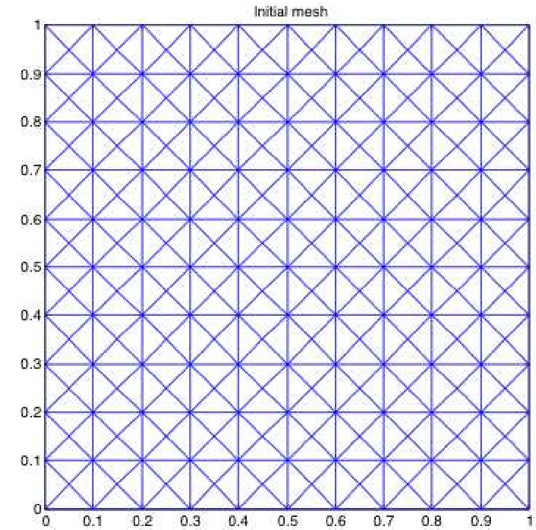
Anisotropic refinement (Conjecture):

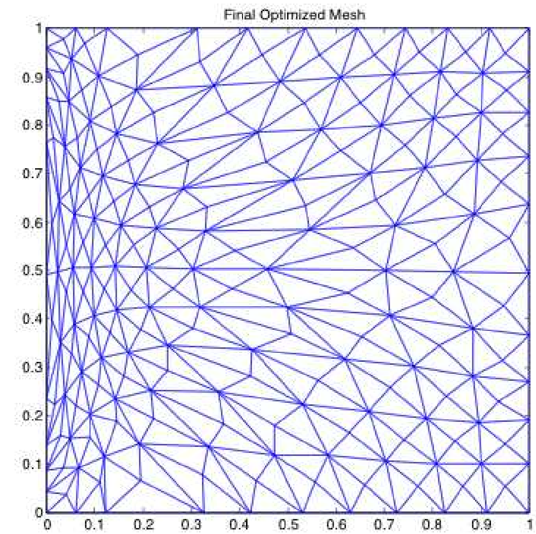
For an anisotropic adaptive mesh method, the error satisfies at best

$$\|u - \Pi(u)\|_2 \leq C \cdot N^{-2/3}$$

Mesh & Accuracy







Anisotropic meshes allow

- **optimal distribution of nodes**
- **better accuracy than isotropic ones**

Anisotropic Mesh Adaptation

Existing techniques:

- **Anisotropic mesh generation (**Frey, Hecht, Huang, …**)**
  - **Anisotropic metric map**
  - **Generate unit mesh in transformed space**
  - **Approximation of solution Hessian**

- **PDE-Dependent R-Adaptive Schemes**
  - **Node movement to minimize objective function**
  - **Energy functional (**Tourigny, Jimack, …**)**
  - **Least-squares norm of residuals (**Tourigny, Jimack, Roe, …**)**

- **PDE-Independent R-Adaptive Schemes**
  - **Node movement to minimize objective function**
  - **Approximation of solution Hessian**

Hessian-based R-Adaptation

Goal: Minimize the L^2-norm of interpolation error

- Equidistribute edge length in transformed space
- Require additional geometric criterion (sliver)
- Works of Berzins, Chen, Jimack, Remacle, …

Very few works check the H^1 semi-norm of error

- Long and thin elements are good for L^2-norm of interpolation error (Rippa, Chen) but can be undesirable for H^1 semi-norm of error (Babuska and Aziz).
- The H^1 norm is related to the accuracy of the solution and its gradient (Céa’s lemma).

Hessian-based R-Adaptation

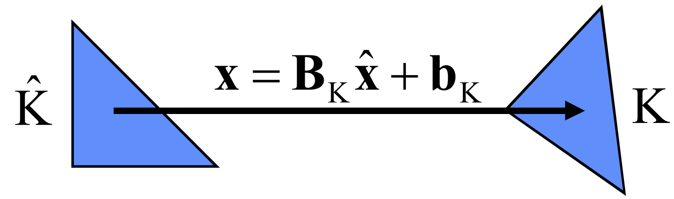
Goal: **Minimize the H^1-norm of interpolation error**

- **Bank & Smith (1997):**
  - Linear Triangles
  - Error between quadratic function and linear interpolant

- **Lague (2006):**
  - Linear Triangles
  - Express error from a Taylor expansion.

- **Propose an approach based on interpolation bounds**
  - Easy extension for 3D
  - Works with affine equivalent elements

Interpolation Error Bounds



Theorem (Formaggia, Huang, …):

$$\sqrt{\int_K |u - \Pi u|^2} \le C \times \sqrt{\int_K \left\| \mathbf{B}_K^T \cdot D^2 u \cdot \mathbf{B}_K \right\|_F^2}$$

$$\sqrt{\int_K |\nabla u - \nabla \Pi u|^2} \le C \times \left\| \mathbf{B}_K^{-1} \right\|_2 \times \sqrt{\int_K \left\| \mathbf{B}_K^T \cdot D^2 u \cdot \mathbf{B}_K \right\|_F^2}$$

- **Πu is the linear interpolant of u.**
- **Constants C depend only on $\hat{K}$.**
- **The bound for H^1 semi-norm contains a barrier term.**
- **Flexibility for choice of reference element.**
    - Choose equilateral triangle

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R-Adaptive Scheme

Objective function: $$\sum_{K} \det(\mathbf{B}_K) \left\| \mathbf{B}_K^{-1} \right\|_F^2 \left\| \mathbf{B}_K^{T} \cdot D^2 u(\mathbf{G}_K) \cdot \mathbf{B}_K \right\|_F^2$$

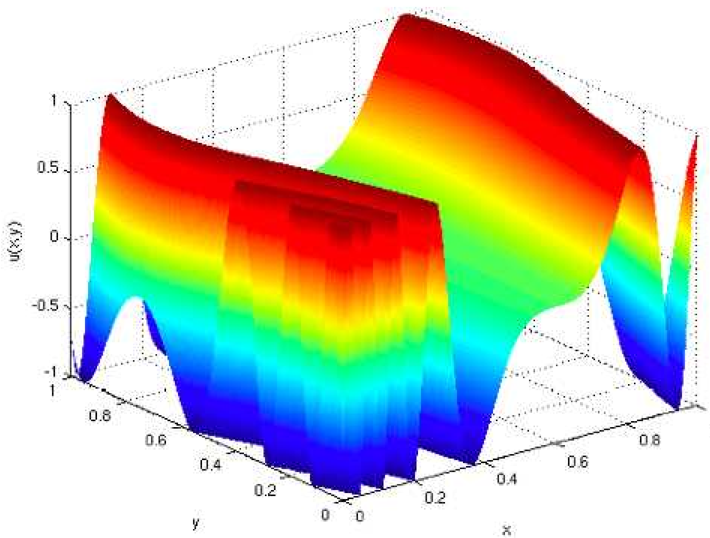
Minimization Algorithm:

1. Get the Hessian matrices at the nodes
2. Sweep through edges to check for swapping
3. Interpolate the Hessian matrices in each element
4. Sweep through nodes to optimize position

- Use analytical formula for the Hessian matrices.
- Do only a few sweeps.
- Do only one pass.

R-Adaptive Scheme

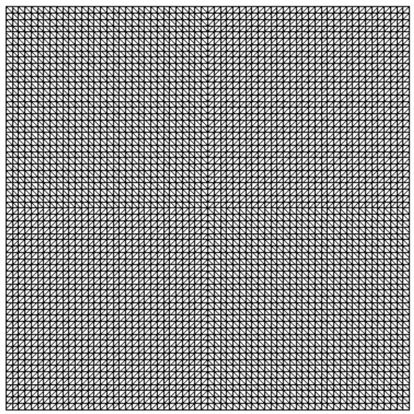
$$u(x,y) = \sin\left[5\left(2x - \frac{6}{5}\right)^3 \left(4y^2 - 6y + 3\right)\right]$$



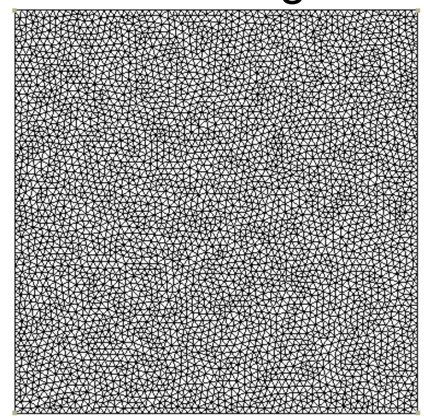
- **F. Alauzet and P. Frey (2003)**

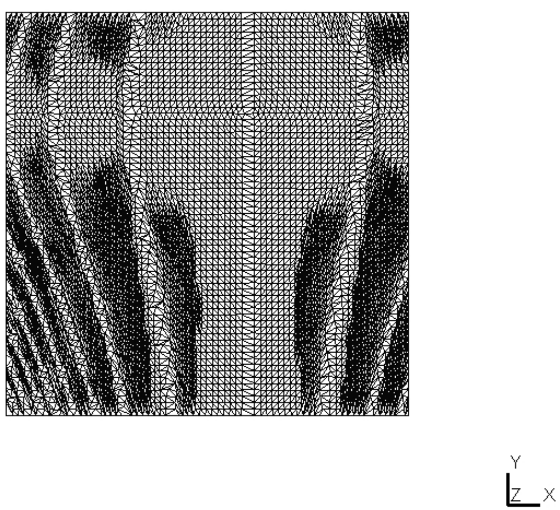
R-Adaptive Scheme

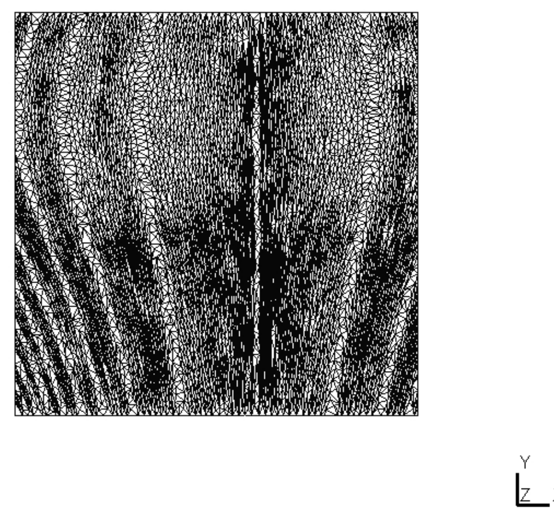
4225 points
8192 triangles



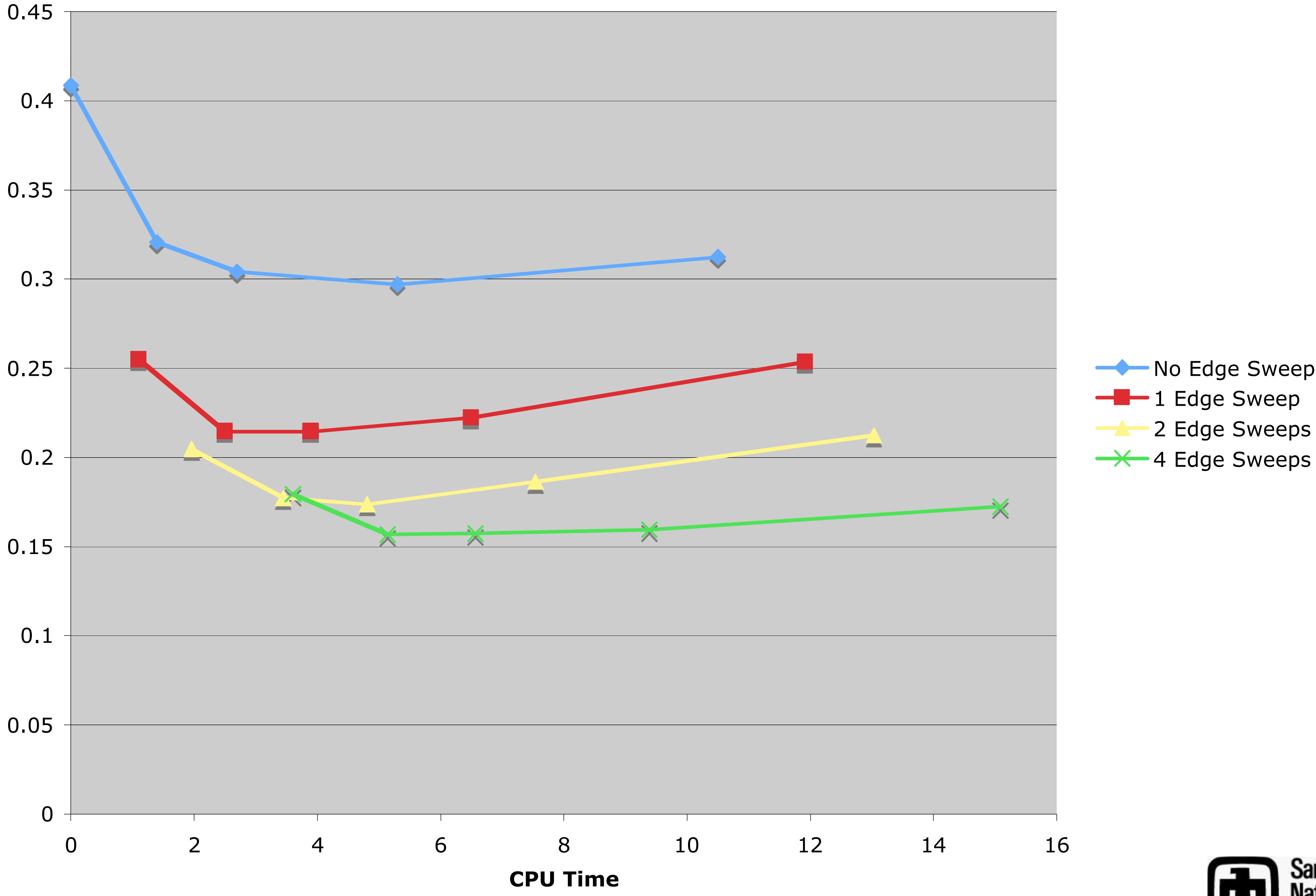
4035 points
8072 triangles







R-Adaptive Scheme



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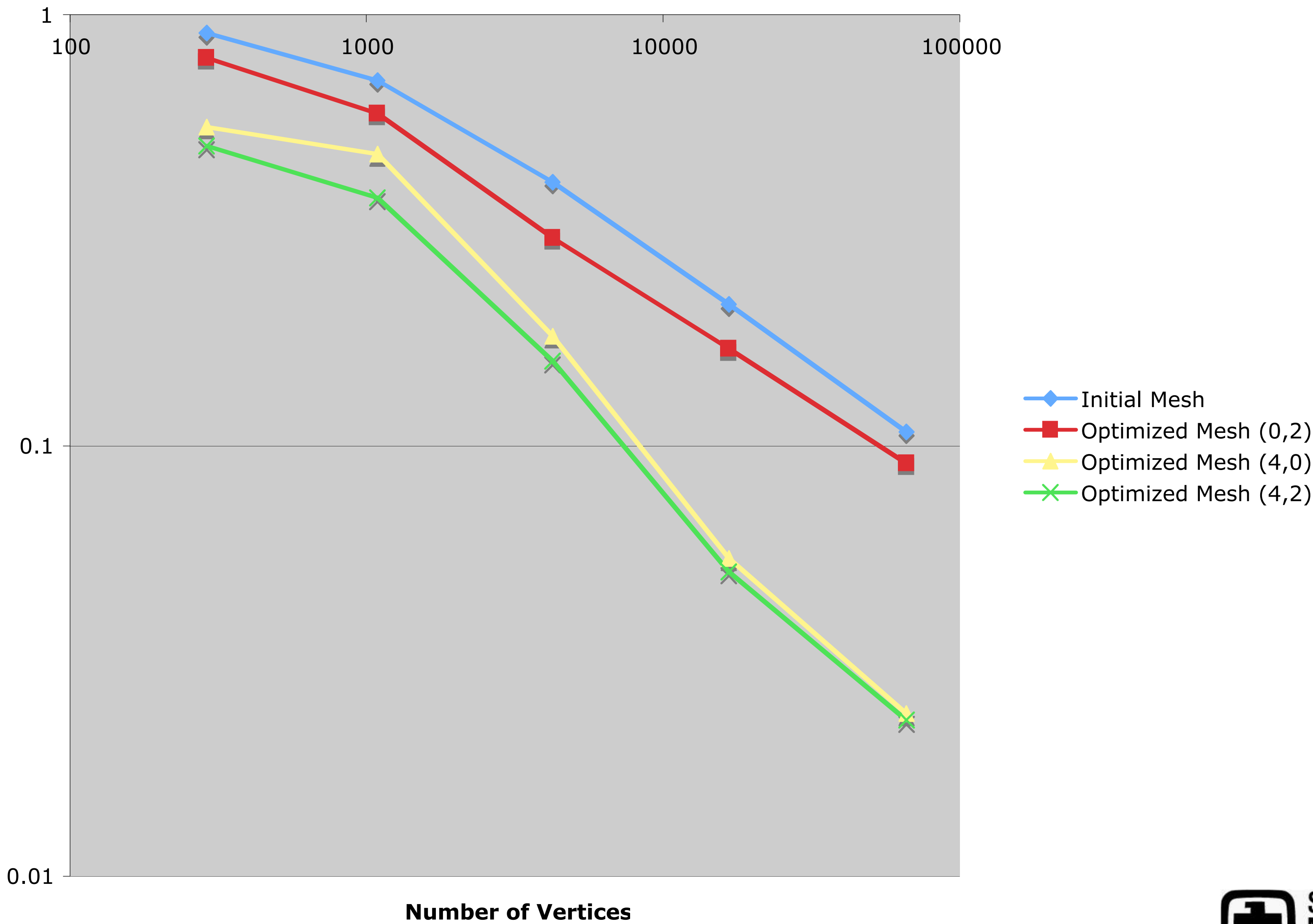
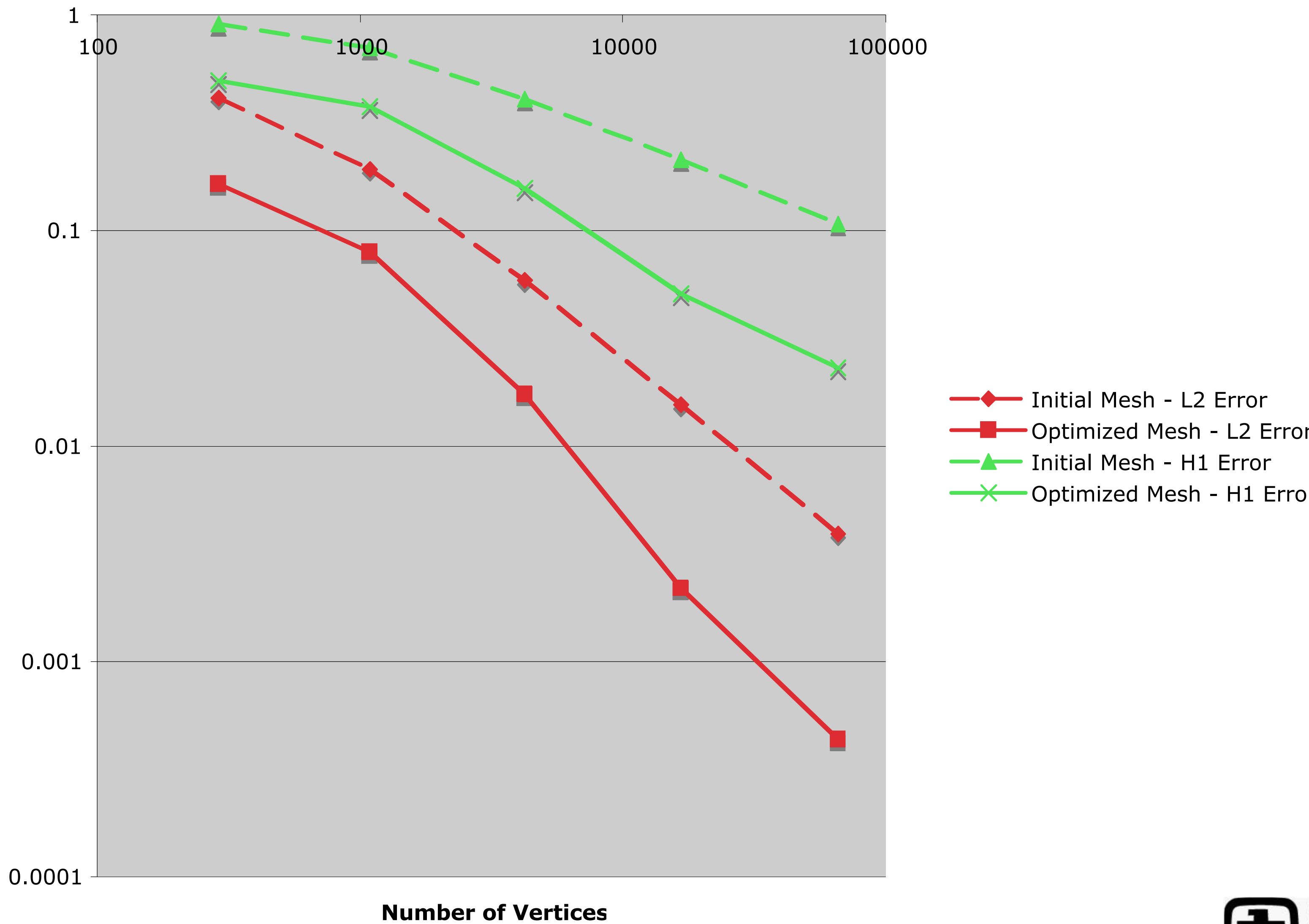
R-Adaptive Scheme

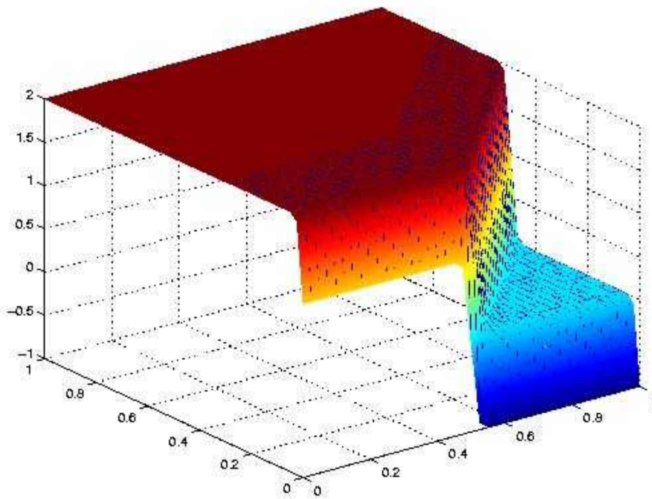


R-Adaptive Scheme



R-Adaptive Scheme

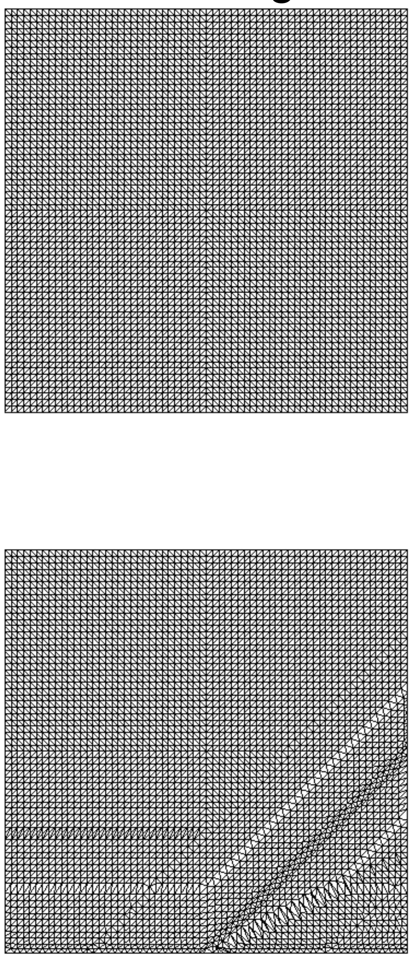
$$u(x,y) = \tanh(64y) - \tanh\left[64\left(x - y - \frac{1}{2}\right)\right]$$



- **Example from W. Huang**

R-Adaptive Scheme

4225 points
8192 triangles



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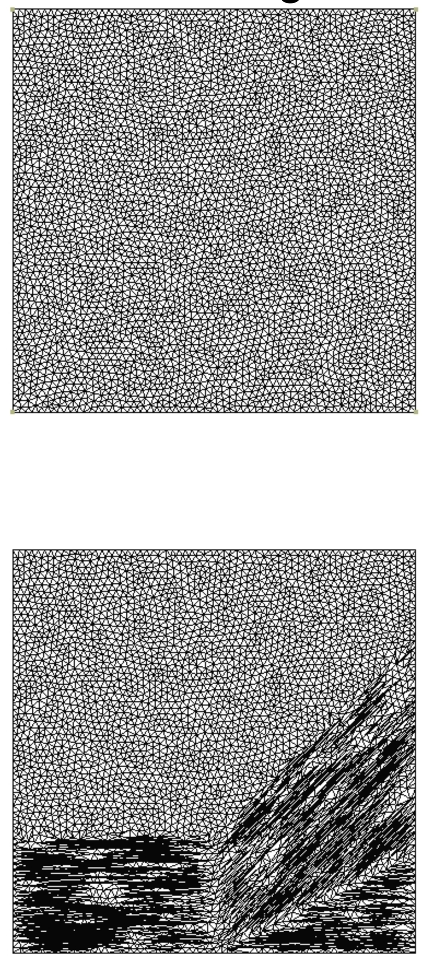

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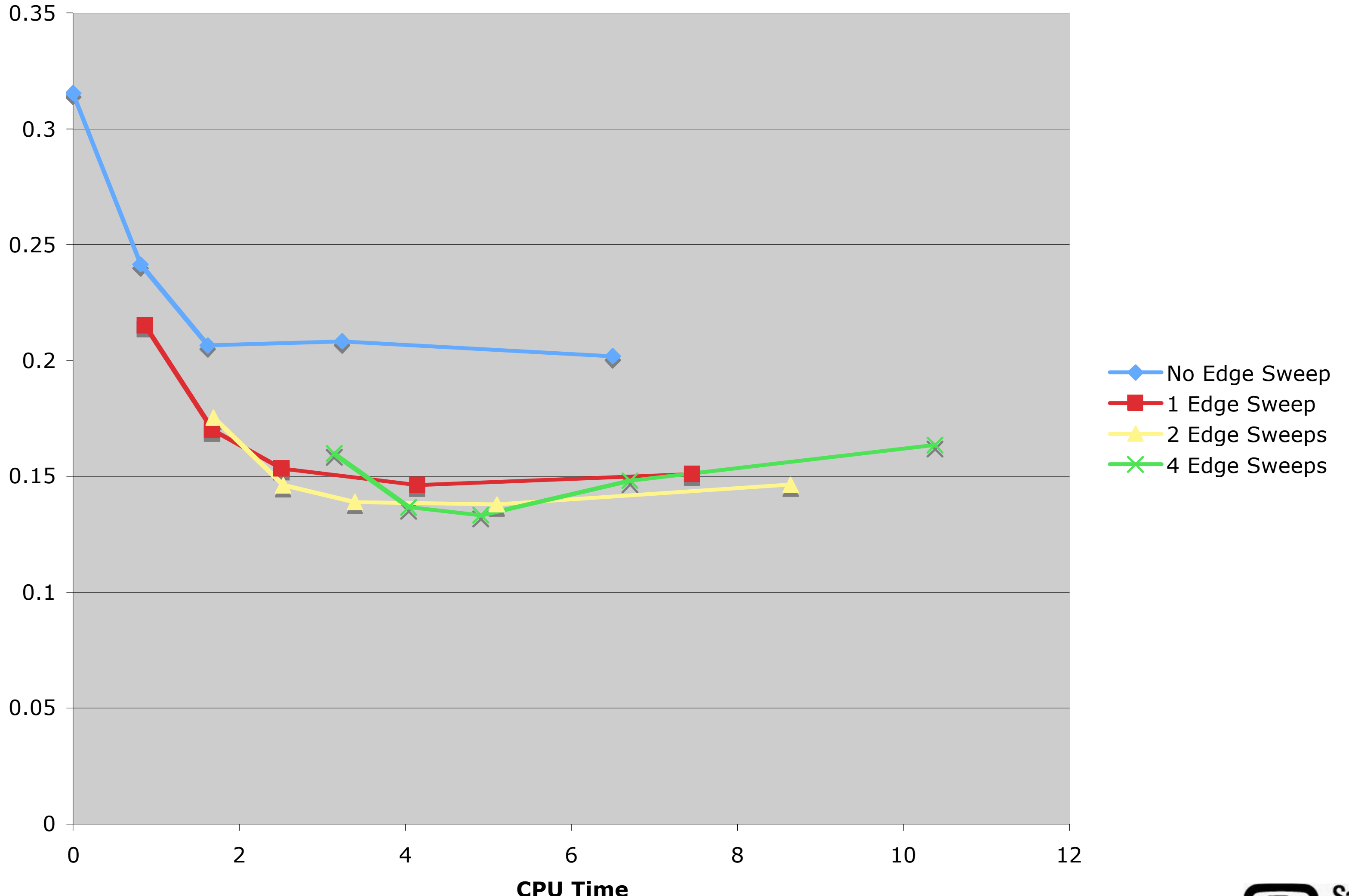

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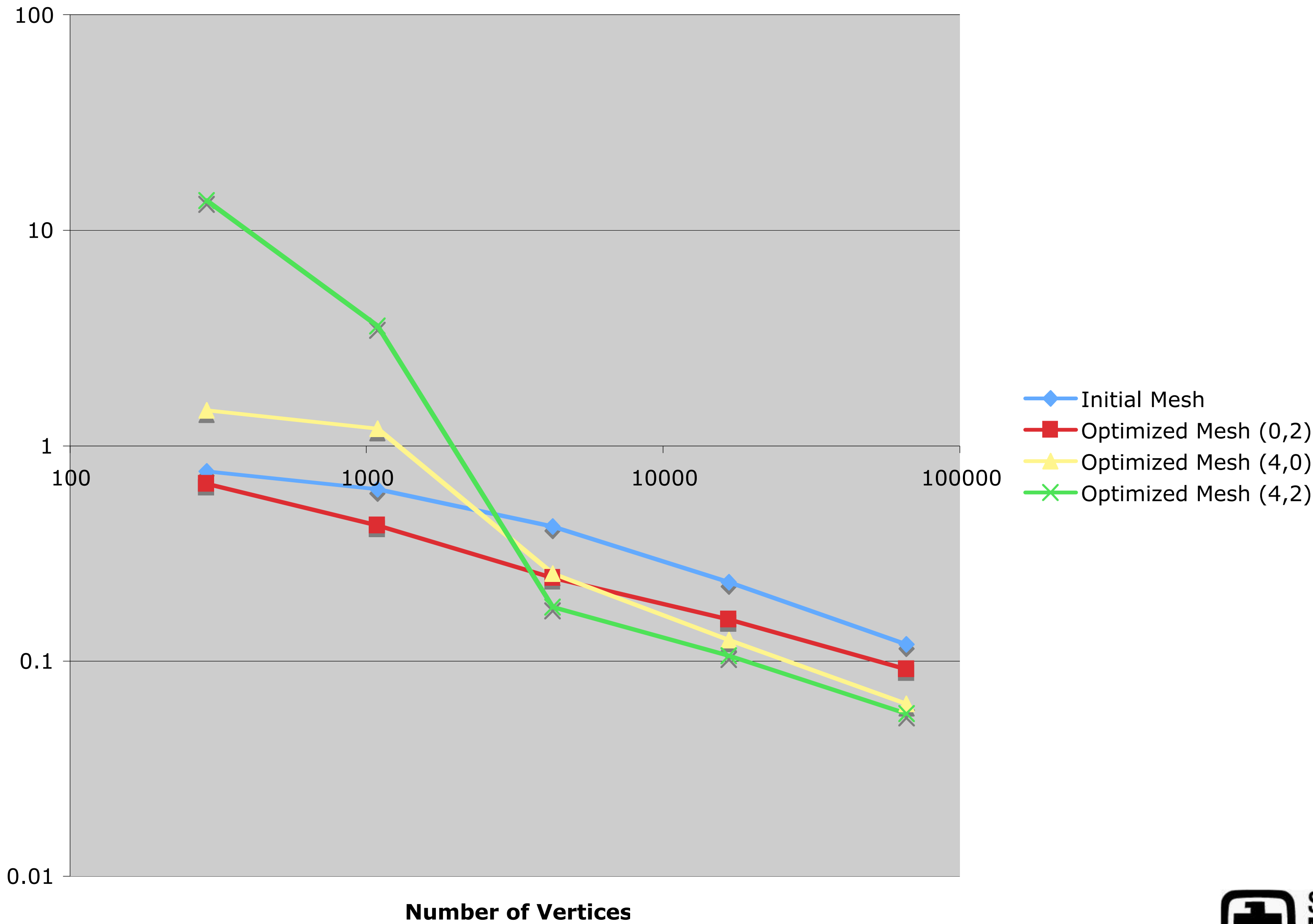
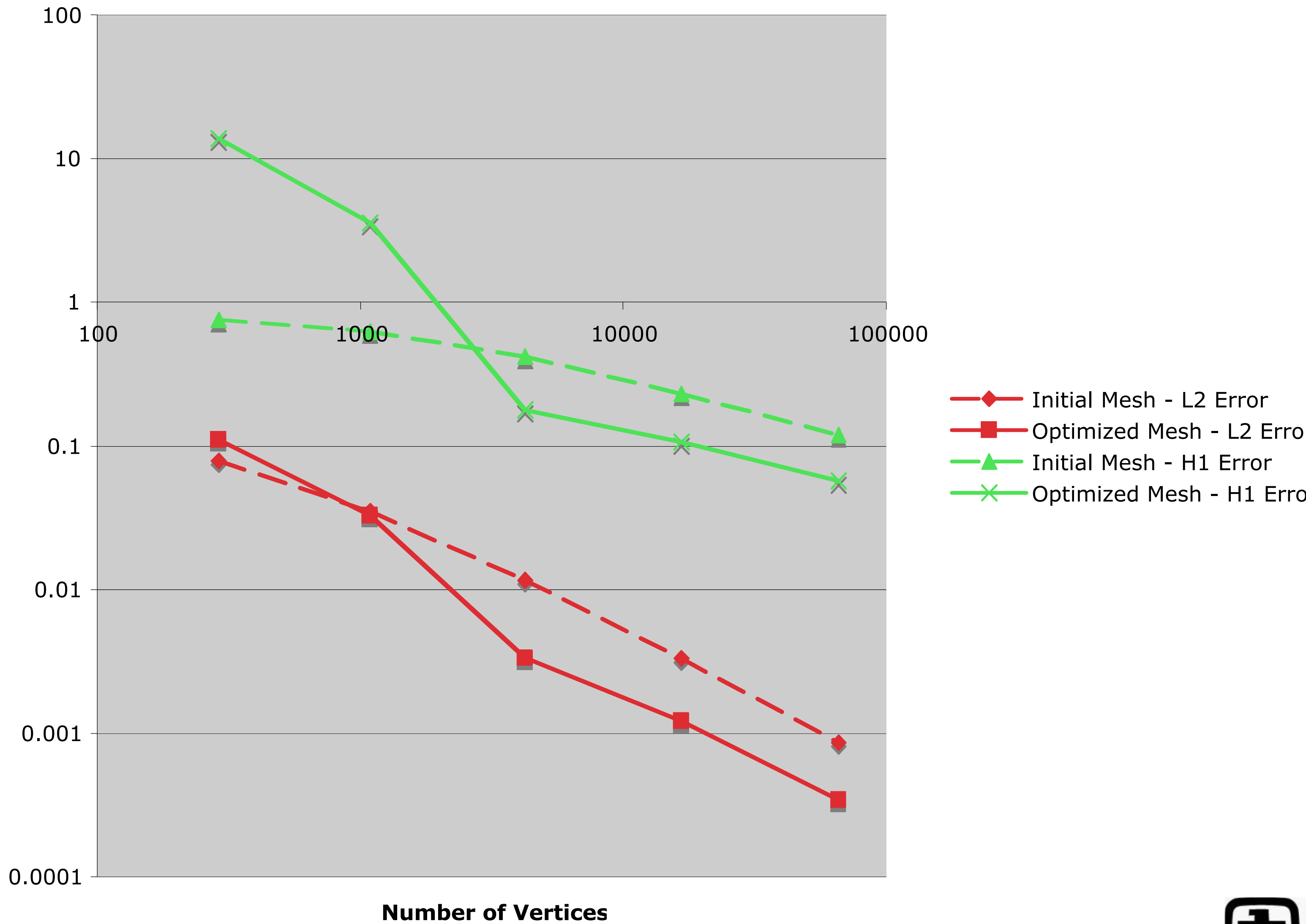
R-Adaptive Scheme

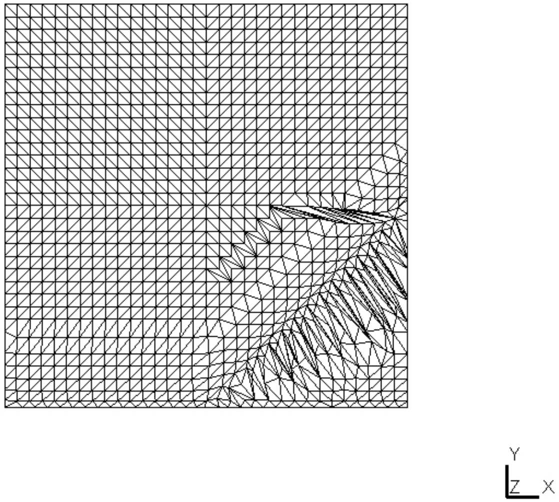


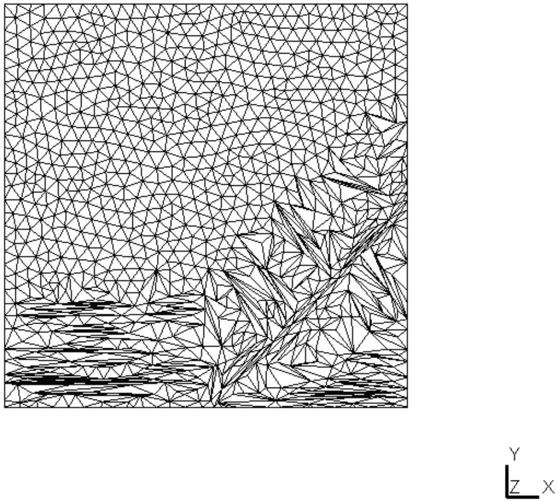
R-Adaptive Scheme



R-Adaptive Scheme

- **On coarse meshes, the error increased.**





How do we know when the mesh is too coarse?

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R-Adaptive Scheme

Summary:

- **Introduce an RS-adaptive scheme.**
- **Objective function based on H^1 interpolation bounds.**
- **On fine meshes, the scheme reduces the H^1-norm of error.**
- **On coarse meshes, there is no guarantee.**
- **Extend to tetrahedra, affine equivalent elements.**

Questions:

- **How to treat the case of coarse meshes?**
- **Extension to quadrilateral elements?**
- **Interpolation formula for Hessian matrix?**

R-Adaptive Scheme