SAND2007-6110C

A Short Course on Mesh Quality

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16th International Meshing RoundTable
Seattle, WA
October 14, 2007

SAND2007-XXXX

Outline:

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   2.1 Measurement & Assessment

   2.2 Advanced Topics

3. Mesh Quality Improvement

Questions or Comments….

1. Impact of Mesh Quality on Solution Accuracy & Efficiency

Applications that use meshes want accuracy & efficiency.

Many factors go into accuracy & efficiency such as the governing equations, the solution, the method of discretization, linear solvers & preconditioners, and the quality of the mesh.

A Definition of Mesh Quality:

"The characteristics of a mesh that permit a numerical PDE simulation to be performed with fidelity to the underlying physics, accuracy, and efficiency."

Fidelity: Preservation of Equation Type,
and Solution Symmetry

Accuracy: 'Small' Discretization Error

Efficiency: Mesh results in reasonable matrix conditioning and 'small' maximum eigenvalues.

The quality of a mesh depends on the discretization scheme, the solver, the PDE, and the physical solution.

1.1 Finite Differences

A body of literature exists on the relationship between truncation error and the quality of structured meshes (Mastin & others).

Mastin, C.W. (1982) "Error induced by coordinate systems," in J.F. Thompson (editor), Numerical Grid Generation, North-Holland, New York, p31-40.

Thompson, J.F., Mastin, C.W. (1983) "Order of difference expressions in curvilinear coordinate systems," in Ghia and Ghia (editors), Advances in Grid Generation, ASME, New York, p17-28.

Lee, D., Tsuei, Y. (1992) "A Formula for Estimation of Truncation Errors of Convection Terms in a Curvilinear Coordinate System," *J. Comp. Phys*., **98**, p90-100.

Huang, H., Prosperetti, A. (1994) "Effect of Grid Orthogonality on the solution accuracy of the two-dimensional convection diffusion equation," *Num. Heat Transfer*, Part B, 26:1-20.

The Jacobian Matrix (Structured Grids)

Mapping from square or cube to physical domain

$$x_i = x_i(\xi_j)$$

Jacobian Matrix

$$J_{ij} = \partial x_i / \partial \xi_j$$

Jacobian Determinant (measure of local volume)

$$J = \det[J_{ij}]$$

Fidelity: Preservation of Type (structured)

Elliptic (diffusion), Parabolic, Hyperbolic (waves) type determined by discriminant of PDE coefficients.

Example: $$\Delta f = div_x \bullet grad_x f = \frac{1}{J} div_\xi \bullet \left(J G^{-1} grad_\xi f \right)$$

Laplacian transformed to general coordinates.

Laplacian elliptic requires J>0.

Meshes with locally negative volumes will cause the PDE to change type over the region, resulting in non-physical calculations.

Mesh Quality affects Accuracy

Local Error Analysis: Dominant error term is Ch^p

where h is the representative size of a element and C is a constant independent of h.

For structured meshes, $h = \max_k (\Delta\xi_k)$

C contains derivatives of the solution and derivatives of the grid.

Example: For centered finite-differences in 1D, the error in f_x is $err \cong f_x (x_{\xi\xi\xi} - f_{\xi\xi\xi}) h^2 / 6$

The third derivative of the mapping is related to geometric mesh properties.

Accuracy: Element Size

Local Error Analysis: Dominant error term is Ch^p

where h is the representative size of a element and C is a constant independent of h.

Element size is thus critical in reducing the truncation error. (h-refinement exploits this.)

C contains derivatives of the solution.

Therefore, one can have h large in parts of the mesh where the solution lacks structure because C will be small. Generally want smallest h where solution gradient or curvature is largest.

Accuracy: Smoothness, Skew, Orientation

Truncation error can also be decreased by reducing C.

Because C contains derivatives of the grid, C depends on combinations of grid smoothness, skew, orientation.

The dependency of C on the grid properties is complicated.

Questions or Comments?

1.2 Finite Elements

A highly developed but incomplete theory exists that presents bounds on interpolation error that necessarily include mesh quality.

There are also results for efficiency.

1.2.1 Accuracy

Accuracy is addressed through bounds on Interpolation Error

Traditional emphasis in FEM is on the asymptotic behavior of the bounds, but one can also study them from the perspective of mesh quality.

We present some selected bounds and show how they depend on mesh quality.

Conforming affine-equivalent finite elements

Two elements K and K' are affine equivalent if there exists an invertible affine mapping

$$F : x' \in R^d \rightarrow F(x') = B_K x' + b_K$$

such that K=F(K').

B is a d x d matrix that is constant over the element and describes mesh quality in terms of shape, size, and orientation.

Definition of Interpolation Error

Let v be a function belonging to $W^{m,p}(K)$

Norm: $$|v|_{m,p,K}^{p} = \sum_{|\alpha|=m} \int_K \left| \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}(x) \right|^p dx$$

Then the local interpolation error is

$$|v - \Pi v|_{m,p,K}$$

where π (Pi) is the interpolation operator.

General Approch due to Ciarlet

For affine-equivalent elements, Ciarlet derives the following bound on the interpolation error:

$$\left|v - \Pi v\right|_{m,p,K} \leq C \left\|B_K\right\|_2^r \left\|B_K^{-1}\right\|_2^m \left|\det\left(B_K\right)\right|^{\frac{1}{p}-\frac{1}{q}} \left|v\right|_{r,q,K}$$

The bound entails a combination of the norm of v and mesh quality expressed in terms of the matrix B. For example, if q=p and r=m, the bound is expressed in terms of the element Condition Number.

$$\left|v - \Pi v\right|_{m,p,K} \leq C\, \kappa_2^m\left(B_K\right)\left|v\right|_{r,q,K}$$

Tight error bounds on Simplices (Shewchuk)

Let v be a continuous scalar function with a bounded second deriviative:

$$\left\| v - \Pi v \right\|_{0,\infty,K} \le \frac{r^2}{2} v_{2,\infty,K}$$

$$\left\| \nabla v - \nabla \Pi v \right\|_{0,\infty,K} \le \frac{\ell_{\max} \ell_{med} \left(\ell_{\min} + 4 \rho_K \right)}{4 meas(K)} \left| v \right|_{2,\infty,K}$$

Where r is the circumradius and ρ is the inradius.
These bounds hold even in the non-assymptotic case!

The bounds show that the gradient error can be arbitrarily large as the element becomes badly shaped, whereas this is not the case for the solution error.

Solution-dependent Bounds

Additional bounds can be derived based on special knowledge of the solution.

Examples:

1. If v is a quadratic function, then
   - Nadler bounds the L2 norm of the interpolation error in terms of the length of the sides of a triangle
   - D'Azevedo & Simpson derive the exact formula for the max-norm of the interpolation error,

2. If v is a smooth function in H2(K), then Formaggia & Perotto bound the function and gradient norms in terms of the singular values of B and the Hessian of v.

Bounds Based on PDE-Coefficients

If the PDE operator is given in terms of a symmetric positive definite matrix A, then the Ciarlet bounds can be extended.

For example,

$$\left| A^{1/2} \nabla (v - \Pi v) \right|_{0,2,K} \leq C \left\| B_K \right\|_2^{k+1} \left\| B_K^{-1} A^{1/2} \right\|_2 |v|_{k+1,2,K}$$

Thus, if A is anisotropic, then the mesh should account for this in order to maintain accuracy.

Non-affine Elements

If the affine element map Bx+b is replaced by a sufficiently smooth 1-1 mapping F with sufficiently smooth inverse, then the Ciarlet theory is much more complex.

For example, the interpolation error on the function is bounded by a sum of 3 terms involving the supremum over the element of the norms of the first, second, and third derivatives of the map. These, in turn, depend on 'mesh quality', but cannot be expressed in terms of simple, well-known quality metrics.

If the map F is iso-parametric, then the following bound can be derived provided F is 'close' to the affine map involving B-tilde:

$$\left| v - \Pi v \right|_{m,p,K} \le C \left\| \left(\tilde{B} \right) \right\|_2^{k+1} \left\| \left(\tilde{B} \right)^{-1} \right\|_2^{m} \sum_{\ell=1}^{k+1} \left| v \right|_{\ell,p,K}$$

Convex Quadrilaterals

Jamet & Acosta: $\left\|v - \Pi v\right\|_{0,2,K} \leq C\, h_K^2 \left|v\right|_{2,2,K}$

where C is independent of the element geometry.

Ciarlet & Raviart give a similar bound on the Norm of the gradient:

$$\left\|v - \Pi v\right\|_{1,2,K} \leq C\, h_K \left|v\right|_{2,2,K}$$

But now C depends on bounds to the radius ratio and the element angle.

Additional Observations on Accuracy:

- Accuracy is decreased if there is a discontinuity in the solution which resides in the interior of the element, as opposed to being located at a node (Carey),
- The optimal grid for best accuracy will be different depending on the choice of norm (Carey)
- The results of Shewchuk, Jamet, and others show that for the interpolation error in the function, mesh quality is equivalent to controlling only h, whereas for the error in the gradient, mesh quality involves not only h, but also element condition number (shape).

Questions or Comments?

1.2.2 Efficiency in Finite Elements

Recall rate of convergence of iterative linear system solver is determined by the largest eigenvalue of the stiffness matrix (steady-state).

Largest Eigenvalue

For a membrane discretized by triangular elements with piecewise linear finite elements, Fried proved:

$$\frac{1}{\sin\theta_{\min}} \le \lambda_{\max} \le 3\frac{p_{\max}}{\sin\theta_{\min}}$$

where λ-max is the largest eigenvalue of the stiffness matrix, θ-min is the smallest angle in the triangle, and p-max is the largest valence in the mesh.

Hence, *small* or *large* angles create large maximum eigenvalues and thus decrease the rate of convergence.

Additional Comments on Efficiency

- There also exist bounds on the condition number of the mass & stiffness matrices which depend on mesh properties; a poor mesh could thus adversely impact roundoff errors,
- Du et. al. presented numerical experiments which showed that large angles could hamper solver efficiency when preconditioned with AMG,
- Batdorf showed that convergence rate of GMRES and multigrid solvers is adversely affected by large angles; smoothing & swaping of meshes can allow a convergent solution to be obtained.

1.2.3 The Discrete Maximum Principle

For a linear finite element approximation of a quasi-linear elliptic operator, the condition that all mesh angles be acute is sufficient to guarantee that the discrete maximum principle holds (Ciarlet & Raviart).

For diffusion operators, a sufficient condition for triangle meshes is:

$$\cot(\theta_1) + \cot(\theta_2) \geq 0$$

where these are the opposite angles of any two triangles sharing an edge (Xu & Zikatanov); this condition allows the mesh to have obtuse angles and still obey the discrete maximum principle.

A result for tetrahedral meshes involving edge lengths and dihedral angles is also given by Xu.

1.2.4 Summary of Impact of Mesh Quality

- Accuracy depends on mesh quality, along with other factors,
- Truncation Error depends on derivatives of the map from which a structured mesh is generated,
- Error bounds depend on mesh properties,
- The mesh properties which appear in the expressions for the error often do not correspond to any traditional mesh quality metrics,
- Condition number (angles and aspect ratios) does appear in FEM error bounds, while angles are important in efficiency,
- Gaps remain in the theory and its appliication to mesh quality (non-simplicial elements, high-order elements)
- We did not discuss finite volume case, nor the case of impact of mesh on `a posteriori error estimates

Questions or Comments?

2. Measuring Initial Mesh Quality

Mesh Quality is also of interest in the generation of initial meshes that may later be adapted to the solution.

Many measures, metrics, and functions exist which measure geometric mesh quality. Widely used in industry.

2.1 Measuring/Assessing Mesh Quality

-Visual Inspection of the Mesh (not practical in 3D unless one is using advanced visualization techniques),

-Mesh or Element Quality Metrics (automatic)

Definition: Element Quality Metric

“A scalar function of element nodal positions that measures some geometric or other property of an element.”

Quality Metrics measure local quality.

Global mesh quality can be measured in terms of norms or p-means of local quality metrics.

Uses of Mesh Quality Metrics

1. Mesh Quality Requirement Specifications,
2. Defect detection (catch problems early),
3. Quality Control (good meshes are hard to make),
4. Mesh Improvement:
   Edge swapping, Element swapping
   Node-movement Strategies

h-adaptivity works best when starting with good initial mesh quality (quality can degrade in some local refinement schemes.)

Threshold Criteria for Automatic Quality Control

Metrics are compared to quantitative threshold criterion.

Example: Let M be the value of some diagnostic metric, with range min M<M<max M over all possible element sizes & shapes. This interval can be subdivided as follows:

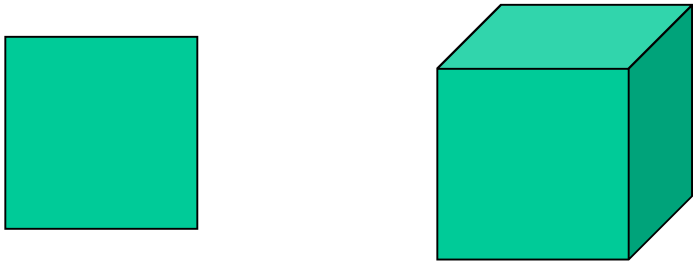
Reject - Must fix problem elements (automatically report an error),
Red Zone - Investigate problem elements (automatically report a warning)
Adequate - Use-able elements (do not report)
Good - Elements close to Ideal (do not report)

T1 & T2 are critical to determine and yet no theory exists on how they should be determined.

Questions or Comments?

Properties of Quality Metrics

Dimension



Definition

A metric is dimension-free if its definition in 3D is an unambiguous natural generalization of its definition in 2D.

Example: Volume metrics are dimension-free, while angle metrics are dimension-specific.

Properties of Quality Metrics

Element type

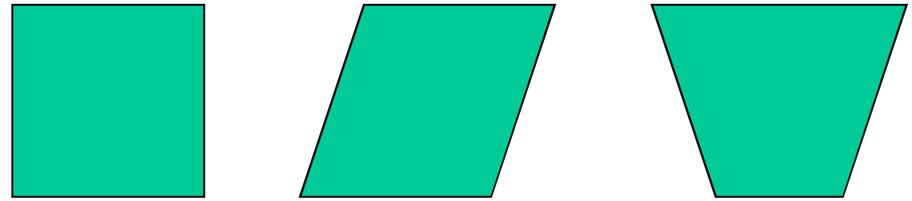
Definition

A metric is element-free if its definition on one element type is an unambiguous natural generalization of its definition on another element type.

Example: Maximum angle is element-free on two-dimensional elements. The ratio of diagonals is element-specific.

Properties of Quality Metrics

Element Shape



Definitions

A metric defined for a fixed element type is shape-specific if it is meaningful for only a particular shape of the element type.

Example: Aspect ratio is shape-specific. It may be defined for any quadrilateral (Robinson, 1987), but loses its meaning as one departs from a rectangle.

A metric defined for a fixed element type is shape-general if it is meaningful over a wide range of possible shapes of the element.

Example: Angles of a Quadrilateral is shape-general

Properties of Quality Metrics

<u>Versatility</u>

Definitions

A metric defined for a fixed element type is <u>versatile</u> if it is sensitive to more than one quality attribute (e.g., skew, aspect ratio, shape, size, orientation)

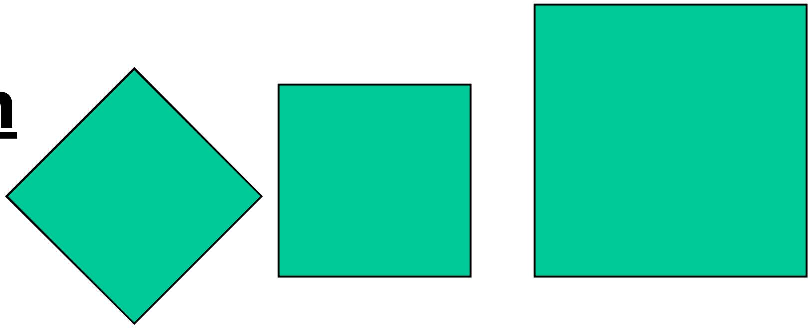
A metric is <u>specialized</u> if it is sensitive to only one quality attribute.

Example: Tetrahedral shape measures are versatile because they are sensitive to both angle and length ratios, while rectangle aspect ratio is specialized.

Versatile metrics reduce the number of metrics needed but provide less detailed information.

Properties of Quality Metrics

Scale & Orientation



Definitions

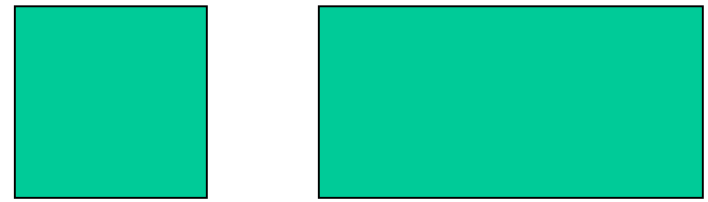
A metric is scale-invariant if it's value does not depend on the size of the element.

A metric is orientation-invariant if it's value does not depend on the orientation of the element.

Example: Aspect Ratio of a Rectangle is scale-invariant & orientation-invariant. Volume is orientation-invariant, but not scale-invariant.

Properties of Quality Metrics

Reference Element



Definition:

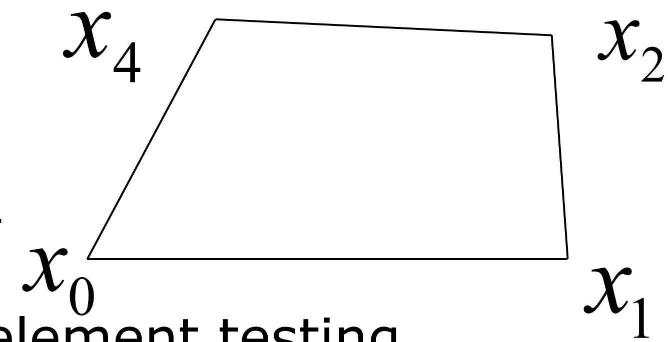
A metric is referenced if it incorporates a comparison to a reference element to account for inhomogeneity or anisotropy.

Examples: Referenced aspect ratio, h / (s w), is referenced to a rectangle with aspect ratio s. The aspect ratio h/w is implicitly referenced to a unit square. Area (h w) is not referenced.

By necessity, referenced metrics are unit-less.

Questions or Comments?

Examples of Metrics:



Robinson Quadrilateral Metrics

J.Robinson (1987) "CRE Method of element testing and the Jacobian shape parameters," Eng. Comput., Vol. 4.

$$e_x = \tfrac{1}{2}\{(x_1 - x_0) + (x_2 - x_3)\}$$

$$e_y = \tfrac{1}{2}\{(x_3 - x_0) + (x_2 - x_1)\}$$

Aspect Ratio: $$AR = \max\left(\frac{|e_x|}{|e_y|}, \frac{|e_y|}{|e_x|}\right)$$

Skew: $$\frac{|\, e_x \bullet e_y \,|}{|e_x| |e_y|}$$

Both squares & 'Kites' have AR=1

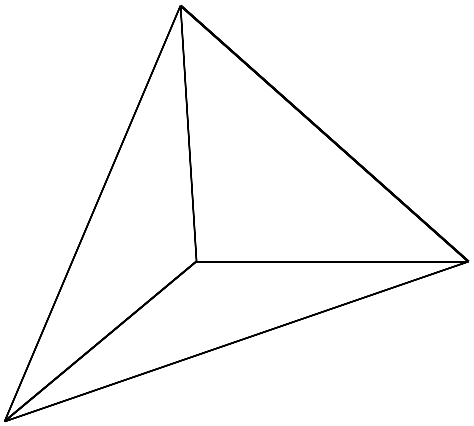
Examples of Metrics:

<u>Tetrahedral Shape Measures</u>

Radius Ratio: $\rho = 3\,\rho_{in} / \rho_{out}$

Mean Ratio: $\eta = \mu^{2/3} / \|M\|_F^2$

'Aspect Ratio' $\gamma = \dfrac{12}{\sqrt{6}} \dfrac{\rho_{in}}{\max\limits_{0 \le i < j \le 3} \ell_{ij}}$



Shape measures are nearly zero for 'sliver' tetrahedra.

Equivalence of Shape Measures

Shape measures with values μ_1 and μ_2 are *equivalent* if there exist positive constants a,c, p,q such that

$$a\mu_1^p \leq \mu_2^q \leq c\mu_1^p$$

for all shapes of the element (Liu & Joe, 1994).

Example: Radius Ratio, Mean Ratio, & Sine of Solid Angle are equivalent.

Equivalent metrics sense the same distortions, grow large together, and grow small together.

Need a good reason to use both of two metrics if they are equivalent.

Status of Widely-Used Mesh Quality Metrics

Redundant: Large number of metrics have been proposed, (e.g., half dozen shape metrics), only a few needed.

Ineffective: Some fail to detect certain bad elements, some measure things that have no clear connection to the physics.

Not-General: Tied to particular dimension, element-type, or element shape.

Implicitly-referenced: Assume isotropic physics.

Absence of Defined Ranges, Ideals, and Meaningful Threshholds.

Lack of correlation between geometric properties and analysis effects.

Questions or Comments?

2.2 Advanced Topics

Many of the metrics currently in use suffer from one defect or another. A mathematical analysis of quality metrics sheds light on the issues and suggests more advanced metrics which have better properties.

Shape Measures

Definition: “A tetrahedral shape measure is a continuous scalar function of the nodal coordinates that is translation, **scale & orientation independent**, which ranges from 0 to 1, with 0 signifying a degenerate tetrahedron and attaining 1 only for the unit equilateral tetrahedron. “

J. Dompierre, et.al., “Proposal of Benchmarks for 3D Unstructured Tetrahedral Mesh Optimization,” p459-478, 7th Intl. Meshing Roundtable, Dearborn MI, 1998

This is an **abstract definition**:

- does not specify a particular form of the metric,
- isolates essential properties of shape metrics,
- relatively non-controversial

Algebraic (Matrix-based) Mesh Quality Metrics

- Incorporates shape definition into Algebraic matrix-based framework,

- Provides abstract definitions of other important mesh quality metrics,

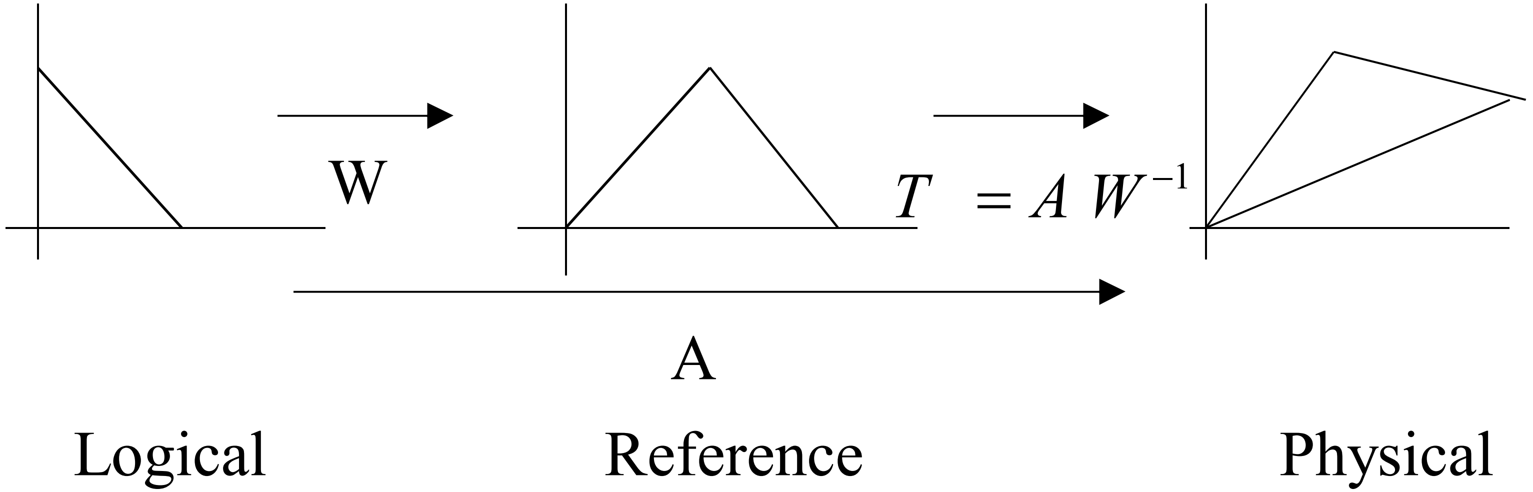
- Introduces the idea of Target Matrices for Referenced Metrics.

P.Knupp, "Algebraic Mesh Quality Metrics," SIAM J. Sci. Comput., Vol. 23, No. 1, pp193-218, 2001.

P. Knupp, “Algebraic Mesh Quality Metrics for Unstructured Initial Meshes,” Finite Elements in Design & Analysis, p 217-241, Vol. 39, No. 3, 2003

Referenced Metrics

All metrics should be explicitly referenced to an ideal element.



If f(X) is maximized by X=I, then f(T) is maximized by A=W.

Determination of the Reference Matrix

How is W determined? Ideally, W is determined from either à priori or à posteriori knowledge of the solution.

<u>Examples</u>:

-element SHAPE may be isotropic or an-isotropic depending on material properties or boundary layers,

- element ORIENTATION may be determined from flow-lines or internal interfaces,

- element relative SIZE may be determined from solution gradients.

W may also take into account certain features of the geometry such as regions of high curvature (determinant of W small where curvature is high).

Algebraic Shape Metric

<u>Definition</u>. Let f be an algebraic mesh quality metric. Then f is an algebraic *Shape* metric if

- the domain of f is restricted to T,
- f is scale and orientation invariant, $f(\mu RS)=f(S)$
- $0<= f(T)<=1$, for all T,
- $f(T)=1$ if and only if $T \in SR(n)$, i.e., $A=\mu RW$,
- $f(T)=0$ if and only if T is degenerate

(T is degenerate if $\det(T)=0$, but $|T|>0$.)

Condition Number Shape Metric

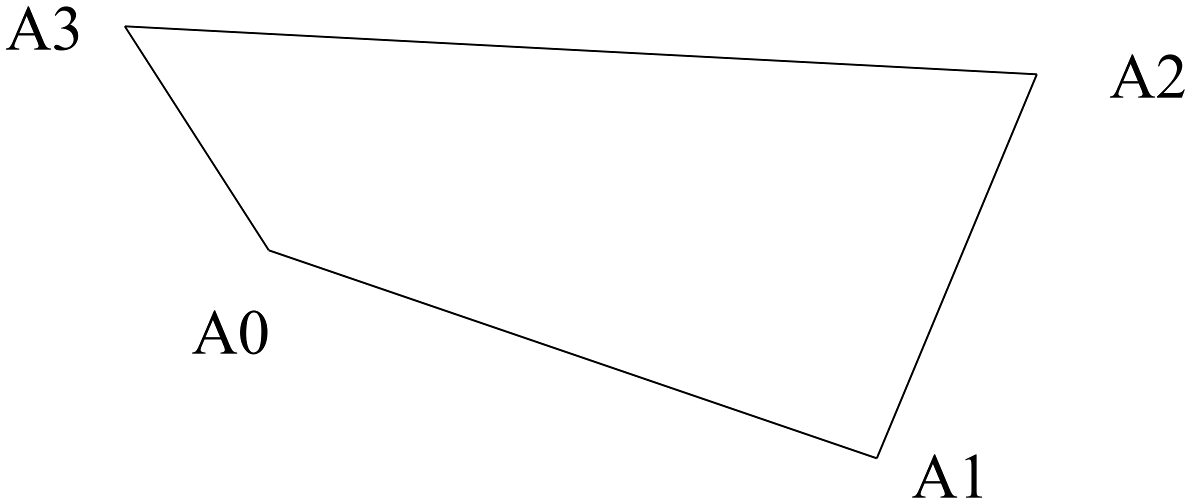
$n / \kappa(T)$ is an algebraic shape metric, where

$\kappa(T) = |T|\,|T^{-1}|$ is the element *condition number*.

Measures distance to set of singular matrices, i.e., degenerate elements.

There are other algebraic metrics which are shape measures (e.g., mean ratio & dimensionless-Winslow), i.e., they are equivalent.

Non-Simplicial Elements



Element geometry cannot be represented by a single matrix.
Use multiple matrices to define algebraic metrics.

Questions or Comments?

3. Improving Mesh Quality

If a mesh defect is detected, how can one remove it?

What to do when the Quality is Insufficient?

Remesh:

A. Change geometry decomposition
B. Re-block
C. Size settings, intervals
D. Change meshing scheme

The above can be time-consuming (non-automatic).

Post-Process:

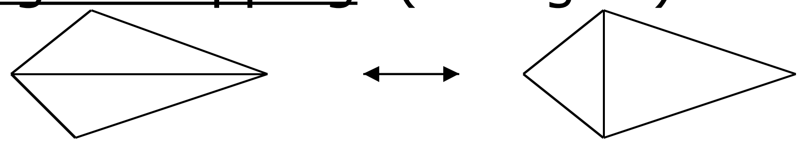
E. Reconnect (flipping & swapping)
F. Node-Movement: Smoothing
G. Node-Movement: Optimization

Adapt:

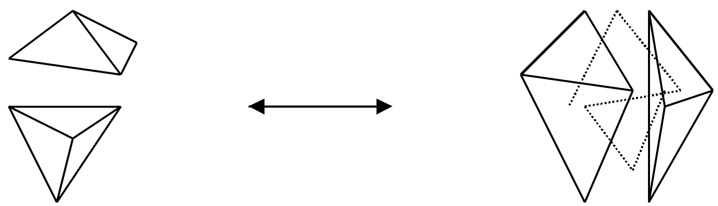
H. h-Refinement
I. r-Refinement
J. hr-Refinement

Local Reconnection Procedures

Edge Swapping (triangles)



Element Swapping (tetrahedra)



Element Swapping (hexahedra)

Node-Movement: Smoothing

Definition:

"Smoothing is a procedure for improving mesh quality via a node-movement strategy in which a non-linear system of equations is solved."

Smoothing Equation: $f(\ldots, x_{ij}, y_{ij}, \ldots) = 0$

Iterative Form:

$$x_{ij}^{n+1} = g(\ldots, x_{ij}^n, y_{ij}^n, \ldots)$$

$$y_{ij}^{n+1} = h(\ldots, x_{ij}^n, y_{ij}^n, \ldots)$$

Node-Movement: Mesh Quality Optimization

Definition:

"Mesh quality optimization is the process of changing nodal positions to find the extremae of some scalar objective function that measures one or more aspects of mesh quality."

Two parts:

1) Choosing an objective function (what)
2) Choosing an optimization procedure (how)

<u>Connection between Optimization & Smoothing</u>: extremae of the objective function occur where the gradient is zero. Setting the gradient of the objective function to zero yields a non-linear set of equations that result in a smoothing scheme.

Advantages of Optimization vs.Smoothing

Case Study: Laplace Smoothing as an Optimization Problem

Minimize: $$\frac{1}{2}\sum_k |A_k|^2$$

Gradient of this objective function gives the Laplace smoothing equation.

Geometrically, $|A_k|^2$ is the sum of the edge-lengths squared.

Not a shape metric! (not scale-invariant)
Not a volume metric! (not zero for degenerate elements)

Thus, optimization approach reveals why Laplace smoothing
is not always effective (does not directly control shape, size, or orientation).

To be meaningful & effective, objective functions should be built from mesh quality metrics.

Questions or Comments?

Local Metrics from an Optimization Viewpoint

Local metrics measure the local relationship between A and W.

Let '*d*' be dimension (2 or 3) and B in M_d (R), the set of real 'd x d' matrices.

Local metrics $\mu = \mu(T)$ are continuous functions from M_d (or some subset) to the real numbers. Examples are |T|, det(T), and trace(T).

For mesh optimization, it is assumed additionally that μ is <u>bounded</u> <u>below</u> since we are going to <u>minimize</u> our objective functions. Thus det(T)^2 or Trace(T)^2 would be used instead of det(T) or trace(T).

In general, our local metrics will have one of two <u>forms</u>:

$$\mu = \mu(T) \qquad \mu = \|T - I\|^2$$

$$\mu = \mu(A, W) \qquad \mu = \|A - W\|^2$$

Either way, these quantities ultimately are a <u>function</u> of mesh <u>vertex</u> <u>positions</u>.

Barrier Metrics

A formal definition of a (weak or strong) barrier metric is given in the paper.

Example: Inverse Mean Ratio (d=2).

$$\mu = \frac{\|T\|^2}{2|\det(T)|}$$

Defined on the set of matrices with positive determinants.
μ undefined for $\det(T) \leq 0$.

$\mu \rightarrow \infty$ as $\det(T) \rightarrow 0$, except when $T = sI$ (with $s \rightarrow 0$), so mean ratio has a weak barrier at $\det(T)=0$. A metric with no exceptions would have a strong barrier.

In general, a barrier metric cannot be used if the initial mesh is inverted. In that case, one must use non-barrier metrics.

If the initial mesh is non-inverted, optimization with a barrier metric should result in a non-inverted optimal mesh.

Specific Local Metrics for 2D Meshes

Some existing metrics, such as condition number, already use target matrices and thus fit within the TMP.

Additional *metrics are needed to enforce particular relationships between the Active and Target metrics.*

Metrics based on 2x2 matrices will be different than metrics based on 3x3 matrices.   We concentrate on the former here.

We consider both non-barrier and barrier forms of the new metrics.

Ideal Properties of Local Metrics

1. Metrics are continuous functions of T on D, and except for a few isolated points, they are differentiable with respect to T on D.

2. If a metric has no barrier, D is all 2x2 matrices. If the metric has a barrier the domain is all 2x2 matrices with positive determinant.

3. Metrics are bounded below by a non-negative constant, but unbounded above.

4. There exists a finite global minimizer at which the lower bound is attained.

5. *The global minimizer belongs to one of the four canonical matrix sets.*

6. If the metric has more than one global minimizer, they all belong to the same canonical set.

7. *The set of stationary points of the metric coincides with the set of global minimizers. (No local minima or saddle points).*

Canonical Relationships Between Active & Target Matrices

If	$T \in M_d^{(i)}$	then	A = W		Shape, Size, Orient
If	$T \in M_d^{(si+)}$	then	A = s W,	with s > 0	Shape, Orient
If	$T \in M_d^{(o+)}$	then	A = R W,	R a rotation	Shape, Size
If	$T \in M_d^{(so+)}$	then	A = s R W.		Shape

Why do this? If only want to measure Shape & Size, for example, then one does not need the target construction algorithm to consider orientation.

Three Non-barrier Metrics

$$\mu_2 = \|T\|^2$$

Global minimizer and stationary point is T=0, not canonical

(Must fix boundary vertices; then can be ok. Laplace.)

$$\mu_d^{(i)} = \|T - I\|^2$$

Global minimizer is T=I, the Identity set. Both d = 2 & 3 derivative is 2(T-I), so stationary point is T=I. (coincide) Equivalent to |A-W| metric used in alignment papers. Target controls shape, size, and orientation of A. Used in TMP deforming mesh method.

$$\mu_d^{(si+)} = \left\| T - \frac{1}{d} |trace(T)| I \right\|^2$$

Global minimizer is T=sI, the scaled-identity set.
Coincident stationary points. New.
Target controls shape & orientation of A.
Size invariant. Both d = 2 & 3.
The key in this metric is to indirectly specify s in terms of T.

The Size-Shape Non-Barrier Metric (o+)

$$\mu = (1-s)\left\| T^t T \;\; - I \right\|^2 + s\left(\det(T)-1\right)^2$$

$0<s<1$ is a trade-off parameter between orthogonality (first term) and unit determinant (second term). Global minimizer is T=R, as desired.

But, the metric has stationary points at T=R and at T=0, so not all are coincident with the global minimizers (T=R).

The Size-Shape Non-Barrier Metric (o+)

$$\mu_2^{(o+)} = \|T\|^2 - 2\psi + 2 = \|T - R\|^2$$

$$\psi = \sqrt{\|T\|^2 + 2\det(T)}$$

$$R = \left(T + adj\left(T^t\right)\right) / \psi$$

$$\frac{\partial \psi}{\partial T} = R$$

Global minimizers & stationary points coincide and belong to the set of Rotations. New!

R is a function of T, so no particular R need be specified.

Only good for d=2. Target controls Shape & Size, but not Orientation.

Non-barrier Shape Metric (so+) $\mu_2^{(so+)} = \left\|T - \tfrac{\psi}{2} R\right\|^2 = \frac{1}{2}\|T\|^2 - \det(T)$

Bibliography on Mesh Quality

http://www.cs.sandia.gov/optimization/knupp/Bibliography.htm

Questions or Comments?