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Validation of Random Vibration Environments

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Abstract

The prediction of structural response to external excitation requires models for (1) the structure and, (2) the excitation environment. The accuracy of the response predictions is limited by the accuracy of the models for both (1) and (2). However, studies in model validation have typically been limited to models for the structure; research efforts on procedures to identify and validate models for the environment have been extremely limited. One reason for this is because, for many applications, *e.g.*, random vibration, the environment is modeled as a stochastic process in space and/or time, and methods for identification and validation of stochastic models are not well established. Herein, we develop methods for the identification and validation of environment models for applications in random vibration. The environment model is a Gaussian, vector stochastic process. Some relevant examples are considered to illustrate the method.

Nomenclature

$\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}$	Mass, damping, stiffness matrices of a structural dynamic system
$\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t), \ddot{\boldsymbol{x}}(t)$	Deterministic displacement, velocity, acceleration responses of structure
$\boldsymbol{X}(t), \dot{\boldsymbol{X}}(t), \ddot{\boldsymbol{X}}(t)$	Random displacement, velocity, acceleration responses of structure
$\boldsymbol{q}(t)$	Deterministic structural excitation
$\boldsymbol{Q}(t)$	Random structural excitation
$\boldsymbol{H}(\omega)$	Frequency response function matrix or vector
$\boldsymbol{S}(\omega), \boldsymbol{G}(\omega)$	Two-sided and one-sided spectral density matrices
v_0	Velocity of propagation of a structure through a medium
$\boldsymbol{\xi}(\omega), \boldsymbol{\theta}(\omega)$	Fourier transforms of $\boldsymbol{x}(t)$ and $\boldsymbol{q}(t)$
$\boldsymbol{\Lambda}(\omega)$	Factor in the model for the excitation spectral density
α	Parameter in the terms of $\boldsymbol{\Lambda}(\omega)$

Introduction

All mechanical systems are subjected to stochastic dynamic environments at some times during their lives, and under certain circumstances these environments can lead to system failure or malfunction. Therefore, it is important to characterize structural response quantitatively. This quantification might, in principle, be

accomplished experimentally, but the impracticality of testing in certain extreme regimes and testing sufficient structures to cover the range of part-to-part variability usually prevents that. Specifically, (1) it is usually quite expensive to build sufficient replicates of the structure under study, (2) it is expensive to perform sufficient field testing, and (3) there is normally a great deal of uncertainty associated with subjecting a small number of structural replicates to severe environments. In view of this, the option of designing structures based on mathematical models is attractive.

To satisfactorily predict the responses of structures using models two things are required – an accurate model for the structural system under consideration, and an accurate model for the environment to which the structure is subjected. The first requirement – an accurate model for a structure – can be met using the methods of validation for structural systems. Validation of a mathematical model involves the quantitative demonstration that a mathematical model can predict response with accuracy that is satisfactory for a specific purpose. When the mathematical model for a structural system is validated, then it is assured that response predictions obtained using the model will be adequate for the class of excitations and responses covered by the validation comparisons. General ideas of model validation are presented in references [1,2]. Some specific techniques for the validation of structural dynamic models are presented in references [3,4].

To assure response prediction accuracy for the actual class of environments to which the structure will be subjected, we must assure that the environmental characterization, itself, is accurately modeled. In other words, a validation of the environment model is required. A procedure for accomplishing this type of validation has not, to our knowledge, been specified at this time, but we will propose one in this paper.

Different types of stochastic dynamic environments excite structures, and they can be classified as stationary (random steady state) or nonstationary. When the random vibration environment has duration that is relatively long compared to the duration of the fundamental period of the system under consideration, and when the signal power and frequency content of the exciting forces are steady, then the random excitation can be approximated as a stationary random process. The stationary environments are particularly important and have been studied extensively in the analytical and experimental frameworks. Stationary random environments will be considered here. Examples of stationary random environments are road transportation environments, lift-off environments, and re-entry environments [5]. In this paper we pursue methods for the identification and validation of spatially distributed stationary random environments like the ones encountered during re-entry of an aerospace vehicle. In particular, we seek to identify and validate stationary random loads applied to one dimensional structures that can be adequately modeled as linear. It is assumed that the loads to be modeled cannot be directly measured.

The first part of the problem – identification of spatially distributed stationary random environments – is solved using approaches that relate to the form of input. When the input excitation can be modeled as Gaussian, it is sufficient to identify the first and second moments of the excitation random process because Gaussian random processes can be completely specified in terms of their second moment information. That is, all the joint probability distributions of the Gaussian excitation random process can be specified when the second order moments are available. This is the problem we will consider here.

The second part of the problem – validation of the stationary random load model – has not previously been described in the literature. An approach for performing validation of the mathematical model of a stochastic load will be described in this report.

For purposes of clarity the entire development presented here is framed in terms of a specific example. Section 1 describes the structure. Section 2 is a brief review of some results from the classical theory of linear random vibrations. The methods described form the foundation for the identification of the second order moments of excitation and response random processes. Section 3 develops a method for the identification of random excitation based on limited output data. Section 4 presents a method for validating the mathematical model of a stochastic structural dynamic excitation. Some factors that affect the potential to validate a stochastic excitation model are discussed. Finally, conclusions are presented.

Several assumptions are used here to facilitate the identification and validation analyses:

- The structure under consideration is satisfactorily modeled as linear.
- The structural model is known and validated.
- Measured structural responses free of measurement error are available.

- Measured excitations are unavailable because they cannot be measured. (Therefore, validation comparisons must be made on indirect measures of the structural excitation model; these indirect measures of the excitation are structural responses or measures of structural responses.)
- The excitation is Gaussian.

1 A Linear One-Dimensional Structure

The structure to be used in the developments to follow is a one-dimensional, non-prismatic, linear beam, excited by forces in the lateral direction; for a detailed description of the model, see [6]. The beam is 50 inches long. Attached to the beam is a three degree-of-freedom (DOF) substructure. Figure 1 is a schematic of the structure. The beam is pinned at its left end and supported on a linear spring at its right end. The beam and substructure are modeled via finite elements (FEs); the beam is modeled using Euler-Bernoulli beam elements and the three DOF substructure is modeled with springs and masses. The beam, itself, has 21 rotational and 21 translational DOF. (The first translational DOF is eliminated because of the pin-end boundary condition, therefore, the beam has 20 external DOF excitable in translation.) The even numbered beam DOF, two through forty, are translational. The odd numbered beam DOF, one through forty-one, are rotational. Beam nodes 1, 2, and 21, are shown as circles in the schematic. The first few and final few DOF on the beam are also shown on the schematic.

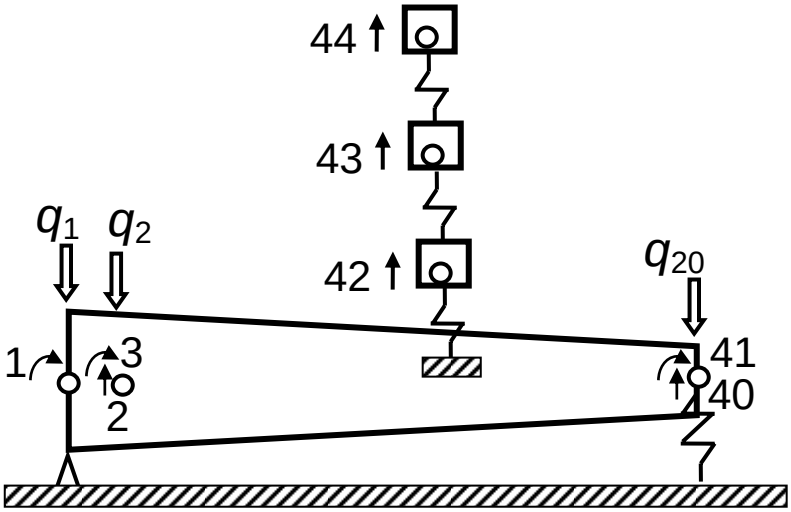


Figure 1. Schematic of the one-dimensional beam analyzed in this work.

The beam has a linear, three DOF substructure attached at translational DOF number 30. The detailed characteristics of the three DOF substructure are known. The three DOF subsystem is attached to the beam at its own DOF 42, and DOFs 43 and 44 are in series with DOF 42. The DOFs of the three-mass substructure are shown in Figure 1.

The beam, itself, has modal frequencies starting at 315 Hz, and its modal damping factors are near two percent. Some of the structural frequency response functions (FRFs) will be shown in the following section.

The structural excitation, whether deterministic or random, is modeled by a sequence of lateral forces denoted by $q_i(t), i = 1,...,20, t \geq 0$. The block arrows in Figure 1 indicate where and in which direction the forces are applied.

2 Some Background from Linear Random Vibration of Force-Excited, Multiple-Degree-of-Freedom Structures

The behaviors of many structures subjected to most of their normal loads can be modeled using a linear finite element model. As well, many of the shock and vibration environments to which structures are subjected are random processes, and we consider the random vibration of a linear, one dimensional structure. The system has multiple-degrees-of-freedom (MDF), and the system response is governed by a matrix, linear ordinary differential equation.

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = \boldsymbol{q}(t) \qquad -\infty \le t \le \infty \tag{1}$$

where t is time, $\boldsymbol{M}$ is the system mass matrix, $\boldsymbol{C}$ is the system viscous damping matrix, $\boldsymbol{K}$ is the system stiffness matrix. All these matrices are symmetric, non-negative definite, and they are assumed to be deterministic. Their dimension is $N \times N$ where N is the number of DOF in the system model. The equations are written for deterministic excitation $\boldsymbol{q}(t), -\infty < t < \infty$, where $\boldsymbol{q}(t)$ is an $N \times 1$ vector of deterministic functions of time. The excitation forces a response with generalized displacement measure, $\boldsymbol{x}(t), -\infty < t < \infty$, velocity measure, $\dot{\boldsymbol{x}}(t), -\infty < t < \infty$, and acceleration measure, $\ddot{\boldsymbol{x}}(t), -\infty < t < \infty$. These are the motions at the structural DOFs. All these are $N \times 1$ vectors of deterministic functions of time. When the excitation $\boldsymbol{q}(t), -\infty < t < \infty$, is specified, a standard structural dynamic FE program can be used to solve for the response.

It is sometimes convenient to express the matrix equation of motion in the frequency domain. This can be accomplished by taking the Fourier transform of both sides of Eq. (1). The result is

$$\left[-\omega^2\boldsymbol{M} + i\omega\boldsymbol{C} + \boldsymbol{K}\right]\boldsymbol{\xi}(\omega) = \boldsymbol{\theta}(\omega) \qquad -\infty \le \omega \le \infty \tag{2}$$

where ω denotes circular frequency, $i = \sqrt{-1}$ denotes the imaginary unit, $\boldsymbol{\xi}(\omega), -\infty < \omega < \infty$, is the Fourier transform of $\boldsymbol{x}(t), -\infty < t < \infty$, and $\boldsymbol{\theta}(\omega), -\infty < \omega < \infty$, is the Fourier transform of $\boldsymbol{q}(t), -\infty < t < \infty$. Equation (2) is simply the frequency domain equivalent of Eq. (1), and knowing the elements of one equation is the same as knowing the elements of the other. Note that the Fourier transform of the acceleration, $\ddot{\boldsymbol{x}}(t), -\infty < t < \infty$, is $-\omega^2\boldsymbol{\xi}(\omega), -\infty < \omega < \infty$.

The Fourier transform of the absolute acceleration response can be obtained by pre-multiplying both sides of Eq. (2) by the matrix inverse of the coefficient on the left side, then multiplying both sides by $-\omega^2$. The result is

$$-\omega^2\boldsymbol{\xi}(\omega) = \boldsymbol{H}_{\ddot{\boldsymbol{X}}\boldsymbol{Q}}(\omega)\boldsymbol{\theta}(\omega) \qquad -\infty < \omega < \infty \tag{3}$$

where $\boldsymbol{H}_{\ddot{\boldsymbol{X}}\boldsymbol{Q}}(\omega), -\infty < \omega < \infty$, is the matrix of frequency response functions (FRFs) of the linear system for force input and acceleration output. For the linear system it is given by

$$\boldsymbol{H}_{\ddot{\boldsymbol{X}}\boldsymbol{Q}}(\omega) = -\omega^2\left[-\omega^2\boldsymbol{M} + i\omega\boldsymbol{C} - \boldsymbol{K}\right]^{-1} \qquad -\infty < \omega < \infty \tag{4}$$

The matrix FRF relates Fourier transforms of the excitations to Fourier transforms of the responses. Specifically, the i^{th} row j^{th} column element of the matrix $\boldsymbol{H}_{\ddot{\boldsymbol{X}}\boldsymbol{Q}}(\omega)$ equals the coefficient of the harmonic response of the structure at DOF i to a harmonic force excitation at DOF j. Equation (3) has been written for the acceleration responses, but it could also be written for the displacement or velocity responses. The practical rationale for using acceleration here is that if the methods described in this document are to be used in a practical setting, they will be used in connection with measured response accelerations. The matrix FRF can be computed from the FE model.

Figure 2 shows the complex moduli of some of the elements of the matrix FRF of the system shown in Figure 1. These and others will be used later in the excitation identification example.

When the structural force excitation is a real-valued vector random process it is denoted $\{\boldsymbol{Q}(t), -\infty \le t \le \infty\}$. This is a vector of N potentially dependent scalar random processes. Because the forcing term is a vector random process, the responses it excites are vector random processes. Therefore, the absolute displacement response $\{\boldsymbol{X}(t), -\infty \le t \le \infty\}$, the absolute velocity response $\{\dot{\boldsymbol{X}}(t), -\infty \le t \le \infty\}$, and the absolute acceleration response $\{\ddot{\boldsymbol{X}}(t), -\infty \le t \le \infty\}$ are $N \times 1$, real-valued vector random processes whose probabilistic characteristics are related

to one another. Equation (1) governs the randomly excited response, as well, and can be written with $\boldsymbol{Q}(t)$ replacing $\boldsymbol{q}(t)$, $\boldsymbol{X}(t)$ replacing $\boldsymbol{x}(t)$, etc. One interpretation is that the equation holds for every realization of the excitation random process and its corresponding realizations of the response random processes.

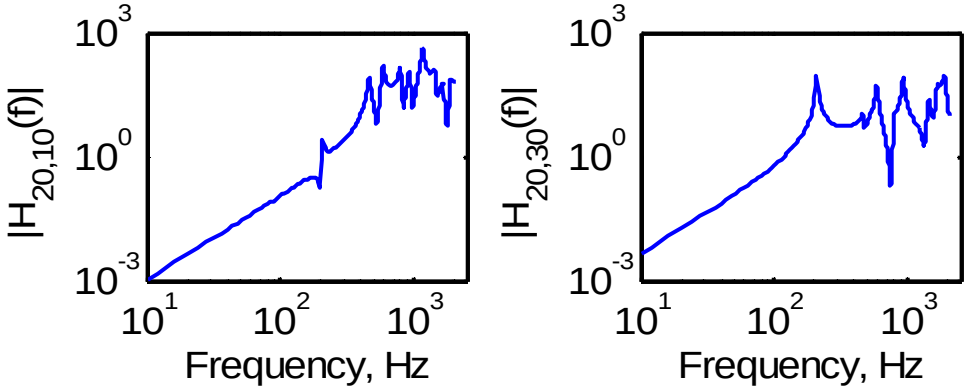


Figure 2. Complex moduli of some elements of the matrix FRF of the structure defined in Section 1.

Some fundamental characteristics of the random vibration excitation and response are contained in the lower order (first and second) moments. In fact, these completely define a Gaussian random process. (The following definitions and developments are provided in more detail in Wirsching, et al. [7].)

To simplify the development to follow, we start with the assumption that the excitation random process has zero mean. That is

$$E\left[\boldsymbol{Q}(t)\right] = \boldsymbol{0} \qquad -\infty < \boldsymbol{t} < \infty \tag{5}$$

where $\boldsymbol{0}$ is the $N \times 1$ vector of zeros. (If, in an application, this is not the case, then the derivation to follow can be easily adjusted to accommodate that fact.)

A general result from the theory of random vibrations of linear structures relates excitation to response spectral densities. (The autospectral density (ASD) of a stationary random process characterizes the distribution of mean square signal content of the random process in the frequency domain. The cross-spectral density (CSD) between a pair of stationary random processes characterizes, in the frequency domain, the degree of linear relation between, and the average phase between the random processes.) Let the spectral density matrix of the force excitation vector random process be constructed from scalar force ASDs and CSDs as follows:

$$\boldsymbol{S_{QQ}}(\omega) = \begin{bmatrix} S_{Q_1Q_1}(\omega) & S_{Q_1Q_2}(\omega) & \cdots & S_{Q_1Q_N}(\omega) \\ S_{Q_2Q_1}(\omega) & S_{Q_2Q_2}(\omega) & \cdots & S_{Q_2Q_N}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ S_{Q_NQ_1}(\omega) & S_{Q_NQ_2}(\omega) & \cdots & S_{Q_NQ_N}(\omega) \end{bmatrix} \qquad -\infty < \omega < \infty \tag{6}$$

$\boldsymbol{S_{QQ}}(\omega), -\infty < \omega < \infty$, has dimension $N \times N$. Its diagonal elements are real-valued, symmetric, non-negative functions. Matching pairs of functions on opposite sides of the diagonal are complex conjugate pairs. The diagonal terms are the two-sided, force excitation ASDs, and the off-diagonal terms are the two-sided CSDs between pairs of force excitation random processes. Let the spectral density matrix of the response random processes, $\left\{\ddot{\boldsymbol{X}}(t), 0 \le t \le T\right\}$, be defined similarly. Then a fundamental relation between excitation and response spectral densities of a linear system can be written

$$\boldsymbol{S_{\ddot{X}\ddot{X}}}(\omega) = \boldsymbol{H_{\ddot{X}Q}}(\omega)\boldsymbol{S_{QQ}}(\omega)\left[\boldsymbol{H^*_{\ddot{X}Q}}(\omega)\right]^T \qquad -\infty < \omega < \infty \tag{7}$$

where $\boldsymbol{H}_{\ddot{\boldsymbol{X}}\boldsymbol{Q}}(\omega), -\infty < \omega < \infty$, is the matrix of FRFs defined by Eq. (4). The relation specifies that when the spectral density of a vector excitation is known, and when the matrix FRF of a linear structure is known, the spectral density matrix of the vector response random process can be established.

3 Identification of Some Stationary Random Excitations

The problem is to identify then validate a model for stationary random vibration excitation, when (1) field excitations cannot be measured directly, (2) the excitations force a linear structure with known characteristics, and (3) measured responses are available. The measured responses will be treated as noise-free. In practical applications, only a finite number of measured response signals will be available, and, usually, that number will be relatively small. For example, often, only one, two, or a few response measurements will be available. In such circumstances only some of the equations in matrix Eq. (7) can be written. We consider the case of one and two available measured responses in Sections 3.1 and 3.2, respectively. It is shown that the properties of the input that can be identified increase as the number of available measurements increases, but the complexity of the identification procedure also increases.

3.1 One Measured Response Available

Consider, first, the problem where response at only one location is measured. Let the location of response measurement be DOF *i*. To develop the pertinent equation, we express only one of the elements in Eq. (7); it is

$$S_{\ddot{X}_i \ddot{X}_i}(\omega) = \boldsymbol{H}_{\ddot{X}_i \boldsymbol{Q}}(\omega) \boldsymbol{S}_{\boldsymbol{Q}\boldsymbol{Q}}(\omega) \left[\boldsymbol{H}^*_{\ddot{X}_i \boldsymbol{Q}}(\omega) \right]^T \qquad i = 1,....,N, -\infty < \omega < \infty \tag{8}$$

where $S_{\ddot{X}_i \ddot{X}_i}(\omega), i = 1,....,N, -\infty < \omega < \infty$, is the i^{th} function along the diagonal in the matrix $\boldsymbol{S}_{\boldsymbol{Q}\boldsymbol{Q}}(\omega), -\infty < \omega < \infty$, the ASD of acceleration response at DOF *i*, and $\boldsymbol{H}_{\ddot{X}_i \boldsymbol{Q}}(\omega), i = 1,....,N, -\infty < \omega < \infty$, is the row vector of FRFs for response at DOF *i* to excitations at all the DOFs.

The problem of excitation random process identification for the Gaussian vector random process reduces to the problem of estimating the elements of the matrix $\boldsymbol{S}_{\boldsymbol{Q}\boldsymbol{Q}}(\omega), -\infty < \omega < \infty$. The problem is usually framed in discrete time and space, and, of course, when the ASD of acceleration response is known only at one location, only limited characteristics of the excitation spectral density matrix can be identified.

The form of Eq. (8) in discrete frequency space is

$$G_{\ddot{X}_i \ddot{X}_i}(f_k) = \boldsymbol{H}_{\ddot{X}_i \boldsymbol{Q}}(f_k) \boldsymbol{G}_{\boldsymbol{Q}\boldsymbol{Q}}(f_k) \left[\boldsymbol{H}^*_{\ddot{X}_i \boldsymbol{Q}}(f_k) \right]^T \qquad i = 1,....,N, k = 0,...,n_f - 1 \tag{9}$$

where the *G*'s that replace the *S*'s in this equation refer to one-sided spectral densities instead of two-sided spectral densities, and the frequencies $f_k = k\Delta f, k = 0,...,n_f - 1$, are the frequencies that arise in the discrete Fourier transform (DFT) with Δf denoting the frequency increment of the analysis. Because of the difference in frequency dependence and the one-sidedness versus two-sidedness, the spectral densities are scaled differently. Still, Eqs. (8) and (9) are equivalent, alternate representations.

We consider the problem of identification of the parameters of an excitation random process in a parametric framework. The excitation is a zero-mean, Gaussian, vector random process with spectral density matrix that has the form

$$\boldsymbol{G}_{\boldsymbol{Q}\boldsymbol{Q}}(f_k) = G_0(f_k) \boldsymbol{\Lambda}(f_k) \qquad k = 0,...,n_f - 1 \tag{10}$$

where $G_0(f_k), k = 0,...,n_f - 1$, is the scalar ASD of the force excitations applied at all structural DOFs, and $\boldsymbol{\Lambda}(f_k)$ is a Hermetian matrix whose i^{th} row j^{th} column element is given by

$$\Lambda_{ij}(f_k) = exp\left(-\alpha\left|u_i - u_j\right| + \left(\sqrt{-1}\right)2\pi f_k \Delta u / v_0\right) \qquad k = 0,...,n_f - 1, i, j = 1,...,N \tag{10a}$$

The quantity α is a parameter of the force excitation random process that establishes how quickly the coherence between pairs of random processes in the force excitation vector decays. Physically, α is related to the reciprocal of the correlation distance of the input. For small/large α the correlation length of the input is long/short. The quantities $u_j = j\Delta u, j = 0,...,N-1$, denote spatial locations along the one dimension of the beam where excitations are applied, and Δu denotes the space between locations on the beam where the excitation is applied. The quantity v_0 is the velocity of propagation of force along the beam. The imaginary term in the exponential establishes the average phase change from one location to the next as the structure moves through an exciting medium at velocity v_0. In most applications, the parameters in the second term of the exponential are known, and the unknowns are the scalar spectral density $G_0(f_k), k = 0,...,n_f - 1$, at the DFT analysis frequencies, and the decay rate of coherence, α.

In this particular case, when the ASD of only one response acceleration is known, only one feature of the excitation spectral density model can be identified. That feature is normally taken to be $G_0(f_k), k = 0,...,n_f - 1$. It must be assumed that the parameter α is known. To see how the parameter identification is performed substitute Eq. (10) into Eq. (9).

$$G_{\ddot{X}_i \ddot{X}_i}(f_k) = G_0(f_k)\boldsymbol{H}_{\ddot{X}_i \boldsymbol{Q}}(f_k)\boldsymbol{\Lambda}(f_k)\left[\boldsymbol{H}^*_{\ddot{X}_i \boldsymbol{Q}}(f_k)\right]^T \qquad i = 1,....,N, k = 0,...,n_f - 1 \tag{11}$$

The spectral density $G_0(f_k), k = 0,...,n_f - 1$, is the only unknown quantity, and it can be identified. The quantity

$$\left|H_{i,eff}(f_k)\right|^2 = \boldsymbol{H}_{\ddot{X}_i \boldsymbol{Q}}(f_k)\boldsymbol{\Lambda}(f_k)\left[\boldsymbol{H}^*_{\ddot{X}_i \boldsymbol{Q}}(f_k)\right]^T \qquad i = 1,....,N, k = 0,...,n_f - 1 \tag{12}$$

on the right side in Eq. (11) might be referred to as the modulus squared of an effective FRF because it relates input ASD to response ASD.

The parameter identification described above is now performed for the system of Figure 1. The DOF where response is measured is number 20. The number of input DOF is $N = 20$; force excitations are applied at all the beam translational DOFs. Parameters $\alpha = 0.1 in^{-1}$, $\Delta u = 2.5 in$, and $v_0 = 5000 in/sec$ were used for all calculations. The maximum frequency of the analysis is $f_{max} = 2048 Hz$, and the analyses are performed with a frequency increment of $\Delta f = 4 Hz$. The spectral density of the force excitations used to generate the response, $G_0(f_k), k = 0,...,n_f - 1$, is shown in Figure 3. The modulus squared of the effective FRF, $\left|H_{20,eff}(f_k)\right|^2, k = 0,...,n_f - 1$, was computed using the FE model and is shown in Figure 4. The ASD of the acceleration response at DOF 20 was computed using the FE code and is shown by the blue curve in Figure 5.

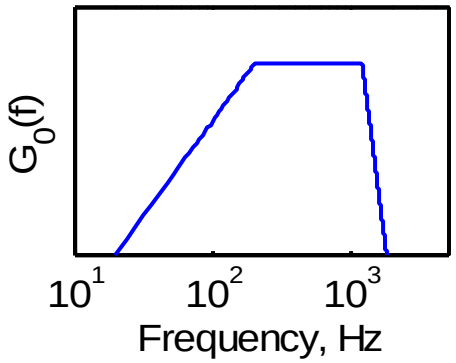


Figure 3. ASD (in units of lb^2/Hz) of the random force excitations that excite the beam in Figure 1.

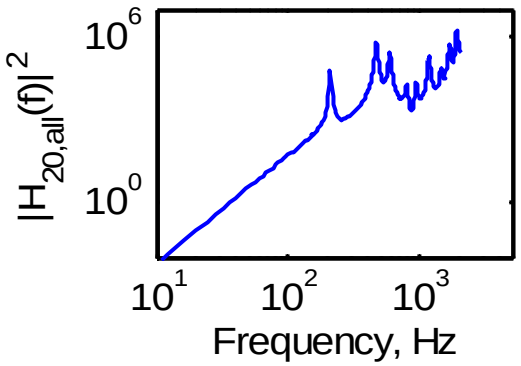


Figure 4. Effective FRF (in units of $\left(\left(in/sec^2\right)/lb\right)^2$) at DOF 20, of the system shown in Figure 1, due to force excitations applied at all translational DOF.

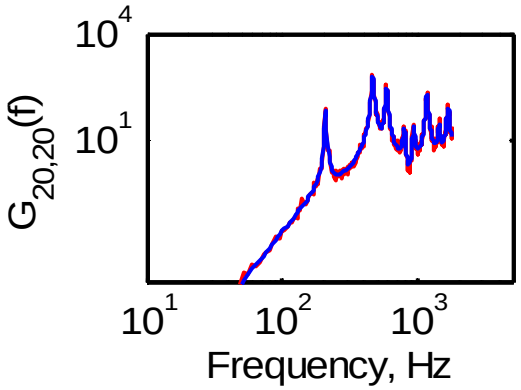


Figure 5. ASD (in units of $\left(in/sec^2\right)^2/Hz$) of acceleration response at DOF 20 excited by the force random process defined in Eqs. (10) and (10a). Noise-free (blue) and noisy (red).

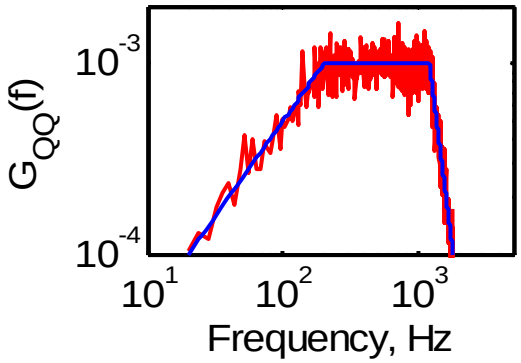


Figure 6. Estimated ASD (in units of lb^2/Hz) of the random force excitations that excite the beam in Figure 1 (red) and the theoretical force ASD (blue).

The computation of the response ASD using Eq. (11) is executed without the insertion of noise (except for round-off), therefore, if the blue curve in Figure 5 is divided by the curve in Figure 4, Eq. (11) is inverted and we recover the ASD of the force excitation in Figure 3. A problem would occur if there were one or more zeros in the modulus squared of the effective FRF, but there are none.

In fact, though we have assumed that the system is linear and the model is validated, we would never obtain a noise-free estimate of response ASD at DOF 20, because every ASD estimate must be based on finite measured data. The noise level depends on the length of data sample used to estimate the ASD and how the data are used. Autospectral density estimates are obtained by (1) dividing the measured data into M equal segments, (2) computing the DFT of each data segment, (3) computing the modulus squared of each DFT, (4) averaging the moduli squared of the DFTs, and (5) scaling the average to obtain units of (units of the signal)^2/Hz. (More details

are given in Wirsching, et al. [7].) Every ASD estimate obtained as specified is a realization of a random variable $(2M)G_{theor}(f)\chi^2_{2M}$, where $G_{theor}(f)$ is the theoretical ASD whose estimate is sought, and χ^2_{2M} is a chi squared distributed random variable with $(2M)$ statistical degrees of freedom. In view of this, noise-contaminated realizations of the estimated ASD can be created by generating realizations of χ^2_{2M} random variables. Assuming $M = 31$ averages, one such realization was generated, and it is shown (in red) along with the theoretical ASD in Figure 5. In the following ASDs and CSDs that include the influence of estimation error, or estimation noise, will be called spectral densities based on "measured" data.

When the noise-contaminated estimate of the acceleration response ASD at DOF 20 is divided by the modulus squared of the effective FRF at DOF 20, the noise-contaminated estimate of the excitation ASD is obtained. It is shown in Figure 6, and of course, includes the effects of noise.

If fewer averages, *M*, were used to obtain the ASD estimate of the acceleration response at DOF 20, then the estimated results would contain more noise. And if the model is imperfect then the estimated ASD of the force excitations may be much worse because of frequency mismatch problems between the model and the physical structure.

3.2 Two Output Signals Available

The second problem we consider is the one in which responses are measured at two locations. Suppose that responses are measured at DOFs *i* and *j*. Then two diagonal and two off-diagonal terms from Eq. (7) can be written; they are

$$\boldsymbol{G}_{\ddot{\boldsymbol{X}}_{(i,j)}\ddot{\boldsymbol{X}}_{(i,j)}}(f_k) = \boldsymbol{H}_{\ddot{\boldsymbol{X}}_{(i,j)}\boldsymbol{Q}}(f_k)\boldsymbol{G}_{\boldsymbol{QQ}}(f_k)\left[\boldsymbol{H}^*_{\ddot{\boldsymbol{X}}_{(i,j)}\boldsymbol{Q}}(f_k)\right]^T \qquad i \neq j = 1,...,N, k = 0,...,n_f - 1 \qquad (13)$$

where the equation has been written in the one-sided spectral density and discrete frequency form, and $\ddot{\boldsymbol{X}}_{(i,j)}$ refers to the matrix of two acceleration response random processes at the DOFs where response measurements were taken. $\boldsymbol{H}_{\ddot{\boldsymbol{X}}_{(i,j)}\boldsymbol{Q}}(f_k), i \neq j = 1,...,N, k = 0,...,n_f - 1$, is the matrix that includes the i^{th} and j^{th} rows from the complete FRF matrix.

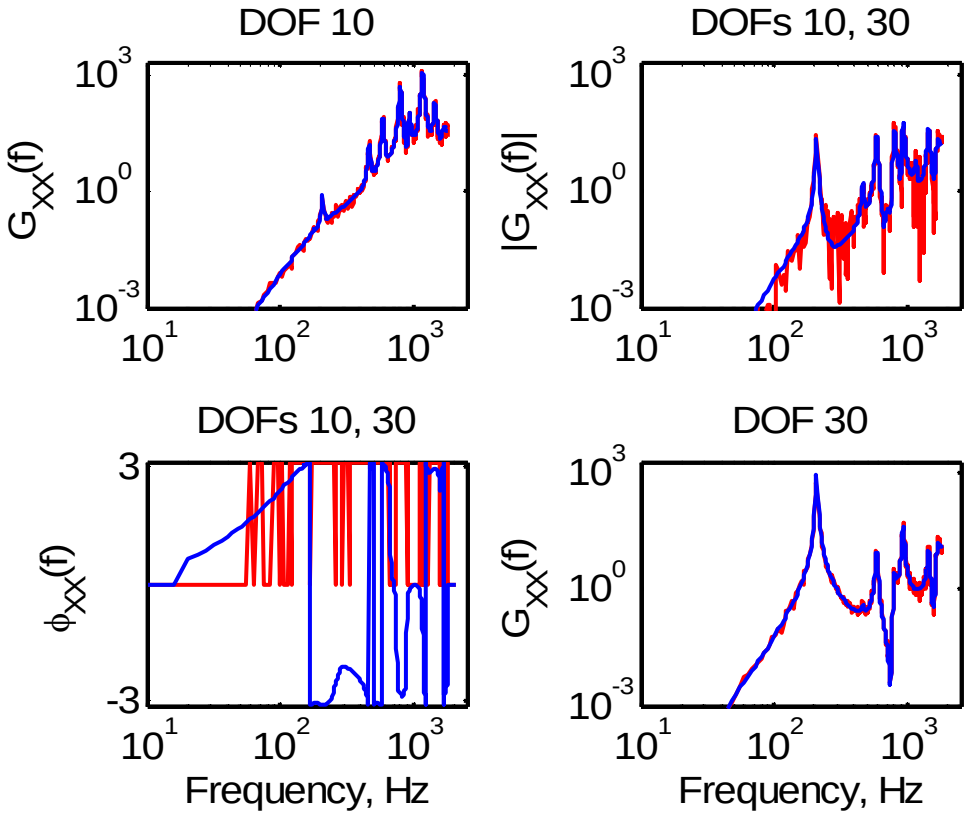
The vector force excitation random process is Gaussian with zero-mean, and with the spectral density matrix defined in Eqs. (10) and (10a). When Eq. (10) is used in Eq. (13) we obtain

$$\boldsymbol{G}_{\ddot{\boldsymbol{X}}_{(i,j)}\ddot{\boldsymbol{X}}_{(i,j)}}(f_k) = G_0(f_k)\boldsymbol{H}_{\ddot{\boldsymbol{X}}_{(i,j)}\boldsymbol{Q}}(f_k)\boldsymbol{\Lambda}(f_k)\left[\boldsymbol{H}^*_{\ddot{\boldsymbol{X}}_{(i,j)}\boldsymbol{Q}}(f_k)\right]^T \qquad i \neq j = 1,...,N, k = 0,...,n_f - 1 \qquad (14)$$

There is hope of identifying both the scalar spectral density $G_0(f_k), k = 0,...,n_f - 1$, and the decay parameter α, in this case, and an example showing how is presented. The example is similar to the previous one. The FE model used previously is used again here. The DOFs where responses are measured are numbers 10 and 30. (See Figure 1.) The number of input DOFs is $N = 20$; force excitations are applied at all the beam translational DOFs. As before, the parameters $\alpha = 0.1 in^{-1}$, $\Delta u = 2.5 in$, and $v_0 = 5000 in/sec$. The maximum frequency of the analysis is $f_{max} = 2048 Hz$, and the analyses are performed with a frequency increment of $\Delta f = 4 Hz$. The spectral density of the force excitations used to generate the response, $G_0(f_k), k = 0,...,n_f - 1$, is the same as before, and is shown in Figure 3. The spectral density matrix of the acceleration responses at DOFs 10 and 30 were computed using the FE code and are shown in Figures 7a through 7d in blue.

The argument made previously regarding estimation noise on the response spectral densities holds here, also. In addition to adding noise to the ASDs, though, we need to add noise to the CSD. It is shown in [7] that the real and imaginary parts of the estimated CSD have a variance that is approximately equal to the theoretical CSD divided

by $\left(2M\gamma^2_{theor}(f)\right)$ where M is the number of averages used to obtain the CSD estimate and $\gamma^2_{theor}(f)$ is the theoretical coherence between the random processes at frequency f. The errors were assumed to be Gaussian distributed, and realizations were generated for this example. The curves shown in red in Figures 7a through 7d are the noisy realizations of ASD and CSD estimates, assuming $M = 31$ averages. Clearly, noise affects the estimation of CSD much more than it affects estimation of ASD.



Figures 7a through 7d. ASD (in units of $\left(in/sec^2\right)^2/Hz$) of acceleration response at DOF 10 (top, left). Modulus of CSD (in units of $\left(in/sec^2\right)^2/Hz$) between acceleration responses at DOFs 10 and 30 (top, right). Phase of CSD (in units of radians) between acceleration responses at DOFs 10 and 30 (bottom, left). ASD (in units of $\left(in/sec^2\right)^2/Hz$) of acceleration response at DOF 30 (bottom, right). Noise-free (blue), noise-contaminated (red).

Because of the form of the matrix $\boldsymbol{\Lambda}(f_k)$ the identification of the function $G_0(f_k), k = 0,...,n_f - 1$, and the decay parameter α, must be accomplished through an optimization procedure. To perform the optimization we create an objective function that is a metric involving differences between the terms in the estimated response spectral density matrix and the terms in the response spectral density matrix computed with arbitrary $G_0(f_k)$ and α, at each frequency f_k. Denote the terms in the former matrix

$$\hat{\boldsymbol{G}}_{\ddot{\boldsymbol{X}}_{(10,30)}\ddot{\boldsymbol{X}}_{(10,30)}}(f_k) = \begin{bmatrix} \hat{G}_{\ddot{X}_{10}\ddot{X}_{10}}(f_k) & \hat{G}_{\ddot{X}_{10}\ddot{X}_{30}}(f_k) \\ \hat{G}_{\ddot{X}_{30}\ddot{X}_{10}}(f_k) & \hat{G}_{\ddot{X}_{30}\ddot{X}_{30}}(f_k) \end{bmatrix} \qquad k = 0,...,n_f - 1 \tag{15}$$

And denote the terms in the latter matrix

$$G_0(f_k)\boldsymbol{H}_{\ddot{\boldsymbol{X}}(10,30)\boldsymbol{Q}}(f_k)\boldsymbol{\Lambda}(f_k)\left[\boldsymbol{H}^*_{\ddot{\boldsymbol{X}}(10,30)\boldsymbol{Q}}(f_k)\right]^T = G_0(f_k)\begin{bmatrix}\Gamma_{11}(\alpha,f_k) & \Gamma_{12}(\alpha,f_k)\\ \Gamma_{21}(\alpha,f_k) & \Gamma_{22}(\alpha,f_k)\end{bmatrix} \qquad k=0,...,n_f-1 \qquad (16)$$

The objective function is defined

$$\varepsilon^2(G_0(f_k),\alpha) = \sum_{j=k-k_0}^{k+k_0} w_j \left\{ \left(\hat{G}_{X_{10}X_{10}}(f_j) - G_0(f_j)\Gamma_{11}(\alpha,f_j)\right)^2 + \left(\hat{G}_{X_{20}X_{20}}(f_j) - G_0(f_j)\Gamma_{22}(\alpha,f_j)\right)^2 \right.$$
$$\left. + \left(Re\left[\hat{G}_{X_{10}X_{20}}(f_j)\right] - G_0(f_j)Re\left[\Gamma_{12}(\alpha,f_j)\right]\right)^2 + \left(Im\left[\hat{G}_{X_{10}X_{20}}(f_j)\right] - G_0(f_j)Im\left[\Gamma_{12}(\alpha,f_j)\right]\right)^2 \right\}$$
$$k=0,...,n_f-1 \qquad (17)$$

where $w_j, j=k-k_0,...,k+k_0$ is a weighting function. That is, the objective function is the weighted sum of squared differences between terms in the estimated spectral density and the modeled spectral density. At each frequency line, f_k, the objective function is minimized to establish optimal estimates for the spectral density and decay rate. A quadratic search was used to establish the estimates.

When the optimization is performed as described above, we obtain best-fit estimates of the forcing random process parameters $G_0(f_k)$ and α, one frequency at a time. The results are contaminated with noise because the response ASDs and CSDs at DOFs 10 and 30 are contaminated with noise (red curves in Figures 7a through 7d). When we know that noise is present we would normally try to alleviate its effects, for example, by smoothing (filtering) the ASDs and the real and imaginary part of the CSDs in Figure 7. But to show the effects of noise we did not follow that course, here. Rather, we smoothed the estimates of $G_0(f_k)$ and α, following their estimation. The optimization was performed using $k_0 = 3$ and a haversine window in Eq. (17). The smoothing was accomplished by low-pass filtering the parameter estimates with a digital filter at five percent of the Nyquist frequency. The approximations so obtained are shown in Figures 8 and 9.

Both parameter estimates are rather poor up to about 150 Hz, and better beyond that frequency, but this would not be known in the realistic case where the actual excitation is unknown. The reason for the imprecision of the estimate is that the amplification of the excitation in the response is low at low frequencies, and this feature amplifies the estimation errors. If fewer averages, *M*, were used to obtain the estimate of the acceleration response spectral density matrix at DOFs 10 and 30, then the estimated results would contain more noise. And if the model is imperfect then the estimated ASD of the force excitations may be much worse because of frequency mismatch problems.

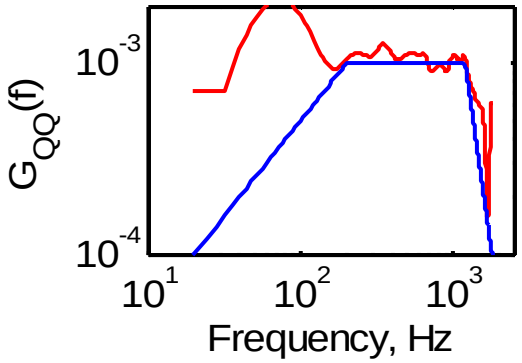


Figure 8. Estimated spectral density (in units of lb^2 / Hz) of the force excitation random process (red) and the spectral density of the random process that served as the actual excitation (blue).

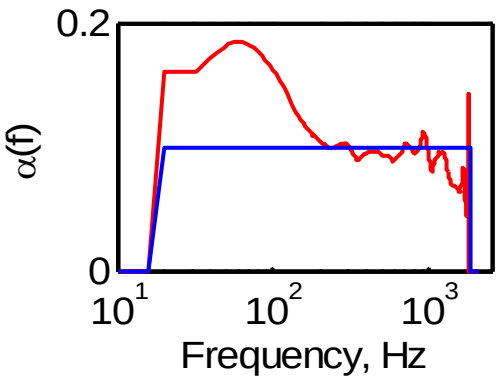
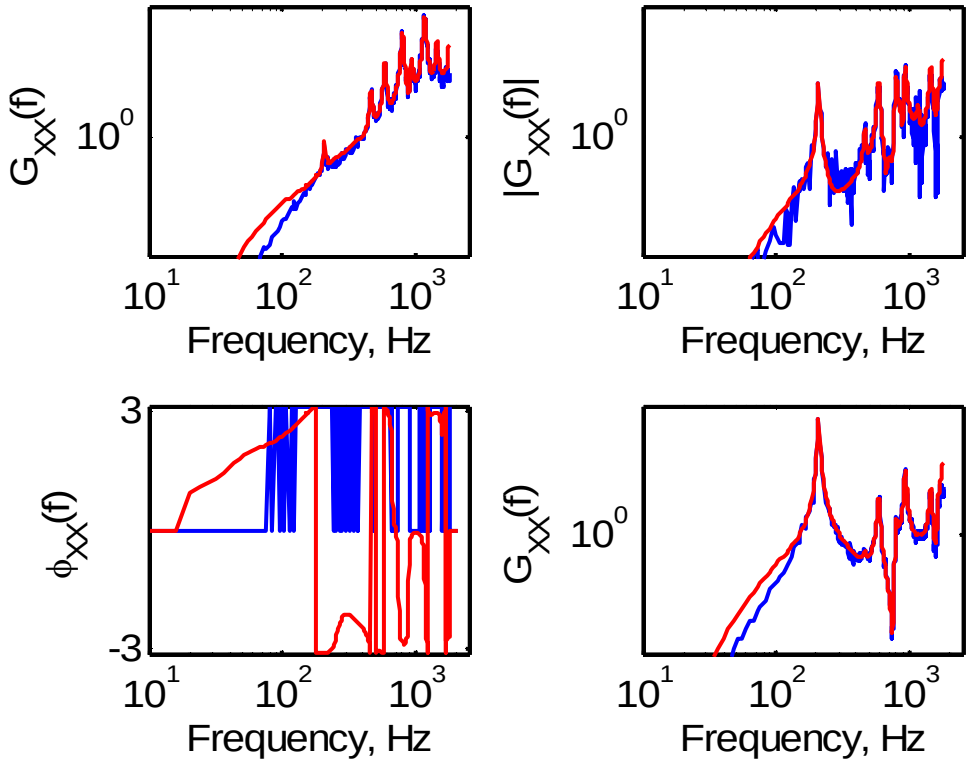


Figure 9. Estimated decay rates $\hat{\alpha}_k$ (in units of in^{-1}), of the off-diagonal terms in the excitation spectral density matrix (red), and the value of α in the actual excitation random process (blue).

The spectral density matrix of structural responses at DOFs 10 and 30 excited by the force random process with parameters in Figures 8 and 9 can be computed using Eq. (14). That was done and the results are shown in Figure 10, along with the noise-contaminated results used to identify the parameters of the force excitation (red curves in Figures 7a through 7d).



Figures 10a through 10d. ASD (in units of $\left(in/sec^2\right)^2/Hz$) of acceleration response at DOF10 (top, left). Modulus of CSD (in units of $\left(in/sec^2\right)^2/Hz$) between acceleration responses at DOFs 10 and 30 (top, right). Phase of CSD (in units of radians) between acceleration responses at DOFs 10 and 30 (bottom, left). ASD (in units of $\left(in/sec^2\right)^2/Hz$) of acceleration response at DOF 30 (bottom, right). Estimates including estimation noise – red in Figures 7 (blue), estimates based on identified force characteristics of Figures 8 and 9 (red).

For the same reason that the estimated characteristics of the forcing random process are rather poorly estimated in Figures 8 and 9, the inaccuracy of their estimates has relatively little influence on the model-predicted response

spectral densities. Except for the phase between the responses at DOFs 10 and 30, the response spectral densities are quite well-estimated by the model.

4 Model Validation of Stationary Random Excitation

Current practice in model validation of structural dynamics is to assess the validity of mathematical models of structures, and not the validity of stochastic excitations. Yet, as mentioned in the Introduction, no prediction of structural response can be accurate as long as the excitation model is inaccurate. Further, our confidence in these predictions cannot be quantified unless models for both the structure and the environment have been validated. When structural excitations cannot be measured, then the validation of the mathematical model of a stochastic excitation must be performed via comparison of statistical measures of structural response obtained from model predictions and from experiments. We begin this section with a summary of the elements that any scheme for the validation of stochastic environment models must include, then apply these elements to the validation of stochastic model considered above.

Figure 11 is a flow chart of some elements that any scheme for validation of stochastic excitation must include.

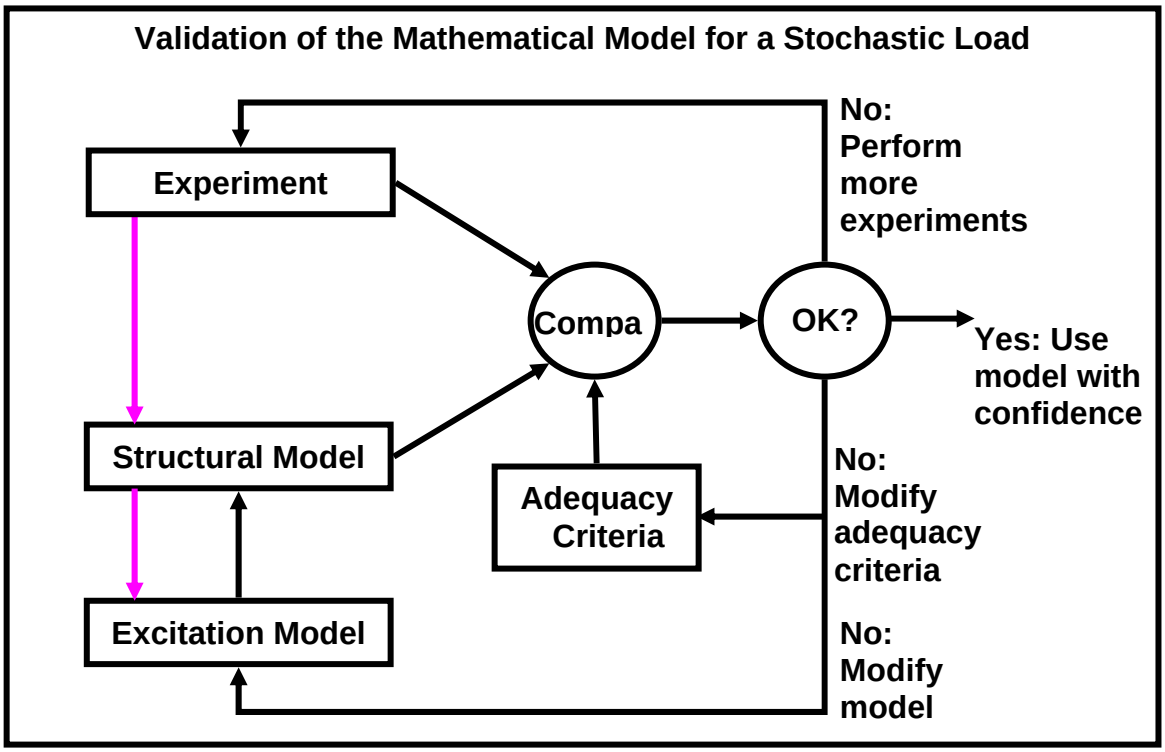


Figure 11. Flowchart for validation of stochastic excitation models.

Below is a possible sequence of activities for performing a validation assessment on the mathematical model for a stochastic load.

1. Prior to experimentation and modeling
    a. Specify the structural response measures of interest – the quantities to be used in the comparison circle of the flowchart of Figure 11. For the current example we use spectral densities and measures related to them.
    b. Specify a validation metric – the precise means for comparing response measures in the comparison circle.
    c. Specify the domain of comparison – the region of environment space within which experiment responses will be measured and excitation model predictions will be made. For the current example this might be random excitations with ASDs in the neighborhood of $10^{-3}\,(lb)^2 / Hz$.
    d. Specify the calibration experiments – the experiments to be performed and used with the structural model and the experiment model to identify excitation stochastic model parameters. For

the current example, the parameters identified are elements of the spectral density matrix of Gaussian excitation. The calibration experiments are those described in Section 3.

e. Specify an adequacy criteria – the values that validation metrics must assume in order for stochastic excitation model to be judged valid.

2. After above definitions have been established
    a. Perform calibration experiments defined in 1d. The calibration experiments might be performed in the laboratory. For the current example the calibration experiments are performed numerically, but the usual procedure requires parameters to be obtained from physical experiments.
    b. Use calibration data to calibrate (identify) parameters of stochastic excitation model. The examples provided in the previous section are examples of how calibration can be performed. The magenta arrows in the flowchart of Figure 11 refer to the use of experimental data, via the structural model, to perform parameter calibration.
3. Generate validation data - experimental
    a. Perform physical validation experiments
    b. Measure system responses to random environments
    c. Compute measures of experimental structural responses specified in 1a
4. Generate validation data – model predictions
    a. Generate random excitations using calibrated model.
    b. Compute structural responses. The black arrow connecting the “Excitation Model” box to the “Structure Model” box refers to this operation.
    c. Compute measure of model-predicted structural response specified in 1a.
5. Validation comparison
    a. Compute validation metrics using experimental and model-predicted response measures. This is part of the operation referred to as “Compare” in the Figure 11 flowchart.
    b. Judge stochastic excitation model adequacy relative to adequacy criteria. This is another part of the operation referred to as “Compare” in the Figure 11 flowchart.
6. Subsequent actions
    a. If stochastic excitation model is valid it can be used within the domain of comparison specified in 1c.
    b. If stochastic excitation model is not valid perform more (perhaps, different) experiments, or
    c. If stochastic excitation model is not valid modify adequacy criteria, or
    d. If stochastic excitation model is not valid modify model.

Any validation comparison performed using the guidelines specified here clearly leaves much to the judgment of (1) analysts, (2) experimentalists, and (3) those performing the formal validation comparison. The three activities should ideally be pursued by independent groups or individuals. The requirements, mostly specified in step 1 above, should be planned and agreed to by all parties. Importantly, validation experiment results and validation model predictions should not be shared between experimentalists and analysts prior to the formal validation comparison so as to avoid the appearance of collusion and the actual temptation to tune the excitation model.

The second part of the example, started above, is continued. Section 3 shows how the force excitation model can be calibrated via excitation identification. We now seek to validate the identified excitation random process model. There are scores of structural response measures that can be used to compare the spectral characteristics of the realized and the model-estimated loads. For the sake of the example, we compare the probability distributions of the peak response random variables for response at DOF 44 (i.e., the top mass on the three DOF substructure of Figure 1) for the measured response and the model-predicted response. The validation requirement is that the mean value and standard deviation of the peak displacement response predicted by the model, differ from the corresponding moments computed using the measured data by ten percent or less. These are the model adequacy criteria, and the validation metrics are the ratios of model-predicted to experimentally obtained means and standard deviations. The probability density function (PDF) for peak displacement response, Z, of a wide-band Gaussian random process is [7]

$$f_Z(z) = \left(1-\beta^2\right)\frac{1}{\sqrt{2\pi\left(1-\beta^2\right)}\sigma_X} exp\left(-\frac{1}{2}\frac{z^2}{\left(1-\beta^2\right)\sigma_X^2}\right) + \beta\Phi\left(\frac{\beta}{\left(1-\beta^2\right)}\frac{z}{\sigma_X}\right)\frac{z}{\sigma_X^2} exp\left(-\frac{1}{2}\frac{z^2}{\sigma_X^2}\right) \qquad -\infty < z < \infty \tag{18}$$

where β is the irregularity factor of a wide-band random process, defined

$$\beta = \frac{\sigma_{\dot{X}}^2}{\sigma_X \sigma_{\ddot{X}}} \tag{18a}$$

with $\sigma_X, \sigma_{\dot{X}}, \sigma_{\ddot{X}}$, the standard deviations of the random displacement, velocity, and acceleration responses. The standard deviations are defined

$$\sigma_X = \left[\int_0^\infty G_{XX}(f)df\right]^{1/2} \quad \sigma_{\dot{X}} = \left[(2\pi)^2\int_0^\infty f^2 G_{XX}(f)df\right]^{1/2} \quad \sigma_{\ddot{X}} = \left[(2\pi)^4\int_0^\infty f^4 G_{XX}(f)df\right]^{1/2} \tag{18b}$$

where $G_{XX}(f), 0 \le f < \infty$, is the one-sided spectral density of the displacement response random process. Equation (18) expresses the PDF of all peaks in the random process, not just the highest one. All these quantities were computed from (1) the directly estimated spectral density of the measured response at DOF 44, and (2) the response spectral density predicted by the FE model at DOF 44 excited by the force random process identified in Section 3.2 – Two Output Signals Available. The former results are

$$\sigma_X = 6.675\times10^{-5}\,in \quad \sigma_{\dot{X}} = 8.273\times10^{-2}\,in/sec \quad \sigma_{\ddot{X}} = 254.6\,in/sec^2 \quad \beta = 0.4739 \tag{19a}$$

The latter results are

$$\sigma_X = 5.938\times10^{-5}\,in \quad \sigma_{\dot{X}} = 8.565\times10^{-2}\,in/sec \quad \sigma_{\ddot{X}} = 281.2\,in/sec^2 \quad \beta = 0.4394 \tag{19b}$$

The differences are relatively small, and the PDFs generated using the results in Eqs. (18a) and (18b) are shown in Figure 12. The PDFs are practically indistinguishable.

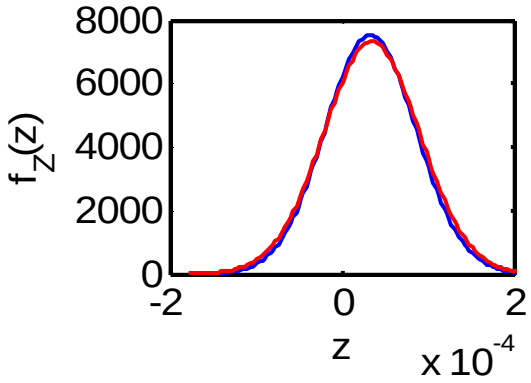


Figure 12. PDFs (in units of in^{-1}) of peak values in the displacement response random process at DOF 44 on the structure of Figure 1. Based on measured data (blue) and model prediction (red).

The mean values and standard deviations of the peak response random variables were computed from the PDFs in Figure 12. The results based on the response spectral density obtained from the measured signals are

$$\mu_Z = 3.369 \times 10^{-5}\ in \quad \sigma_Z = 5.294 \times 10^{-5}\ in \tag{20a}$$

The results based on the response spectral density obtained from model predictions are

$$\mu_Z = 3.525 \times 10^{-5}\ in \quad \sigma_Z = 5.695 \times 10^{-5}\ in \tag{20b}$$

The latter moments in Eq. (20b) differ from the former ones in Eq. (20a) by less than ten percent, therefore, the force excitation model is judged valid.

Of course, more stringent validation adequacy criteria than those used above could have been specified, and the excitation model would not have been validated. However, judging from Figures 10a through 10d and Figure 12 the force excitation model is relatively accurate. So the conclusion reached here is probably reasonable. For the present illustration the validation adequacy criteria could only be specified late in the analysis, but they should be specified in step 1 of the validation procedure.

Discussion and Conclusions

This paper summarizes techniques for (1) estimation of the model for the spectral density matrix of a force excitation random process, and (2) validation of the model. When the excitation is stationary and jointly Gaussian distributed the spectral density matrix is sufficient to completely specify the random excitation. The developments were framed in terms of a particular parametric model for the spectral density of the excitation, and in terms of a specific physical system. It was assumed that the structure under consideration is linear, with known parameters, and validated. It was shown, at least for the structure and form of excitation considered, that it is possible to identify the parameters of a force excitation random process with sufficient accuracy to validate the load model in terms of the spectral density of responses the load excites. Though the effects of estimation error (or noise) were realistically included in the analysis, it was also assumed, implicitly, that the form of the spectral density matrix of the excitation was known, and this latter assumption may not always be realistic. The process of force excitation random process identification and validation needs to be considered in the case where the actual input spectral density matrix varies in ways other than that considered here, and when the form assumed for the input spectral density matrix is not known exactly. The problems of environment model identification and validation need to be extended to nonlinear structures subjected to non-Gaussian excitations.

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