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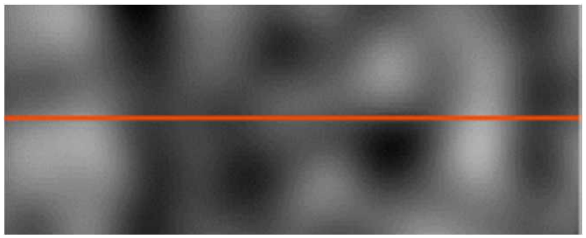

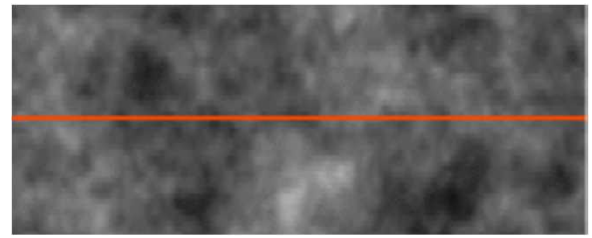

Front Propagation in Random Media: An Application of Burgers Turbulence and Directed Polymers

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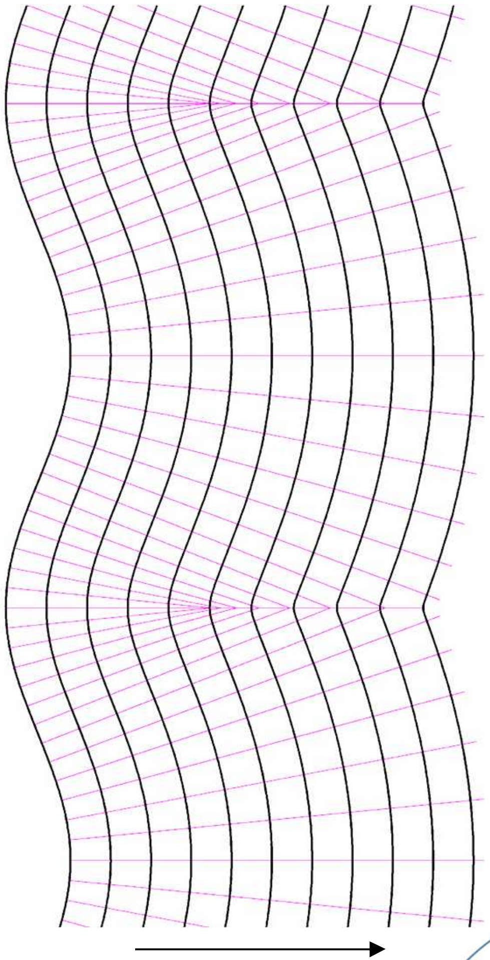
Huygens' principle idealizes the physics of front propagation

- Many phenomena (light, sound, combustion) spread at a characteristic speed
- A "front" comprises points to which the fastest path (first passage) takes time t
- The leading paths are "rays" perpendicular to the front
- Concave regions shrink to "cusps" that consume rays and flatten the front
- Random variations in local speed wrinkle the front and increase its surface area, resulting in faster propagation



Weak perturbations can be rescaled into white noise, making widely studied models relevant

- In a medium with local speed $v(\mathbf{x})$, a path's travel time $t(C) = \int_C ds/v(\mathbf{x})$ looks like the energy of a stretched string in a random potential
- Huygens propagation gives the absolute minimum energy (zero temperature); the finite-temperature free energy corresponds to smoothed cusps
- Weak randomness is described by the rms value $\epsilon \ll 1$ and normalized spectrum $D(k)$ of relative fluctuations in $v(\mathbf{x})$
- Front evolution occurs over a large distance $\propto \epsilon^{-2/3}$; a longitudinal rescaling transforms the medium into white noise with transverse spectrum $D(k)$, and extracts $\epsilon^{4/3}$ scaling of the speedup (which can be interpreted heuristically)
- The string becomes the standard directed polymer with longitudinally white potential, equivalent to the Burgers equation stirred by a white-in-time force
- The correspondence elevates these "toy models" and motivates detailed calculations of their nonuniversal properties

The replica method reduces the problem to the variational quantum mechanics of zero particles

- The front speedup is $\Delta\epsilon^{4/3}$, where Δ is minus the directed polymer's free energy per unit length (equal to the energy density of the Burgers fluid)
- The free energy is proportional to the logarithm of the partition function Z and can be averaged over the noise using $\ln Z = \lim_{n\to 0}(Z^n - 1)/n$
- $\langle Z^n \rangle$ is the partition function for n interacting polymers, and also the imaginary-time Feynman path integral for an n-particle nonrelativistic quantum Hamiltonian $\mathcal{H}_n$ with a pair potential
- If the quantum ground-state energy $E_g(n)$ can be analytically continued in n, then $\Delta = -\lim_{n\to 0} E_g(n)/n$
- A variational bound on $E_g(n)/n$ is found using Gaussian wave functions ψ with hierarchical symmetry breaking, described by a function $z(u)$ on $[0,1]$
- Stationarity of $\langle \psi | \mathcal{H}_n | \psi \rangle$ yields a solution for $z(u)$, generalizing Blum (1994), and an explicit expression for the bound on Δ at zero temperature

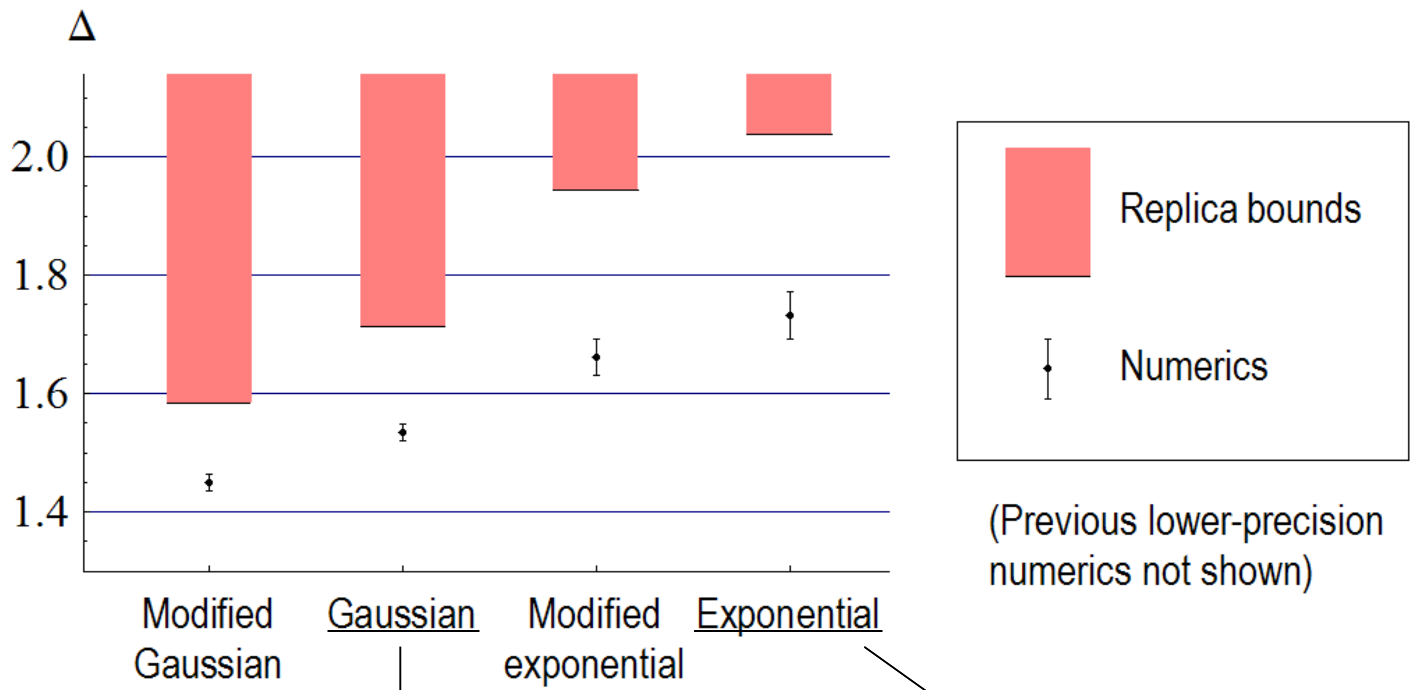
An explicit formula allows calculation of replica bounds and reveals their underlying structure

- For certain media, including those in $d = 2$ with "Gaussian" $\exp(-r^2)$ and "exponential" $\exp(-r)$ correlation functions, the bound is

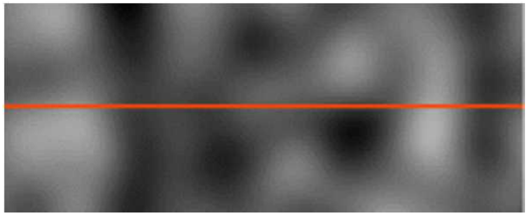
$$\Delta \leq \int_0 \frac{dz}{z} P\,[illegible]^{2/3} \quad P\,[illegible]\;\frac{d\,[illegible]}{2\pi\,[illegible]}\; k\; z^{\,[illegible]\,2}\,[illegible]\; -\tfrac{1}{2}k^2 z \,[illegible]$$

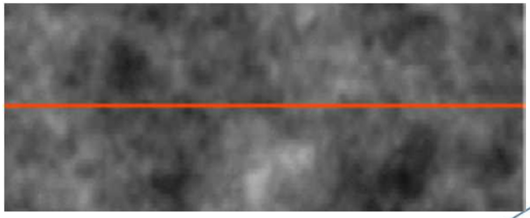
- Other media require specific modifications to the bound formula
- $P(z)$ is the power in an order-unity spectral band around $k \sim z^{-1/2}$
- The front propagation speed is successively renormalized by each finite band's effective $\epsilon^{4/3}$, reflecting a coarse-grained front in a marginally steady state
- Broader spectra have larger Δ because the power is spread over more bands and the $2/3$ exponent gives small power a disproportionate effect
- Previous speed-renormalization calculations obtained misleading results using infinitesimal bands

Numerical simulations confirm that 2D replica bounds are valid and reasonably sharp



(Previous lower-precision numerics not shown)





- The agreement raises confidence in replica predictions for 3D

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Conclusion: Weak-perturbation first passage is now well understood theoretically

- A weakly perturbed Huygens front reduces to an inviscid Burgers fluid driven by white noise, or to the zero-temperature limit of a directed polymer
- In the process, the $\epsilon^{4/3}$ scaling of the front speedup is extracted
- The prefactor of $\epsilon^{4/3}$ is analytically bounded above using a replica analysis that is equivalent to bounding the energy density of Burgers turbulence and the binding energy of the directed polymer
- Replica results for 2D propagation match within $\sim$ 15% the speedup values obtained numerically (new evidence that replica bounds are valid and usefully sharp)
- The success of the replica method implies applications to weakly random optics and acoustics, as well as weakly turbulent combustion
- Finite-band renormalization may lead to improved turbulent-combustion models beyond weak perturbations