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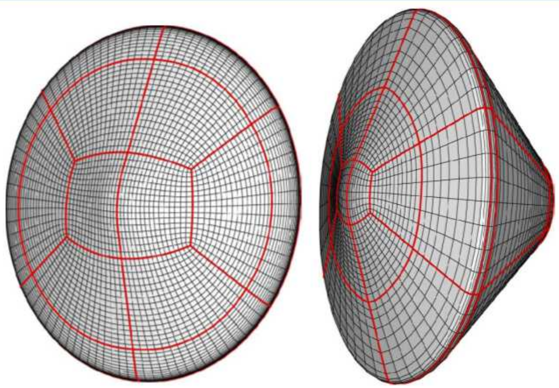

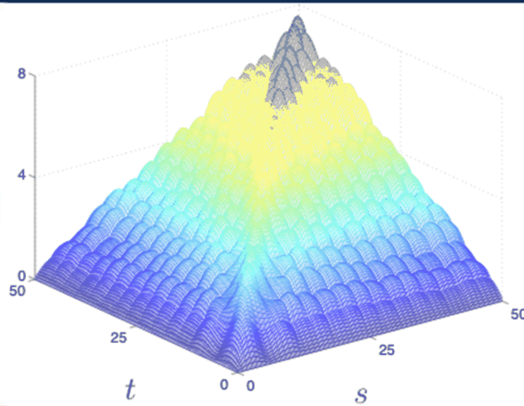

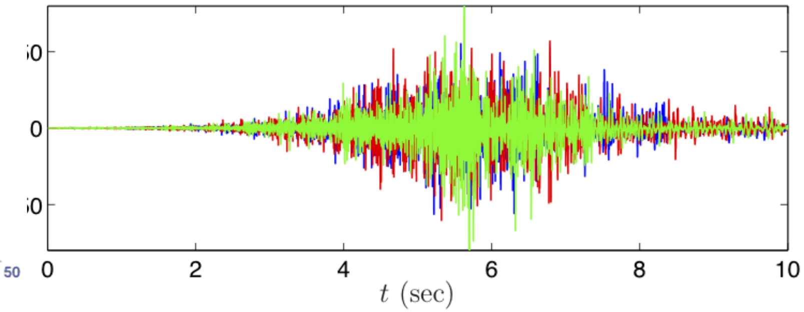

On-the-Fly Generation of Samples of non-Stationary Gaussian Processes

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NNSA National Nuclear Security Administration

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Introduction & Motivation

- Objective: Develop new method for generating samples of non-stationary Gaussian processes
- Target application: Random vibration due to long-time transient excitations
  - Transportation environments, e.g., spacecraft atmospheric entry
- Some popular methods for generating samples of non-stationary Gaussian processes include
  - Oscillatory processes
  - Cholesky decompositions
  - Karhunen-Loève representations
  - Fourier series representations with Gaussian coefficients
  - Generalized version of the spectral representation theorem
  - Filtered Gaussian processes

Introduction & Motivation (cont.)

- However, these approaches typically require that we
  - Calculate entire load history
  - Store to file (can be huge for problems of interest)
  - Input into FEA
- Proposed approach: "On-the-fly" sample generation
  - Given value of the load at time t, compute load at time $t + \Delta t$
  - Based on (1) Shannon's sampling theorem; and (2) Conditional Gaussian random variables
  - Was developed for stationary processes*; herein we provide an extension to non-stationary processes
  - Efficient square-root-like decomposition of a sequence of covariance matrices (not covered)

*M. Grigoriu. Simulation of stationary processes via a sampling theorem. Journal of Sound and Vibration, 166(2):301–313, 1993.

Outline

- Shannon sampling theorem
  - Review for deterministic functions
  - Discuss truncation and aliasing errors
- Sampling theorem for Gaussian processes
  - Definition
  - Properties
  - Focus is on non-stationary processes
- Monte Carlo simulation
  - Algorithm
- Applications
  - Two academic examples
  - Engineering example: Spacecraft planetary entry

Shannon Sampling Theorem

Let $x(t)$ be a real-valued function with frequency content contained in $(-\nu_c, \nu_c)$, $0 < \nu_c < \infty$. The sequence of approximations

$$x_n(t) = \sum_{|k| \leq n} x(k\,t_c)\,\alpha(t - k\,t_c), \quad t \in \mathbb{R}, \quad n = 1, 2, \ldots$$

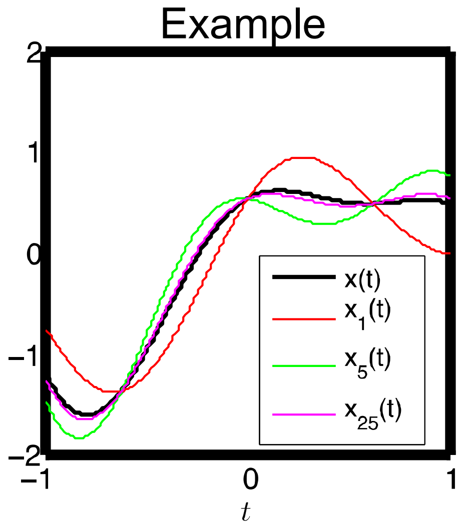
where $\alpha(t) = \dfrac{\sin(\nu_c\,t)}{\nu_c\,t}$, $t_c = \dfrac{\pi}{\nu_c}$

converges to $x(t)$ as $n \to \infty$ at each $t \in \mathbb{R}$.

- Truncation error

$$|x(t) - x_n(t)| \leq \frac{|\sin(\nu_c\,t)|}{\pi\,\nu_c} \left[\left(\frac{1}{n\,t_c + t} \sum_{k=-\infty}^{-(n+1)} x(k\,t_c)^2 \right)^{1/2} + \left(\frac{1}{n\,t_c - t} \sum_{k=n+1}^{\infty} x(k\,t_c)^2 \right)^{1/2} \right]$$

  - Decreases with increasing *n*



Shannon Sampling Theorem (cont.)

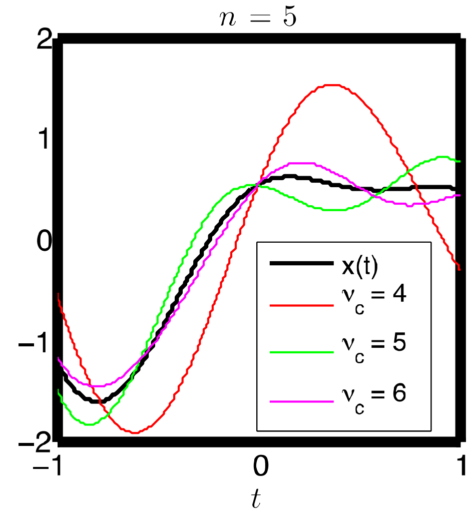
- Aliasing error

  Let $\bar{\nu} < \nu_c$ be the sampling frequency, and let $\bar{x}_n(t)$ be $x_n(t)$ with $\bar{t} = \pi/\bar{\nu}$ in place of t_c. Aliasing occurs because $\bar{x}_\infty(t) = \lim_{n\to\infty} \bar{x}_n(t)$ does not coincide with $x(t)$. Let $x_F(\nu)$ be the Fourier transform of $x(t)$, then

$$|x(t) - x_\infty(t)| \le \frac{1}{\pi} \int_{|\nu|>\bar{\nu}} |x_F(\nu)|\, \mathrm{d}\nu$$

  - Decreases with increasing sampling frequency $\bar{\nu}$

Example



Application to Gaussian Processes

- Let $X(t), -\infty < t < \infty$ be a Gaussian process with zero mean
  - Global approximation for $X(t)$

$$X_n(t) = \sum_{|k| \leq n} X(k\,t_c)\,\alpha(t - k\,t_c), \quad t \in \mathbb{R}, \quad n = 1, 2, \ldots,$$

  - Local approximation for $X(t)$

$$\tilde{X}_n(t) = \sum_{k=n_t-n}^{n_t+n+1} X(k\,t_c)\,\alpha(t - k\,t_c), \quad t \in [n_t\,t_c, (n_t+1)\,t_c], \quad n_t = \lfloor t/t_c \rfloor$$

  - Properties of the approximations

1. $\lim_{n\to\infty} \mathrm{E}[(X(t) - X_n(t))^2] = 0$ at any time t
2. X_n converges almost surely (a.s.) to X as $n \to \infty$
3. X_n becomes a version of X as n increases

** These properties also hold for $\tilde{X}_n$ **

- Bounds on truncation and aliasing errors similar to those for deterministic functions can be constructed

Focus on non-Stationary Processes

- Let $X(t), -\infty < t < \infty$, be a Gaussian process with zero mean, covariance function $c(s,t) = \mathrm{E}[X(s)\,X(t)]$, and generalized spectral density

$$s(\nu,\eta) = \frac{1}{(2\,\pi)^2} \int_{\mathbb{R}^2} c(s,t)\, e^{-i\,(\nu\, s - \eta\, t)}\, \mathrm{d}s\, \mathrm{d}t$$

  - $s(\nu,\eta)$ is complex-valued even for real-valued processes and satisfies $s(\nu,\eta)^* = s(\eta,\nu)$, $\forall \nu, \eta \in \mathbb{R}$
  - We say X is "bandlimited if $s(\nu,\eta) = 0$, $(\nu,\eta) \in D^c$, where $D = [-\nu_c, \nu_c] \times [-\nu_c, \nu_c]$ and $0 < \nu_c < \infty$ is a constant
  - If X is a weakly stationary process with zero mean and spectral density s_0, its generalized spectral density is $s(\nu,\eta) = s_0\left(\frac{\nu+\eta}{2}\right)\,\delta(\nu-\eta)$

Some Properties

- Bandlimited non-stationary processes

The m.s. difference between $X(t)$ and its approximation $X_n(t)$ is such that

$$\lim_{n\to\infty} \mathrm{E}\left[\left(X(t) - X_n(t)\right)^2\right] = 0$$

- Non-bandlimited non-stationary processes

If $s(\nu,\eta)$ is square integrable, that is, if $\int_{\mathbb{R}^2} |s(\nu,\eta)|^2 \, \mathrm{d}\nu \, \mathrm{d}\eta < \infty$, then

$$\lim_{n\to\infty} \mathrm{E}\left[(X(t) - X_n(t))^2\right] \leq 4 \int_{D^c} |s(\nu,\eta)|^2 \, \mathrm{d}\nu \, \mathrm{d}\eta$$

- These properties demonstrate that the Sampling Theorem can be used to approximate non-stationary Gaussian processes

** These properties also hold for $\tilde{X}_n$ **

Monte Carlo Simulation

- Three-step procedure to produce samples of $\tilde{X}_n(t)$, the local approximation for $X(t)$, a non-stationary Gaussian process with zero mean, on $[0, \tau]$

1. Select cutoff frequency ν_c and half window-width n that defines $\{X(k\, t_c)\},\ k = n_t - n, \ldots, n_t + n + 1$
2. Generate independent samples of $\mathbb{R}^{2\,(n+1)}$-valued Gaussian random variable $\{X(k\, t_c)\},\ k = 1, \ldots, 2\,(n+1)$ using classical algorithms; use them to calculate samples of $\tilde{X}_n(t)$ in cells $[0, t_c], \ldots, [(n+1)\, t_c, (n+2)\, t_c]$
3. Extend the samples in the time interval $[0, (n+2)\, t_c]$ to the subsequent time interval by using properties of conditional Gaussian variables

Repeat step 3 until moving window contains endtime τ

Applications

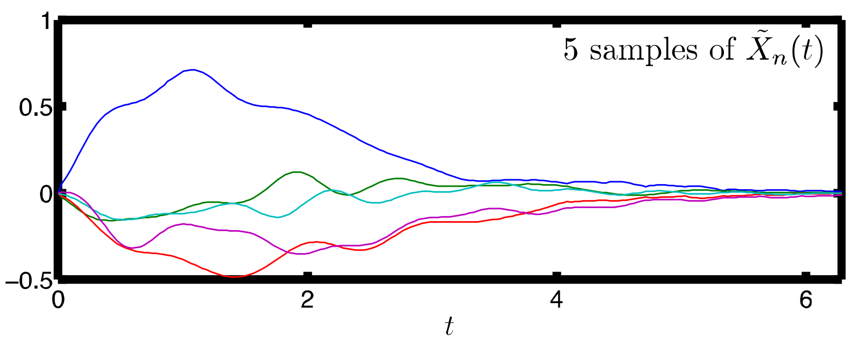
- Example 1: Uniformly modulated process

Let $Y(t)$, $t \in \mathbb{R}$, be a stationary Gaussian process with zero mean and covariance function $c_Y(\tau) = \mathrm{E}[Y(t+\tau)\,Y(t)] = \exp(-\lambda\,|\tau|)$, $\lambda > 0$, $\tau \in \mathbb{R}$. Then

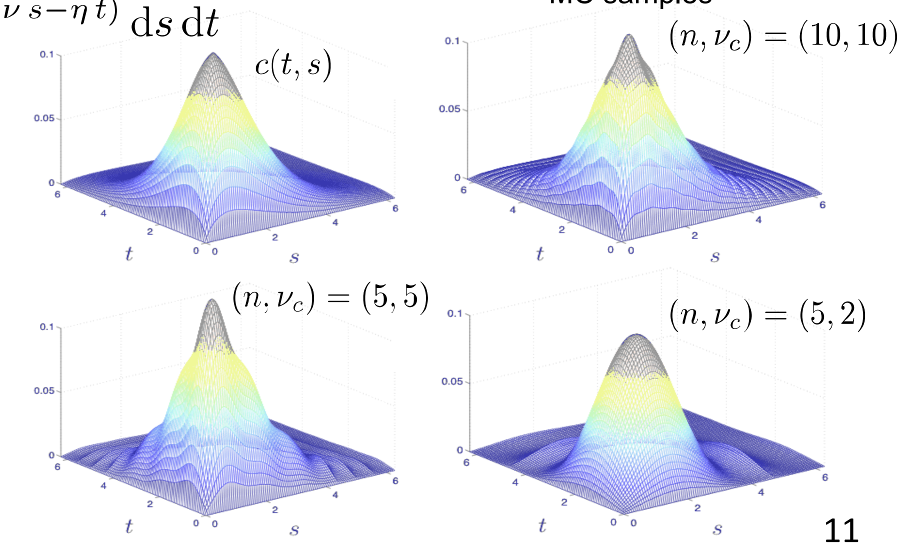
$$X(t) = \beta(t)\,Y(t), \quad t \in \mathbb{R},$$

is a non-stationary Gaussian process with zero mean, covariance function $c(s,t) = \mathrm{E}[X(s)\,X(t)] = \beta(s)\,\beta(t)\,c_Y(s-t)$, and generalized spectral density

$$s(\nu,\eta) = \frac{1}{(2\,\pi)^2} \int_{\mathbb{R}^2} \beta(s)\,\beta(t)\,e^{-\lambda\,|s-t|}\,e^{-i\,(\nu\,s - \eta\,t)}\,\mathrm{d}s\,\mathrm{d}t$$



Estimates using 5,000 MC samples



Applications

- Example 2: Fractional Brownian motion

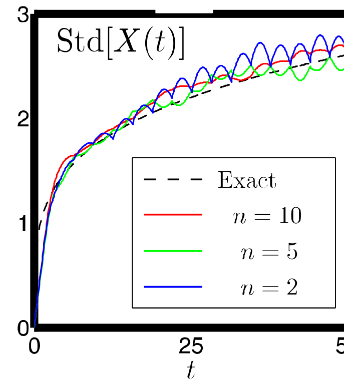
Let $H \in (0,1)$ be a constant and let $B_H(t)$, $t \geq 0$, be a fractional Brownian motion, that is, a non-stationary Gaussian process with zero mean, covariance function

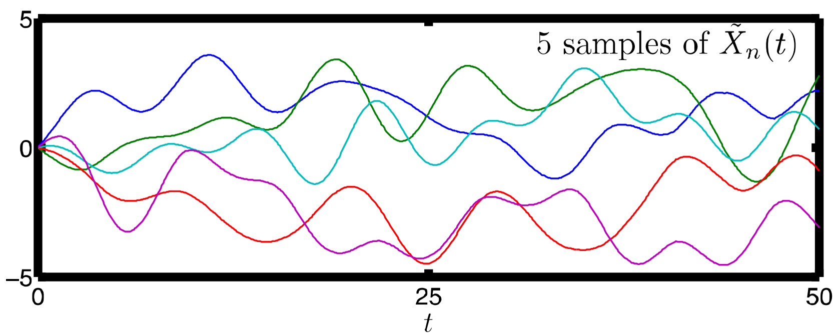
$$c_H(s,t) = \mathrm{E}[B_H(s)\, B_H(t)] = \frac{1}{2}\left[s^{2\,H} + t^{2\,H} - |s-t|^{2\,H}\right], \quad s,t \geq 0,$$

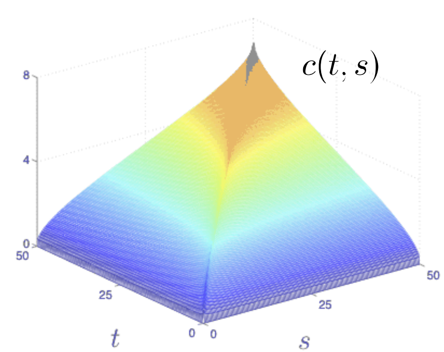
and initial state $B_H(0) = 0$. Define

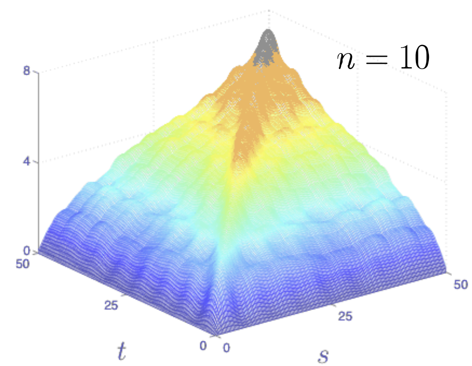
$$X(t) = 1(0 \leq t \leq \tau)\, B_H(t),\; 0 < \tau < \infty$$

Estimates using 5,000 MC samples



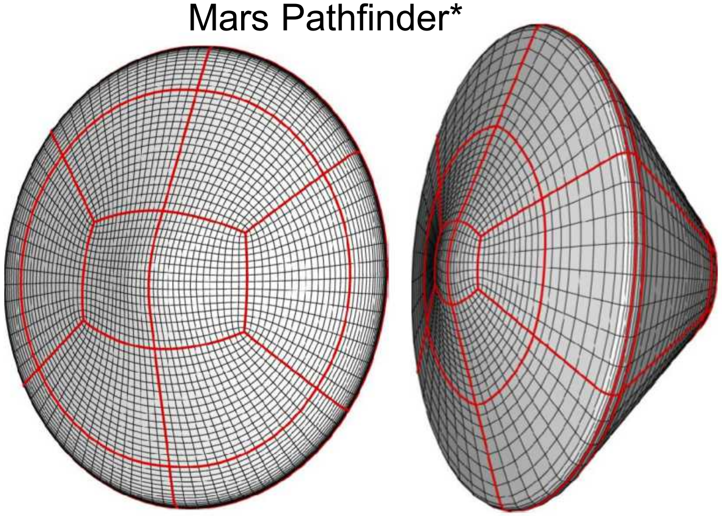




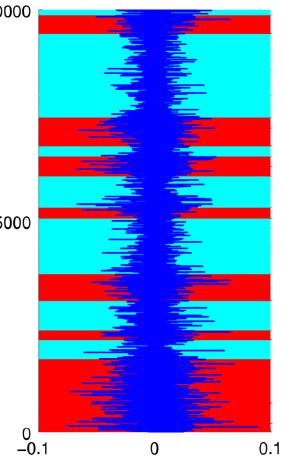


Application – Spacecraft Atmospheric Entry

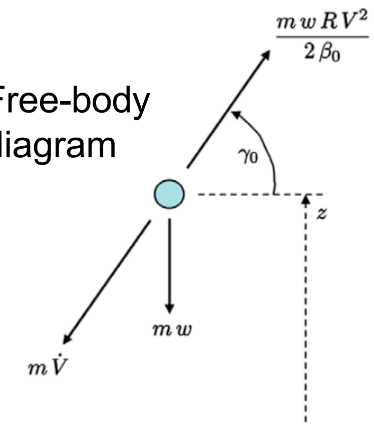
Mars Pathfinder*



One atmospheric variable vs. altitude



Free-body diagram

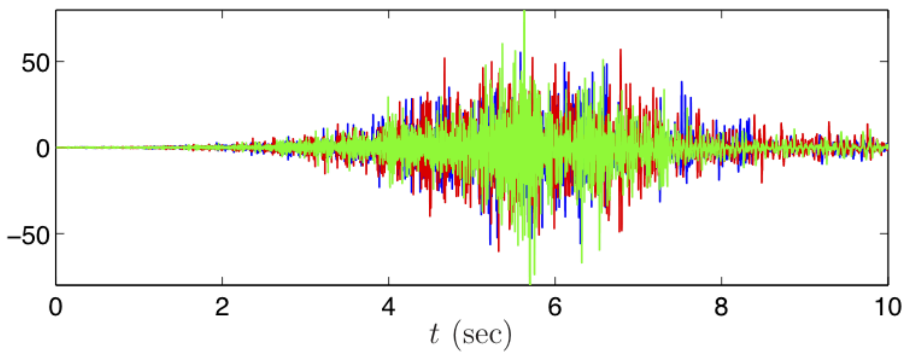


- Objective: Predict vibration response of spacecraft to random fluctuations in atmospheric conditions during planetary entry
  - Random process of atmospheric variables with altitude
  - Mapped to applied force on spacecraft via 6dof trajectory analysis
- Analysis
  - High-fidelity finite element model for spacecraft
  - Applied force is a non-stationary stochastic process (aerodynamic drag)
  - Perfect candidate for proposed on-the-fly method

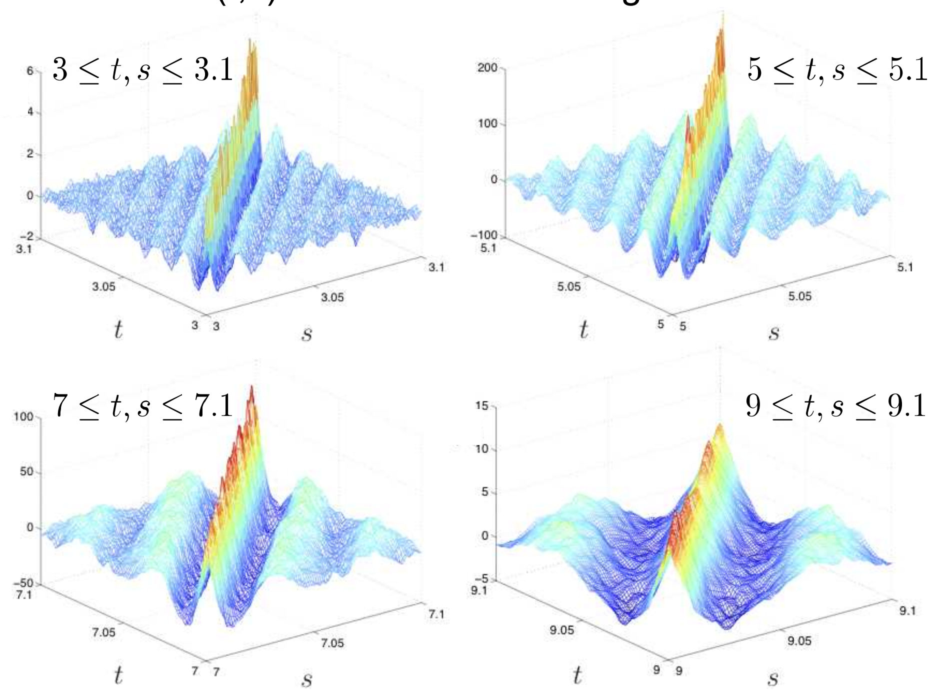
*Taken from W. Dieudonné and M. Spel. Entry probe stability analysis for the Mars Pathfinder and the Mars Premier Orbiter. Proceedings of the International Workshop on Planetary Probe Atmospheric Entry and Descent Trajectory Analysis and Science, October 6-9, 2003, Lisbon,

Application – Spacecraft Atmospheric Entry

*500 samples of (stochastic) drag force, X(t)



Estimates of covariance function $c(t,s)$ over various time segments



- Procedure
  1. Compute 500 samples of $X(t)$, the stochastic drag force*
  2. Estimate covariance function of $X(t)$
  3. Utilize on-the-fly method within FE solver to compute structural vibration response

*Taken from R. V. Field, Jr., T. S. Edwards, and J. W. Rouse. Modeling of atmospheric temperature fluctuations by translations of oscillatory random processes with application to spacecraft atmospheric re-entry. Probabilistic Engineering Mechanics, 26(2):231–239, 2011.

Summary

- Objective was to develop new algorithm for generating samples of non-stationary Gaussian processes
  - Applications involving long-time transient events
  - Avoid the need to store entire load history to file
  - Instead realize the load "on-the-fly"
- Proposed method was based on sampling theorem for stationary processes, extended to non-stationary processes
  - Developed bounds on truncation and aliasing errors
  - Efficient square-root-like decomposition of a sequence of covariance matrices
- Applications
  - Simple examples