SAND2012-0840C

Towards a Unified Swirl Vortex Model

Sal B. Rodriguez[1A, 1B] and Flor Espinoza[2]
[1A]Sandia National Laboratories, P.O. Box 5800, MS 0825, Albuquerque, NM 87185-0825
[1B]University of New Mexico, Chemical and Nuclear Engineering Dept. and Institute for Space and Nuclear Power Studies, Albuquerque, NM 87131

[2]University of New Mexico, Mathematics and Statistics Dept., Albuquerque, NM 87131

Stan L. Steinberg[3]
[3]University of New Mexico, Mathematics and Statistics Dept., Albuquerque, NM 87131

and

Mohamed S. El-Genk[4]
[4]University of New Mexico, Chemical and Nuclear Engineering Dept. and Institute for Space and Nuclear Power Studies, Albuquerque, NM 87131

A review of the literature shows the existence of dozens of single-cell, axisymmetric, Newtonian-fluid vortices that, in one form or another, are solutions to the Navier-Stokes Equation. However, little research has been conducted in the literature to investigate common traits that are shared by the vortices. This research attempts to lay out a foundation that seeks mathematical traits that are shared by the vortices, with a focus on the azimuthal velocity (v_θ). Notwithstanding their diverse mathematical and physical description, it can be shown that the vortices are diverse manifestations of a single vortex family—they are united by a set of at least seven common mathematical traits. As a first common trait, we show that there are only four possible categories. Next, by normalizing and overlaying the vortices into a single figure, three additional traits are noted, namely that $v_\theta(r=0) = 0$; they have similar azimuthal velocity profiles; and the azimuthal velocity is asymmetric about r. Thereafter, a fifth trait is noted by taking the azimuthal velocity for all the vortices (as found in the literature), and expressing them as alternating series' that expand geometrically with odd exponents. Finally, two more traits are noted by taking limit bounds for each series, thus showing that one bound is the Rankine (solid body) vortex, and the other is a Lamb-Oseen sine-like bound. In brief, we note that the vortices have seven traits in common, and hereby propose that the myriad of single-cell, axisymmetric, Newtonian-fluid vortices are essentially diverse manifestations of a single vortex family.

Nomenclature

A_n	=	Coefficient in AVE vortex, n=1, 2, …
AVE	=	Aboelkassem, Vatistas, and Esmail
C_n	=	Coefficients used in vortex expansions, n=-1, 0, 1, 2, …
G	=	Gas mass flux, ρ V (kg/m^{2}s)
J_1	=	Bessel function of order 1
k	=	Coefficient in Chepura and Modified Chepura vortices
n	=	Exponential coefficient
PDE	=	Partial differential equation
q	=	Coefficient in Martynenko's equations
r	=	Radial coordinate for cylindrical system

$\vec{r}$	=	Position vector (m)
RHS	=	Right hand side
SNL	=	Sandia National Laboratories
u	=	Velocity in the r direction (m/s)
UNM	=	University of New Mexico
$\vec{V}$	=	Velocity vector consisting of (u, v, w) components in cylindrical coordinates (m/s)
v	=	Velocity in the θ direction (azimuthal velocity) (m/s)
v_θ	=	Azimuthal velocity (m/s)
w	=	Velocity in the axial direction (m/s)
z	=	Axial coordinate for cylindrical system (direction of jet flow for S = 0)
z_O	=	Planar slice taken at fixed point along z axis (m)
3D	=	Three-dimensional

Greek Symbols

α	=	Coefficient in Loitsyanskiy's equations
α	=	Coefficient in Martynenko's equations
β	=	Coefficient in Loitsyanskiy's equations
β	=	Coefficient in AVE vortex
Γ	=	Circulation, $\int_0^{2\pi} v(r)rd\theta$ (m^2/s)
γ	=	Coefficient in Loitsyanskiy's equations
γ	=	Coefficient in Martynenko's equations
η	=	Coefficient in Loitsyanskiy's equations
η	=	Coefficient in Martynenko's equations
θ	=	Azimuthal coordinate for cylindrical system
λ	=	Coefficient in AVE vortex
μ	=	Dynamic viscosity (Pa-s, kg/m-s)
ν	=	Kinematic viscosity (m^2/s)
ρ	=	Density (kg/m^3)
ζ	=	Coefficient in Martynenko's equations
Ω	=	Coefficient in Rankine vortex (1/s)
ω	=	Angular frequency (1/s)
ω	=	Coefficient in Martynenko's equations

Subscripts

max	=	Maximum
min	=	Minimum
o	=	Property taken at fixed value

Superscripts

.	=	Time derivative
→	=	Vector

I. Introduction

Numerous types of single-cell, axisymmetric, Newtonian-fluid swirling vortex flows have been developed over the years, such as the

- Loitsyanskiy vortex [Loitsyanskiy, 1953],
- Gortler vortex [Gortler, 1954; Khorrami, 1995],
- Sullivan vortex [Sullivan, 1959; Huang *et al*., 2008],
- Newman vortex [Newman, 1959],
- Lamb-Oseen vortex (sometimes referred as "Oseen-Lamb") [Lamb, 1932; Mayer and Powell, 1992; Sipp, Coppens, and Jacquin, 1999; Chadwick, 2006; Facciolo, 2006; Busch, Ryan, and Sheard, 2007; Sereno, Pereira, and Pereira, 2009; Mao and Sherwin, 2009],

- Batchelor vortex (also known as the "q" vortex) [Batchelor, 1964; Tam, 1971; Duck and Foster, 1980; Mayer and Powell, 1992; Olendraru and Sellier, 1999; Delbende, 2002; Facciolo, 2006; Mao and Sherwin, 2009],
- Squire vortex [Squire, 1965],
- Burgers vortex [Burgers, 1948; Rott, 1958; Rott, 1959; Maxworthy, Hopfinger, and Redekopp, 1985; Bazant and Moffatt, 2005], and
- Rankine vortex (also known as the "solid body" vortex) [Rankine, 1858; Loiseleux, Chomaz, and Huerre, 1998; Billant, Chomaz, and Huerre, 1998; Facciolo, 2006; Rossi, 2006; Ortega-Casanova and Fernandez-Feria, 2009].

The radial, azimuthal, and axial velocity equations for all 15 vortices considered in this research are found in Table 1.

There are many other vortices, such as vortices whose tangential velocity is only in the shear layer [Lu and Lele, 1997; Cooper and Peake, 2002] and the Scully vortex [Scully, 1975; Zioutis *et al*., 2010]. The Batchelor vortex ("q" vortex), which is similar to the Lamb-Oseen vortex, is one of the most popular in the literature. The Batchelor vortex has been used extensively to model aircraft vortex wake issues. The Rankine vortex ("solid body") is another prevalent vortex found in the literature, and has found usage in modeling tornadoes and cyclones. The Gortler vortex considers vortices whose tangential velocity is nearly zero everywhere except in the shear layer. The recently-developed helicoid vortex models the swirl field produced by static, Archimedes-screw swirl devices found in heating and cooling applications.

Certainly, the velocity components vary according to which NS terms were included during the original derivation of the solution. Upon a casual search in the literature, one notes that there are not just the 15 axisymmetric vortices mentioned here, but many more, with more are being added at a fast rate. There seems to be a proliferation of vortices, to a degree that baffles (and perhaps even annoys!), rather than provides a higher degree of confidence in vortex modeling. And indeed, such sentiments have already been expressed in the literature [Aboelkassem and Vatistas, 2007]. After all, which vortex is "better" (more fundamental), and based upon what criteria? Many literature reports mention that a given vortex was used because it is "widely used…" [Vatistas, Kozel, and Mih, 1991; Alekseenko, *et al*., 1998; Mao and Sherwin, 2009; Okulov and Sorensen, 2009; Ortega-Casanova and Fernandez-Feria, 2009]. Further, how do the results from a given axisymmetric vortex apply to the others, if at all? Most importantly, are the vortices related to each other, in some form of vortex family, wherein they share certain mathematical characteristics? The subsequent sections in this report discuss a set of seven similar mathematical traits.

Table 1. Vortex Velocity Definitions

Vortex	Radial Velocity	Azimuthal Velocity	Axial Velocity
Rankine 1 field, f(r)	0	$\Omega r,\ r \le a$ ("solid body") $\dfrac{\Omega a^2}{r},\ r > a$ ("free")	0
Lamb-Oseen 1 field, f(r,t)	0	$\dfrac{\Gamma}{2\pi r}\left(1 - e^{-\frac{r^2}{4\nu t}}\right)$	0
Burgers 3 fields, f(r,z)	-ar	$\dfrac{\Gamma}{2\pi r}\left(1 - e^{-\frac{ar^2}{2\nu}}\right)$	2az

Vortex	Radial Velocity	Azimuthal Velocity	Axial Velocity
Loitsyanskiy3 fields, f(r,z)	$\frac{\alpha\sqrt{\nu}}{z}\frac{\alpha\eta\left(1-\frac{\alpha^2\eta^2}{4}\right)}{\left(1+\frac{\alpha^2\eta^2}{4}\right)^2}$	$\frac{1}{z^2}\frac{\alpha\gamma\eta}{\left(1+\frac{\alpha^2\eta^2}{4}\right)^2}$	$\frac{2\alpha^2}{z}\frac{1}{\left(1+\frac{\alpha^2\eta^2}{4}\right)^2}$
Gortler 3 fields, f(r,z)	$\frac{\gamma\nu}{z}\frac{\xi-\frac{\xi^2}{4}}{\left(1+\frac{\xi^2}{4}\right)^2} : \ 0$	$\frac{\gamma\nu}{z}\frac{\xi}{\left(1+\frac{\xi^2}{4}\right)^2}$	$\frac{\gamma^2\nu}{z}\frac{2}{\left(1+\frac{\xi^2}{4}\right)^2}$
Newman 3 fields, f(r,z)	$-\frac{Ar}{2z^2}e^{-\frac{Wr^2}{4\nu z}}$	$\frac{A}{r}\left(1-e^{-\frac{Wr^2}{4\nu z}}\right)$	$\frac{A}{z}e^{-\frac{Wr^2}{4\nu z}}$
Sullivan 3 fields, f(r,z)	$-ar+\frac{6\nu}{r}\left(1-e^{-\frac{ar^2}{2\nu}}\right)$	$\frac{\Gamma}{r}\left(1-e^{-\frac{r^2}{f^2}}\right),\ 0\le r< c$	$2az\left(1-3e^{-\frac{ar^2}{2\nu}}\right)$
Batchelor 2 fields, f(r)	0	$\frac{C_0}{r}\left(1-e^{-\frac{Ur^2}{4\nu z}}\right)$	$W_0 e^{-\left(\frac{r}{a}\right)^2}$
Squire 1 field, f(r,z)	0	$\frac{K}{2\pi r}\left[1-e^{-\frac{W_0 r^2}{4(\nu+aK)z}}\right]$	0
Chepura 1 field, f(r)	0	$\frac{kr}{2}\left[2-\left(\frac{r}{r_t}\right)^2\right],\ 0\le r\le r_t$ $\frac{kr_t}{2r},\ r_t\le r\le r_c$	0
Martynenko 3 fields, f(r,z)	$:\ \frac{1}{z}$	$\frac{\alpha^{\frac{3}{2}}L_0}{4\pi\mu}\left(\frac{K_0}{\pi\rho\nu^2}\right)^{\frac{1}{2}}\frac{(\alpha\eta)^{\frac{1}{2}}}{(1+\alpha\eta)^2}\omega$	$\frac{K_0}{\pi\mu}\frac{2\alpha}{(1+\alpha\eta)^2}zq$
AVE Vortex 1 field, f(r,t)	0	$\beta^2 r-\sum_{n=1}^{\infty}A_n J_1(\lambda_n\beta r)e^{-\lambda_n^2\beta t}$	0
Helicoid 2 fields, f(r)	0	sin(πr), for small r	W_0

Vortex	Radial Velocity	Azimuthal Velocity	Axial Velocity
Modified Chepura 1 field, f(r)	0	$k\left(\frac{r}{r_t}\right) - \frac{k}{3!}\left(\frac{r}{r_t}\right)^3 + \frac{k}{5!}\left(\frac{r}{r_t}\right)^5,$ $0 \le r \le r_t$	0
Modified Newman 3 fields, f(r,z)	$-\frac{Ar}{2z^2} e^{-\frac{Wr^2}{4\nu z}}$	$Ae^{-\frac{Wr^2}{4\nu z}}$ "free vortex" domain	$\frac{A}{z} e^{-\frac{Wr^2}{4\nu z}}$

II. Trait 1: Four Categories of Axisymmetric Vortices

Note that a 3D generalized velocity field in cylindrical coordinates can be expressed as

$$\vec{V}(r,\theta,z) = \vec{V}\left[u(r,\theta,z),v(r,\theta,z),w(r,\theta,z)\right]. \tag{1}$$

If the flow is symmetric about the axial coordinate z, then it moves about in circles around the z axis [Lamb, 1932]. Thus, the flow field is still 3D, but is axisymmetric about z. In such case, the above equation reduces to

$$\vec{V}(r,z) = \vec{V}\left[u(r,z),v(r,z),w(r,z)\right] = u(r,z)\vec{\delta}_r + v(r,z)\vec{\delta}_\theta + w(r,z)\vec{\delta}_z. \tag{2}$$

Even swirling flows that have a θ-dependence eventually become axisymmetric as a result of viscous effects [Loitsyanskiy, 1953].

Even swirling flows that have a θ-dependence eventually become axisymmetric as a result of viscous effects [Loitsyanskiy, 1953]. In any case, when axisymmetry is assumed, the Navier-Stokes PDE is simplified considerably, as all partials with respect to θ vanish. Note that the overall flow field must retain the azimuthal v velocity component in order to induce a vortex (swirl), but that none of the velocity components retain the θ dependency. Due to the number of possible permutations for axisymmetric vortices (see Table 2), then there are at most four distinct types of flows:

$$\vec{V}(r,z) = \vec{V}\left[u(r,z),v(r,z),w(r,z)\right], \tag{3}$$

$$\vec{V}(r,z) = \vec{V}\left[0,v(r,z),w(r,z)\right], \tag{4}$$

$$\vec{V}(r,z) = \vec{V}\left[u(r,z),v(r,z),0\right], \text{ and} \tag{5}$$

$$\vec{V}(r,z) = \vec{V}\left[0,v(r,z),0\right]. \tag{6}$$

Table 2. The Four Categories of Axisymmetric Vortices

Vortex Category	Radial Velocity, u	Azimuthal Velocity, v	Axial Velocity, w
$\vec{V}(r,z) = \vec{V}\left[u(r,z),v(r,z),w(r,z)\right]$ Example: Burgers	-ar	$\frac{\Gamma}{2\pi r}\left(1 - e^{-\frac{ar^2}{2\nu}}\right)$	2az

$\vec{V}(r,z)=\vec{V}\left[0,v(r,z),w(r,z)\right]$ Example: Batchelor	0	$\frac{C_0}{r}\left(1-e^{-\frac{Ur^2}{4\nu z}}\right)$	$W_0 e^{-\left(\frac{r}{a}\right)^2}$
$\vec{V}(r,z)=\vec{V}\left[u(r,z),v(r,z),0\right]$ Example: *none found in the literature*	u(r,z)	v(r,z)	0
$\vec{V}(r,z)=\vec{V}\left[0,v(r,z),0\right]$ Example: Rankine	0	$\Omega r,\ r \leq a$ $\frac{\Omega a^2}{r},\ r > a$	0

It is noteworthy that many examples were found in the literature for the first, second, and fourth categories, but none for the third. This should be investigated further.

III. Three Traits of the Azimuthal Velocity Profile

Despite the varied developmental origins and applications of vortex models, when normalized and overlayed into a single figure, the vortices display three similar traits, as shown in Figure 1.

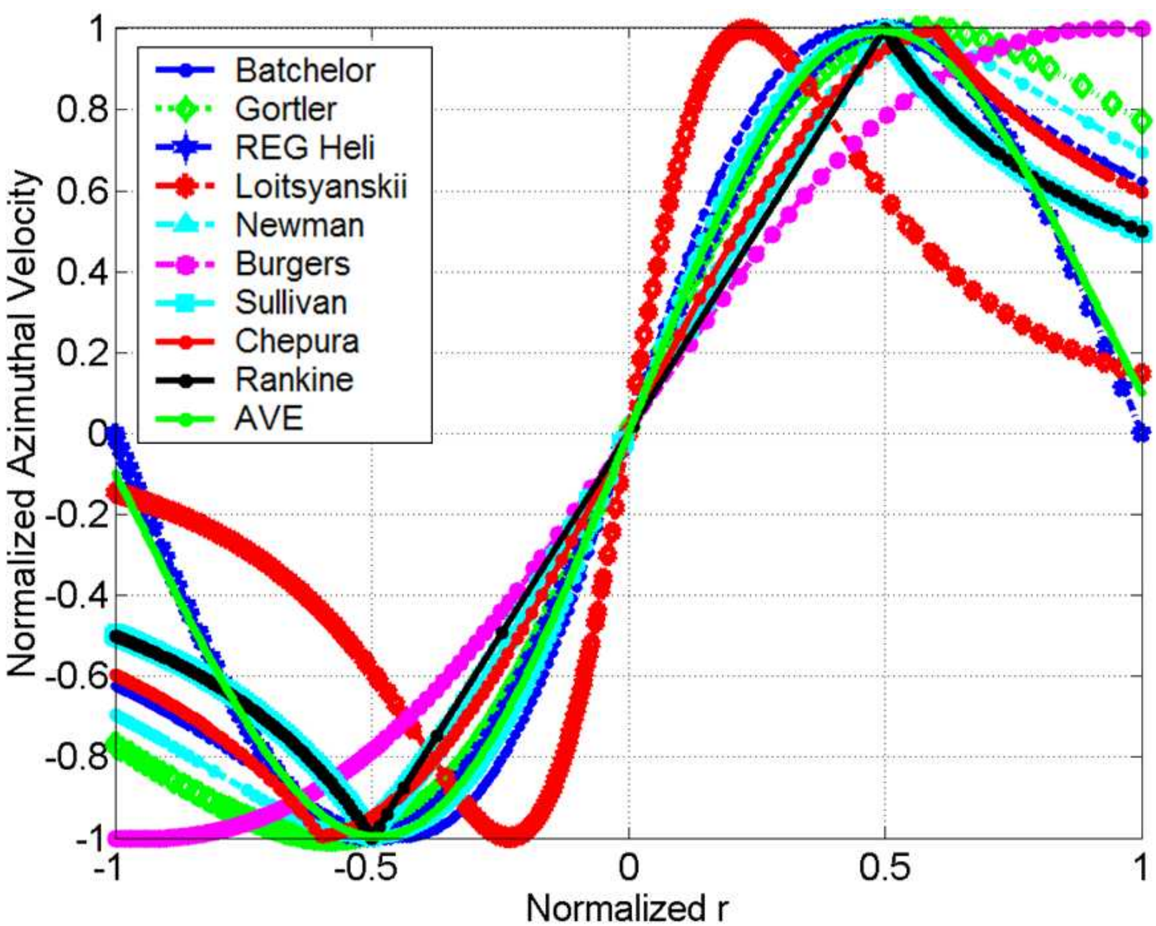

Figure 1. Rankine, Burgers, Loitsyanskiy, Gortler, Newman, Sullivan, Batchelor, Chepura, AVE, and Rodriguez/El-Genk Helicoid (REG): Normalized Azimuthal Velocity Overlay.

Figure 1 expands the work of the previous researchers just discussed by showing that the Rankine, Burgers, Loitsyanskiy, Gortler, Newman, Sullivan, Batchelor, Chepura, AVE, and the Rodriguez/El-Genk helicoid (REG) vortex, and indeed all 15 vortices discussed here, are part of a vortex family whose normalized azimuthal velocity is sine-like; so as to not overwhelm the Figure, only 10 vortices are displayed. It is also interesting that sinusoidal azimuthal velocity distributions have already been used in recent experiments found in the literature [Semaan and Naughton, 2010].

First, it is noted that at r=0, $v_\theta(r) = 0$ for all the vortices. Secondly, they share a similar sine-like profile when the vortex equation parameters are adjusted (a discussion for the sine-like profile will be provided later). Third, the $v_\theta(r)$ profile is asymmetric about r. That is, $v_\theta(r) = v_\theta(-r)$, and so the vortices have odd azimuthal velocity functions.

IV. Trait 5: Series Formulation of Vortices

Despite the varied mathematical forms that the azimuthal velocities have in the literature, all can be expressed as alternating series that expand geometrically with odd exponents, as noted in Table 3. We used Maple to obtain the series expansions.

Table 3. Mathematical Form and Series Expansion

Vortex Name	Azimuthal Velocity	Mathema-tical Form ($z=z_0$; $t=t_0$, const.)	Series Expansion
Rankine (“solid body”) [Rankine, 1858]	$\Omega r,\ r \le a$ $\dfrac{\Omega a^2}{r},\ r > a$ $\Omega = \dfrac{\Gamma}{2\pi a^2}$	Series	$\dfrac{C_0}{r}$ $C_1 r$
Lamb-Oseen [Lamb, 1932]	$\dfrac{\Gamma}{2\pi r}\left(1 - e^{-\frac{r^2}{4\nu t}}\right)$	Exponential	$C_1 r - C_3 r^3 + C_5 r^5 - C_7 r^7 ...$
Burgers [Burgers, 1948]. (The vortex has also been referred as the “Burgers-Rott”, though Rott’s paper was published 10 years later) [Rott, 1958]	$\dfrac{1}{r}\left(\dfrac{1-e^{-\frac{1}{2}\mathrm{Re}\, r^2}}{1-e^{-\frac{1}{2}\mathrm{Re}}}\right)$	Exponential	$C_1 r - C_3 r^3 + C_5 r^5 - C_7 r^7 ...$
Loitsyanskiy [Loitsyanskiy, 1953]	$\dfrac{1}{z^2}\dfrac{\alpha\gamma\eta}{\left(1+\dfrac{\alpha^2\eta^2}{4}\right)^2}$, $\eta = \dfrac{1}{\sqrt{\nu}}\dfrac{r}{z}$	Rational	$C_1 r - C_3 r^3 + C_5 r^5 - C_7 r^7 ...$

Vortex Name	Azimuthal Velocity	Mathema-tical Form (z=z_0; t=t_0, const.)	Series Expansion
Gortler [Gortler, 1954]	$\frac{C_1 z_0^2}{z^2}\frac{\frac{2r}{r_0}}{\left[1+\left(\frac{r}{r_0}\right)^2\right]^2}$	Rational	$C_1 r - C_3 r^3 + C_5 r^5 - C_7 r^7 \ldots$
Newman [Newman, 1959]	$\frac{\Gamma}{2\pi r}\left(1 - e^{-\frac{Wr^2}{4\nu z}}\right)$	Exponential	$C_1 r - C_3 r^3 + C_5 r^5 - C_7 r^7 \ldots$
Sullivan [Sullivan, 1959]; Using alternate form as a combination of two vortices: [Chuah and Kushida, 2007]	$\frac{\Gamma}{r}\left(1 - e^{-\frac{r^2}{f^2}}\right), 0 \le r < c$ and $\left[\frac{(1-a)RV_R}{r} + \frac{a\Gamma\left(1-e^{-\sigma}\right)}{r}\right]$, $c \le r < R$	Exponential	$C_1 r - C_3 r^3 + C_5 r^5 - C_7 r^7 \ldots$
Batchelor ("q") [Batchelor, 1964]	$\frac{C_0}{r}\left(1-e^{-\frac{Ur^2}{4\nu z}}\right)$	Exponential	$C_1 r - C_3 r^3 + C_5 r^5 - C_7 r^7 \ldots$
Squire [Squire, 1965]	$\frac{K}{2\pi r}\left[1 - e^{-\frac{W_0 r^2}{4(\nu + aK)z}}\right]$	Exponential	$C_1 r - C_3 r^3 + C_5 r^5 - C_7 r^7 \ldots$
Chepura [Chepura *et al*., 1969; Chepura 1971]	$\frac{kr}{2}\left[2 - \left(\frac{r}{a}\right)^2\right], 0 \le r \le a$ $\frac{ka}{2r}$, $a \le r \le r_c$	Series	$C_1 r - C_3 r^3$
Martynenko [Martynenko, 1989]	$\frac{\alpha^{\frac{3}{2}} L_0}{4\pi\mu}\left(\frac{K_0}{\pi\rho\nu^2}\right)^{\frac{1}{2}}\frac{(\alpha\eta)^{\frac{1}{2}}}{(1+\alpha\eta)^2}\omega$	Rational	$C_1 r - C_3 r^3 + C_5 r^5 - C_7 r^7 \ldots$
AVE Vortex [Aboelkassem, Vatistas, and Esmail, 2005]	$\beta^2 r - \sum_{n=1}^{\infty} A_n J_1(\lambda_n \beta r) e^{-\lambda_n^2 \beta t}$	Bessel	$C_1 r - C_3 r^3 + C_5 r^5 - C_7 r^7 \ldots$
REG helicoid vortex [Rodriguez and El-	sin(πr), for small r	Sine	$C_1 r - C_3 r^3 + C_5 r^5 - C_7 r^7 \ldots$

Vortex Name	Azimuthal Velocity	Mathema-tical Form ($z=z_0$; $t=t_0$, const.)	Series Expansion
Genk, 2008, 2010]			
Modified Chepura [Rodriguez, 2011]	$\frac{k}{1!}\left(\frac{r}{a}\right) - \frac{k}{3!}\left(\frac{r}{a}\right)^3 + \frac{k}{5!}\left(\frac{r}{a}\right)^5$ $0 \le r \le a$	Series	$C_1 r - C_3 r^3 + C_5 r^5$
Modified Newman [Rodriguez, 2011]	$C_2 e^{-C_3 \nu r^2}$	Exponential	$-\frac{C_{-1}}{r}$ $+ C_1 r - C_3 r^3 + C_5 r^5 - C_7 r^7 \ldots$

From Table 3, it is noted that the v_θ vortex distributions can be classified according to five mathematical functions:

1. Series (Rankine, Chepura, Modified Chepura),
2. Exponential (Lamb-Oseen, Burgers, Newman, Sullivan, Batchelor, Squire, Modified Newman),
3. Rational (Loitsyanskiy, Gortler, Martynenko),
4. Bessel (AVE), and
5. Sine (REG helicoid).

A casual inspection of the five mathematical forms suggests that they are quite different. However, that is not the picture suggested by Figure 1, with its 10 vortices that seemingly share a similar sine-like profile. Upon further thought, the Rankine, modified Chepura, and REG are shown to have the following in common: the formulation for Rankine vortex for $r \le a$ and the modified Chepura for $r \le a$ have similar series terms when compared to an expansion of the REG helicoid. In particular, for the Rankine Domain 1,

$$v_{Rankine}(r) = C_1 r. \tag{7}$$

Here, Domain 1 refers to the "solid body" part of the vortex, where the fluid essentially behaves as if attached to a rotating, solid body. Domain 2 refers to the "free" vortex, away from the solid body region, where the vortex velocity decays due to viscous effects.

The modified Chepura for Domain 1 is [Rodriguez, 2011]

$$v_{Chepura}(r) = \frac{k}{1!}\left(\frac{r}{a}\right) - \frac{k}{3!}\left(\frac{r}{a}\right)^3 = C_1 r - C_3 r^3. \tag{8}$$

and a series expansion of sine for the REG helicoid yields

$$v(r) = \sin(\pi r) = \pi r - \frac{\pi^3}{3!} r^3 + \frac{\pi^5}{5!} r^5 - \frac{\pi^7}{7!} r^7 \ldots = C_1 r - C_3 r^3 + C_5 r^5 - C_7 r^7 \ldots \tag{9}$$

Next, by noting from Figure 1 that 1) the REG helicoid and the AVE vortex [Aboelkassem, Vatistas, and Esmail, 2005] are very similar in profile and 2) that the Bessel function of the first kind has long been observed to behave as the sine function, we are tempted to also do a series expansion for it. By searching for the most sine-like series expansion for the Bessel function (see Figure 2), and setting t=to and n=1, Maple calculated the series expansion as

$$v_{AVE} = series\left[\beta^2 r - \sum_{n=1}^{\infty} A_n J_1(\lambda_n \beta r) e^{-\lambda_n^2 \beta t}\right] \tag{10}$$
$$= 3.173r - 5.5r^3 + 3.06r^5 - 0.869r^7 ... = C_1 r - C_3 r^3 + C_5 r^5 - C_7 r^7 ...$$

Indeed, the AVE vortex can uniquely be expressed as Fourier-Bessel series [Aboelkassem, Vatistas, and Esmail, 2005].

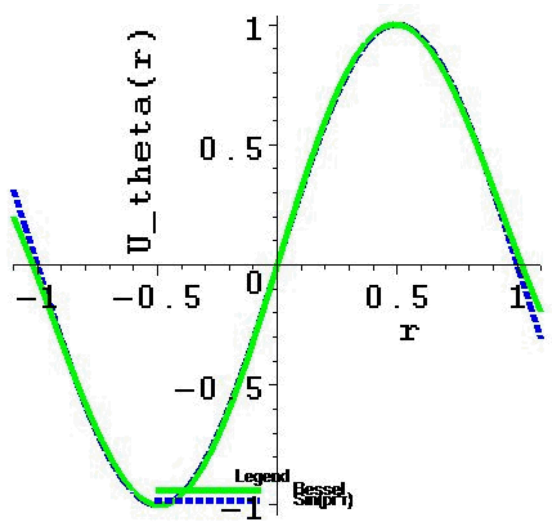

Figure 2. Bessel and Sine Azimuthal Velocities.

Therefore, this shows that the Rankine, modified Chepura, REG helicoid, and the AVE vortex can be collapsed into a single mathematical series form. By using Maple, we easily obtain series expansions for the rational and exponential forms. Not surprisingly, it is found that

$$v_{\text{exponential}} = series\left[\frac{C_1}{r}\left(1 - e^{-C_2 r^2}\right)\right] = C_1 r - C_3 r^3 + C_5 r^5 - C_7 r^7 ... \tag{11}$$

and likewise,

$$v_{\text{rational}} = series\left[\text{rational}\left(\text{Loitsyanskiy, Gortler, or Martynenko}\right)\right] \tag{12}$$
$$= C_1 r - C_3 r^3 + C_5 r^5 - C_7 r^7 ...$$

Figure 3 shows that Loitsyanskiy vortex with parameters that make it the most "sine-like" (LHS). In fact, it is easy to observe that such is the case for small r, say $r \leq 0.7$. The RHS of Figure 3 shows the relative error between the most sine-like Loitsyanskiy and $\sin(\pi r)$. The same analysis was performed for the Newman vortex, as shown in Figure 4. The figure shows that same conclusions can be made with the Newman vortex as was done with the Loitsyanskiy vortex.

Therefore, all the vortices presented herein can be represented as alternating geometric series with odd exponents!

Interestingly, for non-Newtonian swirling fluids, the axial and azimuthal velocities were assumed to be representable as series expansions with alternating signs exponential power [Som, 1983]:

$$v_{\text{non-Newtonian,Som}}(r) = -C_0 + C_1 r - C_2 r^2 \tag{13}$$

and

$$w_{\text{non-Newtonian,Som}}(r) = -C_0 + C_1 r - C_2 r^2 + C_3 r^3 . \tag{14}$$

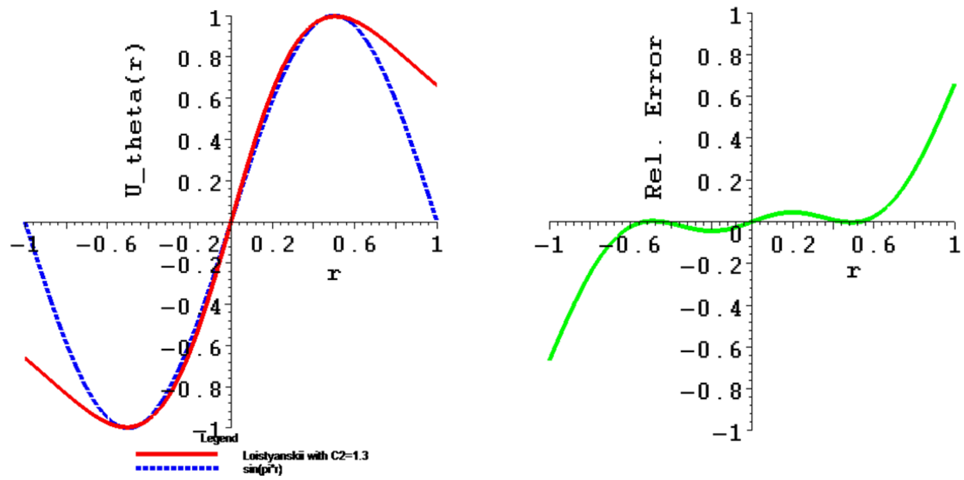

Figure 3. LHS: Loitsyanskiy and Sine Azimuthal Distributions. RHS: Relative Error between Both Distributions.

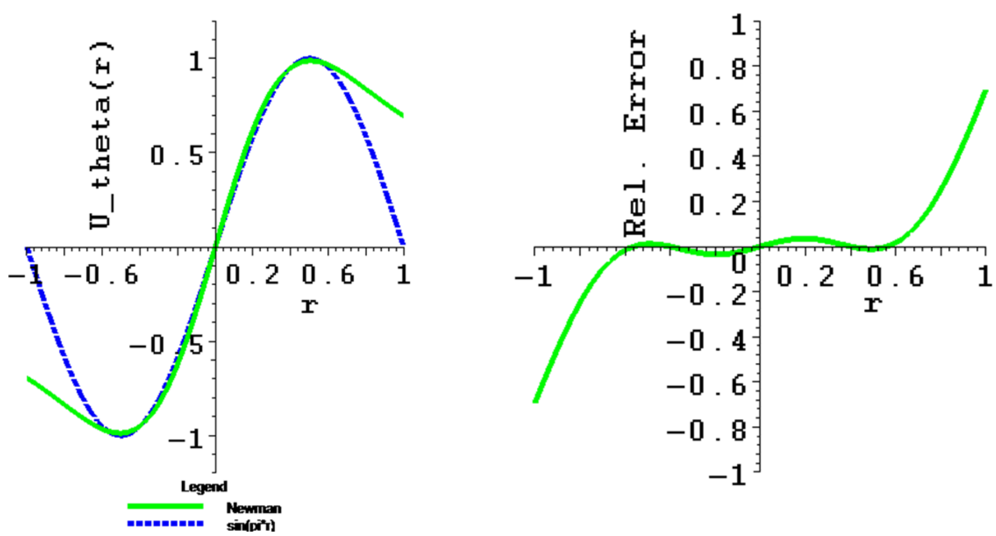

Figure 4. LHS: Newman and Sine Azimuthal Distributions. RHS: Relative Error between Both Distributions.

V. Traits Six and Seven: Series Bounds

Some researchers have noted that some of the most "popular" vortices share some basic characteristics. Perhaps the earliest example of this was noted by Chepura, where he overlayed the normalized azimuthal velocity of his newly-formulated vortex (the Chepura vortex) with the Rankine vortex [Chepura *et al.*, 1969]. This showed that the two velocity profiles were similar (e.g. both had zero velocity at r=0 and their velocity profiles were quite similar), as noted in Figure 4. In the figure, the dashed and solid lines are the Rankine and Chepura vortices, respectively. Further, it has also been shown that a vortex formulation with azimuthal velocity represented as a function of r and n very closely matched the Burgers vortex for n=2 and was Rankine-like for n=∞ [Vatistas, Kozel, and Mih, 1991]. In this context, n is an exponential value for r. As another example, the azimuthal velocity for the Rankine, Scully, and Lamb vectors were normalized and overlayed, only to show that their profiles were quite similar [Alekseenko *et al.*, 1999]. Finally, the normalized azimuthal velocity for the Rankine, Oseen-Lamb, Burgers-Rott, and Sullivan vortices were also plotted [Batterson, Maicke, and Majdalani, 2007]; see Figure 5. Interestingly, the authors called such plot a "unified normalization".

Note that there are two versions of the Sullivan vortex, depending on the number of "cells". The single-cell Sullivan vortex is included in this research. The "two-cell" model and (also called "bidirectional) [Batterson, Maicke, and Majdalani, 2007], consists of an inner down flow region coexists with a downward-flowing region. In other words, the swirling flow has "nested" regions with reversed flow interspersed upon the swirling field [Donaldson and Sullivan, 1960]. Note that the focus of this work is for "one-cell" vortices, not the more complex "two-cell" types.

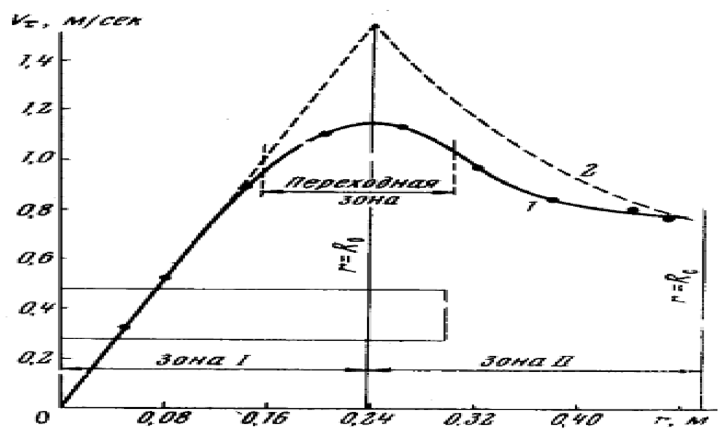

Рис. 1. Зависимость v_τ от r: 1 — для сосуда с радиально-лопастной мешалкой, по данным работы [1]: $d_м$ = 600 мм; D_c = 1000 мм; $H_ж = D_c$; n = 63 об/мин; вода; 2 — расчет по модели вихря Рэнкина

Figure 4. Chepura and Rankine Vortices: Normalized Azimuthal Velocity Overlay [Chepura, 1969].

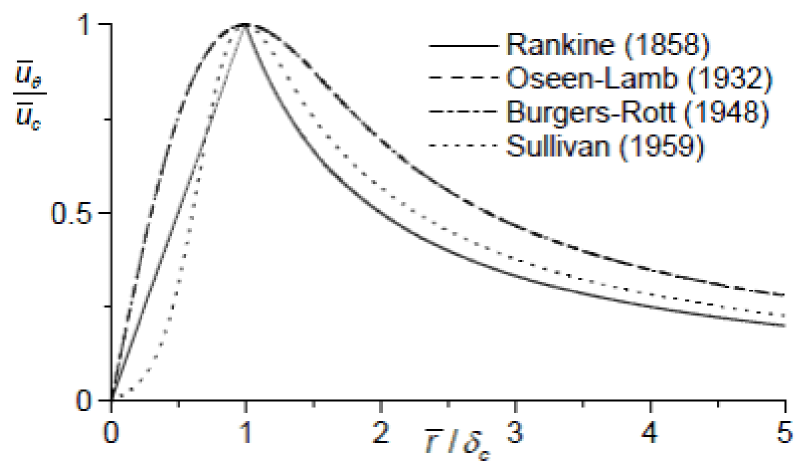

Figure 5. Rankine, Oseen-Lamb, Burgers-Rott, and Sullivan Vortices: Normalized Azimuthal Velocity Overlay [Batterson, Maicke, and Majdalani, 2007].

Finally, we note that the azimuthal velocity approaches the Rankine solid-body azimuthal velocity for large C_1 and small C_3, C_5, etc.

$$\lim_{\substack{C_1 \to \text{large} \\ C3, C5, C7 \ldots \to 0}} \left[v_\theta(r) = C_1 r - C_3 r^3 + C_5 r^5 - C_7 r^7 \ldots \right] = C_1 r. \tag{15}$$

This explains why the "solid body" Rankine vortex served as a bound for Chepura's work. On the other hand, the Lamb-Oseen-like sine limit is approached when

$$\begin{aligned} &C_1 \to \pi, \\ &C_3 \to \frac{\pi^3}{3!}, \\ &C_5 \to \frac{\pi^5}{5!}, \\ &C_7 \to \frac{\pi^7}{7!}, \ldots \end{aligned} \tag{16}$$

Thus, this sheds light as to why the vortices in Figure 1 have a "sine-like" form.

When the limits in Equations 15 and 16 are applied to the series forms found in Table 3, it is noted that all the vortices collapse to the Rankine vortex bound on the one hand, and on the other extreme, all the vortices collapse to a Lamb-Oseen-like sine bound.

VI. Conclusion

Steps were taken towards establishing that 15 single-cell, axisymmetric, Newtonian vortices share seven mathematical characteristics, and are therefore essentially diverse manifestations of a single vortex family that unites them. If so, then it is likely that a given mathematical analysis afforded to a particular vortex also applies to the others.

References

Aboelkassem, Y., G. H. Vatistas, and N. Esmail, "Viscous Dissipation of Rankine Vortex Profile in Zero Meridional Flow", *Acta Mech. Sinica*, Vol. 21, 550 – 556, 2005.

Alekseenko, S. V. *et al*., "Helical Vortices in Swirl Flow", *J. Fluid Mech*., Vol. 382, 195 – 243, 1999.

Batterson, J. W., B. A. Maicke, and J. Majdalani, "Advancements in Theoretical Models of Confined Vortex Flowfields", University of Tennessee Space Institute, Tullahoma, Tennessee, 2007.

Batchelor, G. K., "Axial Flow in Trailing Line Vortices", *J. Fluid Mech*., Vol. 20, Part 4, 645 – 658, 1964.

Bazant, M. Z. and H. K. Moffatt, "Exact Solutions of Navier-Stokes Equations Having Steady Vortex Structures", *J. Fluid Mech*., Vol. 541, 55 – 64, 2005.

Billant, P., J.-M. Chomaz, and P. Huerre, "Experimental Study of Vortex Breakdown in Swirling Jets", *J. Fluid Mech*., Vol. 376, 183 – 219, 1998.

Burgers, J. M., *Advances in Applied Mechanics*, Vol. 1, Academic Press, New York, 171, 1948.

Busch, H., K. Ryan, and G. J. Sheard, "Strain-Rate Development Between a Co-Rotating Lamb-Oseen Vortex Pair of Unequal Strength", 16th Australasian Fluid Mechanics Conference, Gold Coast, Australia, December 2-7, 1470 – 1474, 2007.

Chadwick, E., "The Vortex Line in Steady, Incompressible Oseen Flow", *Proc. R. Soc. A*, Vol. 462, 391 – 401, 2006.

Chepura, I. B. *et al*., "On the Tangential Component of the Velocity Field in a Smooth-Wall Vessel Equipped with a Radial-Blade Mixer", *Teoreticheskie Osnovy Khimichesko i Tekhnologii*, Vol. 3, No. 3, 404 – 411, 1969.

Chepura, I. B., "Certain Principles of Varying Circumferential Velocity in Smooth-Walled Vessels with Radial Blade Agitators", *Teoreticheskie Osnovy Khimichesko i Tekhnologii*, Vol. 5, No. 5, 764 – 767, 1971.

Chuah, K. H. and G. Kushida, "The Prediction of Flame Heights and Flame Shapes of Small Fire Whirls", *Proceedings of the Combustion Institute,* Vol. 32, 2599 – 2606, 2007.

Cooper, A. J. and N. Peake, "The Stability of a Slowly Diverging Swirling Jet", *J. Fluid Mech.,* Vol. 473, 389 – 411, 2002.

Delbende, I. *et al*., "Various Aspects of Fluid Vortices", C. R. Acad. Sci. Paris, France, 2002.

Donaldson, C. duP. and R. D. Sullivan, "Behaviour of Solutions of the Navier-Stokes Equations for a Complete Class of Three-Dimensional Viscous Vortices", Proceedings of the 1960 Heat Transfer and Fluid Mechanics Institute, 16 – 30, 1960.

Duck, P. W. and M. R. Foster, "The Inviscid Stability of a Trailing Line Vortex", *J. of Applied Mathematics and Physics*, Vol. 31, 524 – 532, 1980.

Facciolo, L., "A Study on Axially Rotating Pipe and Swirling Jet Flows", Royal Institute of Technology, Department of Mechanics, S-100 44 Stockholm, Sweden, Ph.D. Diss., 2006.

Gortler, H., "Decay of Swirl in an Axially Symmetrical Jet, Far from the Orifice", *Revista Matematica Hispano-Americana*, Vol. 14, 143 – 178, 1954.

Huang, S.-L. *et al*., "On the Transition Process of a Swirling Vortex Generated in a Rotating Tank", *Exp. Fluids*, Vol. 45, 267 – 282, 2008.

Khorrami, R. M., "Stability of a Compressible Axisymmetric Swirling Jet", *AIAA Journal*, Vol. 33, No. 4, 650 – 658, 1995.

Lamb, H., *Hydrodynamics*, 6th Ed., Cambridge Univ. Press, 1932.

Loiseleux, T., J. M. Chomaz, and P. Huerre, "The Effect on Swirl on Jets and Wakes: Linear Instability of the Rankine Vortex with Axial Flow", *Physics of Fluids*, Vol. 10, No. 5, 1120 – 1134, 1998.

Loitsyanskiy, L. G., "The Propagation of a Twisted Jet in an Unbounded Space Filled with the Same Fluid", *Prikladnaya Matematika i Mekhanika*, Vol. 17, No. 1, 3 – 16, 1953.

Lu, G. and S. K. Lele, "Inviscid Instability of Compressible Swirling Mixing Layers", *Physics of Fluids*, Vol. 11, No. 2, 450 – 461, 1999.

Mao, X. and S. J. Sherwin, "Spectra of Swirling Flow", under consideration for publication in *J. Fluid Mech*., received November 15, 2009.

Maxworthy, T., E. J. Hopfinger, and L. G. Redekopp, "Wave Motions on Vortex Cores", *J. Fluid Mech.*, Vol. 151, 141 – 164, 1985.

Mayer, E. W. and K. G. Powell, "Viscous and Inviscid Instabilities of a Trailing Vortex", *J. Fluid Mech*., Vol. 245, 91 – 114, 1992.

Newman, B. G., "Flow in a Viscous Trailing Vortex", *The Aeronautical Quarterly*, 149 – 162, May 1959.

Okulov, V. L. and J. N. Sorensen, "Applications of 2D Helical Vortex Dynamics", *Theor. Comput. Fluid Dyn*., Vol. 24, 395 – 401, 2010.

Olendraru, C. and A. Sellier, "Inviscid Instability of the Batchelor Vortex: Absolute-Convective Transition and Spatial Branches", *Physics of Fluids*, Vol. 11, No. 7, 1805 – 1820, 1999.

Ortega-Casanova, J. and R. Fernandez-Feria, "Three-Dimensional Transitions in a Swirling Jet Impinging Against a Solid Wall at Moderate Reynolds Numbers", *Physics of Fluids,* Vol. 21, 2009.

Rankine, W. J., *A Manual of Applied Mechanics*, 9th Ed., C. Griffin and Co., London, UK, 1858.

Rodriguez, S. B., "Swirling Jets for the Mitigation of Hot Spots and Thermal Stratification in the VHTR Lower Plenum", University of New Mexico, ChNE Department, Ph.D. Diss., 2011.

Rodriguez, S. B. and M. S. El-Genk, "Using Helicoids to Eliminate 'Hot Streaking' and Stratification in the Very High Temperature Reactor Lower Plenum", *Proceedings of ICAPP '08*, American Nuclear Society, Paper 8079, Anaheim, CA, 2008.

Rodriguez, S. B. and M. S. El-Genk, "Heat Transfer and Flow Field Characterization of a Triangular Array of Swirling Jets Impinging on an Adiabatic Plate," *Proc. 14th Int. Heat Transfer Conference*, Washington DC, 2010.

Rossi, M., "Of Vortices and Vortical Layers: An Overview", Laboratoire de Modelisation en Mecanique, Paris, France, 2006.

Rott, N., “On the Viscous Core of a Line Vortex”, *Zeitschrift fur Angewandte Mathematik und Physik*, Vol. 9b, 543 – 553, 1958.

Rott, N., “On the Viscous Core of a Line Vortex II”, *Zeitschrift fur Angewandte Mathematik und Physik*, Vol. 10, 73 – 81, 1959.

Semaan, R. and J. W. Naughton, “Three Component LDA Measurements in the Near and Far Field of Swirling Jets”, 46th AIAA Aerospace Sciences Meeting and Exhibit, Reno, Nevada, January 7-10, 2010.

Sereno, J. L., J. M. C. Pereira, and J. C. F. Pereira, “A Stochastic Lamb-Oseen Vortex Solution of the 2D Navier-Stokes Equation”, *Commun. Numer. Meth. Engng*., 2009.

Sipp, D., F. Coppens, and L. Jacquin, “Theoretical and Numerical Analysis of Wake Vortices”, Third International Workshop on Vortex Flows and Related Numerical Methods, ESAIM: Proceedings, Vol.7, 397 – 407, 1999.

Som, S. K., “Theoretical and Experimental Studies on the Formation of an Air Core in a Swirl Spray Pressure Nozzle Using a Power Law Non-Newtonian Liquid”, *Applied Scientific Research*, Vol. 40, 71 – 91, 1983.

Squire, H. B., “The Growth of a Vortex in a Turbulent Flow”, *The Aeronautical Quarterly*, Vol. 16, Part 1, 302 – 306, 1965.

Sullivan, R. D., “A Two-Cell Vortex Solution of the Navier-Stokes Equations”, *J. of the Aerospace Sciences*, Vol. 26, No. 11, 767 – 768, 1959.

Tam, K. K., “A Note on the Flow in a Trailing Vortex”, *J. of Engineering Mathematics*, Vol. 7, No. 1, 1 – 6, 1973.

Vatistas, G. H., V. Kozel, and W. C. Mih, “A Simpler Model for Concentrated Vortices”, *Experiments in Fluids*, Vol. 11, 73 – 76, 1991.

Zioutis, C. K. *et al*., “Influence of Helicopter Rotor Wake Modeling on Blade Airload Predictions”, *Int. J. of Engineering*, Vol. 3, Is. 6, 521 – 537, 2010.