Analysis of the volume-constrained peridynamic Navier equation of linear elasticity

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Goal: Peridynamic Navier equation

- Use nonlocal vector calculus to show that

$$\mathcal{L} := -\mathcal{D}\Big(\frac{15}{m}\mu\varpi\big(\mathcal{D}^*\big)^T\Big) - \mathcal{D}_\omega\Big(\big(\lambda - \frac{13}{3}\mu\big)\mathrm{Tr}\big(\mathcal{D}_\omega^*\big)\mathbf{I}\Big)$$

where $m(\mathbf{x}) = \int_{\mathbb{R}^3} |\boldsymbol{\xi}|^2 \varpi(\mathbf{x}, \boldsymbol{\xi})\, d\boldsymbol{\xi}$ converges to

$$\mathcal{N} := \mu\nabla\cdot\nabla\mathbf{u} + \mu\nabla\,\nabla\cdot\mathbf{u} + \lambda\nabla\,\nabla\cdot\mathbf{u}$$

as the nonlocality vanishes
- Integral operator $\mathcal{L}$ valid on functions with jump discontinuities in contrast to the differential operator $\mathcal{N}$
- Volume constraints are the nonlocal analogue of boundary conditions

Peridynamic Navier problem has a unique solution

- Dirichlet volume-constrained equilibrium equation

$$\begin{cases} -\mathcal{L}\mathbf{u} = \mathbf{b} & \text{on } \Omega \\ \mathbf{u} = \mathbf{0} & \text{on } \Omega_{\mathcal{I}} \end{cases}$$

- Dirichlet volume-constrained Navier equation

$$\begin{cases} \mathbf{u}_{tt}(\mathbf{x},t) - \mathcal{L}\mathbf{u}(\mathbf{x},t) = \mathbf{b}(\mathbf{x},t) & \forall\, (\mathbf{x},t) \in \Omega \times (0,T) \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0 & \forall\, \mathbf{x} \in \Omega \\ \mathbf{u}_t(\mathbf{x},0) = \mathbf{v}_0 & \forall\, \mathbf{x} \in \Omega \\ \mathbf{u}(\mathbf{x},t) = \mathbf{0} & \forall\, (\mathbf{x},t) \in \Omega_{\mathcal{I}} \times (0,T) \end{cases}$$

- Develop a variational formulation and demonstrate that the variational problem is well-posed
- Basis for a finite element method; appropriate numerical solutions are converging to *the* solution

Kinematics

- The extension state

$$\underline{e}\langle \mathbf{x}' - \mathbf{x} \rangle = |\underline{\mathbf{Y}}\langle \mathbf{x}' - \mathbf{x} \rangle| - |\mathbf{x}' - \mathbf{x}|$$

represents the change in the length of a bond $\mathbf{x}' - \mathbf{x}$ due to

- deformation where

$$\begin{aligned} \underline{\mathbf{Y}}\langle \mathbf{x}' - \mathbf{x} \rangle &= \mathbf{y}(\mathbf{x}') - \mathbf{y}(\mathbf{x}) && \forall \mathbf{x}' \\ &= \mathbf{u}(\mathbf{x}') + \mathbf{x}' - \big(\mathbf{u}(\mathbf{x}) + \mathbf{x}\big) && \forall \mathbf{x}' \end{aligned}$$

is the "deformation state" about $\mathbf{x}$

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Linearized kinematics

- Suppose $\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})$ is small (relative to the horizon, or extent of nonlocal interactions)
- Linearize the extension state with respect to $\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})$

$$\frac{\underline{e}\langle \mathbf{x}' - \mathbf{x} \rangle}{|\mathbf{x}' - \mathbf{x}|} = \left(\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})\right) \cdot \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^2} + R\left(\frac{|\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})|}{|\mathbf{x}' - \mathbf{x}|}\right)$$

and if we assume that the displacement $\mathbf{u}$ is differentiable

$$= \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|} \cdot \mathbf{E} \, \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|} + o\left(\frac{|\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})|}{|\mathbf{x}' - \mathbf{x}|}\right)$$

- $\mathbf{E} = \frac{1}{2}\left(\nabla \mathbf{u}(\mathbf{x}) + \nabla^T \mathbf{u}(\mathbf{x})\right)$ is the symmetric strain tensor

Energy density

Let $\xi := \mathbf{x}' - \mathbf{x}$ denote a bond; strain energy density function for an elastic constitutively linear anisotropic peridynamic heterogenous solid

$$W(\underline{e}) := \frac{\kappa}{2}\vartheta^2 + \frac{\eta}{2}\int_{\mathbb{R}^3} \varpi(\mathbf{x},\xi)\left(\underline{e}\langle\xi\rangle - \vartheta(\mathbf{x})\frac{|\xi|}{3}\right)^2 d\xi,$$

$\kappa = \kappa(\mathbf{x}), \mu = \mu(\mathbf{x})$ are the bulk, shear moduli, and

$$\eta(\mathbf{x}) \to \frac{15}{m(\mathbf{x})}\mu(\mathbf{x}) \qquad \text{(nonlocal shear)}$$

$$\vartheta(\mathbf{x}) = \frac{3}{m(\mathbf{x})}\int_{\mathbb{R}^3} |\xi|\,\varpi(\mathbf{x},\xi)\,\underline{e}\langle\xi\rangle\,d\xi \qquad \text{(nonlocal dilatation)}$$

$$m(\mathbf{x}) = \int_{\mathbb{R}^3} |\xi|^2\,\varpi(\mathbf{x},\xi)\,d\xi \qquad \text{(second moment)}$$

Influence function ϖ

- Influence function ϖ is a weighting function
- Select the horizon, or extent of the nonlocal interactions, to be the region over which ϖ is nonzero (think of a ball)
- If $\varpi(\mathbf{x}, \boldsymbol{\xi}) = \varpi(|\boldsymbol{\xi}|)$ then we have an isotropic body
- Ultimately, ϖ determines the amount of smoothing associated with the peridynamic Navier operator (how many derivatives the displacement field gains over the force field)
- For example, $m(\mathbf{x}) = \int_{\mathbb{R}^3} |\boldsymbol{\xi}|^2 \, \varpi(\mathbf{x}, \boldsymbol{\xi}) \, d\boldsymbol{\xi}$ may be finite even if $\varpi(\mathbf{x}, \boldsymbol{\xi})$ is not integrable, e.g.,

$$\varpi(\mathbf{x}, \boldsymbol{\xi}) \sim \frac{1}{|\boldsymbol{\xi}|^{3+2s}} = \frac{1}{|\mathbf{x}' - \mathbf{x}|^{3+2s}} \qquad 0 < s < 1$$

Quadratic energy density

- Let $\boldsymbol{\xi} := \mathbf{x}' - \mathbf{x}$ denote a bond; strain-energy density

$$W(\underline{e}) := \frac{\kappa}{2}\vartheta^2 + \frac{\eta}{2}\int_{\mathbb{R}^3} \varpi(\mathbf{x}, \boldsymbol{\xi})\left(\underline{e}\langle\boldsymbol{\xi}\rangle - \vartheta(\mathbf{x})\frac{|\boldsymbol{\xi}|}{3}\right)^2 d\boldsymbol{\xi},$$

- Linearize strain-energy density (Silling J Elast 2010)
- A quadratic energy approximation occurs when the extension state $\underline{e}$ and nonlocal dilatation ϑ are replaced by their linearizations with respect to $\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})$ grants

$$\tilde{W}(\mathbf{u}) = \frac{\kappa}{2}\big(\mathrm{Tr}(\mathcal{D}^*_\omega \mathbf{u})\big)^2 + \frac{\eta}{2}\int_{\mathbb{R}^3} \varpi(\mathbf{x}, \boldsymbol{\xi})\left(\mathrm{Tr}(\mathcal{D}^* \mathbf{u}) - \mathrm{Tr}(\mathcal{D}^*_\omega \mathbf{u})\frac{|\boldsymbol{\xi}|}{3}\right)^2 d\boldsymbol{\xi}$$

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Nonlocal vector calculus

- Nonlocal analogue of the relation $\text{div}^* = -\text{grad}$
- Nonlocal divergence of a tensor $\boldsymbol{\Psi}$

$$\mathcal{D}(\boldsymbol{\Psi})(\mathbf{x}) = \int_{\mathbb{R}^3} \big(\boldsymbol{\Psi}(\mathbf{x}, \mathbf{x}') + \boldsymbol{\Psi}(\mathbf{x}', \mathbf{x})\big) \cdot \boldsymbol{\alpha}(\mathbf{x}, \mathbf{x}')\, d\mathbf{x}'$$

Special choice of $\boldsymbol{\Psi}(\mathbf{x}, \mathbf{x}') = \omega(\mathbf{x}, \mathbf{x}')\mathbf{U}(\mathbf{x})$ leads to

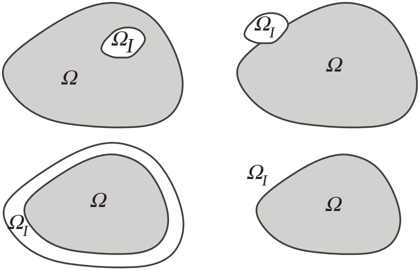
$$\mathcal{D}_\omega\big(\mathbf{U}\big)(\mathbf{x}) := \mathcal{D}\big(\omega(\mathbf{x}, \mathbf{x}')\mathbf{U}(\mathbf{x})\big)(\mathbf{x})$$

- Adjoint of the nonlocal divergence of a vector function $\mathbf{v}$

$$\mathcal{D}^*(\mathbf{v})(\mathbf{x}, \mathbf{x}') = -\big(\mathbf{v}(\mathbf{x}') - \mathbf{v}(\mathbf{x})\big) \otimes \boldsymbol{\alpha}(\mathbf{x}, \mathbf{x}')$$

$$\mathcal{D}^*_\omega\big(\mathbf{v}\big)(\mathbf{x}) = \int_{\mathbb{R}^3} \mathcal{D}^*(\mathbf{v})(\mathbf{x}, \mathbf{x}')\, \omega(\mathbf{x}, \mathbf{x}')\, d\mathbf{x}'$$

Interaction region $\Omega_{\mathcal{I}}$



- Four of the possible configurations for $\Omega_{\mathcal{I}}$, the nonlocal analogue of the boundary $\partial\Omega$
- $\Omega_{\mathcal{I}}$ is (typically) the union of spheres about $\mathbf{x} \in \Omega$; peridynamic horizon $\delta_{\mathbf{x}}$ is the radius of each sphere

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Potential energy

- Potential energy given a body force density $\mathbf{b}(\mathbf{x})$ is given by

$$E(\mathbf{u};\mathbf{b},\mathbf{g}) = \int_{\Omega\cup\Omega_{\mathcal{I}}} \tilde{W}(\mathbf{u})\,d\mathbf{x} - \int_{\Omega} \mathbf{u}\cdot\mathbf{b}\,d\mathbf{x}$$

- Displacement $\mathbf{u}$ can be characterized as the solution of the constrained optimization problem

$$\min_{\mathbf{u}\in U_0(\Omega\cup\Omega_{\mathcal{I}})} E(\mathbf{u};\mathbf{b},\mathbf{g}) \quad \text{subject to} \quad \mathbf{u}=\mathbf{0} \text{ for } \mathbf{x}\in\Omega_{\mathcal{I}},$$

where $U_0(\Omega\cup\Omega_{\mathcal{I}})$ is the peridynamic strain energy space

Euler-Lagrange equations

- A tedious calculation shows

$$\frac{d}{d\epsilon}E(\mathbf{u}+\epsilon\mathbf{v};\mathbf{b},\mathbf{g})\bigg|_{\epsilon=0} = \int_{\Omega\cup\Omega_{\mathcal{I}}} \mathcal{D}\big(\eta\varpi(\mathcal{D}^*\mathbf{u})^T\big)\cdot\mathbf{v}\,dx + \int_{\Omega\cup\Omega_{\mathcal{I}}} \mathcal{D}_\omega\big(\sigma\mathrm{Tr}(\mathcal{D}_\omega^*\mathbf{u})\mathbf{I}\big)\cdot\mathbf{v}\,dx - \int_{\Omega}\mathbf{b}\cdot\mathbf{v}\,dx \qquad \forall\mathbf{v}\in U_0(\Omega\cup\Omega_{\mathcal{I}})$$

- This gives the peridynamic Navier operator

$$-\mathcal{L} := \mathcal{D}\Big(\eta\varpi\big(\mathcal{D}^*\big)^T\Big) + \mathcal{D}_\omega\Big(\sigma\mathrm{Tr}\big(\mathcal{D}_\omega^*\big)\mathbf{I}\Big), \qquad \sigma = \kappa - \frac{\eta m}{9}$$

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Isotropic peridynamic material

- If $\frac{\eta}{15}\int_{B_\varepsilon(0)}|\mathbf{x}|^2\varpi(|\mathbf{x}|)\,d\mathbf{x}\to\mu$ as $\varepsilon\to 0$ where $\varpi(|\mathbf{x}'-\mathbf{x}|)$ is positive on $B_\varepsilon(0)$, the sphere of radius ε centered at the origin, then

$$\mathcal{L}\mathbf{u}\to\mu\nabla\cdot\nabla\mathbf{u}+(\mu+\lambda)\nabla\,\nabla\cdot\mathbf{u}\quad\text{in}\quad H^{-1}(\mathbb{R}^3)\text{ as }\varepsilon\to 0$$

- **Take home message**: $\mathcal{L}$ is the peridynamic analogue of the Navier operator

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Peridynamic Navier operator

The idea of the proof is to establish

$$\mathcal{D}\Big(\eta\varpi\big(\mathcal{D}^*\big)^T\Big) \to -\mu\nabla\cdot\nabla\mathbf{u} - \mu\nabla\,\nabla\cdot\mathbf{u}$$

$$\mathcal{D}_\omega\big(\sigma\mathrm{Tr}\big(\mathcal{D}_\omega^*\big)\mathbf{I}\big) \to -\lambda\nabla\,\nabla\cdot\mathbf{u}$$

Both these facts use results about the various components established in

Du, Gunzburger, Lehoucq, Zhou, *A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws* Mathematical Models and Methods in Applied Sciences (M3AS), Volume 23, pp. 493-540, 2013, DOI:10.1142/S0218202512500546

Peridynamic Navier problem

- $-\mathcal{L} := \mathcal{D}\big(\eta\varpi\big(\mathcal{D}^*\big)^T\big) + \mathcal{D}_\omega\big(\sigma \mathrm{Tr}\big(\mathcal{D}_\omega^*\big)\mathbf{I}\big), \qquad \sigma = \kappa - \frac{\eta m}{9}$
- Dirichlet volume-constrained equilibrium equation

$$\begin{cases} -\mathcal{L}\mathbf{u} = \mathbf{b} & \text{on } \Omega \\ \mathbf{u} = \mathbf{0} & \text{on } \Omega_{\mathcal{I}} \end{cases}$$

- Dirichlet volume-constrained Navier equation

$$\begin{cases} \mathbf{u}_{tt}(\mathbf{x},t) - \mathcal{L}\mathbf{u}(\mathbf{x},t) = \mathbf{b}(\mathbf{x},t) & \forall\, (\mathbf{x},t) \in \Omega \times (0,T) \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0 & \forall\, \mathbf{x} \in \Omega \\ \mathbf{u}_t(\mathbf{x},0) = \mathbf{v}_0 & \forall\, \mathbf{x} \in \Omega \\ \mathbf{u}(\mathbf{x},t) = \mathbf{0} & \forall\, (\mathbf{x},t) \in \Omega_{\mathcal{I}} \times (0,T) \end{cases}$$

Variational problem

Given $\mathbf{b} \in U_0^*(\Omega \cup \Omega_{\mathcal{I}})$ and $\mathbf{u} = \mathbf{0}$ on $\Omega_{\mathcal{I}}$: Find the displacement $\mathbf{u} \in U_0(\Omega \cup \Omega_{\mathcal{I}})$ such that

$$B(\mathbf{u}, \mathbf{v}) = F(\mathbf{v}) \qquad \forall\, \mathbf{v} \in U_0(\Omega \cup \Omega_{\mathcal{I}}),$$

where the internal and external works are given by

$$\begin{aligned} B(\mathbf{u}, \mathbf{v}) &= \int_{\Omega\cup\Omega_{\mathcal{I}}} \int_{\Omega\cup\Omega_{\mathcal{I}}} \eta\varpi \operatorname{Tr}(\mathcal{D}^*\mathbf{u}) \operatorname{Tr}(\mathcal{D}^*\mathbf{v})\, d\mathbf{x}' d\mathbf{x} \\ &\quad + \int_{\Omega\cup\Omega_{\mathcal{I}}} \sigma \operatorname{Tr}(\mathcal{D}_\omega^*\mathbf{u}) \operatorname{Tr}(\mathcal{D}_\omega^*\mathbf{v})\, d\mathbf{x} \\ F(\mathbf{v}) &= \int_{\Omega} \mathbf{b} \cdot \mathbf{v}\, d\mathbf{x} \end{aligned}$$

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Internal work

- Peridynamic internal work

$$
\begin{aligned}
B(\mathbf{u},\mathbf{v}) = & \int_{\Omega\cup\Omega_{\mathcal{I}}}\int_{\Omega\cup\Omega_{\mathcal{I}}} \eta\varpi \mathrm{Tr}(\mathcal{D}^{*}\mathbf{u})\mathrm{Tr}(\mathcal{D}^{*}\mathbf{v})\, d\mathbf{x}' d\mathbf{x} \\
& + \int_{\Omega\cup\Omega_{\mathcal{I}}} \sigma \mathrm{Tr}(\mathcal{D}^{*}_{\omega}\mathbf{u})\mathrm{Tr}(\mathcal{D}^{*}_{\omega}\mathbf{v})\, d\mathbf{x} \\
& \to \int_{\Omega} \mu \nabla\mathbf{u} : \nabla\mathbf{v}\, d\mathbf{x} + \int_{\Omega} (\mu+\lambda)(\nabla\cdot\mathbf{u})\,(\nabla\cdot\mathbf{v})\, d\mathbf{x}
\end{aligned}
$$

- When the constraint $\mathbf{u},\mathbf{v}=\mathbf{0}$ on $\Omega_{\mathcal{I}}$ is imposed (the Dirichlet volume-constraint), then the internal work is positive
- Remove the Dirichlet volume constraint and there are six rigid displacements $\mathbf{u}(\mathbf{x}) = \mathbf{S}\mathbf{x}+\mathbf{c}, \quad \mathbf{S} = -\mathbf{S}^{T}$; $B(\mathbf{S}\mathbf{x}+\mathbf{c},\mathbf{v}) = B(\mathbf{u},\mathbf{S}\mathbf{x}+\mathbf{c}) = 0$

Well-posed problem

Theorem: The variational problem: Given $\mathbf{b} \in U_0^*(\Omega \cup \Omega_{\mathcal{I}})$ and $\mathbf{u} = \mathbf{0}$ on $\Omega_{\mathcal{I}}$: Find $\mathbf{u} \in U(\Omega \cup \Omega_{\mathcal{I}})$ such that

$$B(\mathbf{u}, \mathbf{v}) = F(\mathbf{v}) \qquad \forall\, \mathbf{v} \in U_0(\Omega \cup \Omega_{\mathcal{I}}),$$

is well-posed and the peridynamic energy space is the same as the space of square integrable functions, i.e.
$U_0(\Omega \cup \Omega_{\mathcal{I}}) \equiv L^2(\Omega \cup \Omega_{\mathcal{I}})$
Proof: Use the Lax-Milgram theorem.
See Du, Gunzburger, Lehoucq, Zhou, *Analysis of the volume-constrained peridynamic Navier equation of linear elasticity*, Journal of Elasticity, Online First, 2012, DOI:10.1007/s10659-012-9418-x

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Smoothing of the peridynamic Navier operator

- Classical Navier equation renders the displacement field two more derivatives smoother than the force field
- Because we assumed that the square of the influence function is integrable, the peridynamic Navier equation renders the displacement field no smoother than the force field—is this at odds with elastic behavior?
- Peridynamic Navier equation can render the displacement field up to $2s$, $0 < s < 1$ derivatives smoother than the force field but the influence function must be singular and non-integrable
- Non-integrable influence functions are a challenge to the so called particle discretization; replace with linear elements (or some other appropriate choice) discontinuous across elements

Volume-constraints and boundary conditions

- The space of square integrable functions allows jump discontinuous functions and so boundary conditions are ill-defined
- Using a non-integrable influence function where the degree of singularity is $0 < s < \frac{1}{2}$ renders a function space with jump discontinuities
- Using a non-integrable influence function where the degree of singularity is $\frac{1}{2} < s < 1$ renders a function space **without** jump discontinuities—but then the role of peridynamic mechanics is not clear

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