Stabilization of Reduced Order Models (ROMs) via Controllers and ROM-Based Control

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Florida State University (FSU) Visit
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ROM Stabilization via Pole Placement: Problem Statement

Full Order Model (FOM)

$$\begin{aligned}\dot{\mathbf{x}}_N &= \mathbf{A}_N \mathbf{x}_N + \mathbf{B}_N \mathbf{u}_N \\ \mathbf{y}_N &= \mathbf{C}_N \mathbf{x}_N\end{aligned}$$

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- Approximate FOM solution $\mathbf{x}_N \in \mathbb{R}^N$ by ROM solution $\mathbf{x}_M \in \mathbb{R}^M$, with $M << N$:

$$\mathbf{x}_N = \mathbf{\Phi}_M \mathbf{x}_M$$

  where $\mathbf{\Phi}_M$ =reduced basis (e.g., POD basis).

- ROM system matrices are given by:

$$\mathbf{A}_M = \mathbf{\Phi}_M^T \mathbf{A}_N \mathbf{\Phi}_M, \quad \mathbf{B}_M = \mathbf{\Phi}_M^T \mathbf{B}_N, \quad \mathbf{C}_M = \mathbf{C}_N \mathbf{\Phi}_M$$

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  - Pick control matrix $\mathbf{B}_C$. Add control $\mathbf{B}_C \mathbf{u}_C$ to ROM system.
  - Pick desired poles of $\mathbf{A}_M$. Assume control $\mathbf{u}_C = -\mathbf{K}_C \mathbf{x}_M$.

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Reduced Order Model (ROM)

$$\begin{aligned} \dot{\mathbf{x}}_M &= \underbrace{(\mathbf{A}_M - \mathbf{B}_C \mathbf{K}_C)}_{\tilde{\mathbf{A}}_M} \mathbf{x}_M + \mathbf{B}_M \mathbf{u}_M \\ \mathbf{y}_M &= \mathbf{C}_M \mathbf{x}_M \end{aligned}$$

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  - Pick desired poles of $\mathbf{A}_M$. Assume control $\mathbf{u}_C = -\mathbf{K}_C \mathbf{x}_M$.
  - Compute feedback $\mathbf{K}_C$ such that $\mathbf{A}_M - \mathbf{B}_C \mathbf{K}_C$ has desired poles.

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Naïve Algorithm

1. Pick a matrix $\mathbf{B}_C$.
2. Use Kalman decomposition to isolate controllable and observable part of $\mathbf{A}_M$ and $\mathbf{B}_C$, call them $\mathbf{A}_M^{co} = \mathbf{U}\mathbf{A}_M\mathbf{U}^T$ and $\mathbf{B}_C^{co} = \mathbf{U}\mathbf{B}_C$.
3. Compute eigenvalues $\lambda_1, ..., \lambda_{N_{co}}$ of $\mathbf{A}_M^{co}$.
4. For $i = 1$ to N_{co}, set[1]

$$\bar{\lambda}_i = \min\{Re(\lambda_i), -Re(\lambda_i)\} + i \cdot Im(\lambda_i)$$

5. Solve pole placement problem: find $\mathbf{K}_C$ such that $\mathbf{A}_M^{co} - \mathbf{B}_C^{co}\mathbf{K}_C$ has eigenvalues $\bar{\lambda}_i$.
6. Set $\mathbf{A}_M = \mathbf{U}^T(\mathbf{A}_M^{co} - \mathbf{B}_C^{co}\mathbf{K}_C)\mathbf{U}$.
7. Run ROM with this new (stable) $\mathbf{A}_M$.

[1] If $\bar{\lambda}_i = \min\{Re(\lambda_i), 0\} + i \cdot Im(\lambda_i)$, there seem to be issues placing poles with multiplicity > 1.

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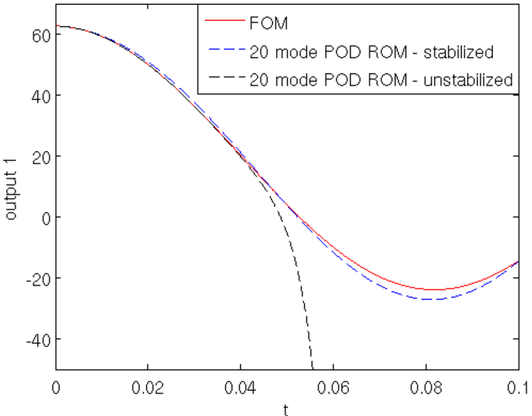
Numerical Experiment: ISS Structural Model [1]

- FOM = stable LTI system, 1 input, 1 output.
- Input: $\mathbf{u}(t) = (1 \times 10^4)\delta_{t=0}$.
- POD basis of size $M = 20$ constructed from 2000 snapshots until $t = 0.1$.
- $\mathbf{B}_C = \mathbf{1}_M$.
- $M = 20$ ROM has 4 unstable eigenvalues, which are modified.



Open Questions

- How to pick control matrix $\mathbf{B}_C$?
- How to pick desired eigenvalues of $\mathbf{A}_M - \mathbf{B}_C\mathbf{K}_C$?
- Need to ensure that modified ROM system dynamics are not "too far" from FOM system dynamics, e.g., through optimization problem:

$$\min_{\mathbf{B}_C, \mathbf{K}_C} \sum_{k=1}^{K} (\mathbf{y}_M^k - \mathbf{y}_M^k)^T (\mathbf{y}_N^k - \mathbf{y}_M^k)$$

subject to constraints:

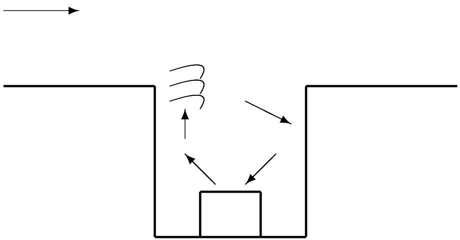
$$\begin{aligned} \dot{\mathbf{x}}_M &= \tilde{\mathbf{A}}_M \mathbf{x}_M + \mathbf{B}_M \mathbf{u}_M \\ \mathbf{y}_M &= \mathbf{C}_M \mathbf{x}_M \end{aligned}$$

and

$$\tilde{\mathbf{A}}_M \equiv \mathbf{A}_M - \mathbf{B}_C \mathbf{K}_C$$

is stable.

Target Cavity Flow Control Problem



- **Configuration/Plant:** compressible non-linear fluid flow over open cavity containing components.

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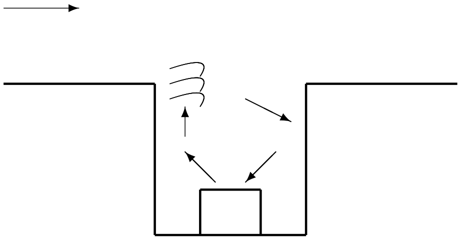

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- **Physical Control Problem:** using upstream actuation, control oscillations within cavity caused by pressure fluctuations propagating between downstream wall and shear layer.

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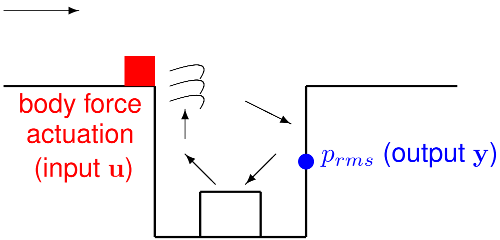


- **Configuration/Plant:** compressible non-linear fluid flow over open cavity containing components.
- **Physical Control Problem:** using upstream actuation, control oscillations within cavity caused by pressure fluctuations propagating between downstream wall and shear layer.
- **Mathematical Control Problem:** compute optimal body-force actuation input $\mathbf{u}_{opt}$ to minimize the RMS pressure halfway up the downstream wall.

$$\text{input } \mathbf{u}: \quad \mathbf{q}^T = \begin{pmatrix} 0, & f(t), & 0 & 0 & 0 \end{pmatrix}^T$$

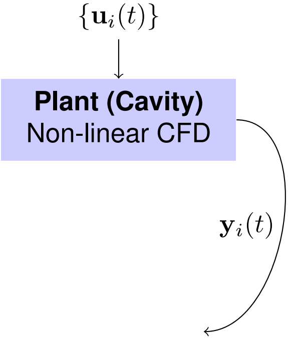
$$\text{output } \mathbf{y}: \quad p_{rms} = \sqrt{\frac{1}{K}\sum_{i=1}^{K}(p(t_k) - \bar{p})^2}$$

ROM-Based Cavity Flow Control Road Map

1. Collect snapshots from non-linear high-fidelity CFD cavity simulation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}_i), \qquad \mathbf{y}_i = \mathbf{h}(\mathbf{x}, \mathbf{u}_i)$$

for some set of inputs $\{\mathbf{u}_i(t)\}$, and construct empirical basis (POD, BPOD) from this snapshot set.



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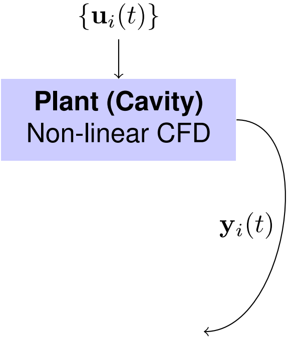
2. Build a ROM for the fluid system, or approximation of fluid system.



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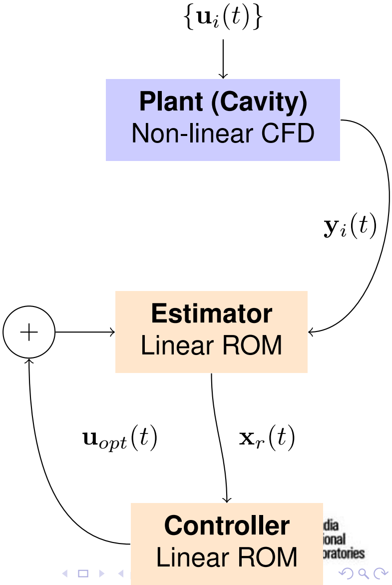

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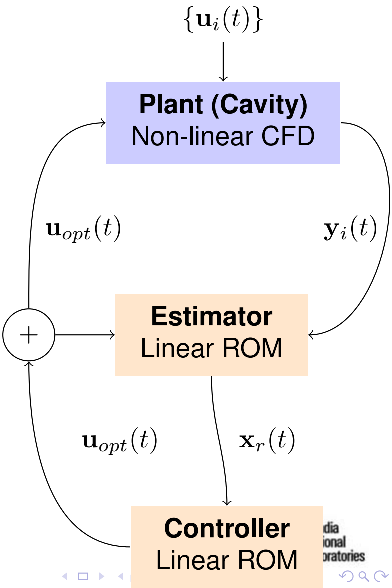
4. Apply ROM-based controller to non-linear cavity problem.



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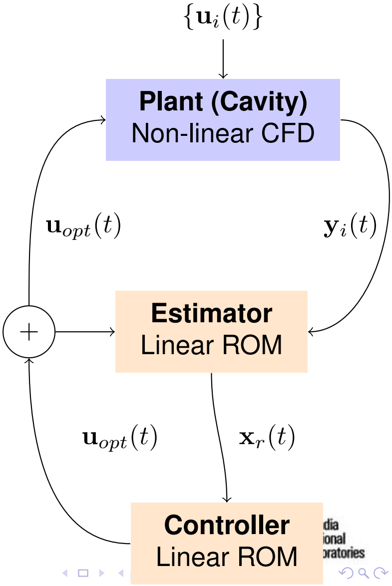

Questions

- Rules of thumb for checking controllability?
- Rules of thumb for tuning of controller parameters?

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[1] A.C. Antoulas, D.C. Sorensen, S. Gugercin. A survey of model reduction methods for large-scale systems. *Contemporary Mathematics* **280** 2001.

[2] K.J. Astrom, R.M. Murray. Feedback systems: an introduction for scientists and engineers. Princeson University Press, 2008.

[3] S.J. Illingworth, A.S. Morgans, C.W. Rowley. Feedback control of flow resonances using balanced reduced-order models. *J. Sound and Vibration* **330**(8) 1567–1581 (2001).