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Analysis of linear systems driven by non-Gaussian noise

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Abstract

A method is developed for approximating the properties of the state of a linear dynamic system driven by a broad class of non-Gaussian noise, namely, by polynomials of filtered Gaussian processes. The method involves four steps. First, the mean and correlation functions of the state of the system are calculated from those of the input noise. Second, higher order moments of the state are calculated based on Itô's formula for continuous semimartingales. It is shown that equations governing these moments are closed, so that moment of any order of the state can be calculated exactly. Third, a conceptually simple technique, which resembles the Galerkin method for solving differential equations, is proposed for constructing approximations for the marginal distribution of the state from its moments. Fourth, translation models are calibrated to representations of the marginal distributions of the state as well as its second moment properties. The resulting models can then be utilized to estimate properties of the state, such as the mean rate at which the state exits a safe set. The proposed method is applied to assess the turbulence-induced random vibration of a flexible plate motivated by problems in wind engineering.

1 Introduction

Classical linear random vibration theory provides equations for calculating the first two moments of the state $\mathbf{X}(t)$ of a linear system subjected to input or driving noise characterized by its first two moments. The theory provides no information beyond the second moment properties of $\mathbf{X}(t)$ unless the noise is Gaussian, in which case the state $\mathbf{X}(t)$ is a Gaussian process. There are no efficient methods for calculating properties of $\mathbf{X}(t)$, and functionals of this process, for the general case of non-Gaussian driving noise. This study develops a practical and efficient method for constructing approximate representations for the state $\mathbf{X}(t)$ of a linear dynamic system driven by a class of non-Gaussian noise that can be used to calculate properties or functionals of $\mathbf{X}(t)$. Developments are based on linear random vibration (Soong and Grigoriu, 1993, Chapter 5), Itô's formula for continuous semimartingales (Grigoriu, 2002, Section 4.6), an elementary solution for the problem of moments (Shohat and Tamarkin, 1943), and translation models $\mathbf{X}_{\mathrm{T}}(t)$ for $\mathbf{X}(t)$ (Grigoriu, 1995, Section 3.1.1). Herein, we consider the driving noise to be from the class of non-Gaussian processes defined by polynomials of filtered Gaussian processes (Grigoriu, 1986; Grigoriu and Ariaratnam, 1988).

In Grigoriu and Ariaratnam (1988), the objective was to calculate the mean upcrossing rate of level x for scalar process $X(t)$ driven by a polynomial of a Gaussian process. Hermite approximations were developed for the joint density of $(X, \mathrm{d}X/\mathrm{d}t)$, and the approximations were used to find the mean upcrossing rate of level x. In the current study, the objective is to estimate the probability law of $\mathbf{X}(t)$, and the development is not limited to scalar-valued processes. It is assumed herein that $\mathbf{X}(t)$ can be approximated by a translation process, so that the marginal distribution of $\mathbf{X}(t)$ and its correlation function are needed. The construction of the marginal distribution is as in Grigoriu and Ariaratnam (1988), generalized for the case of vector-valued processes. The construction of the correlation function for the input involves novel aspects of linear random vibration. Translation models are very flexible and have been

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used in a wide variety of modeling applications, including wind pressure fluctuations on bluff bodies (Gioffrè et al., 2000), the description of irregular masonry walls (Spence et al., 2008), and damage in glass plates (Gioffrè and Gusella, 2002).

For demonstration purposes, the vibration response of a flexible plate subjected to turbulent flow is presented. The random pressure fluctuations applied to the plate surface are proportional to the square of the velocity field, which is assumed to be Gaussian. Hence the applied pressure field is non-Gaussian. Muscolino and co-workers worked on a related problem assuming a SDOF oscillator (Benfratello et al., 1996; Gullo et al., 1998).

2 Correlation function and moments

Let $\mathbf{X}(t)$ be an $\mathbb{R}^d$-valued stochastic process defined by the following linear differential equation

$$\dot{\mathbf{X}}(t) = \mathbf{a}(t)\,\mathbf{X}(t) + \mathbf{b}(t)\,\mathbf{Z}(t),\ t \geq 0, \tag{1}$$

where $\mathbf{a}(t)$ and $\mathbf{b}(t)$ are $d \times d$ and $d \times d'$ matrices with real-valued, time-dependent entries, $\mathbf{Z}(t)$ denotes an $\mathbb{R}^{d'}$-valued input process, and $\mathbf{X}(0)$ is the initial state specified by its mean vector $\boldsymbol{\mu}_0 = \mathrm{E}[\mathbf{X}(0)]$ and covariance matrix, $\boldsymbol{\gamma}_0 = \mathrm{E}[(\mathbf{X}(0)-\boldsymbol{\mu}_0)\,(\mathbf{X}(0)-\boldsymbol{\mu}_0)']$. Vector $\dot{\mathbf{X}}(t)$ has coordinates $\dot{X}_k(t) = \mathrm{d}X_k(t)/\mathrm{d}t$, $k = 1, \ldots, d$. It is assumed that input $\mathbf{Z}(t)$ is a weakly stationary process with mean $\boldsymbol{\mu}_{\mathbf{Z}} = \mathrm{E}[\mathbf{Z}(t)]$ and covariance function $\mathbf{c}_{\mathbf{Z}}(\tau) = \mathrm{E}[(\mathbf{Z}(t+\tau) - \boldsymbol{\mu}_{\mathbf{Z}})\,(\mathbf{Z}(t+\tau) - \boldsymbol{\mu}_{\mathbf{Z}})']$.

It can be shown (Soong and Grigoriu, 1993, Section 5.2.1) that the time evolution of $\boldsymbol{\mu}(t) = \mathrm{E}[\mathbf{X}(t)]$ and $\mathbf{c}(t,s) = \mathrm{Cov}[\mathbf{X}(t), \mathbf{X}(s)]$, the mean and covariance of state $\mathbf{X}(t)$ described by Eq. (1), are given by

$$\dot{\boldsymbol{\mu}}(t) = \mathbf{a}(t)\,\boldsymbol{\mu}(t) + \mathbf{b}(t)\,\boldsymbol{\mu}_{\mathbf{Z}},\ t \geq 0, \tag{2}$$

and

$$\frac{\partial}{\partial t}\mathbf{c}(t,s) = \mathbf{a}(t)\,\mathbf{c}(t,s) + \mathbf{d}(t,s),\ t > s, \tag{3}$$

where

$$\mathbf{d}(t,s) = \mathbf{b}(t)\int_0^s \mathbf{c}_{\mathbf{Z}}(t,u)\,\mathbf{b}(u)'\,\boldsymbol{\theta}(s,u)'\,\mathrm{d}u, \tag{4}$$

and $\boldsymbol{\theta}(t,s)$ is a system property satisfying the differential equation $\frac{\partial}{\partial t}\boldsymbol{\theta}(t,s) = \mathbf{a}(t)\,\boldsymbol{\theta}(t,s)$, $t \geq s$, with $\boldsymbol{\theta}(s,s)$ equal to the identity matrix $\forall s \geq 0$.

Next let $\mathbf{Y}(t)$ be an $\mathbb{R}^n$-valued Gaussian process defined by

$$\mathrm{d}\mathbf{Y}(t) = \boldsymbol{\alpha}(t)\,\mathbf{Y}(t)\,\mathrm{d}t + \boldsymbol{\beta}(t)\,\mathrm{d}\mathbf{B}(t),\ t \geq 0, \tag{5}$$

where $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ are $n \times n$ and $n \times n'$ matrices, respectively, with real-valued, time-dependent entries, $\mathbf{B}(t)$ is an $\mathbb{R}^{n'}$-valued Brownian motion with independent coordinates, and $\mathbf{Y}(0) \sim N(\boldsymbol{\mu}_{\mathbf{Y},0}, \boldsymbol{\gamma}_{\mathbf{Y},0})$ is a Gaussian random vector independent of $\mathbf{B}(t)$ that defines the (random) value for $\mathbf{Y}$ at time $t = 0$. We refer to $\mathbf{Y}(t)$ as a filtered Gaussian process since it is the output of a linear filter to Gaussian white noise. It is assumed that process $\mathbf{Z}(t)$ that serves as the driving noise in Eq. (1) is expressed as a polynomial function of $\mathbf{Y}(t)$, the solution to Eq. (5).

The coordinates $\{Z_k(t)\}$ of $\mathbf{Z}(t)$ are polynomials of $\mathbf{Y}(t)$ that have the form

$$Z_k(t) = \sum_{m_1,\ldots,m_n \geq 0} \lambda_{k;m_1,\ldots,m_n} \prod_{r=1}^{n} Y_r(t)^{m_r},\ k = 1, \ldots, d', \tag{6}$$

where $\lambda_{k;m_1,\ldots,m_n}$ are real-valued coefficients and $m_r \geq 0$ are integer powers. Indeed, the class of input processes defined by Eq. (6) are very general and can represent a broad range of non-Gaussian probability laws. The mean and correlation functions of $\mathbf{Z}(t)$ have the expressions

$$\mathrm{E}[Z_k(t)] = \sum_{m_1,\dots,m_n \geq 0} \lambda_{k;m_1,\dots,m_n} \mathrm{E}\Big[\prod_{r=1}^{n} Y_r(t)^{m_r}\Big] \text{ and}$$

$$\mathrm{E}[Z_k(s)\, Z_l(t)] = \sum_{m_1,\dots,m_n \geq 0} \lambda_{k;m_1,\dots,m_n} \sum_{m'_1,\dots,m'_n \geq 0} \lambda_{l;m'_1,\dots,m'_n} \mathrm{E}\Big[\prod_{r=1}^{n} Y_r(s)^{m_r} \prod_{q=1}^{n} Y_q(t)^{m'_q}\Big]. \tag{7}$$

The expectations in the second formula of Eq. (7) can be calculated from the observation that, for $t > s$,

$$Y_q(t) = \sum_{i=1}^{n} \theta_{qi}(t,s)\, Y_i(s) + G_q = V_q(t,s) + G_q, \tag{8}$$

where $(G_1, \dots, G_n)$ is a Gaussian vector independent of $\mathbf{Y}(s)$ with mean $\mathbf{0}$ and covariance matrix $\int_s^t \boldsymbol{\theta}(t,u)\, \boldsymbol{\beta}(u)\, \boldsymbol{\beta}(u)'\, \boldsymbol{\theta}(t,u)'\, \mathrm{d}u$. Accordingly, the expectations in the second formula of Eq. (7) have the form

$$\begin{aligned}\mathrm{E}\Big[\prod_{r=1}^{n} Y_r(s)^{m_r} \prod_{q=1}^{n} Y_q(t)^{m'_q}\Big] &= \mathrm{E}\Big[\prod_{r=1}^{n} Y_r(s)^{m_r} \prod_{q=1}^{n} \Big(V_q(t,s) + G_q\Big)^{m'_q}\Big] \\ &= \sum_{p_1=0}^{m'_1} \cdots \sum_{p_n=0}^{m'_n} \frac{m'_1!}{p_1!\,(m'_1 - p_1)!} \cdots \frac{m'_n!}{p_n!\,(m'_n - p_n)!} \times \\ &\qquad \mathrm{E}\Big[\prod_{r=1}^{n} Y_r(s)^{m_r} \prod_{i=1}^{n} V_i(t,s)^{p_i}\Big] \mathrm{E}\Big[\prod_{i=1}^{n} G_i^{m'_i - p_i}\Big].\end{aligned} \tag{9}$$

The expectations in the expression of $\mathrm{E}\big[\prod_{r=1}^{n} Y_r(s)^{m_r} \prod_{q=1}^{n} Y_q(t)^{m'_q}\big]$ represent higher order moments of Gaussian variables, that is, the coordinates of $\mathbf{Y}(s)$ and of the coordinates of $(G_1, \dots, G_n)$. Properties of Gaussian variables can be used to calculate these expectations; see, for example, Papoulis (1991, Section 5.4).

3 Approximate model for the state

The method in Section 2 delivers the marginal moments of any order of the state $\mathbf{X}(t)$ of a linear system driven by a polynomial of a filtered Gaussian process, as well as the correlation function of this process. In this section, we use this information to construct a translation model $\mathbf{X}_{\mathrm{T}}(t)$ for $\mathbf{X}(t)$. The construction involves two steps. First, approximations are developed for the marginal distributions of $\mathbf{X}(t)$ based on higher order moments of this process. Second, $\mathbf{X}_{\mathrm{T}}(t)$ is selected to match the marginal distributions of $\mathbf{X}(t)$ and approximate, or match whenever possible, the correlation function of this process.

3.1 Marginal distributions

The construction of approximations for the marginal distributions of $\mathbf{X}(t)$ from their moments constitutes the solution of an inverse problem, referred to as the problem of moments (Shohat and Tamarkin, 1943). Our objective is to develop a conceptually simple and efficient method for solving the problem of moments for the marginal distributions of $\mathbf{X}(t)$. The posed problem has a solution since the prescribed moments correspond to distributions. However, it does not have a unique solution since the available information on the probability law of $\mathbf{X}(t)$ is incomplete. We construct approximations for the marginal distributions of $\mathbf{X}(t)$ that are optimal in some sense. For clarity, we limit the following discussion to stationary, scalar-valued processes $\mathbf{X}(t) = X(t)$; the extension to the case of non-stationary $\mathbf{R}^d$-valued processes is direct and will be used in Section 4.

Suppose $X(t)$ is a real-valued stationary process with unknown marginal distribution $F(x)$ and known first $n \geq 1$ marginal moments $\mu(r) = \mathrm{E}[X(t)^r]$, $r = 1, \dots, n$. As previously stated, our objective is to construct an approximation $\tilde{F}(x)$ for $F(x)$ based on the available information, that is, the moments $\mu(r)$, $r = 1, \ldots, n$, of $X(t)$. Various approximations can be constructed for $F(x)$; we shall consider approximations $\tilde{F}(x)$ and $\tilde{f}(x)$ for the marginal distribution and density functions, $F(x)$ and $f(x)$, of $X(t)$ that resemble the Galerkin method (Grigoriu and Lind, 1980). It is assume that $\tilde{f}(x)$ is a member of the linear space spanned by a finite collection of densities $\{f_k(x)\}$, $k = 1, \ldots, b$, that is,

$$\tilde{f}(x) = \sum_{k=1}^{b} p_k \, f_k(x), \tag{10}$$

where the constants $\{p_k\}$ are such that $p_k \geq 0$, $k = 1, \ldots, b$, and $\sum_{k=1}^{b} p_k = 1$, so that $\tilde{f}(x)$ is a density for any parameters $\{p_k\}$ satisfying the above constraints. The corresponding approximate distribution function has the expression

$$\tilde{F}(x) = \sum_{k=1}^{b} p_k \, F_k(x) \tag{11}$$

with $F_k(x) = \int_{u \leq x} f_k(u) \, \mathrm{d}u$, $k = 1, \ldots, b$. The densities $\{f_k(x)\}$ may or may not be completely specified; for example, some densities may depend on a collection of unknown parameters. However, we will consider the case in which the densities $\{f_k(x)\}$ are fully specified, so that the probabilities $\{p_k\}$ are the only uncertain parameters of $\tilde{f}(x)$ and $\tilde{F}(x)$.

The optimal values for $\{p_k\}$ defined by Eqs. (10) and (11) minimize the discrepancy

$$e(p_1, \ldots, p_b) = \sum_{r=1}^{m} v(r) \left(\mu(r) - \sum_{k=1}^{b} p_k \, \mu_k(r) \right)^2 \tag{12}$$

between the exact moments $\mu(r) = \mathrm{E}[X(t)^r]$ of $X(t)$ and their approximations

$$\tilde{\mu}(r) = \int x^r \, \tilde{f}(x) \, \mathrm{d}x = \int x^r \sum_{k=1}^{b} p_k \, f_k(x) \, \mathrm{d}x = \sum_{k=1}^{b} p_k \int x^r \, f_k(x) \, \mathrm{d}x = \sum_{k=1}^{b} p_k \, \mu_k(r), \tag{13}$$

under the constraints $p_k \geq 0$, $k = 1, \ldots, b$, and $\sum_{k=1}^{b} p_k = 1$. Further, $m \geq 1$ is an integer denoting the largest moment considered in the analysis and $v(r) \geq 0$ denotes a weighting function, for example, $v(r) = 1/\mu(r)^2$. As in the Galerkin method, the accuracy of $\tilde{f}(x)$ is essentially controlled by the properties of the densities $\{f_k(x)\}$.

Heuristic arguments and/or prior information may be used to select the basis $\{f_k(x)\}$ for $\tilde{f}(x)$. For example, we may require that the skewness and kurtosis coefficients of $\{f_k(x)\}$ bracket the corresponding coefficients of $f(x)$ and that the first two moments of $\tilde{f}(x)$ coincide with those of $f(x)$ for all value of $\{p_k\}$. In addition, knowledge regarding the behavior of dynamic systems may provide information on the properties of $f(x)$, for example, the state of linear systems under driving noise with a symmetric density is known to have an even density.

3.2 Translation model

Let $\mathbf{X}(t)$ be a stationary $\mathbb{R}^d$-valued stochastic process specified partially by its second moment properties and marginal distributions $F_i(x)$, or approximations $\tilde{F}_i(x)$, $i = 1, \ldots, d$, of these distributions. Methods for calculating these properties of $\mathbf{X}(t)$ have been discussed above. In this section, we develop translation models for $\mathbf{X}(t)$, denoted by $\mathbf{X}_{\mathrm{T}}(t)$, which can be used to approximate properties of $\mathbf{X}(t)$. For example, let $\mathbf{X}(t)$ denote the state of a dynamic system, and suppose that whenever the state resides within a "safe set" $\mathcal{S}$, the system performance is known to be satisfactory. A typical property of interest for this scenario is the mean rate at which $\mathbf{X}(t)$ exits out of $\mathcal{S}$, since this property can be used to estimate system reliability (Soong and Grigoriu, 1993, Chapter 7).

Translation models are defined by nonlinear transformations of Gaussian random functions (Grigoriu, 1995, Section 3.1.1). By careful selection of the properties of the Gaussian function and the functional form of the transformation, it is possible to calibrate the translation model to match a wide variety of prescribed marginal CDFs and second moment properties. However, translation models matching exactly specified marginal distributions and second moment properties may not exist. If there exists no translation model matching these properties, we construct an $\mathbb{R}^d$-valued translation model $\mathbf{X}_{\mathrm{T}}(t)$ that matches exactly the marginal distributions $\{F_i(x)\}$ of $\mathbf{X}(t)$, or their approximate representations $\{\tilde{F}_i(x)\}$, and characterizes approximately the covariance functions of $\mathbf{X}(t)$.

Let

$$X_{\mathrm{T},i}(t) = F_i^{-1} \circ \Phi\big(G_i(t)\big), \quad i = 1, \ldots, d, \tag{14}$$

denote the coordinates of $\mathbf{X}_{\mathrm{T}}(t)$, where $\mathbf{G}(t) = (G_1(t), \ldots, G_d(t))$ is an $\mathbb{R}^d$-valued stationary Gaussian process with $\mathrm{E}[G_i(t)] = 0$, $\mathrm{E}[G_i(t)^2] = 1$, and $\mathrm{E}[G_i(t+\tau)\, G_j(t)] = \rho_{ij}(\tau)$. The marginal distributions of $\mathbf{X}_{\mathrm{T}}(t)$ in Eq. (14) coincide with those of $\mathbf{X}(t)$. The covariance function of $\mathbf{X}_{\mathrm{T}}(t)$ depends on the mapping in Eq. (14) and the covariance function of $\mathbf{G}(t)$. If there are no correlation functions $\{\rho_{ij}(\tau)\}$ such that the scaled covariance functions

$$\xi_{\mathrm{T},ij}(\tau) = \frac{\mathrm{E}\{(X_{\mathrm{T},i}(t+\tau) - \mathrm{E}[X_{\mathrm{T},i}(t+\tau)])\,(X_{\mathrm{T},j}(t) - \mathrm{E}[X_{\mathrm{T},j}(t)])\}}{\mathrm{Std}[X_{\mathrm{T},i}(t+\tau)]\,\mathrm{Std}[X_{\mathrm{T},j}(t+\tau)]} \tag{15}$$

of $\mathbf{X}_{\mathrm{T}}(t)$ match the corresponding scaled covariance functions $\{\xi_{ij}(\tau)\}$ of $\mathbf{X}(t)$, we select $\{\rho_{ij}(\tau)\}$ such the discrepancy between $\{\xi_{T,ij}(\tau)\}$ and $\{\xi_{ij}(\tau)\}$ is minimized in some sense.

4 Engineering application: turbulent flow over a flat plate

In this section, we apply the proposed method to study the flow-induced random vibration of a flexible plate. Consider Fig. 1, which illustrates a thin flexible plate that is simply supported along all four edges. Fluid flows from left to right at constant speed v_0 over the surface of the plate, resulting in a turbulent boundary layer. Random pressure fluctuations within this layer, denoted by Z, provide a time-varying excitation to the top surface of the plate; the resulting (random) displacement response at the neutral axis of the plate is denoted by W. This class of problems has seen much attention; see, for example, Blake (1986), Corcos (1963), Strawderman and Brand (1969), and Tack and Lambert (1962). Most of these studies assume the applied pressure field is Gaussian. Herein we study non-Gaussian excitations. The velocity field is assumed to be composed of a collection of processes that each satisfy a Langevin equation. The surface pressure driving the plate response is then assumed to be the dynamic pressure, which is proportional to the square of the velocity field. Properties of the plate response, such as the mean rate at which W crosses a prescribed level, are then calculated and can be used to estimate time-dependent reliability of the plate structure (Soong and Grigoriu, 1993, Chapter 7). These estimates are not possible by classical linear vibration theory unless the applied pressure field is assumed Gaussian.

4.1 The fluctuating pressure field

Recall Fig. 1 and consider flow moving at constant speed v_0 in the $+u_1$-direction over the surface of a thin plate with length l, width h, and thickness $\epsilon \ll l, h$. Pressure fluctuations in the boundary layer, modeled by a space-time random field, induce structural vibration. As per Lin and Cai (1995, Section 2.3.2), we assume the flow moves uniaxially over the plate, and the only component of the velocity field driving the plate response is that component which acts normal to the plate surface. Let $V(t)$ denote fluid particle velocity in turbulent flow; its motion can be modeled by the Langevin equation:

$$\mathrm{d}V(t) = -\alpha\, V(t)\, \mathrm{d}t + \beta\, \mathrm{d}B(t), \tag{16}$$

where $\alpha > 0$ is the "inverse integral time scale" (Pope, 2000, Section 12.3.1), and $\beta > 0$ denotes the scale of the driving noise.

To model fluid velocity over the entire plate, we construct a velocity field composed of a collection

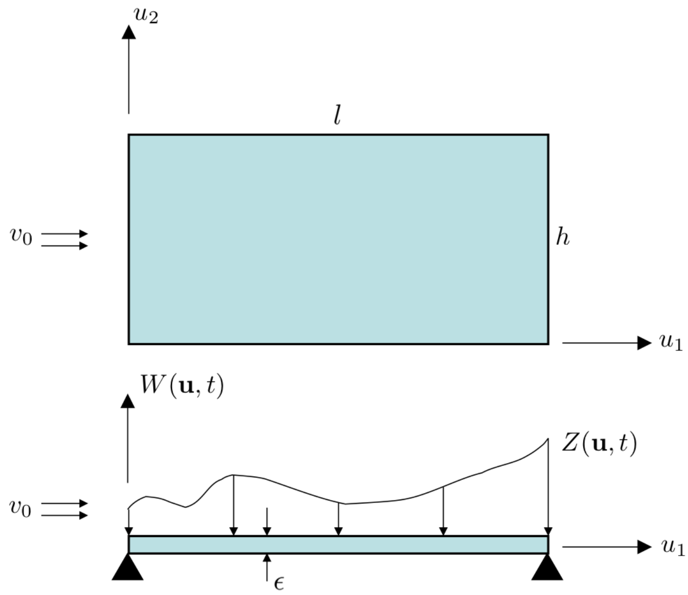

Figure 1. Flexible plate subject to flow-induced vibration.

$\{V_k(t)\}$ of iid copies of $V(t)$ defined by Eq. (16) with associated parameters $\{\alpha_k, \beta_k\}$. Let

$$Y(\mathbf{u},t) = \sum_{k\geq 1} \sqrt{b_k}\, \phi_k(\mathbf{u})\, V_k(t), \tag{17}$$

denote the velocity of the fluid at location $\mathbf{u} = (u_1, u_2)' \in [0,l]\times[0,h]$ on the surface of the plate, where $\{b_k > 0\}$ and $\{\phi_k(\mathbf{u})\}$ are deterministic constants and basis functions, respectively. The second-moment properties of Y, assuming each $V_k(0) = 0$ almost surely, are

$$\begin{aligned} \mathrm{E}[Y(\mathbf{u},t)] &= 0 \\ \mathrm{Cov}(Y(\mathbf{u},t), Y(\mathbf{v},s)) &= \sum_{k\geq 1} b_k\, \phi_k(\mathbf{u})\, \phi_k(\mathbf{v})\, \mathrm{Cov}(V_k(t), V_k(s)) \\ &= \sum_{k\geq 1} b_k\, \phi_k(\mathbf{u})\, \phi_k(\mathbf{v})\, \frac{\beta_k^2}{2\alpha_k}\left(1 - e^{-2\alpha_k \min(t,s)}\right) e^{-\alpha_k|t-s|} \end{aligned} \tag{18}$$

demonstrating that the covariance function of Y is time/space separable. Further, it is clear by Eq. (18) that the spatial covariance of Y is controlled by $\{b_k > 0\}$ and $\{\phi_k(\mathbf{u})\}$ defined by Eq. (17). The corresponding dynamic pressure applied to the top surface of the plate is given by

$$Z(\mathbf{u},t) = \kappa_0\, \frac{\rho_0}{2}\, Y(\mathbf{u},t)^2, \tag{19}$$

where ρ_0 denotes the density of air, and κ_0 is an empirical parameter that depends on the speed and temperature of the flow (Laganelli et al., 1983).

4.2 The structural model

Let κ, γ, and ρ denote the elastic modulus, Poisson's ratio, and mass density of the plate material, respectively. The equation of motion of an undamped thin flat plate with thickness ϵ driven by applied pressure field Z, based on the theory by Kirchhoff (Leissa, 1993), is

$$\delta\, \nabla^4 W(\mathbf{u},t) + \rho\,\epsilon\, \frac{\partial^2}{\partial t^2} W(\mathbf{u},t) = -Z(\mathbf{u},t),\ \mathbf{u} = (u_1,u_2) \in [0,l]\times[0,h],\ t \geq 0, \tag{20}$$

where W denotes the vertical displacement of the neutral plane of the plate, $\delta = \kappa\,\epsilon^3/(12(1-\gamma^2))$ is a parameter defining the effective plate flexural rigidity, and $\nabla^4(\,\cdot\,) = \frac{\partial^4}{\partial {u_1}^4}(\,\cdot\,) + 2\,\frac{\partial^2}{\partial {u_1}^2}\frac{\partial^2}{\partial {u_2}^2}(\,\cdot\,) + \frac{\partial^4}{\partial {u_2}^4}(\,\cdot\,)$ is the biharmonic operator. We assume the plate is simply supported along each edge and initially at rest.

Denote by $\psi_{mn}(\mathbf{u})$ and ω_{mn}, $m,n \geq 1$, the undamped natural modes and frequencies of free vibration of the structure, *i.e.*, the solution to

$$\rho\,\epsilon\,\omega_{mn}^2\,\psi_{mn}(\mathbf{u}) = \delta\,\nabla^4\psi_{mn}(\mathbf{u}),\ \mathbf{u} \in [0,l]\times[0,h]. \tag{21}$$

For a simply supported plate with length l and width h, we have (Leissa, 1993)

$$\omega_{mn} = \sqrt{\frac{\delta}{\rho\,\epsilon}}\left[\left(\frac{m\pi}{l}\right)^2 + \left(\frac{n\pi}{h}\right)^2\right] \text{ and } \psi_{mn}(\mathbf{u}) = \frac{2}{\sqrt{\rho\,\epsilon\,h\,l}}\,\sin\left(\frac{m\pi u_1}{l}\right)\,\sin\left(\frac{n\pi u_2}{h}\right), \tag{22}$$

where the mode shapes form an orthonormal basis with respect to the mass of the plate, *i.e.*,

$$\int_0^l\int_0^h \rho\,\epsilon\,\psi_{mn}(\mathbf{u})\,\psi_{qr}(\mathbf{u})\,\mathrm{d}u_2\,\mathrm{d}u_1 = \begin{cases} 1 & m=q \text{ and } n=r \\ 0 & \text{else}\end{cases} \tag{23}$$

The displacement of the plate can be expressed as

$$W(\mathbf{u},t) = \sum_{mn\geq 1}\psi_{mn}(\mathbf{u})\,Q_{mn}(t), \tag{24}$$

where $\{Q_{mn}(t)\}$ form a set of generalized modal coordinates. Applying a constant damping factor $0<\zeta<1$ to each mode, each $Q_{mn}(t)$ is the solution to the following ordinary differential equation

$$\ddot{Q}_{mn}(t) + 2\,\zeta\,\omega_{mn}\,\dot{Q}_{mn}(t) + \omega_{mn}^2\,Q_{mn}(t) = A_{mn}(t),\ t\geq 0, \tag{25}$$

where $Q_{mn}(0) = \dot{Q}_{mn}(0) = 0$ are the initial conditions, and

$$\begin{aligned} A_{mn}(t) &= \int_0^l\int_0^h Z(\mathbf{u},t)\,\psi_{mn}(\mathbf{u})\,\mathrm{d}u_2\,\mathrm{d}u_1 = \int_0^l\int_0^h \kappa_0\,\frac{\rho_0}{2}\,Y(\mathbf{u},t)^2\,\psi_{mn}(\mathbf{u})\,\mathrm{d}u_2\,\mathrm{d}u_1 \\ &= \sum_{k,l\geq 1}\kappa_0\,\frac{\rho_0}{2}\sqrt{b_k\,b_l}\,V_k(t)\,V_l(t)\int_0^l\int_0^h \phi_k(\mathbf{u})\,\phi_l(\mathbf{u})\,\psi_{mn}(\mathbf{u})\,\mathrm{d}u_2\,\mathrm{d}u_1 \\ &= \sum_{k,l\geq 1}\lambda_{kl}\,V_k(t)\,V_l(t), \end{aligned} \tag{26}$$

where the third line follows from Eq. (17), and parameters $\{\lambda_{kl}\}$ are introduced to simplify notation.

4.3 Moments and correlation function of state vector

For numerical illustration, we truncate Eq. (26) at two terms and assume the plate displacement can be approximated by its first structural mode. Therefore

$$W(\mathbf{u},t) \approx \psi_{11}(\mathbf{u})\,Q(t), \tag{27}$$

where $Q(t) = Q_{11}(t)$ satisfies

$$\ddot{Q}(t) + 2\,\zeta\,\omega\,\dot{Q}(t) + \omega^2 Q(t) = \sum_{k,l=1}^{2}\lambda_{kl}\,V_k(t)\,V_l(t), \tag{28}$$

$\omega = \omega_{11}$ is the first resonant frequency of the plate given by Eq. (22), and $\{V_k(t)\}$ are iid copies of $V(t)$ defined by Eq. (16) with associated parameters $\{\alpha_k, \beta_k\}$, driven by Brownian motions $\{B_k(t)\}$. We limit the discussion that follows to the case of a single structural mode and two fluid particles for clarity;

additional structural modes and fluid particles can be added to the formulation if necessary.

Let $\mu(t; p, q, r, w) = \mathrm{E}[X_1^p(t)\, X_2^q(t)\, V_1^r(t)\, V_2^w(t)]$ define the moments of the state vector. It can be shown that (Grigoriu and Field, 2014)

$$\begin{aligned}\dot{\mu}(t; p, q, r, w) = {}& p\,\mu(t; p-1, q+1, r, w) + q\big[-\omega^2\,\mu(t; p+1, q-1, r, w) - 2\,\zeta\,\omega\,\mu(t; p, q, r, w) + \\ & \lambda_{11}\,\mu(t; p, q-1, r+2, w) + (\lambda_{12} + \lambda_{21})\mu(t; p, q-1, r+1, w+1) + \\ & \lambda_{22}\,\mu(t; p, q-1, r, w+2)\big] - r\,\alpha_1\,\mu(t; p, q, r, w) - w\,\alpha_2\,\mu(t; p, q, r, w) + \\ & \beta_1^2\,\frac{r(r-1)}{2}\,\mu(t; p, q, r-2, w) + \beta_2^2\,\frac{w(w-1)}{2}\,\mu(t; p, q, r, w-2)\end{aligned} \tag{29}$$

is a family of differential equations describing the time evolution of the moments of the state. For calculations, we solve this system of equations for $m = 1$ and increasing r, w, then solve the system for $m = 2$ and increasing r, w. Further, it can be shown that the covariance function of the state is, for $t > s$,

$$\begin{aligned}\mathbf{c}(t, s) = {}& \boldsymbol{\theta}(t-s)\,\mathbf{c}(s, s) + \\ & \sum_{k,l=1}^{2} \lambda_{kl} \int_s^t \boldsymbol{\theta}(t-u) \begin{bmatrix} 0 & 0 \\ \mathrm{E}[V_k(u)\,V_l(u)\,X_1(s)] & \mathrm{E}[V_k(u)\,V_l(u)\,X_2(s)] \end{bmatrix} \mathrm{d}u\end{aligned} \tag{30}$$

where (Soong and Grigoriu, 1993, p. 177)

$$\boldsymbol{\theta}(t) = e^{-\zeta\,\omega\,t} \begin{bmatrix} \cos(\omega_d\,t) + \dfrac{\zeta\,\omega}{\omega_d}\,\sin(\omega_d\,t) & \dfrac{1}{\omega_d}\,\sin(\omega_d\,t) \\ \dfrac{-\omega^2}{\omega_d}\,\sin(\omega_d\,t) & \cos(\omega_d\,t) - \dfrac{\zeta\omega}{\omega_d}\,\sin(\omega_d\,t) \end{bmatrix} \tag{31}$$

4.4 Translation model

In this section, we apply the techniques from Section 3 to approximate process $Q(t)$ defined by Eq. (28) by a translation model, denoted by $Q_{\mathrm{T}}(t)$. The corresponding approximation for the plate displacement response is then given by Eq. (27), *i.e.*, $W_{\mathrm{T}}(\mathbf{u}, t) = \psi_{11}(\mathbf{u})\, Q_{\mathrm{T}}(t)$.

Following the discussion from Section 3.1, let

$$\tilde{f}(x; t) = \sum_{k=1}^{b} p_k(t) f_k(x; q_k, \eta_k) \tag{32}$$

denote an approximation for the marginal PDF of $Q(t)$, where each trial density f_k is the PDF of a gamma random variable with parameters (q_k, η_k), $k = 1, \ldots, b$. We note that $\tilde{f}(x; t)$ is time-varying and is a generalization to Eq. (10), and $\tilde{F}(u; t) = \int_0^\infty \tilde{f}(y; t)\,\mathrm{d}y$ is the corresponding marginal CDF.

Figure 2(a) illustrates the optimal solution $\{p_k(t)\}$ to Eq. (32) based on the moment calculations from Section 4.3. The solution minimizes the discrepancy defined by Eq. (12) at each time t, assuming weight function $v(t; r) = 1/\mu(t; r)^2$ and $b = 10$; the time-evolution of the discrepancy is illustrated by Fig. 2(b). We note that, in general, the solution improves with increasing t.

$W_{\mathrm{T}}(\mathbf{u}, t)$, the translation model for the plate displacement response $W(\mathbf{u}, t)$, is completely defined by the second-moment properties and marginal CDF of Q_{T}, as well as $\psi_{11}(\mathbf{u})$, the first mode shape of the plate defined by Eq. (22). We now can use this model for W to approximate various output properties of interest. For example,

$$\nu_{\mathrm{T}}(x; \mathbf{u}) = \frac{\sigma}{2\pi}\, \exp\left[-\frac{1}{2}\left(\Phi^{-1} \circ \tilde{F}_{\mathrm{s}}\left(\frac{x}{\psi_{11}(\mathbf{u})}\right)\right)\right] \tag{33}$$

is the mean x-upcrossing rate of process $W_{\mathrm{T}}(\mathbf{u}, t)$, where $\tilde{F}_{\mathrm{s}}(x) = \lim_{t\to\infty} \tilde{F}(x; t)$ denotes the stationary version of the marginal CDF of Q_{T}, and parameter $\sigma > 0$ denotes the standard deviation of the process defined as the mean square time derivative of $\Phi^{-1} \circ \tilde{F}(Q_{\mathrm{T}}(t))$. Quantity $\nu_{\mathrm{T}}(x; \mathbf{u})$ defined by Eq. (33) is illustrated by Fig. 3 for $0 \le u_1 \le l$ and $u_2 = h/2$ as a function of level x. This quantity can be used,

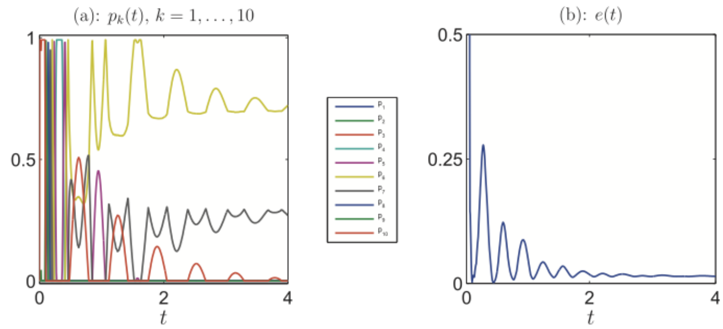

Figure 2. Optimal solution for translation model of process $Q(t)$: (a) probabilities $p_k(t)$, $k = 1, \ldots, 10$, and (b) discrepancy $e(t)$ defined by Eq. (12).

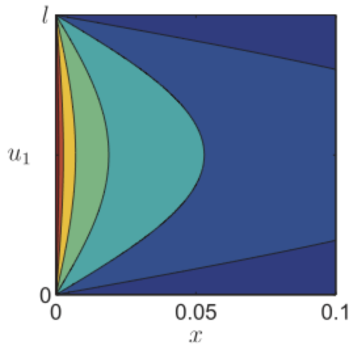

Figure 3. Mean x-upcrossing rate, ν_{T}, of plate displacement as a function of level x and spatial coordinate u_1.

for example, to make estimates of the probability that W will exceed level x within a prescribed time interval (Soong and Grigoriu, 1993, Chapter 7).

5 Conclusions

A method has been developed for approximating the properties of the state of a linear dynamic systems driven by a broad class of non-Gaussian noise, namely, by polynomials of filtered Gaussian processes. The method involved four steps. First, the mean and correlation functions of the state $\mathbf{X}(t)$ of the linear system were calculated from those of the input. Second, equations were developed for higher order moments of $\mathbf{X}(t)$ based on Itô's formula for continuous semimartingales. It was shown that these equations are closed, so that moment of any order of $\mathbf{X}(t)$ can be calculated exactly. Third, a conceptually simple method was proposed for constructing approximations for the marginal distributions of $\mathbf{X}(t)$ from its moments. The method resembles the Galerkin method for solving differential equations. Fourth, translation models were calibrated to representations of the marginal distributions of $\mathbf{X}(t)$ and the second moment properties of this process. The resulting models were then utilized to estimate properties of $\mathbf{X}(t)$, such as the mean rate at which the state exits a safe set. The implementation of the proposed method was demonstrated by numerous examples, including the turbulence-induced random vibration of a flexible plate.

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