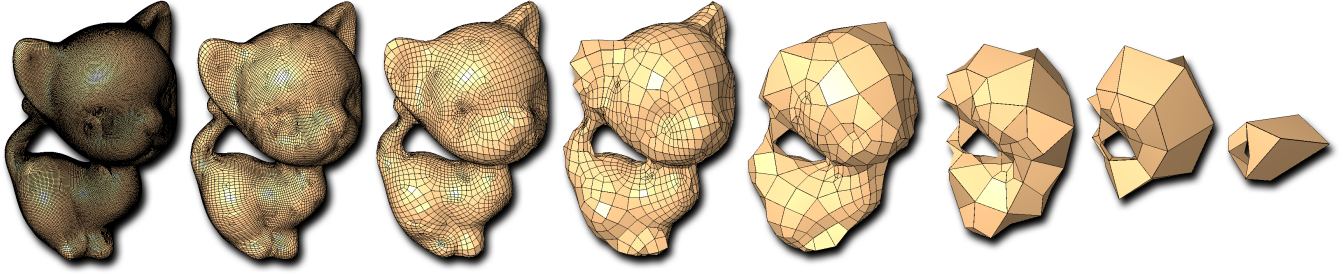# Quadrilateral Mesh Simplification

Figure 1: *Our simplification algorithm can be used to generate a pure quad level-of-detail hierarchy. The algorithm preserves topology during simplification, and attempts to optimize geometric fidelity and quad structure (vertex valences near 4) throughout the process.*

Abstract

We introduce a simplification algorithm for meshes composed of quadrilateral elements. Our technique can handle models with sharp features, and can be used for re-meshing polygonal, i.e. tri- and quad-dominant, meshes into fully quad-meshes. It is reminiscent of edge-collapse based methods for triangle meshes, but takes a novel approach to the challenging problem of maintaining the quad structure during level-of-detail creation. The method consists of a set of unit operations applied to the dual of the mesh, each designed to maintain the quadrilateral structure and topology of the mesh. Geometric shape is maintained by a simple extension of a quadric error metric to quad meshes. The technique is straightforward to implement and efficient enough to be applied to real-world models.

1 Introduction

Although in the past geometry processing methods have been dominated by techniques that operate on triangule meshes, there is a growing interest in developing algorithms that operate natively on quad meshes (i.e., meshes composed of only quadrilateral elements). The motivation for the use of this type of mesh has been well articulated in pioneering work on quad remeshing, e.g., [Lai et al. 2008; Dong et al. 2006; Marinov and Kobbelt 2006; Alliez et al. 2003]. Our attraction to this surface representation can be attributed to the ability of quadrilateral elements to describe the principal curvature directional fields of a surface naturally and to share a common domain with surface parameterization solutions. These characteristics enable a number of important applications, including texturing and modeling with splines. An additional motivation for our work in quad-based meshing algorithms is the use of quad meshes in finite element analysis.

The goal of mesh simplification, analogous to downsampling in digital signal processing, is to gracefully remove elements while maintaining mesh fidelity. Mesh simplification is an important geometry processing operation that has been used as a building block for many higher-level processing steps, including mesh compression, rendering, progressive transmission, editing operations, smoothing, parameterization, and shape reconstruction. It is for this reason that triangle mesh simplification techniques have been some of the most useful operations developed.

A major challenge associated with quadrilateral simplification, unlike triangle-based techniques, is the consideration of the structured nature of the quadrilateral elements that force global constraints on

the mesh connectivity. For instance, it is not possible to create a quadrangulation of a planar surface region bounded by a polyline with an odd number of vertices. For triangle meshes it is possible to limit the attention to local ooperations, that is, to collapse an edge in a triangle mesh, one only needs to consider the triangles in its one-neighborhood. In contrast, the deletion of a single quad element in a quad mesh might require a much larger collection of elements to be removed.

Our algorithm includes operations that are related to edge-collapse based methods for triangle meshes, but addresses the challenging problem of maintaining the quad structure at all levels-of-detail. The method consists of a set of unit operations applied to the dual of the mesh, each designed to maintain the quadrilateral structure and topology of the mesh. The geometric shape is maintained by a simple extension of a quadric error metric to quad meshes. In addition our technique preserves sharp features in models and can be used for re-meshing tri- and quad-dominant meshes into fully quad-meshes.

Contributions. In this paper, we introduce a technique for quadrilateral mesh simplication that we term QMS. Our technique is based on the novel idea of exploiting the dual structure of the mesh. We propose simplification operators and a prioritization scheme based on an extension of the quadric error metric to sort the collapse operators. We also show how our simplification algorithm can be used for re-meshing tri- and quad-dominant meshes into fully quad-meshes. Our experimental results demonstrate the efficiency and effectiveness of our technique on a variety of models.

2 Related Work

To our knowledge, there is no literature that directly addresses quadrilateral mesh simplification. Some research has investigated quadrilateral improvement methods [Staten and Canann 1997; Kinney 1997]; however, it does not provide a controllable means to reduce element counts in the final mesh. On the other hand, many quadrilateral remeshing algorithms can achieve varying level-of-detail meshes, thus mimicking simplification in a bottom-up approach. Quadrilateral generation algorithms often have control parameters that tune the sizes of the created elements, with which they can construct meshes with varying levels of detail.

Tracing iso-curves over orthogonal vector fields that form a chart covering a surface yields the connectivity that defines a quadrilateral mesh [Kalberer et al. 2007]. This technique extends earlier anisotropic quadrilateral meshing algorithms that use principal curvature to define the frames over the surface [Marinov and Kobbelt

2004; Alliez et al. 2003]. These techniques generate high quality, feature aware quadrilateral meshes, and can achieve level-of-detail meshes by adapting the spacing between the traced parameter lines. A coarse-to-fine approach for generating quad shape approximation is proposed in [Guskov et al. 2002].
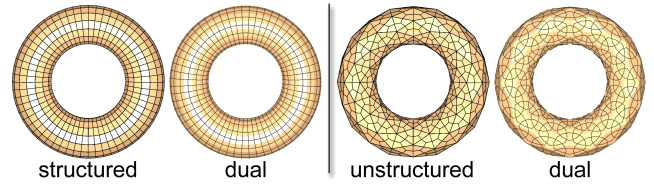
Divide-and-conquer quadrilateral remeshing schemes segment the input model then individually remesh each region. Paving [Blacker and Stephenson 1991] is an early advancing front scheme that generates quadrilateral meshes of segmented and bounded regions of a model. [Krishnamurthy and Levoy 1996] proposed a method based on using a grid for resampling regions extracted by a user-guided segmentation technique. A two step algorithm [Marinov and Kobbelt 2006] automatically segments the model with a clustering method similar to variational shape approximation [Cohen-Steiner et al. 2004]. Samples are defined uniformly along the boundary edges to obtain water-tight boundaries between adjacent regions; then, the vertices are optimally paired using Bezier curves to define the quadrilateral elements internal to each region. Morse-Smale complexes automatically segment a surface into regions homeomorphic to a disc [Ni et al. 2004; Dong et al. 2005; Tong et al. 2006], and parameterization methods can guide the quadrilateral-based remesh [Dong et al. 2006]. These divide-and-conquer approaches require careful attention along boundary edges to obtain a piece-wise continuous surface representation across the divided regions. By modifying the density of samples during the conquer phase, these techniques are able to output models of varying element counts.

Another method of quadrilateral mesh generation is the direct conversion of triangular mesh elements into quadrilaterals. Splitting schemes based on Catmull-Clark subdivision [Catmull and Clark 1978] and its square root extension [Kobbelt 1996], directly transform the triangles into quadrilaterals. However, these techniques greatly increase the number of elements describing the model and tend to result in unstructured meshes.

Complete matching conversion, another direct conversion scheme, pairs neighboring triangles to define a single quadrilateral, the method seeks pairing combinations that will couple every element with another. Q-Morph [Owen et al. 1999] is an advancing front conversion mechanism that builds on this concept. Bubble packing [Shimada 1993; Shimada and Gossard 1995], in particular rectangular cell based algorithms [Shimada 1999; Viswanath et al. 2000], build quadrilateral-dominant meshes through near complete matchings of Delaunay triangulations. These methods distribute vertices over the surface using repulsion forces, where each rectangular cell encodes forces similar to a clustered group of charged particles [Meyer et al. 2007].

Other quadrilateral mesh generation methods [Eck and Hoppe 1996; Takeuchi et al. 2000] leverage conversion methods in combination with triangular-based simplification methods. Conversion back-and-forth between the quadrilateral and triangular mesh representations allows the use of established triangle mesh simplification algorithms while modeling quadrilateral elements. The control of the final element quality is not supported, and may prove difficult, for such approaches because the triangle-based simplification is unaware of the related quadrilateral elements. Additionally, conversion between representations can become expensive and unstable, as it is not guaranteed to be one-to-one and onto. Instead, we propose a simplification method to directly reduce quadrilateral elements that does not require such conversions.

Simplification methods execute deletion operations to reduce the number of elements until breaching a prescribed error threshold [Luebke et al 2002]. In our work, we propose a generalization of the triangle edge collapse (see Sec. 3) that provides promis-

Figure 2: *The dual structure of a highly structured mesh is itself highly structured (left); whereas, a non-structured mesh contains a complex dual representation (right).*

ing adaptability to quadrilateral meshing; whereas, quadrilateral vertex merging does not maintain quadrilateral-only meshes and neighborhood reconfiguration is an inherently expensive operation. Triangular-based simplification methods generate an automated prioritization method using a quadric error metric (QEM) [Garland and Heckbert 1997], with extended development [Garland and Heckbert 1998; Hoppe 1999; Lindstrom 2002; Garland et al. 2001; Lindstrom and Silva 2001], which defines a measurement tool to compare and minimize the collapse affects on the geometric structure. The QEM matrix encodes the planar equations of the neighboring triangles to store geometric data per vertex or edge. By subdividing each quadrilateral into four triangular elements and connecting the centroid to each corner, we accumulate planar equations for the original vertices using the established triangle-based scheme. For further discussion of triangular mesh simplification, we direct the reader to [Cignoni et al. 1998] and [Luebke et al 2002].

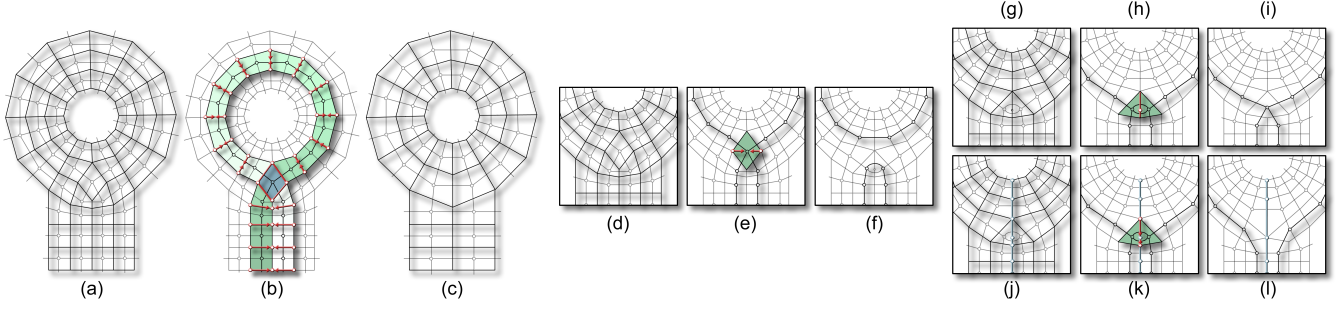## 3 Quadrilateral Mesh Simplification

Our simplification algorithm is based on a set of operations that modify the dual structure of the quad mesh in a controlled manner. The dual representation of a quadrilateral mesh [Borden et al. 2002] (see Figs. 2 and 3) is defined to have the following components:

- the dual of a quadrilateral element is its *centroid*,

- the dual of a quadrilateral edge is the *chord* that connects the centroids of neighboring quadrilaterals,

- the dual of a vertex is the *polygon* formed by connecting the centroids, in a cyclic order, of neighboring quadrilateral elements.

An important structure is the *poly-chord*, a polyline whose adjacent segments are chords that meet at a common centroid and are dual to opposing edges in that quadrilateral. On a closed quadrilateral mesh without boundaries, the *poly-chords* always form closed loops. That is, starting at a single edge on a closed quadrilateral mesh and traversing opposite edges on adjacent quadrilaterals, the traversal will always end at the starting edge.

Many simplification techniques for triangle meshes are based on edge-collapse operations. Unfortunately, if an edge is collapsed in a quad structure, triangles are introduced and structure is lost. We adapt ideas proposed in [Shepherd 2007; Shepherd and Johnson 2007] who showed, in the context of hexahedral mesh generation, that the quality of a hex mesh is related to properties of its dual structure. In our case, we use the dual structure as a tool for designing simplification operators.

In order to sharpen our intuition, consider the dual structure of the two different quad representations of a torus shown in Fig. 2. A high-quality quad mesh of a torus (shown on the left) with uniform valence 4 vertices has a similar dual structure, with chords that tend to meet at approximately right angles, poly-chordal loops

Figure 3: *Three deletion operations utilized throughout this paper: poly-chord (a-c), quadrilateral (d-f) and doublet collapse (g-l). For a quadrilateral mesh with the dual representation (a), deletion of the selected poly-chord (b) merges the vertex groups (red) creating the simplified mesh (c). A quadrilateral collapse (d) merges two opposing vertices thus modifying the topology of two highlighted chords in the simplified mesh (e). Deletion of a doublet, two neighboring quadrilaterals that share two consecutive edges (h), removes the common edges to create a single element, altering the dual structure (i). In the presence of an annotated feature, the doublet quadrilaterals (k) are simultaneously collapsed to preserve the topology of the feature (l).*

that exhibit low curvature, and polygons that are dual to the vertices are generally rectangular (i.e., four-sided). The lower-quality quad mesh (shown on the right) has chords intersecting with non-right angles, poly-chordal loops that exhibit higher curvature, and the polygons that are dual to the vertices of the mesh are typically non-rectangular (i.e., typically, are not four-sided).

Below, we design operators that are used to modify the dual structure of the quad mesh to improve its overall quality. To raise geometric fidelity, we extend the well-known quadric-error metric to assist prioritization of the simplification operators based on the geometric error incurred by each operation, as well as to determine the locations of created vertices.

3.1 Simplification Operations

We introduce three different simplification operators, each designed to modify the dual representation of a quadrilateral mesh, as illustrated in Fig. 3. The *poly-chord collapse* is a global operator while and the *quadrilateral* and *doublet collapses* induce modification of the dual structure through localized deletions.

Poly-Chord Collapse. Removal of a poly-chord from the dual representation simultaneously deletes all quadrilaterals through which it passes by merging the vertices of each edge dual to the selected poly-chord. A poly-chord, as it is related to hexahedral meshing, is equivalent to the intersection of a hexahedral dual sheet [Murdoch et al. 1997] with the boundary surface. Because removal of hexahedral sheets has been shown to preserve the hexahedral connectivity during mesh improvement [Borden et al. 2002], the lower dimensional equivalent poly-chord deletions preserve quadrilateral-only elements.

Quadrilateral Collapse. The quadrilateral collapse is a localized deletion operator that removes a selected element by merging diagonally opposing vertices (also described as a quadrilateral close [Kinney 1997]). By imagining the quadrilateral as two triangle elements connected by an edge between the merging vertices, observe that the described quadrilateral collapse is a generalization of the triangle edge collapse. The elements modified by a quadrilateral collapse are limited to a local neighborhood and maintains our quadrilateral-only constraint. Furthermore, we emphasize the modifications that result to the connectivity of the dual poly-chords, observing that the collapse describes a swap in the connectivity of two poly-chords.

Doublet Collapse. Doublets, neighboring quadrilaterals that share

two consecutive edges, are removed from the quadrilateral mesh following each collapse operation. The valence 2 doublet vertex describes a *degenerate critical point*, associated with a degenerate dual polygon. Under normal circumstances, a doublet is removed by merging the two faces into a single quadrilateral element modifying the connectivity of the highlighted dual chords. The removal of a doublet may generate new doublets, requiring additional deletions, and, similar to the quadrilateral collapse, modifies the structure of two poly-chords.

3.2 Prioritizing Operations

To improve the mesh connectivity and maintain geometric fidelity, it is important that the algorithm selects the elements for deletion intelligently. The prioritization of the collapse operations is achieved by queuing the elements based on the impact of the deletion on the resulting mesh. A combination of factors is considered by the weighting scheme, including the quality of the final vertex valences, the geometric loss, and the area distortion associated with each collapse operator. Consequently, our prioritization scheme is able to reduce the accumulated geometric error while improving the element quality during sequential deletions.
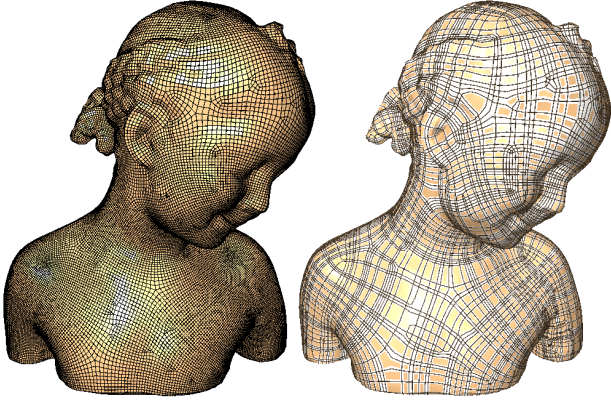
The poly-chord collapse is a more complicated simplification operator than the quadrilateral deletion so the weighting scheme discussion focuses on this operator. The error metric E assigned to a poly-chord p is

$$E(p) = \alpha_q(1 - e^{-E_q(p)}) + \alpha_d(1 - e^{-E_d(p)}) + \alpha_v(1 - e^{-E_v(p)}).$$

where αs are user defined positive scalars that sum to 1. This function considers multiple contributors while weighing the impact of the deletion, similar in purpose to [Smith and Boier-Martin 2005]; however, the user defined α's enable control over the influence of each term. The function $E_q(p)$ returns the worst case QEM error over all of the groups of merging vertices for the poly-chord p. The function $E_d(p)$ evaluates the length of the longest edge collapse (or group of edges) due to the collapse of the poly-chord p. The purpose of E_d is to prioritize collapses in order to construct square elements.

The final weighting term, $E_v(p)$, measures the change in vertex valences from the ideal 4 and penalizes poly-chord collapses that deteriorate the valences of neighboring vertices. The poly-chord p is decomposed into multiple vertex groups $V_i = \{v_{i,j=0}^{N_i}\}_{i=0}^{M}$ that merge to a single new vertex $\tilde{v}_i$, illustrated in Fig. 3b. The valence

Figure 4: *A poly-chord may form a complex knot over a significant portion of the mesh (a **single** poly-chord is shown in the image on the right) complicating, and often invalidating, its collapse. The localized quadrilateral collapse is used to modify the chord structure, unwinding such knots.*

weighting term sums the worst created valence with the average,

$$E_v(p) = \max_i \max_j \beta_{i,j}(|\nu(v_{i,j}) - 4| - |\nu(\tilde{v}_i) - 4|)$$

$$+ 1/M \sum_{i=0}^{M} 1/N_i \sum_{j=0}^{N_i} \beta_{i,j}(|\nu(v_{i,j}) - 4| - |\nu(\tilde{v}_i) - 4|))$$

$$\beta_{i,j} = \begin{cases} 0, & \text{if } |\nu(v_{i,j}) - 4| \leq |\nu(\tilde{v}_i) - 4| \\ 1, & \text{otherwise} \end{cases}$$

where $\nu(v)$ returns the valence of v. Thus, E_v penalizes poly-chord collapses that result in a variation from the ideal vertex and sorts the deletions with equivalent worst cases based on their average created valence.

The weighting metric assigned to each quadrilateral element for the quadrilateral collapse is computed similarly as $E(p)$, allowing the user to vary the influence of the geometric-, area-, and valence-based functions. For a quadrilateral q with vertices (in counter-clockwise order) a, b, c, d, the QEM measurement $E_q(q)$ associated with a quadrilateral collapse is based on the loss of shape fidelity induced by merging the chosen opposing vertices, for example, merging a and c to $\tilde{v}_{ac}$. The area term $E_d(q)$ measures the distance between the two merging vertices of q.

The final weighting term, $E_v(q)$, measures the difference between the current vertex valences and the new configuration and is biased to favor collapses that improve the local connectivity. The valence term sums the difference of the created valences from the originals,

$$E_v(q) = \beta_a(|\nu(a) - 4| - |\nu(\tilde{v}_{ac}) - 4|)$$
$$+ \delta_b(|\nu(b) - 5| - |\nu(b) - 4|)$$
$$+ \beta_c(|\nu(c) - 4| - |\nu(\tilde{v}_{ac}) - 4|)$$
$$+ \delta_d(|\nu(d) - 5| - |\nu(d) - 4|)$$

$$\beta_i = \begin{cases} 0, & \text{if } |\nu(i) - 4| \leq |\nu(\tilde{v}) - 4| \\ 1, & \text{otherwise} \end{cases}$$

$$\delta_i = \begin{cases} 0, & \text{if } |\nu(i) - 5| \leq |\nu(i) - 4| \\ 1, & \text{otherwise} \end{cases}$$

Thus, E_v penalizes quadrilateral collapses that lower the quality of the local vertex valences. Note that for a given quadrilateral two

collapse configurations exist, each with distinct metrics. Finally, doublets are given the highest priority and collapsed when detected.
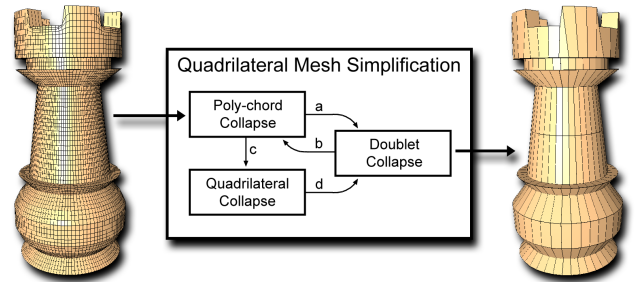
3.3 Naïve Simplification

The repeated iteration of any single collapse operator, while ideal in some circumstances, proves inadequate when applied to a broader spectrum of quadrilateral mesh connectivity. In particular, illustrated in Fig. 6, the poly-chord simplification scheme ideally reduces the structured Pensatore quadrilateral mesh, maintaining the high quality elements at all levels of detail. However, in an unstructured model, poly-chord deletions create many high valence vertices and poorly shaped quadrilaterals. Furthermore, the method terminates early, unable to generate lower resolution models, due to the creation of a complex poly-chord structure. Even for quadrilateral meshes that exhibit a relatively structured connectivity, illustrated in Fig. 4, the poly-chords may exhibit a *knotting* effect, winding over a significant portion of the model, prohibiting further deletions.

On the other hand, the quadrilateral-based scheme is not impeded by the dependence on the dual structure. Given unstructured meshes, the weighting scheme prioritizes the collapses such that the number of ideal vertices significantly improves through simplification, illustrated in Fig. 6. However, this technique increases the number of non-ideal valence vertices undesirably during simplification of structured quadrilateral meshes.

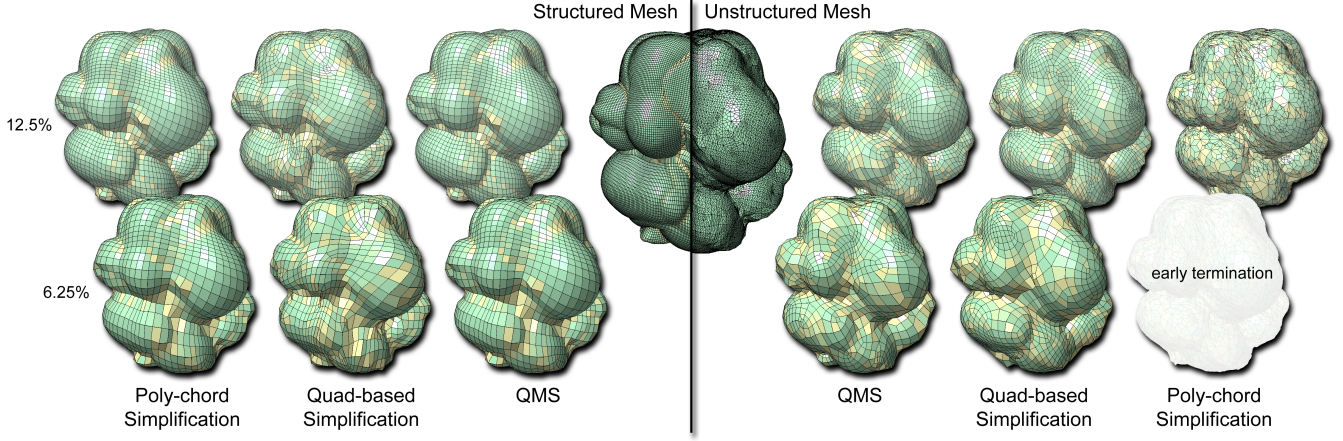### 3.4 Simplification Algorithm

We propose a novel quadrilateral mesh simplification, that we call QMS, algorithm that balances the poly-chord and quadrilateral collapses to achieve a flexibility that generates high quality results independent of the structure of the mesh connectivity, further analyzed in Sec. 4. As shown in the state diagram, illustrated in Fig. 5, the algorithm favors the poly-chord collapse, iteratively choosing the deletion that best improves the mesh's connectivity while least affecting its geometry (looping over state transitions a and b). When no further poly-chord collapses are available, bounded by threshold tolerances, the algorithm selects a quadrilateral element for deletion (state transition c). The quadrilateral collapse modifies the local neighborhood's adjacencies and alters the dual poly-chord structure. Consequently, the algorithm returns to the poly-chord collapse state (via state transitions d and b), to inspect the deletion eligibility of the newly configured poly-chords. By removing the doublets following each collapse (state transitions a and d), as opposed to waiting until the simplification is complete, our experiments indicated a better control over the element count and quality of the final mesh.

The state transition from the poly-chord collapse to the

Figure 5: *Our QMS state diagram.*

Structured Mesh | Unstructured Mesh

12.5%

6.25%

early termination

Poly-chord Simplification | Quad-based Simplification | QMS | QMS | Quad-based Simplification | Poly-chord Simplification

Figure 6: *Simplification results for a* structured *(left) and an* unstructured *(right) quadrilateral mesh using the poly-chord, quadrilateral-based and QMS algorithms. The poly-chord method maintains high quality elements at all levels of detail of a structured mesh, but generates high vertex valences for unstructured meshes at intermediate resolutions and is unable to complete the simplification to lower resolutions. The quadrilateral-based simplification generates additional critical points on the structured mesh, but improves the number of ideal vertices on an unstructured mesh. The QMS algorithm mimics the poly-chord results for structured meshes and the quadrilateral-based results for unstructured meshes, performing well, independent of the underlying connectivity.*

quadrilateral-based collapse enables the adaptive nature of our simplification method. This approach relies on the quadrilateral collapse to *unwind* complex knots that prevent poly-chord collapses, illustrated in Fig. 4, to continue simplification. Furthermore, given unstructured meshes, the algorithm quickly transitions to the quadrilateral collapse, avoiding the poor quality results associated with poly-chord deletions.

For the simplification results shown throughout this paper, mesh connectivity is the dominant factor of the sorting metrics ($\alpha_v \geq 0.95$). The QEMs associated with each collapse are used primarily to determine the new vertex locations to better maintain the surface geometry. To improve element quality and vertex location, we implement a posteriori smoothing procedure described by [Zhang et al. 2007], applied only after the simplification has finished. The mesh vertices are perturbed in the normal direction to reduce mesh noise while preserving the mesh volume; simultaneously, a tangential movement is included to improve the quad aspect ratios.

Topology Preservation. To preserve the topology of the manifold mesh, we use the Link condition [Edelsbrunner 2006]. The quadrilateral collapse inserts an edge between the merging vertices, equating its link condition discussion to that of the triangular edge collapse. Because the poly-chord collapse consists of multiple simultaneous edge contractions, the discussion is more complicated.

Deletion of the poly-chord p executes multiple edge collapses simultaneously, merging the vertex groups V_i into the new vertices $\tilde{v}_i$. In the simple case this involves a single edge, pairing the two endpoints as a vertex group. However, more complex groups are formed when multiple edges merge together as the poly-chord is adjacent to itself or self-intersects, illustrated in Fig. 3. Therefore, while inspecting a vertex group V_i, the remaining vertex groups are temporarily replaced with their collapse vertex $V_k = \tilde{v}_k, \forall k, k \neq i$, to account for the multiple edge collapses occurring. The mesh topology is preserved during poly-chord deletions by comparing the intersection of Link conditions associated with the one-neighborhoods of the vertices in V_i to the union of those associated with the one-neighborhoods of the edges that connect any two vertices of V_i. For further details, we refer the reader to [Edelsbrunner 2006].

Complexity. The complexity of a single call to the different operators differ; while quadrilateral and doublet collapse operators take $\Theta(1)$, a single poly-chord operation can take $\Theta(n)$. Still, it is not hard to use amortized analysis to show that the overall cost of each operation is still $\Theta(1)$. The argument is based on the fact that that once an element is deleted, it is not touched again.

Sharp Features. In some applications, the preservation of user annotated feature edges is important. Our method supports the preservation of feature edges. Included in Fig. 10 are two manufacturable models that contain user annotated features highlighting the sharp edges of the mesh.

Our implementation ensures that the simplified feature edges are homeomorphic to their original counterparts. Conceptually, triangular facets are connected from each feature edge to a temporary transition vertex. In this way, the feature edges belong to three polygonal faces, thus defining *degree*-3 edges. The remaining edges of the 2-manifold quadrilateral mesh are degree-2, shared by only 2 quadrilateral elements. The Link condition supports the preservation of the non-2-manifold edges to ensure topological isomorphism of the annotated features.

The preservation of feature edges defines additional constraints that restrict the simplification operators and invalidate some collapses. While this is terminal for the poly-chord and quadrilateral collapse operators, we describe a special case to handle doublet deletions along feature edges. Illustrated in Fig. 3, when the shared edges of a doublet are annotated as part of a feature edge, the two quadrilaterals are simultaneously collapsed. If the synchronized quadrilateral collapse special case maintains the topology of the mesh and features edges, the doublet is removed from the mesh.

4 Experimental Results and Applications

We have implemented the QMS algorithm as described in Sec. 3 in C++. Our code was written with flexibility in mind, with no particular effort given to optimizations for speed at this time. The running times reported in this section were performed on a dual core AMD Opteron 2.21 GHz processor with 4GB memory. The code is robust, and our implementation has been tested on a large

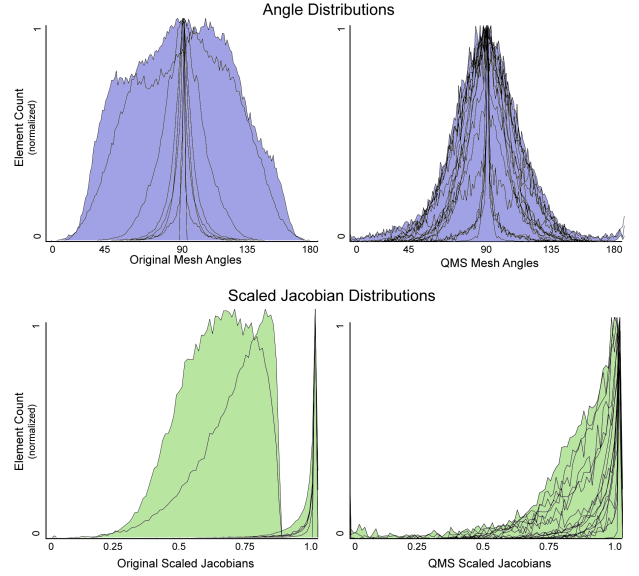| Model	|Quads|	Time (sec)	% Ideal	|Critical|	Worst
	44k	n/a	99.9%	8	3
Pensatore[1]	5.2k	52	99.9%	8	3
	2.5k	53	99.9%	8	3
	46k	n/a	50%	23k	10
Pensatore[2]	5.7k	157	76%	1.3k	7
	2.9k	160	80%	556	6
Torus	95k	n/a	50%	47.6k	13
	23.5k	878	66%	8006	10
	62.8k	n/a	98%	726	6
	15.5k	479	94%	898	6
Bimba	7.8k	500	92%	600	6
	3.9k	505	89%	398	7
	970	506	80%	189	6
	32.4k	n/a	97%	876	6
	8k	115	91%	678	6
Fish	4k	119	88%	475	6
	2k	120	84%	314	6
	490	121	73%	131	6
	20.8k	n/a	98%	405	6
	5.1k	30	90%	491	6
Casting	2.6k	31	86%	344	6
	1.3k	32	79%	259	6
	630	32	67%	205	6
	3.9k	n/a	98%	52	5
	925	1	94%	52	5
Ra	450	1	90%	44	5
	190	1	81%	36	5
	95	1	69%	28	5

Table 1: *Performance and vertex valence analysis of models shown throughout the paper: the structured pensatore and unstructured pensatore ([1,2] respectively, Fig. 6), the bumpy torus (Fig. 9), and the bimba, fish, casting and Ra (Fig. 10) models.*

Figure 7: *A comparison of the angle distributions and scaled Jacobians measured on the original and QMS simplified meshes shown throughout this paper, analyzed in Table 1. The QMS algorithm produces angles near $90°$ and scaled Jacobians near 1.0 despite the input distributions.*

number of quad meshes, including the models used to created the figures in this paper.

For quad meshes, we not only consider the visual quality of the simplified models as shown in Fig. 10, but it is also important to consider quality metrics. Here, we examine the connectivity of the simplified meshes as well as the quality of the quadrilateral elements. An ideal vertex, important in finite element meshing, has a valence of 4. Non-ideal vertices are *critical points* that complicate parameterization solutions and geometric computations. Included in Table 1 is the percentage of ideal vertices, the number of critical points, and the worst case valence for each mesh. To further investigate the quality of the quadrilateral elements, in Fig. 7 we plot a histogram of the original and simplified element angles and scaled Jacobians. The scaled Jacobian is a metric equal to 1.0 for a rectangular element, 0.0 if three vertices (of the four) are co-linear, and negated if the quadrilateral is not convex. Ideally, the angles of a quadrilateral mesh are near $90°$ and the scaled Jacobians near 1.0, corresponding to orthogonal corners in the mesh.

As shown in Table 1, the QMS algorithm generates high quality connectivity. The QMS algorithm maintains low worst case valences and reduces the number of critical points. For unstructured models, i.e. the bumpy torus and the Pensatore[2], the simplification results in significant improvements in the number of ideal valence vertices. Simplification of highly structured models, i.e. the Pensatore[1] and Ra models, generates high quality results. In some cases, the number of critical points increases on the structured models, i.e. the Bimba and casting models. This is the result of a number of quadrilateral collapses executed in order to unwind complex poly-chords, as illustrated in Fig. 4.

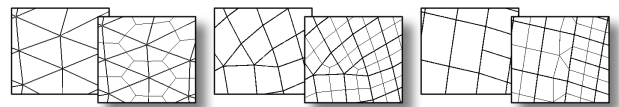The improved connectivity augments the quality of the correspond-

ing quadrilateral elements (Fig 7). Despite the quality of the input models, our QMS algorithm results in simplified meshes with good quality angles and scaled Jacobians. Some degradation is inherent, similar to the geometric error incurred during the deletions. Importantly, the element quality of the two models with poor original distributions in Fig. 7, corresponding to the bumpy torus (Fig. 9) and the Pensatore[2] (Fig. 6), are improved through the QMS simplification. There are no similarly bad distributions evident in the simplified histograms. The low quality mesh angles and their corresponding scaled Jacobians are attributed to the preservation of feature edges, i.e. the casting and Ra models (Fig. 10), further discussed in Sec. 5.
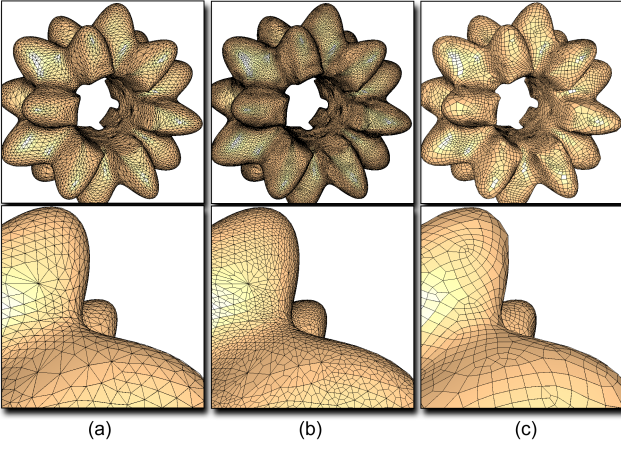
4.1 Quadrilateral Remeshing

We introduce a novel approach to quadrilateral remeshing of arbitrary topological and polygonal surface meshes. Our remeshing pipeline constructs a quad-only mesh by splitting the polygonal elements based on the rules of Catmull-Clark subdivision (Fig. 8) then simplifies to the desired element count, illustrated in Fig. 9. A single iteration of Catmull-Clark subdivision constructs quad-only meshes despite the polygonal types of the original model. In this way, our technique is also able to generate quadrilateral only remeshes of quad-dominant models. Geometric error thresholds during simplification are used to constrain the output remesh within a bounded distance of the original model surface. Additionally, our validation methods maintain mesh topology and user-annotated fea-

Figure 8: *A single iteration of the Catmull-Clark splitting scheme generates quad-only meshes despite the original polygonal type.*

Figure 9: *The elements of a triangular mesh (a) are split using a Catmull-Clark based scheme to generate a quadrilateral-only mesh (b) which is simplified to the original element count (c).*

Figure 10: *The hybrid simplification generates controlled level-of-detail representations for quadrilateral-only models while preserving the topology of the mesh and annotated features.*

ture edges, thus recreating important (i.e. sharp) feature curves from the original model.
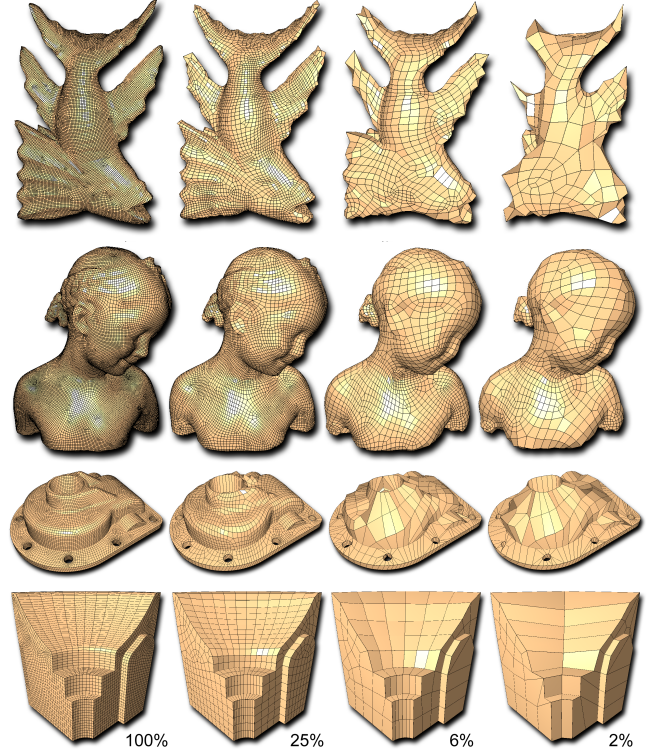
We could further improve the ratio of ideal valence vertices to critical points, the remeshing pipeline can be extended by simplifying to a quarter of the desired element count then subdivide the results via Catmull-Clark. The additional simplification and subdivision increases the geometric error of the final results. Consequently, it may necessitate a projection of the final vertices to the original surface mesh.

5   Discussion & Limitations

Our QMS algorithm balances the poly-chord and quadrilateral-based simplification operators to generate high quality meshes independent of the underlying mesh structure. As illustrated in Fig. 6, the QMS algorithm is an improved approach over any single operation type. For the structured mesh, QMS behaves similarly to the poly-chord simplification method, better maintaining the regularity of the mesh with a minimal number of critical points and near orthogonal edges. However, where the poly-chord collapse performs poorly, terminating early on the unstructured mesh, our QMS algorithm instead mimics the results of the quadrilateral-based simplification scheme.

Quadric error metrics are used to reduce the loss of geometric fidelity experienced during the simplification process. Consequently, even when the weighting metrics used to sort the simplification operators heavily emphasize connectivity improvement, the reduced meshes maintain high shape fidelity. Illustrated in Fig. 10 are several organic and machined models simplified using the QMS algorithm, weighing the deletions solely based on connectivity.

Our simplification algorithm is unable to discern when too many elements have been deleted from constrained regions surrounding the user annotated feature edges. In practice, simplification to coarse representations generates low quality edge angles (Fig. 7), sometimes resulting in negative scaled Jacobians, indicating inverted elements. These poorly shaped elements typically reside in regions of the constrained feature edges. While the mesh topology is preserved, the inverted elements may inhibit computations and visualization of the mesh. Another issue is that when compared to triangle simplification (see Fig. 11), it is harder to maintain small detailed features like the scales of the dragon because of the global nature
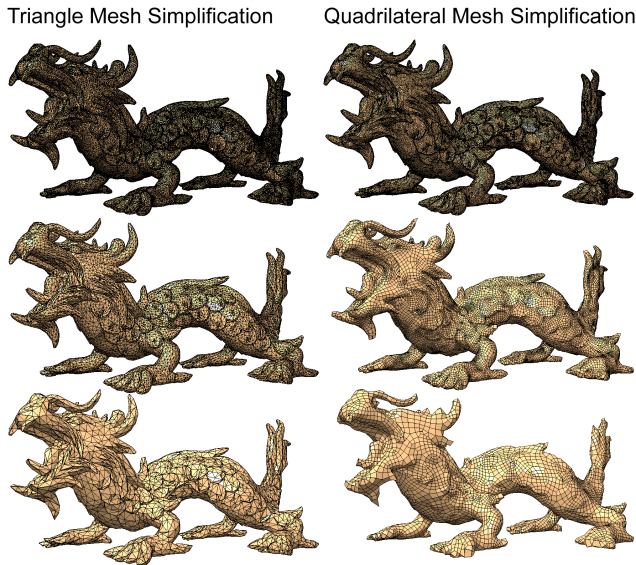
of the poly-chord collapses. It should be possible to add feature constraints to the algorithm to favor creating poly-chords around features.

6   Conclusion

In this paper we show how it is possible to use the dual structure of a quad mesh to develop a quadrilateral mesh simplification (QMS) technique. To the best of our knowledge, QMS is the first technique for simplification of quad meshes. Robustness is one of the key strengths of this approach; we have successfully experimented with our QMS implementation on a large number of models. Another is simplicity; it is relatively easy to implement our algorithm and reproduce our results. We only touched on the potential applications of this framework with our description of a simple quadrilateral remeshing approach. We believe this technique can be used as a building block in many other geometry processing algorithms for quad meshes.

There are many avenues for future research. We plan to investigate improvements on the weighting of the simplification methods, stressing the relationship between the quadrilateral collapse and the dual structure. Currently, the scheme implements a greedy prioritization, that may give too much weight to improving the connectivity of the mesh. It may be possible to improve the final results of the QMS algorithm by intelligently selecting quadrilateral collapses that better improve the poly-chord structure. We would like to further explore other optimizations, including the minimization of critical vertices, controlling element size and scaling, curvature alignment and other related challenges. We are also interested in creating high-efficient implementations of this framework, including out-of-core and streaming approaches. Furthermore, we are

Triangle Mesh Simplification   Quadrilateral Mesh Simplification

Figure 11: *The QMS algorithm generates controlled level-of-detail representations, shown in the comparison between QEM triangle mesh simplification and our method for the dragon model with* 150k *elements (top) at* 37.5k *(middle) and* 9.4k *elements (bottom).*

also interested in other applications, e.g., improved quality remeshing and progressive compression of quad meshes.

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