Saddle-Point Preconditioners Based on Commutators

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Outline

- Incompressible Navier-Stokes
- Differential Commuting
  - Pressure Convection Diffusion (PCD) Preconditioning
  - BFBT method
- Some Computational Results
- Discrete Issues
  - Mass Matrices
    BFBT $\Rightarrow$ Least Squares Commutator (LSC)
  - Stabilization
  - Boundary Conditions
- Modified PCD & LSC Methods
- Computational Results

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Incompressible Navier-Stokes

$$\alpha\, u_t - \nu\, \nabla^2 u + (u \cdot grad\,)u + grad\;\; p = f$$

$$div\, u = 0$$

<u>Continuous</u>

$$\begin{pmatrix} \mathcal{F} & \mathcal{B}^T \\ \mathcal{B} & 0 \end{pmatrix} \begin{pmatrix} u \\ \boldsymbol{p} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} \\ 0 \end{pmatrix}$$

<u>Discrete</u>

$$\begin{pmatrix} \boldsymbol{F} & \boldsymbol{B}^T \\ \boldsymbol{B} & -\boldsymbol{C} \end{pmatrix} \overline{\mathrm{u}} = \boldsymbol{rhs}$$

$C \neq 0$: stabilized

Use preconditioner of form

$$\begin{pmatrix} \boldsymbol{Q}_F & \boldsymbol{B} \\ 0 & -\boldsymbol{Q}_S \end{pmatrix}$$

with Krylov subspace method (GMRES)

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Schur Complement & Commuting

How to precondition the Schur complement, $\mathcal{S} = -\mathcal{B}\,\mathcal{F}^{-1}\,\mathcal{B}^{\mathsf{T}}$?

Suppose

$$\mathcal{B}^{\mathsf{T}}\,\mathcal{F}_p = \mathcal{F}\,\mathcal{B}^{\mathsf{T}} \quad \& \quad \mathcal{M}^{-1} = -\mathcal{F}_p\,(\mathcal{B}\,\mathcal{B}^{\mathsf{T}})^{-1}$$

$\Rightarrow$

$$\mathcal{S}\,\mathcal{M}^{-1} = I$$

Also

$$\mathcal{F}_p\,\mathcal{B} = \mathcal{B}\,\mathcal{F} \quad \& \quad \mathcal{M}^{-1} = -\,(\mathcal{B}\,\mathcal{B}^{\mathsf{T}})^{-1}\,\mathcal{F}_p$$

$\Rightarrow$

$$\mathcal{M}^{-1}\,\mathcal{S} = I$$

Is there hope for $\mathcal{F}_p\,\mathcal{B} = \mathcal{B}\,\mathcal{F}$?

Consider

$$\mathcal{F} = \begin{bmatrix} \mathcal{F}_p & 0 \\ 0 & \mathcal{F}_p \end{bmatrix} \qquad \mathcal{B}^T = \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix}$$

and

$$\mathcal{F}_p = \frac{\partial}{\partial t} + \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + c_1\frac{\partial}{\partial x} + c_2\frac{\partial}{\partial y}$$

$\Rightarrow$

$$\mathcal{F}_p\,\mathcal{B} = \mathcal{B}\,\mathcal{F}$$

Preconditioners

PCD

$$\mathcal{M}^{-1} = -\mathcal{F}_p (\mathcal{B}\mathcal{B}^{\mathrm{T}})^{-1} \quad \Rightarrow \quad M^{-1} = (M_p)^{-1} F_p (A_p)^{-1}$$

where

M_p : pressure mass matrix

A_p : Laplace Operator

F_p : convection-diffusion operator

BFBT

$$M^{-1} = (BB^T)^{-1} B \ F \ B^T (BB^T)^{-1}$$

$$M^{-1} = (A_p)^{-1} B F B^T \ (A_p)^{-1}$$

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Re	Mesh	DD	PC-D	Proc
10	32x32x32	67.0 (634.6)	28.0 (803.2)	1
	64x64x64	159.8 (1507.5)	28.4 (865.2)	8
	128x128x128	356.2 (4529.3)	29.1 (1103.2)	64
100	32x32x32	61.7 (730.7)	56.0 (1232.7)	1
	64x64x64	168.5 (2131.6)	62.1 (1697.8)	8
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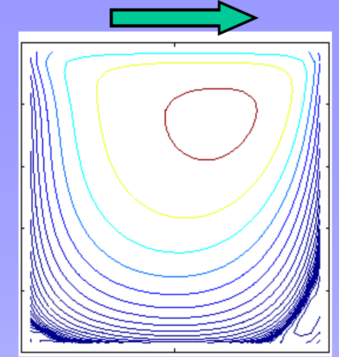
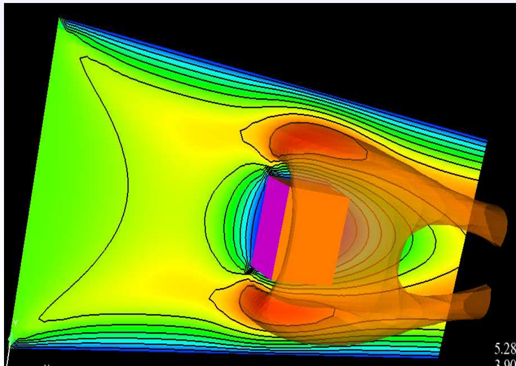


- GMRES
- Inexact F^{-1} & A_p^{-1} via AMG
- DD uses ILUT
- Newton's Method

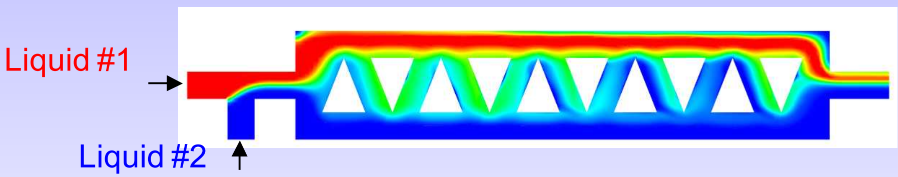
Re	Mesh	DD	PC-D	Nprocs
10	270K	67.2 (859.8)	20.7 (997.7)	1
	2.1 M	151.2 (2004.1)	21.7 (1507.5)	8
	16.8 M	667.2 (20908.0)	24.7 (1997.7)	64
50	270K	69.4 (889.2)	35.9 (1209.1)	1
	2.1 M	132.4 (2676.1)	38.7 (1797.2)	8
	16.8 M	637.2 (18646.0)	44.7 (2397.7)	64

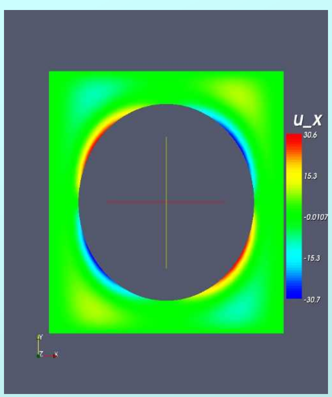
Average outer iterations per Newton step. (*) is the total CPU time for the experiment



Microfluidic Results

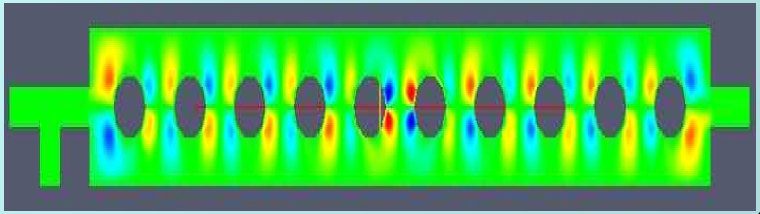
- Bio-briefcase
  - Mix liquids at low Re by induced charge electro osmosis
  - Optimize shape & topology of mixing obstructions





Re	Unkowns	PC-D
1	140K	52.1
	320K	51.2
20	140K	57.2
	320K	56.3

Re	Unknowns	PC-D
1	62K	64.0
	256K	62.1
20	62K	68.0
	256K	67.9
50	62K	77.0
	256K	74.9



Research Topics

- **Discrete vs. Differential Commuting**
  - BFBT $\Rightarrow$ Least Squares Commutator (LSC)

======= Book by Elman, Silvester, Wathen=========

- **Stabilization**
  - S-LSC
- **Boundary Conditions**
  - Weighted LSC & Modified PCD

LSC: Discrete vs. Differential Commuting

$\mathcal{F}_p\mathcal{B} = \mathcal{B}\mathcal{F}$ $\quad \not\Rightarrow \quad$ $F_p B = BF$ $\quad$ as $\quad$ $B \not\approx \mathcal{B}$, $\quad F \not\approx \mathcal{F}$

e.g. F.E's mass matrices play a role

Recall $\quad \mathcal{M}^{-1} = -\mathcal{F}_p (\mathcal{B}\mathcal{B}^T)^{-1}$

- Approximate $\mathcal{B}\mathcal{B}^T$ via $A_p = B\,(M_v)^{-1}\,B^T$
- Approximate $\mathcal{F}_p$ via normal equations solution to

$$\Rightarrow \qquad \left\| [M_v^{-1} F M_v^{-1} B^T]_j - M_v^{-1} B^T [M_p^{-1} F_p]_j \right\|_{M_v}$$

$$M^{-1} = \underbrace{(A_p)^{-1}\, B\, (M_v)^{-1}\, F\, (M_v)^{-1}\, B^T}_{F_p}\, (A_p)^{-1}$$

Fp Fp

Stabilization

$$\begin{pmatrix} F & B^T \\ B & -C \end{pmatrix}$$

- $(BB^T)^{-1}$ is unstable
  - Amplifies high frequencies
  - C effectively stabilizes $(B F^{-1} B^T)$

- Preconditioner $(A_p)^{-1}$ B $(M_v)^{-1}$ F $(M_v)^{-1}$ B^T $(A_p)^{-1}$ needs stabilization
  - Best to design new stabilization matrix
  - Using C for preconditioner stabilization keeps method algebraic
    - Not really designed for BB^T
    - S-LSC:

$$(A_p)^{-1} B (M_v)^{-1} F (M_v)^{-1} B^T (A_p)^{-1} + \alpha D$$

with $A_p = B (M_v)^{-1} B^T + \gamma C$

$\Rightarrow$ simple formulas for α and γ obtained by constant coefficient analysis

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Oseen

Q_2-Q_1

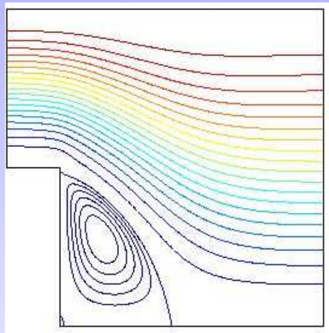
Re	F_p	LSC
10	33	23
100	58	29
200	63	29

Size(n=7)

grid $2^n \times 3(2^n)$

Newton

	F_p			S-LSC		
Size (n)	**4**	**5**	**6**	**4**	**5**	**6**
Q_1-P_0 Re 10	29	35	37	12	16	22
Q_1-P_0 Re 100	62	66	79	21	19	25
Q_1-P_0 Re 200	94	92	100	35	26	27
Q_1-Q_1 Re 10	27	32	35	13	19	26
Q_1-Q_1 Re 100	57	61	76	18	19	28
Q_1-Q_1 Re 200	89	84	93	31	22	29



Oseen

MAC

	F_p		LSC	
	64x64	**128x128**	**64x64**	**128x128**
Re 10	6	6	7	9
Re 100	18	16	18	23
Re 1000	51	52	40	49

Boundary Conditions

PCD (F_p & A_p):

Dirichlet @ inflow

Neumann otherwise

LSC implicitly defines bcs

LSC's $A_p \Rightarrow$ Neumann at inflow

Observation:

$$\mathcal{F} = \begin{bmatrix} \mathcal{F}^{(u_1)} & 0 \\ 0 & \mathcal{F}^{(u_2)} \end{bmatrix}$$

$$\begin{aligned} \mathcal{E} &= \mathcal{B}\mathcal{F} - \mathcal{F}_p\mathcal{B} \\ &= [\mathcal{B}_x\mathcal{F}^{(u_1)} - \mathcal{F}_p\mathcal{B}_x \quad , \quad \mathcal{B}_y\mathcal{F}^{(u_2)} - \mathcal{F}_p\,\mathcal{B}_y] \\ &= [(1)+(2) \quad , \quad (3)+(4)] \end{aligned}$$

(1) $\mathcal{B}_x\mathcal{F}_x^{(u_1)} - \mathcal{F}_x^{(p)}\mathcal{B}_x$ (2) $\mathcal{B}_x\mathcal{F}_y^{(u_1)} - \mathcal{F}_y^{(p)}\mathcal{B}_x$

(3) $\mathcal{B}_y\mathcal{F}_x^{(u_2)} - \mathcal{F}_x^{(p)}\mathcal{B}_y$ (4) $\mathcal{B}_y\mathcal{F}_y^{(u_2)} - \mathcal{F}_y^{(p)}\mathcal{B}_y$

(1) $\mathcal{B}_x \mathcal{F}_x^{(u_1)} - \mathcal{F}_x^{(p)} \mathcal{B}_x$

Composite operators $\mathcal{B}_x \mathcal{F}_x^{(u_1)}$

with $\mathcal{F}_x^{(u_1)} \boldsymbol{u} = -\upsilon \boldsymbol{u}_{xx} + \boldsymbol{c}_1 \boldsymbol{u}$

Now assume

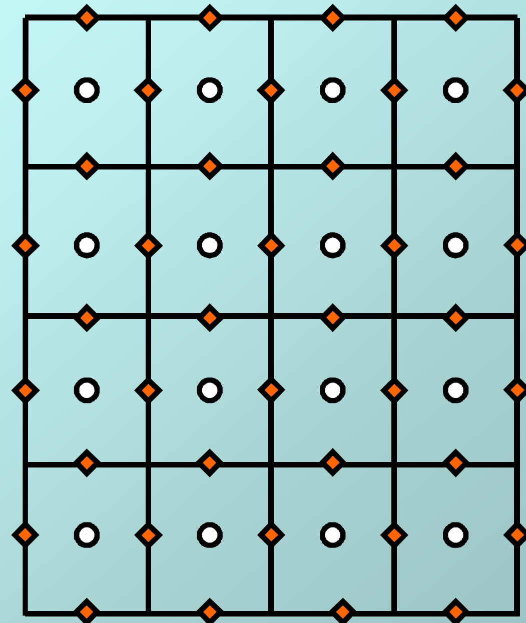
$$\mathcal{F}_x^{(u_1)} \boldsymbol{u} \quad \Rightarrow \quad \boldsymbol{u}(0$$

Observe that $\hat{\boldsymbol{u}} = \mathcal{F}_x^{(u_1)} \boldsymbol{u}$ im

$$\hat{\boldsymbol{u}} = (-\upsilon \frac{\partial}{\partial \boldsymbol{x}} + \boldsymbol{c}_1) \boldsymbol{u}_x = (-\upsilon \frac{\partial}{\partial \boldsymbol{x}} + \boldsymbol{c}_1)\mathcal{B}$$

$B_x F_x^{(u_1)} = F_x^{(p)} B_x$

for MAC scheme



$\Rightarrow \mathcal{F}_x^{(p)}$ should enforce Robin condition $-\upsilon\, \boldsymbol{p}_x + \boldsymbol{c}_1 p = 0$

Notice that $\boldsymbol{u}(0, \boldsymbol{y}) = 0 \quad \not\Rightarrow \quad \boldsymbol{p}(0, \boldsymbol{y}) = 0$ with $\boldsymbol{p} = \mathcal{B}_x \boldsymbol{u}$

High Re ≈ Dirichlet, low & characteristic ≈ Neumann

Dirichlet bcs & 4 commutators

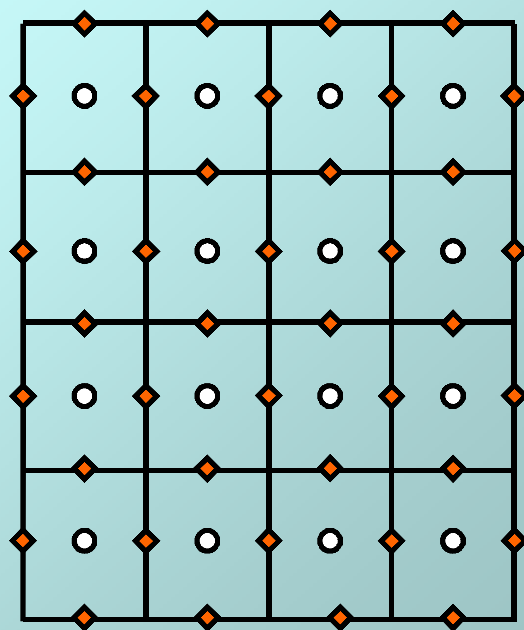
(1) $\mathcal{B}_x \mathcal{F}_x^{(u_1)}$

$\mathcal{F}_x^{(p)} : -\upsilon\, p$

Dirichlet

Conflict

3 out of 4 commutator s can be satisfied for MAC scheme



(3) $\mathcal{B}_y \mathcal{F}_x^{(u_2)}$

$\mathcal{B}_y \mathcal{F}_y \quad \mathcal{F}_y \mathcal{B}_y$

$\mathcal{F}_x^{(p)} : p\Big|_{x=0} = 0$

No requirements

Which commutators are important?

Perturbation Analysis

- Constant co

Let $Y = F_p\, S\, N$

Y

Y

⊥ **more important @ Dirichlet!**

⇒ inflow bcs

*** Robin**

⇒ characteristic bcs

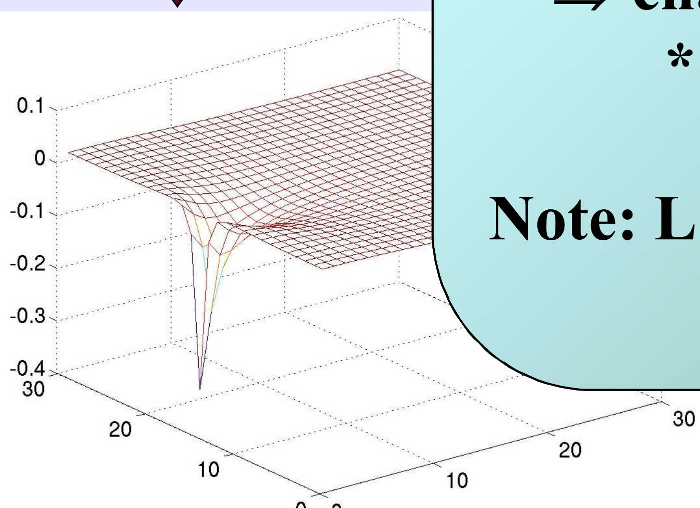
*** Neumann**

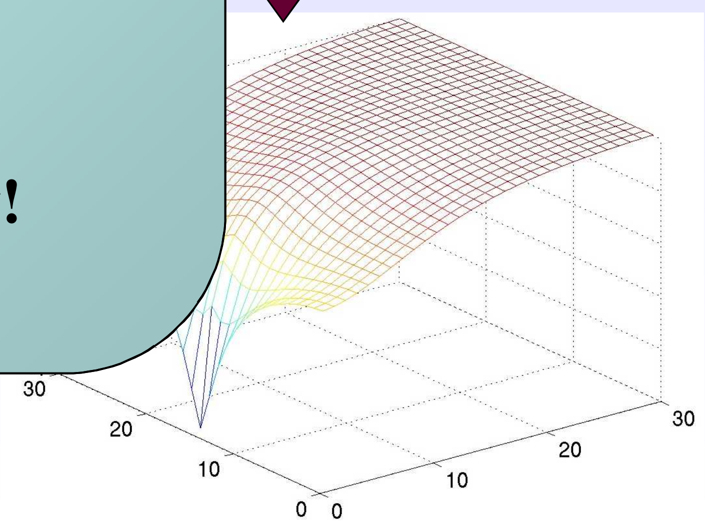
Note: LSC weights equally!

left bc

$Y = \begin{pmatrix} \equiv \\ 0 \\ \equiv \end{pmatrix}$

right bc





What about outflow?

$F \,\&\, B:$ $\qquad \nu(u_1)_x = p \qquad (u_2)_x = 0$

- Perturbation analysis:
- B_x & B_y make no special assumptions @ bc
- unclear whether $Y^{\perp}$ or $Y^{|}$ is more significant

- PDE arguments $\Rightarrow$ satisfy (1) via Dirichlet (3) via Neumann

- Discrete setting must also be considered

- We'll show some experiments

Weighted LSC

$$\min \;\| B\,(Q_v)^{-1} F\,]_j \;-\; [X]_j\, B \,\|_H$$

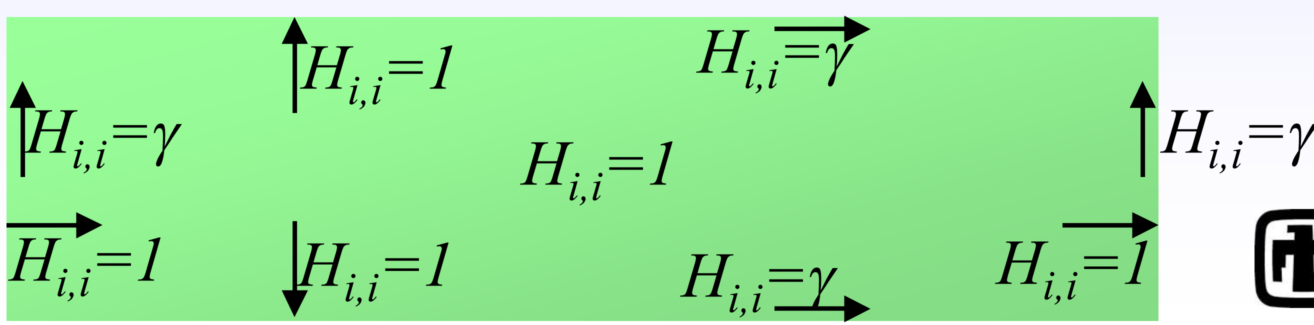
with

$$H = W^{½}\,(Q_v)^{-1}\,W^{½}\,, \quad [X]_j \text{ is } j^{th} \text{ row of } F_p$$

If $Q_v = I$

$$\Rightarrow \quad M^{-1} = (A_p)^{-1}\; B\,H\,F\,B^T\,(B\,H\,B^T)^{-1}$$

Consider diagonal matrix for H with entries 1 or γ ($\gamma < 1$)



Results

Conclusions

- Preconditioners based on commutators can give mesh independent convergence rates

- But … getting the details right can be important
  - Discrete commuting (mass matrix)
  - Stability issues
  - Boundary conditions

- Two new/modified methods
  - New bcs for PCD
  - W-LSC

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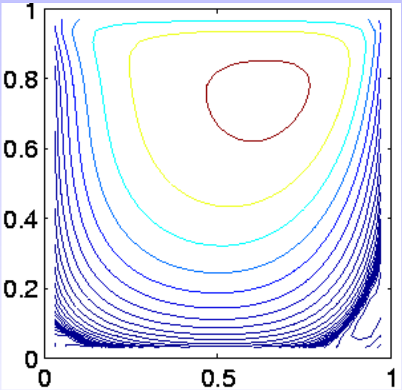
Lid Driven Cavity Results

	Old PCD			New PCD		
Mesh	**32x32**	**64x64**	**128x128**	**32x32**	**64x64**	**128x128**
Re =10	15	15	15	16	16	16
Re = 100	25	26	26	26	25	25
Re = 400	41	41	39	53	39	37
Re = 1000	76	65	55	82	71	53

	Old LSC			New LSC		
Mesh	**32x32**	**64x64**	**128x128**	**32x32**	**64x64**	**128x128**
Re =10	11	14	18	9	9	10
Re = 100	16	21	27	14	14	13
Re = 400	31	30	35	31	26	22
Re = 1000	62	55	45	67	58	40



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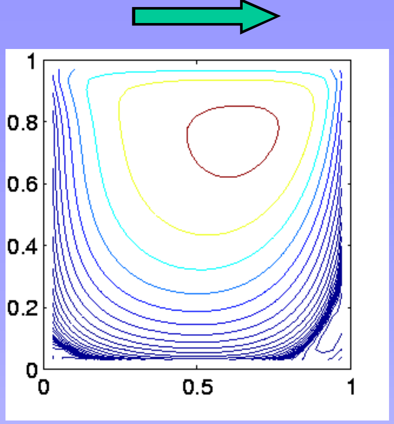
Lid Driven Cavity



Re	Mesh	DD	PC-D	Proc
10	128 x 128	220.6	21.2	4
	256 x 256	467.2	23.0	16
500	128 x 128	334.9	74.5	4
	256 x 256	896.1	76.3	16
1000	128 x 128	352.5	126.4	4
	256 x 256	839.5	126.6	16

- Newton's Method
- GMRES
- Inexact F^{-1} & A_p^{-1} via AMG
- DD uses ILUT

Re	Mesh	DD	PC-D	Proc
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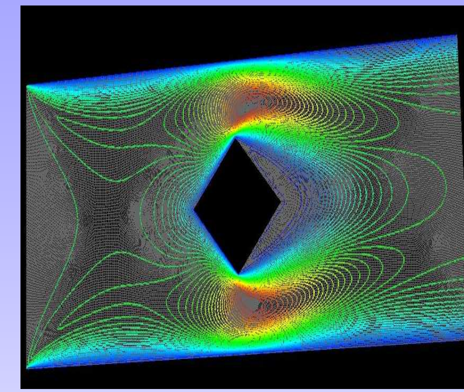
Average outer iterations per Newton step. (*) is the total CPU time for the experiment

Flow over Obstruction

Re	DOFs	DD	PC-D	Nprocs
10	256K	282.6 (1054.9)	22.6 (192.7)	4
	1M	890.2 (6187.4)	25.6 (252.3)	16
	4M	NC (NC)	29.7 (397.5)	64
40	256K	203.9 (1269.3)	68.9 (975.2)	4
	1M	770.0 (6933.5)	72.7 (1039.6)	16
	4M	NC (NC)	78.3 (1528.6)	64



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