SAND2009-4338C

Stabilized Tied Contact

10th US National Congress on Computational Mechanics
Columbus, Ohio
July 16-19, 2009

Clark R. Dohrmann

Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company,
for the United States Department of Energy's National Nuclear Security Administration
under contract DE-AC04-94AL85000.

Acknowledgements

- Rich Lehoucq (Math/Computer Sciences)
- Dan Segalman (Engineering Sciences)

Sandia National Laboratories

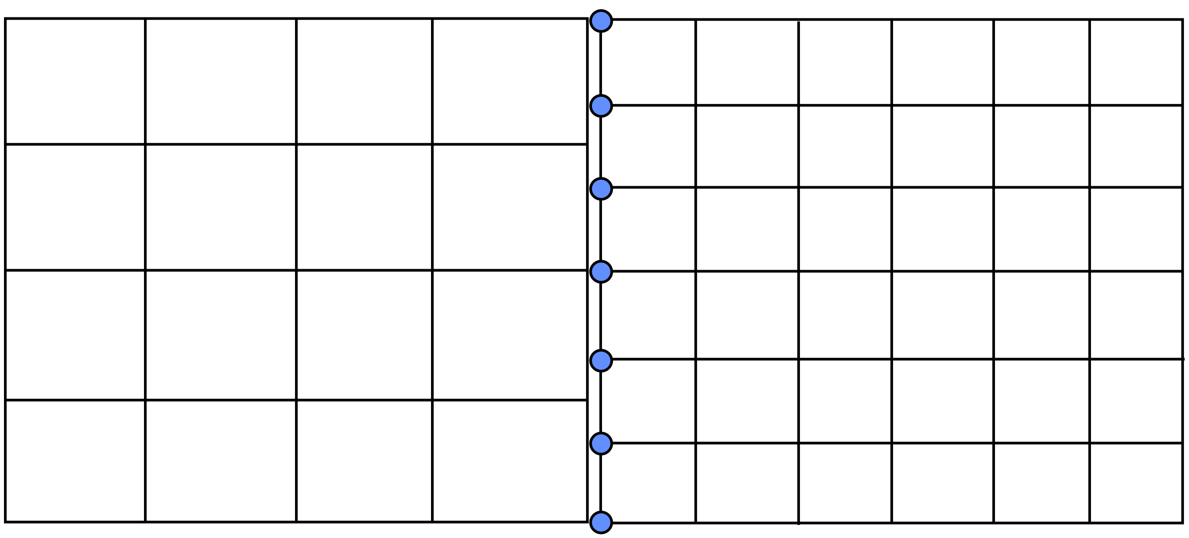
Outline

- **Background**
  - classic tied contact
  - limitations
  - alternatives
- **How Stabilized Method Works**
  - interface springs (penalty method)
  - implementation details (parameter-free)
  - easy “fix” (just add springs)
- **Numerical Results & Theory**
  - convergence rates
  - iterative solver performance
  - supporting theory

Sandia National Laboratories

Background

- **Classic Tied Contact**
  - constrain nodes on one side of interface to other
  - often used for connecting dissimilar meshes
  - standard method available in many FE codes
  - known by many names: node-on-segment, permanent glued contact, node collocation, MPC approach, …



Classic Tied Contact

- **Pros**
  - simplicity
  - constraints have local support
  - elimination of slave dofs efficient
- **Cons**
  - stresses near interface can have large errors
  - these errors may not diminish w/ mesh refinement
  - non optimal global convergence rates

Mortar Method Alternative

- **Overview**
  - constraints based on weak form of interface continuity

$$\int_{\Gamma} \lambda [u] dx = 0$$

  - Lagrange multiplier basis spans constants
  - classic tied contact may be viewed as a mortar method, but LM basis consists of delta functions

- **Pros**
  - optimal convergence rates
  - no spurious oscillations of stresses at interface for smooth solutions
  - available theory

Sandia National Laboratories

Mortar Method Alternative

- **Cons**
  - **3D surface integral calculations are nontrivial***
    - **need intersections of master and slave element faces**
    - **may require large number of quadrature points**
    - **much more complicated than classic tied contact**
  - **constraint equations local, but eliminating dependent dofs may lead to coupling across entire interface ****

* **3D implementation** INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING
Int. J. Numer. Meth. Engng 2004; **59**:315–336 (DOI: 10.1002/nme.865)

* **dual LM basis avoids this problem** SIAM J. NUMER. ANAL.
Vol. 38, No. 3, pp. 989–1012

Sandia National Laboratories

Nitsche Method Alternative

$$a_h(w,v) := \sum_{i=1}^{2}\left[(\nabla w_i, \nabla v_i)_{\Omega_i} + (\gamma/h)\int_\Gamma [w][v]\,ds\right] - \int_\Gamma (\partial w/\partial n)[v]\,ds - \int_\Gamma (\partial v/\partial n)[w]\,ds$$

- **Pros**
  - **optimal convergence rates**
  - **no constraint equations**
  - **“penalty” term scales nicely with mesh refinement**
  - **available theory**

ESAIM: M2AN
M2AN, Vol. 37, N° 2, 2003, pp. 209–225
DOI: 10.1051/m2an:2003023

Sandia National Laboratories

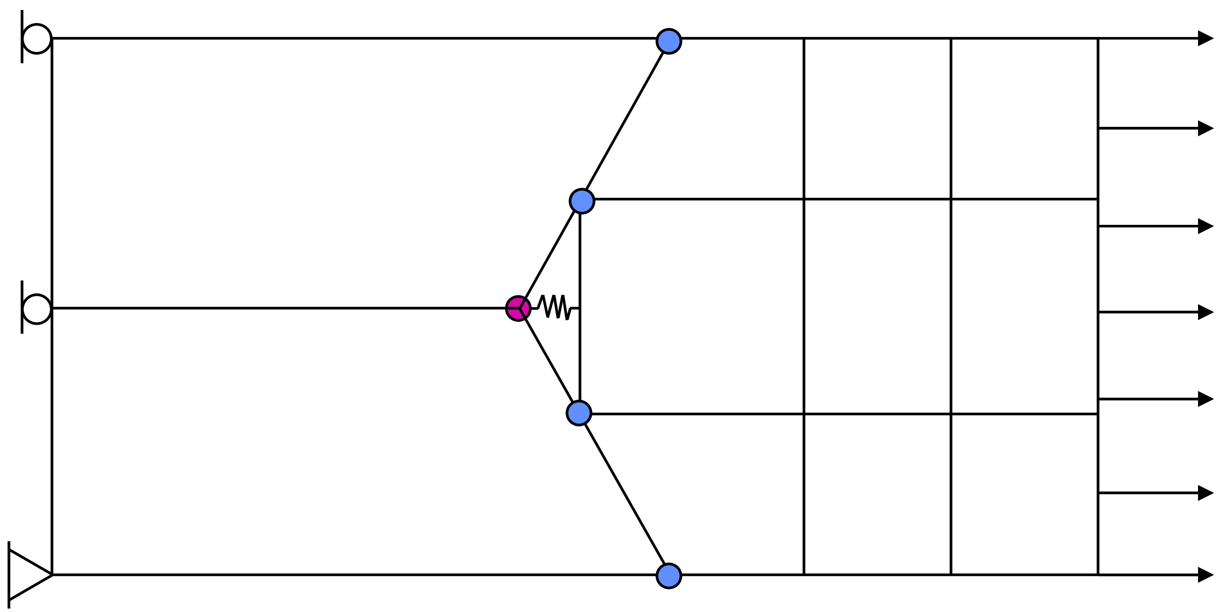
Nitsche Method Alternative

- **Cons**
  - surface integrations required as in mortar methods
  - method unstable if penalty parameter γ too small
  - too large a value of γ can overly stiffen interfaces
  - simple and reliable method needed to determine γ*

* INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING
Int. J. Numer. Meth. Engng 2009; **78**:1009–1036
Published online 11 December 2008 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/nme.2514

How Stabilized Method Works

- Retains classic tied contact constraints
  - symmetric form avoids all constraints
- Introduces springs at interface
- Can be viewed as modification of discrete bilinear form for theoretical purposes

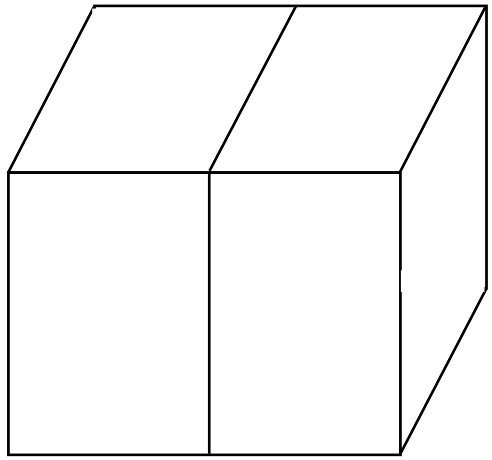


You May Be Wondering

- where are springs located?
  - nodes on master surface (linear elements)
- do I need to pick a penalty parameter?
  - no, this is automated
- how is penalty parameter determined?
  - see next slide
- is theory available, optimal convergence?
  - yes, but theory incomplete (see later slides)
- good performance for iterative solvers?
  - thus far yes (see later slides)
- convergence rate of standard tied contact?
  - $h^{1/2}$ in energy norm (see later slides)
- what about the patch test?

Sandia National Laboratories

Spring Stiffness Calculation



length, width, depth = H
Young's modulus = E
element length = h

- **nodal stiffness k proportional to $E(1/h)^2h^3 = Eh$**
- **split cube in half and connect coincident nodes at interface with springs of stiffness $\beta k = \beta Eh$**
- **cross-sectional stiffness for 1 layer of elements $Eh(H/h)^2 = EH^2/h$ and $\beta EH^2/h$ for spring layer**

Spring Stiffness Calculation

- net axial compliance of original mesh $(H/h)(EH^2/h)^{-1}$
- net axial compliance of mesh with springs $(H/h)(EL^2/h)^{-1}\ (1 + (h/H)/\beta)$
- require relative compliance difference to be proportional to $(h/H)^2$ for linear elements $\Rightarrow \beta = H/h$
- in general can choose $k_{spring} = k_{node}(H_\Gamma/h_{node})$
- scaling consistent with theory

Some Other Connections

- **DG and penalty methods**
- **Thermal contact resistance algorithm**

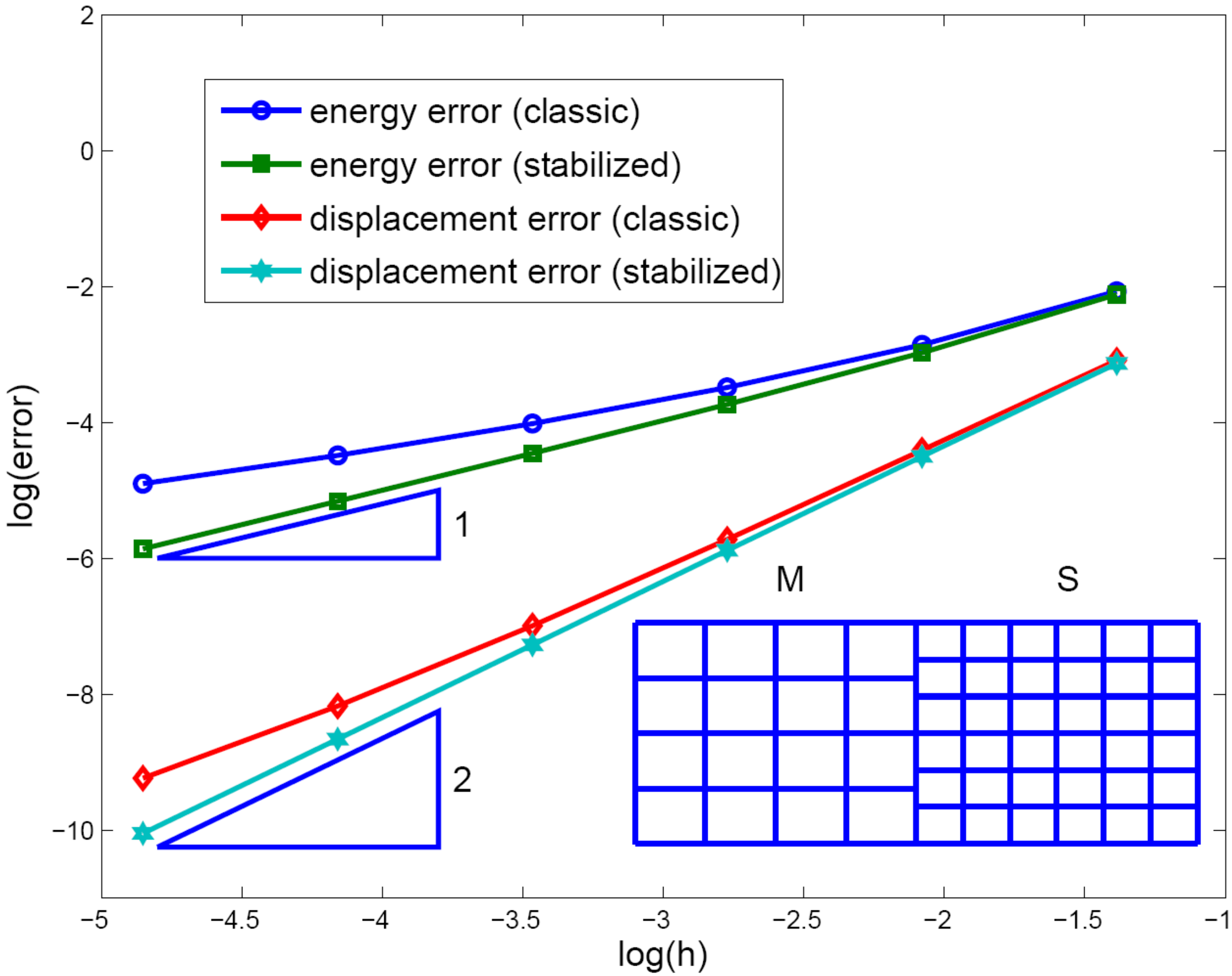
TABLE 3.2
Primal forms for the DG methods in Table 3.1.

Method	$B_h(w,v)$
Bassi–Rebay [10]	$g - \langle\{\nabla_h w\}, [\![v]\!]\rangle - \langle[\![w]\!], \{\nabla_h v\}\rangle + \big(r([\![w]\!]), r([\![v]\!])\big)$
Brezzi et al. [22]	$g - \langle\{\nabla_h w\}, [\![v]\!]\rangle - \langle[\![w]\!], \{\nabla_h v\}\rangle + \big(r([\![w]\!]), r([\![v]\!])\big) + \alpha^{\mathrm{r}}(w,v)$
LDG [41]	see (3.27)
IP [50]	$g - \langle\{\nabla_h w\}, [\![v]\!]\rangle - \langle[\![w]\!], \{\nabla_h v\}\rangle + \alpha^{\mathrm{j}}(w,v)$
Bassi et al. [13]	$g - \langle\{\nabla_h w\}, [\![v]\!]\rangle - \langle[\![w]\!], \{\nabla_h v\}\rangle + \alpha^{\mathrm{r}}(w,v)$
Baumann–Oden [15]	$g - \langle\{\nabla_h w\}, [\![v]\!]\rangle + \langle[\![w]\!], \{\nabla_h v\}\rangle$
NIPG [64]	$g - \langle\{\nabla_h w\}, [\![v]\!]\rangle + \langle[\![w]\!], \{\nabla_h v\}\rangle + \alpha^{\mathrm{j}}(w,v)$
Babuška–Zlámal [7]	$g + \alpha^{\mathrm{j}}(w,v)$
Brezzi et al. [23]	$g + \alpha^{\mathrm{r}}(w,v)$

SIAM J. NUMER. ANAL.
Vol. 39, No. 5, pp. 1749–1779

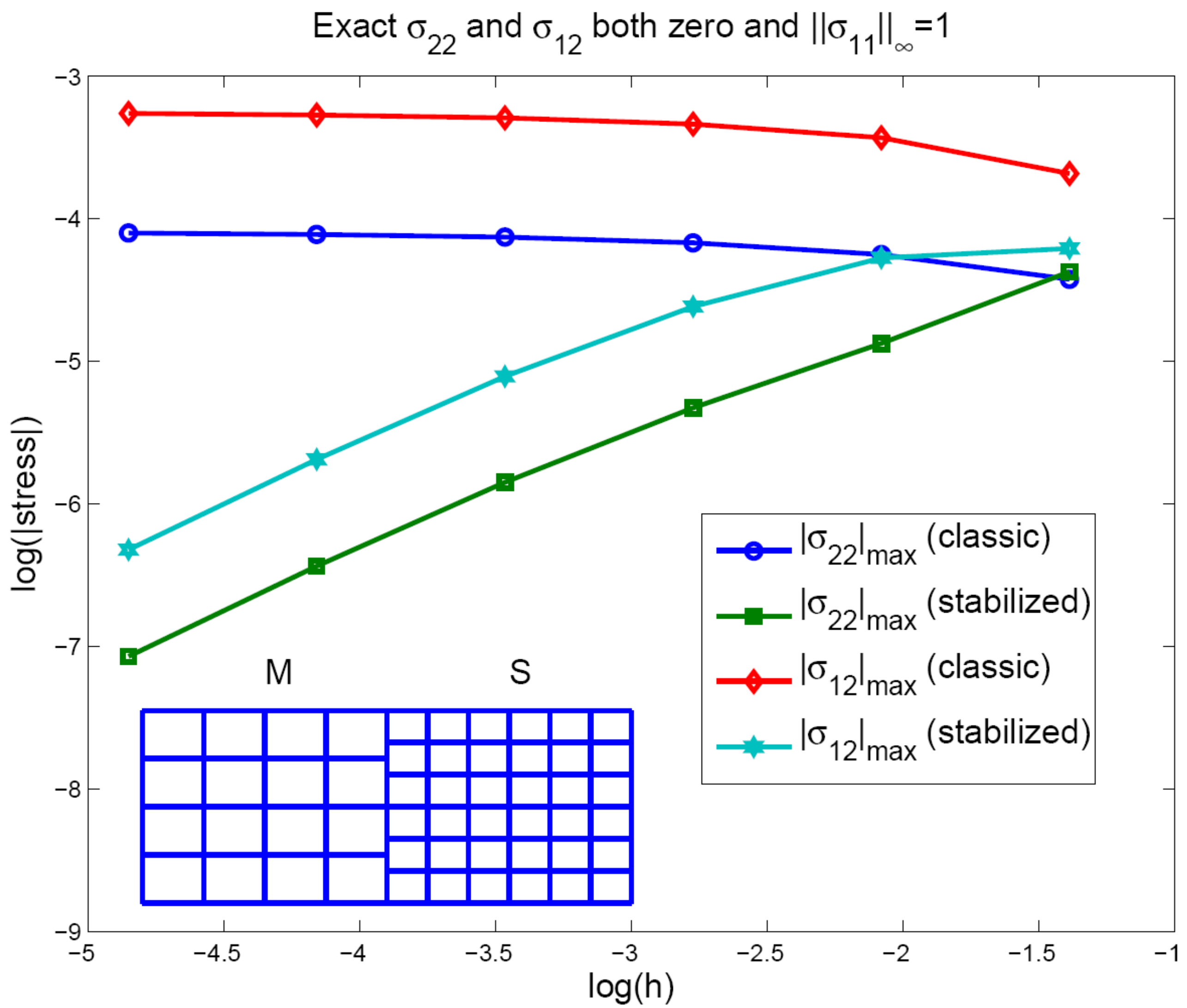
Sandia National Laboratories

2D Plane Stress Bending Example

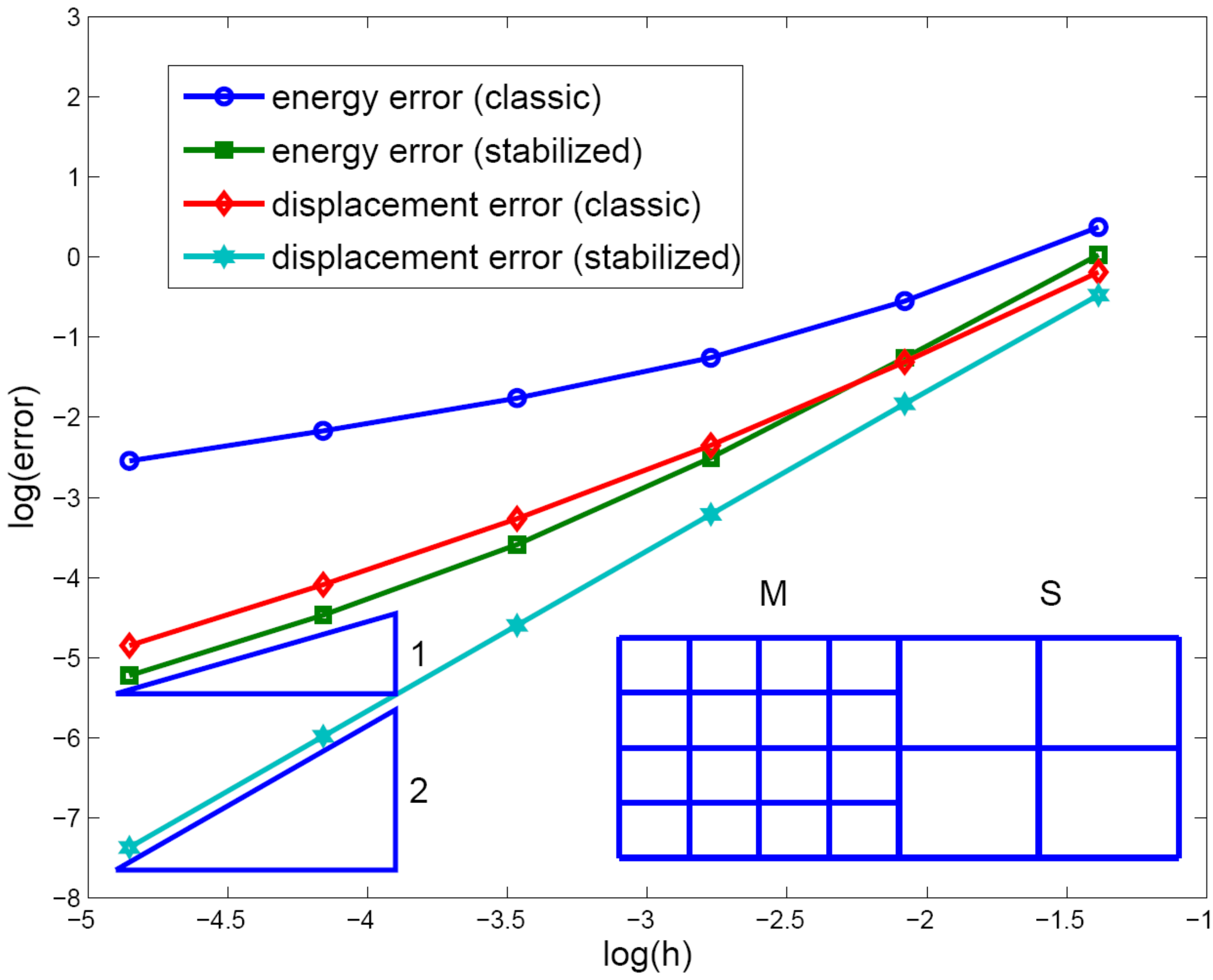


Sandia National Laboratories

2D Plane Stress Bending Example

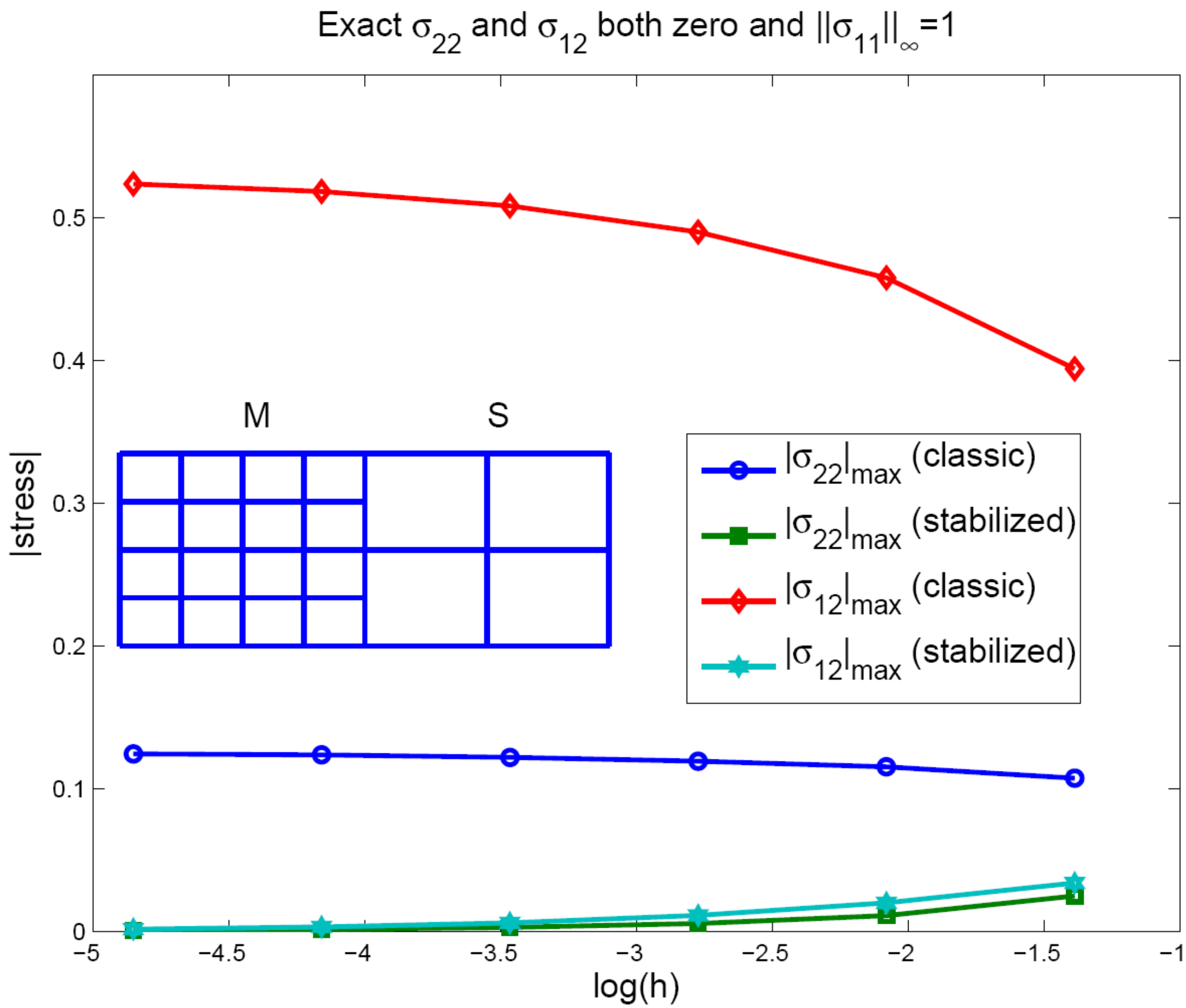


2D Plane Stress Bending Example



Sandia National Laboratories

2D Plane Stress Bending Example



Sandia National Laboratories

Classic Tied Contact Convergence

- **Various published numerical results support convergence rate of $h^{1/2}$ in energy norm, but haven't found related analysis yet.**
- **Two experts consulted, but did not know**
- **Applicable theory not common knowledge**

in elasticity. In standard commercial codes for computing contact between two elastic bodies, the contact condition is only checked at the nodes either on one or on both of the bodies. This corresponds to choosing discrete Lagrange multipliers which is not natural from the perspective of the variational formulation of the problem. The stability and convergence properties of these approaches are in general not known, and the results have to be carefully interpreted, which requires some experience. Furthermore, in our experience,

Numer. Math. (2005) 100: 91–115
Digital Object Identifier (DOI) 10.1007/s00211-005-0587-4

Classic Tied Contact Convergence

- **result for connecting standard/spectral elements**

MATHEMATICS OF COMPUTATION
VOLUME 54, NUMBER 189
JANUARY 1990, PAGES 21-39

- **I think I'm on to something!**

discretized by a consistent interpolation method, for example using Eq. (17) where the fluid interface displacement field is interpolated from the values of the structural interface displacement field, it can be proved [20] that the discretization error of the coupled fluid/structure model problem grows asymptotically as $\mathscr{E}^{\text{interp}}_{\text{F/S}} = \text{O}(\sqrt{h_\text{F}})$. In other words, interpolating the fluid displacement field at the fluid/structure boundary

Comput. Methods Appl. Mech. Engrg. 157 (1998) 95–114

- **Let's go find reference [20]**

Sandia National Laboratories

Classic Tied Contact Convergence

- **result for connecting standard/spectral elements**

MATHEMATICS OF COMPUTATION
VOLUME 54, NUMBER 189
JANUARY 1990, PAGES 21–39

- **I think I'm on to something!**

discretized by a consistent interpolation method, for example using Eq. (17) where the fluid interface displacement field is interpolated from the values of the structural interface displacement field, it can be proved [20] that the discretization error of the coupled fluid/structure model problem grows asymptotically as $\mathscr{E}^{\text{interp}}_{\text{F/S}} = O(\sqrt{h_F})$. In other words, interpolating the fluid displacement field at the fluid/structure boundary

Comput. Methods Appl. Mech. Engrg. 157 (1998) 95–114

- **Let's go find reference [20]**

[20] Y. Maday, Private communication.

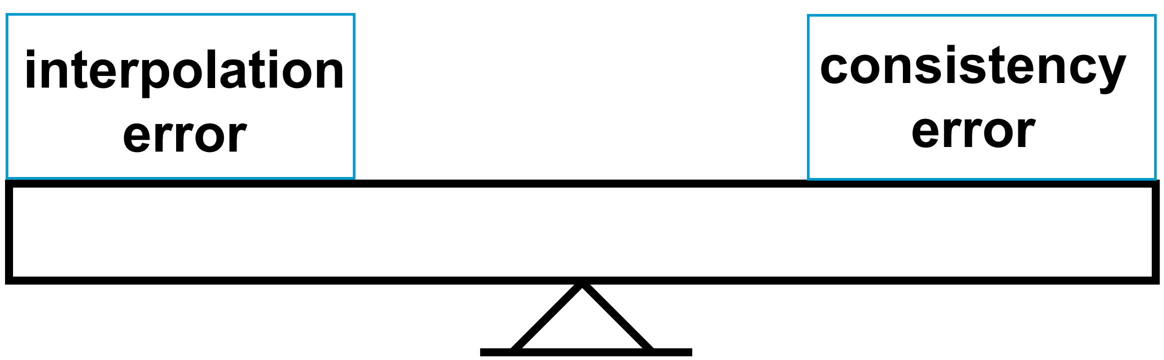
Sandia National Laboratories

Some Theory

- **Discrete Bilinear Form**

$$a_h(u,v) = \sum_{i=1}^{N} \alpha_i \int_{\Omega_i} \nabla u \cdot \nabla v \, dx + s_i(u,v),$$

$$s_i(u,v) = \alpha_i \sum_{j \in \mathcal{S}_i} H_{ij}^{-1} \boxed{\rho_{ij}} \int_{\Gamma_{ij}} [u][v] \, ds.$$



Some Theory

- **Q: How do we avoid integrals in implementation?**

$$s_i(u,v) = \alpha_i \sum_{j \in \mathcal{S}_i} H_{ij}^{-1} \rho_{ij} \boxed{\int_{\Gamma_{ij}} [u][v]\, ds.}$$

- **A: springs at master nodes spectrally equivalent to term in box for linear elements**
- **Convergence theory for linear elements only thus far**

$$|u - u_h|_{H^1} \leq Ch\|u\|_{H^2}$$

- **To do: L^2 error estimates and quadratic elements**

Sandia National Laboratories

Symmetric Variant Without Constraints

- Tied contact constraints no longer included
- Symmetric treatment with springs attached to both master and slave nodes (pure penalty)
- Implementation, numerical results, and theory very similar to stabilized tied contact
- Similar in spirit to two-pass methods, but constraints are enforced via a penalty method

Summary

- Simple fix to classic tied contact
  - discrete springs introduced at interfaces
  - simple physical interpretation
  - avoids complicated surface integral calculations
  - no parameters to choose
  - optimal error estimates for linear elements
  - non-oscillatory stresses near interfaces for smooth solutions
  - iterative solver performance not degraded significantly in numerical experiments thus far

Summary

- **Future Work**
  - **Numerical Studies**
    - **problems with curved surfaces**
    - **problems with material property jumps**
    - **iterative solver performance for larger problems**
  - **Theory Development**
    - **L^2 norm estimates**
    - **quadratic elements**