SAND2009-4608C

An Analytically Enriched XFEM for Cohesive Crack Modeling

USNCCM X

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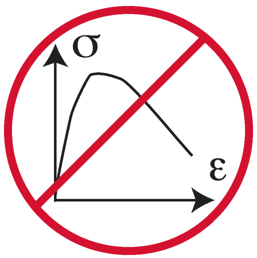
Preview

- Introduction
- PUFEM/XFEM displacement field enrichment
- Numerical formulation issues
- Results for a model problem
- Extensions for mixed mode problems
- Mixed mode examples
- Examining accuracy of the fields
- Observations and conclusions

Study Introduction

Objective: A "valid" means of modeling material localization in finite element analyses (ARL, 2001).



Initial Scope: quasi-brittle fracture using cohesive zone modeling -- consistent with the enrichment functions

Approach: Develop an extended FEM (XFEM) that allows the displacement field to be enriched in the neighborhood of a strong discontinuity.

- can represent a discontinuity without mesh refinement
- can <u>potentially</u> represent the gradients near a surface of localization without mesh refinement

Background

Initial related studies:

- Melenk and Babuska (1996)
  - theory for Partition of Unity FEM (PUFEM)
- Belytschko and Black (1999)
  - developed PUFEM for LEFM $\rightarrow$ XFEM

XFEM/PUFEM-Cohesive Zone Studies

- Wells and Sluys (2001)
- Moes and Belytschko (2002)
- Zi and Belytschko (2003) -- tip function addresses tip position but not the field
- Xiao and Karihaloo (2006) -- asymptotic fields
- …

PUFEM Displacement Field Enrichment

- Standard FEM
- PUFEM/XFEM

Global displacement approximations

Standard FEM:

$$\mathrm{u}(\mathrm{x}) = \sum_{i=1}^{N_\Phi} \Phi_i(\mathrm{x})\mathrm{u}_i$$

PUFEM/XFEM:

$$\mathrm{u}(\mathrm{x}) = \sum_{i=1}^{N_\Phi} \Phi_i(\mathrm{x})\mathrm{u}_i + \sum_{j=1}^{N_\Lambda}\sum_{i=1}^{N_{\Phi^*}} \Lambda_j(\mathrm{x})\Phi_i^*(\mathrm{x})\alpha_{ij}$$

Element displacement approximations

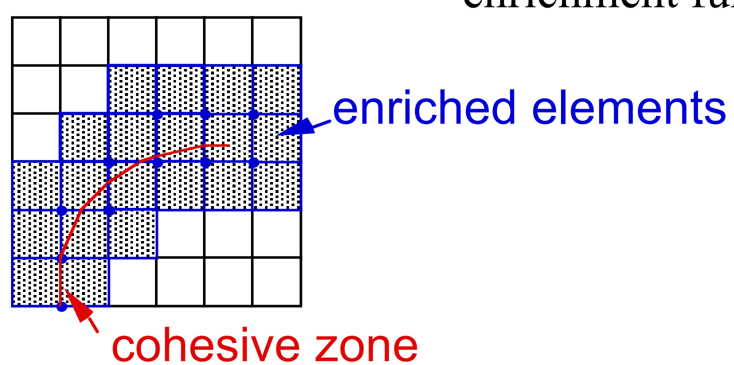
Standard FEM:

$$\mathrm{u}(\mathrm{x}) = \sum_{i=1}^{N_N} \mathrm{N}_i(\mathrm{x})\mathrm{u}_i$$

PUFEM/XFEM:

$$\mathrm{u}(\mathrm{x}) = \sum_{i=1}^{N_N} \mathrm{N}_i(\mathrm{x})\mathrm{u}_i + \sum_{j=1}^{N_\Lambda}\sum_{i=1}^{N_{N^*}} \Lambda_j(\mathrm{x})\mathrm{N}_i^*(\mathrm{x})\alpha_{ij}$$

enrichment functions



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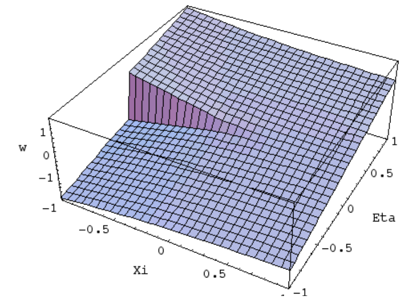
“My Path to Enrichment”

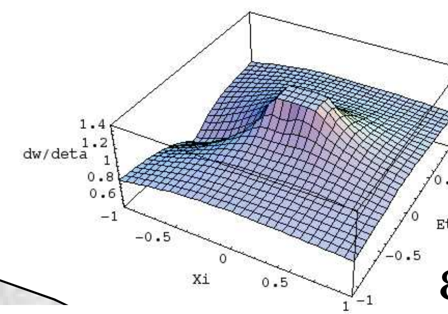
“I am not discouraged, because every wrong attempt discarded is another step forward. I have not failed. I’ve found 10,000 ways that won’t work.” – Thomas Edison

Formulated simple series that incorporated a discontinuity.



u_2

Formulated simple functions that had key features of accurate numerical results.



ε_{22}

Analytically derived enrichment functions based upon the Muskhelishvili formalism.

Enrichment Functions: An Analytical Source

Muskhelishvili formalism (1953)

Hong & Kim (2003) obtained a series solution to the inverse problem

Zhang & Deng (2007) obtained "asymptotic solutions"

– both assumed linear elastic isotropic material (except for cohesive zone)

Additional analysis was used to:

verify the proposed solutions

extend them for field variables required by the XFEM

Displacements

$$u_1 + iu_2 = \frac{1}{2\mu}\left\{\kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}\right\}$$

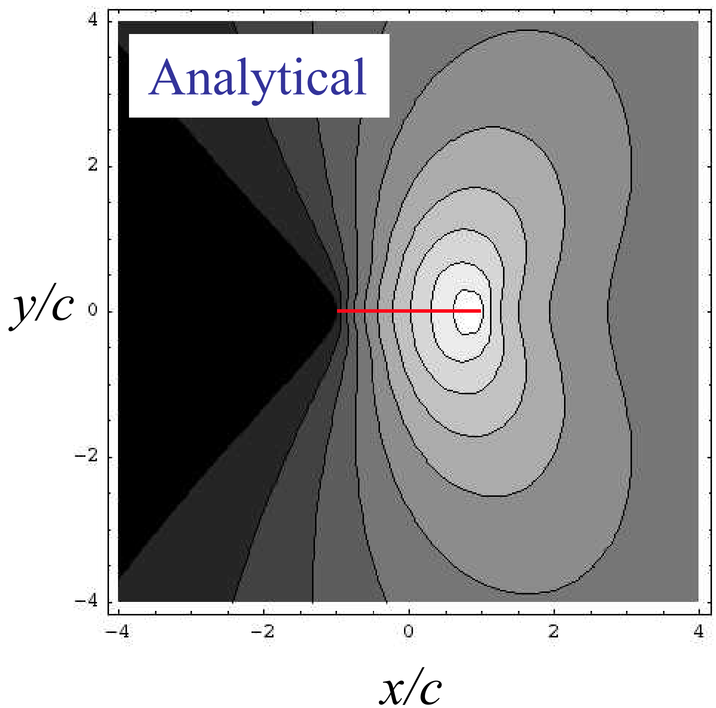
where φ and ψ are analytic functions, and $z = x+iy$.

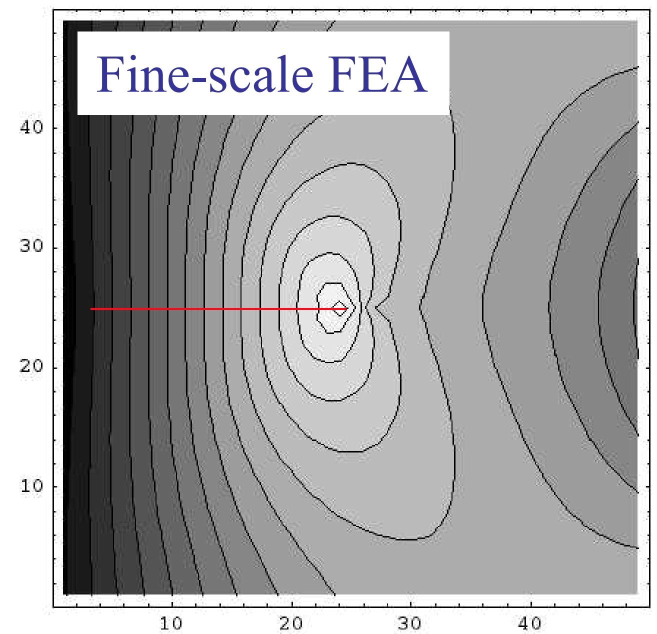
Enrichment Functions: An Analytical Source

Qualitative comparison of σ_{22} with fine-scale FEA

- Analytical ~ First terms in series for Hong & Kim solution
- "Fine-scale" FEA ~ results for finely meshed FEA with interface el.

Note: problems differ and CZ sizes are not to the same scale.





Cohesive zone length = $2c$

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Enrichment Functions: An Analytical Source

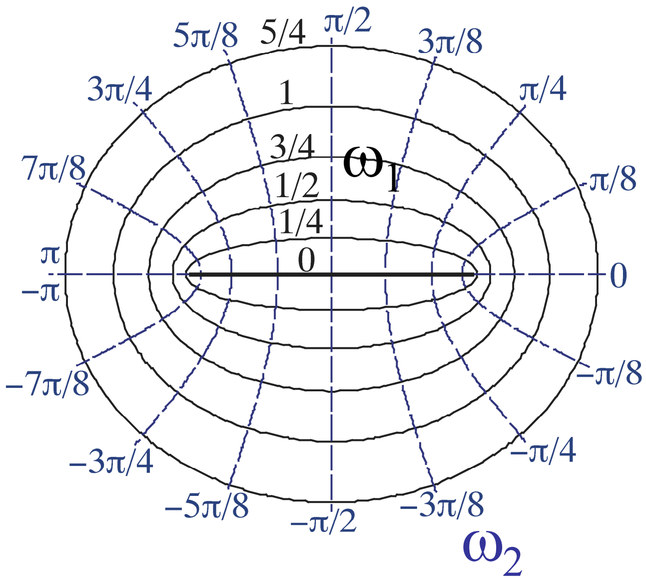
Zhang & Deng (2007) solve the problems in terms of elliptic coordinates (ω)

$$z = c\cosh(\omega)$$

Symbolically the inverse map is give by

$$\omega = \cosh^{-1}(z/c)$$

complex analysis → useful forms.



They (1) adopt a Westergard stress function,

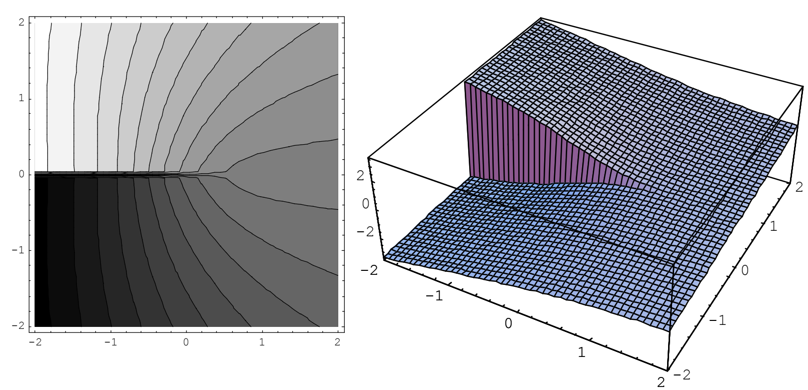
→ one unknown analytic function,

(2) express this in a series, and

(3) define one term of the series to be the asymptotic solution.

Mode-I Enrichment Functions

- Based upon the asymptotic solutions of Zhang & Deng



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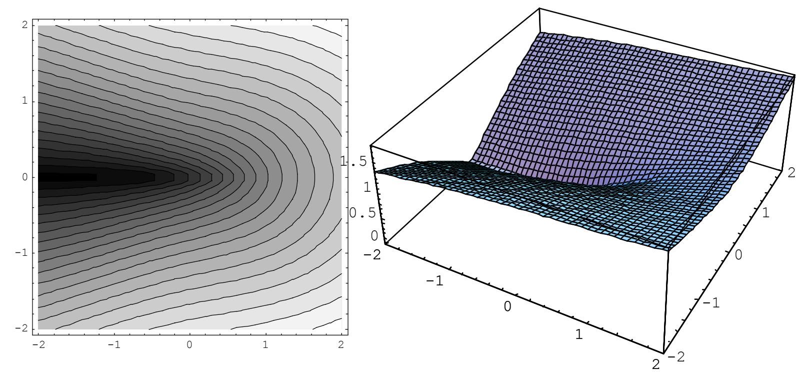
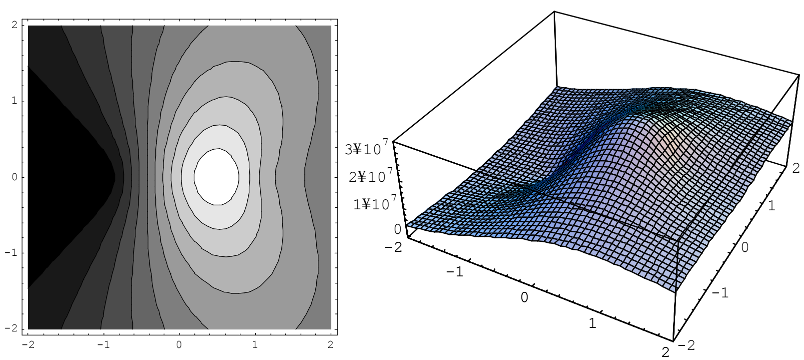
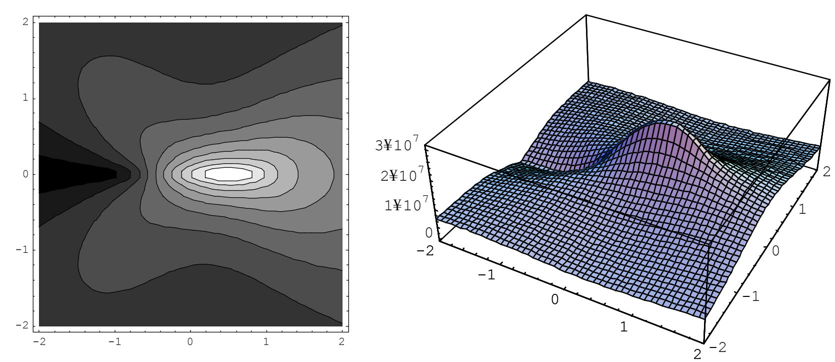


u_1



σ_{22}



σ_{11}

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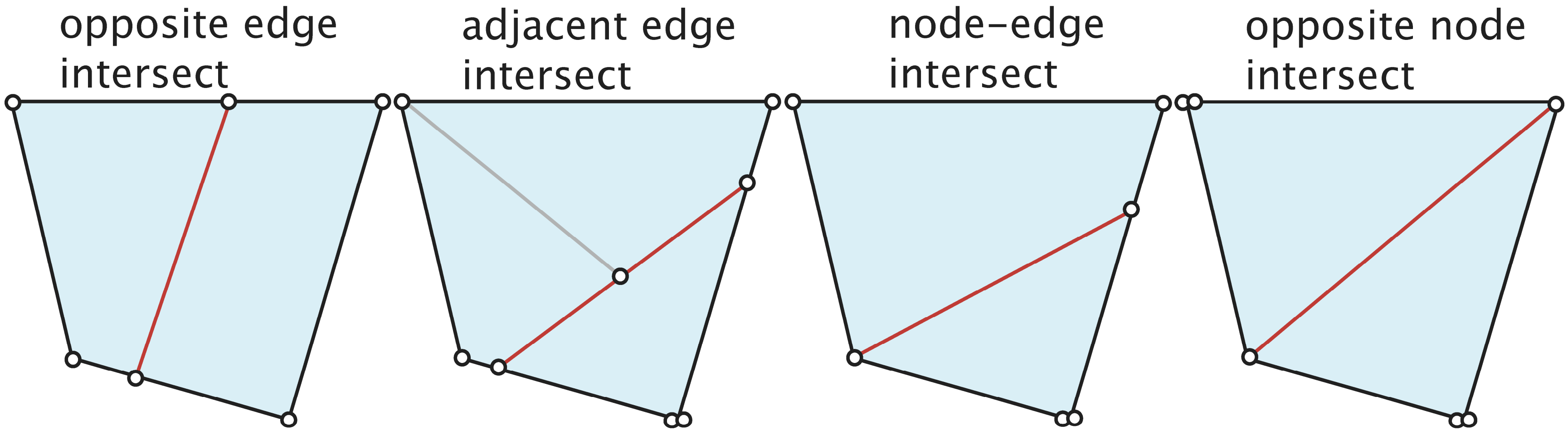
Numerical Formulation Issues

- Element Integration

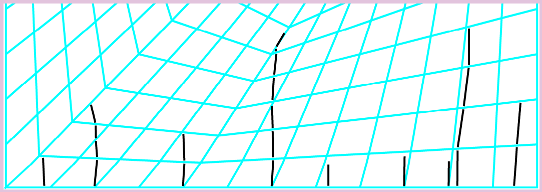


- Key solver issue: new DOFs that result from adding enrichment to nodes do not have a good initial values $\Rightarrow$ nonlinear solver can have problems.

  Solution: Penalty relaxation in a multi-level solver

Numerical Formulation Issues

```
Assign equation numbers;   Determine storage for K;
Repeat (* time increment loop *)
|   Repeat (* outer level solver loop -- aka localization loop *)
|   |   Update K & R
|   |   Reset penalty number to large value when entering a new element, else 0
|   |   …
|   |   Repeat (* penalty reduction loop *)
|   |   |   Relax the penalty number
|   |   |   Reset line search
|   |   |   Repeat (* nonlinear iteration loop *)
|   |   |   |   Factor K
|   |   |   |   Forward eliminate & back substitute to obtain dUiter
|   |   |   |   Repeat (* line search loop *)
|   |   |   |   |   Search line for dU_iter
|   |   |   |   |   …
|   |   |   |   Until (||R||<R_toler) OR (||R||<||R_old||)
|   |   |   |   …
|   |   |   Until ||R||<R_toler
|   |   Until penalty number is reduced to zero
|   |   …
|   Until localization is complete
|   U:= U + dU_step;   dU_step:= 0;   U_old:= U
|   …
Until time stepping is complete
```



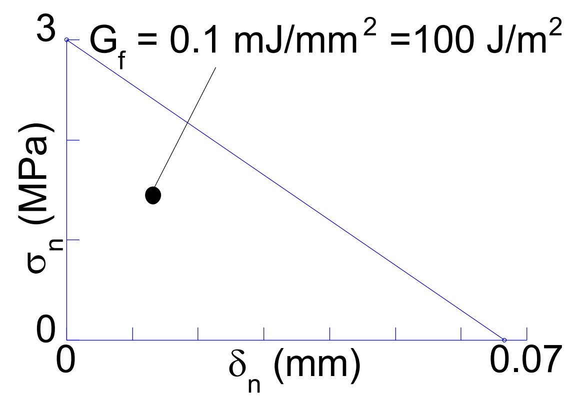
Preview of Results

- Mode-I Model Problems -- emphasis on reproducing the cracking history
  - Results for aligned meshes
  - Results for skewed meshes
- Extensions for “mixed mode”
- Mixed mode examples

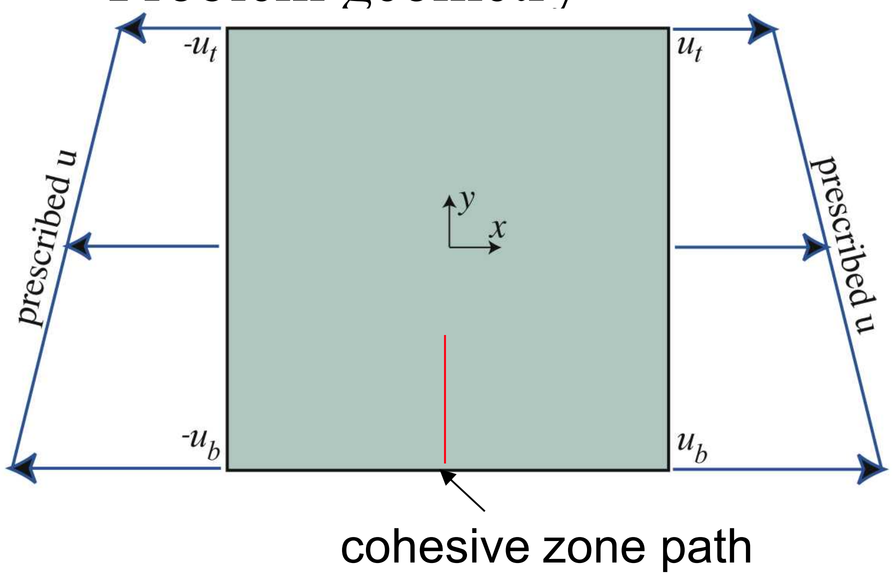
quasibrittle

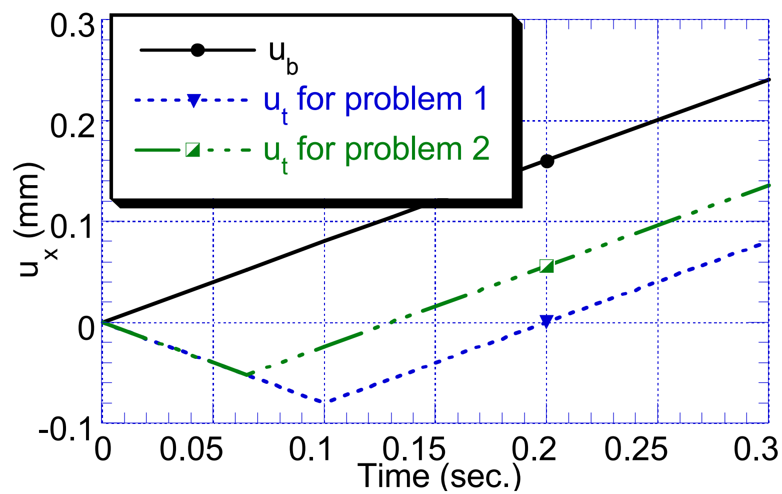
Initial Simple Test Problems

- Concrete test problems
  - relevant to HDBT
  - domain 1 m x 1 m
  - process-zone size ~ O(200 mm)
  - representative concrete tensile properties (except for simplified linear softening)
  - mode I quasistatic crack propagation



Problem geometry





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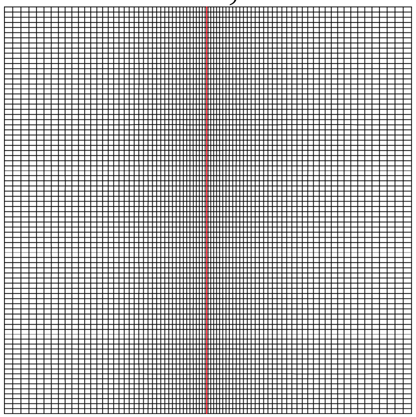
Spacial Discretizations

- Fine FEM mesh

81x80 ~ 13,284 dofs

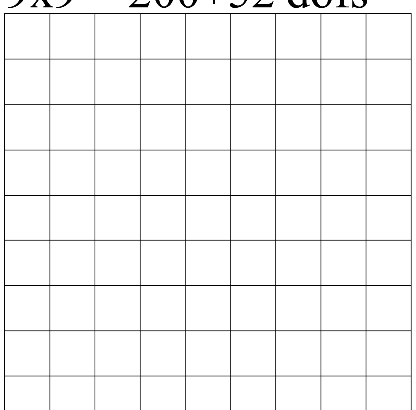


• provides an accurate reference solution

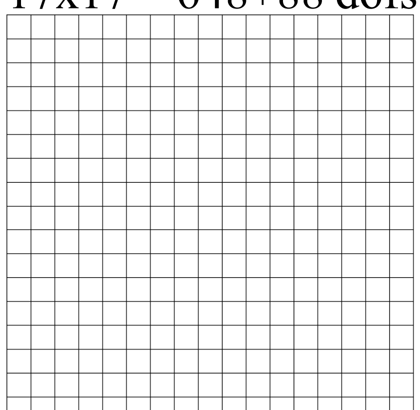
• includes a column of interface elements

- XFEM – Aligned Meshes & Skewed Meshes

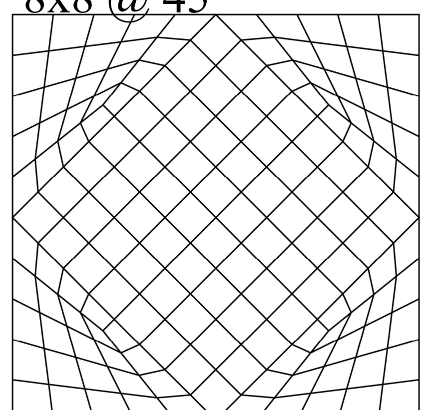
9x9 ~ 200+52 dofs



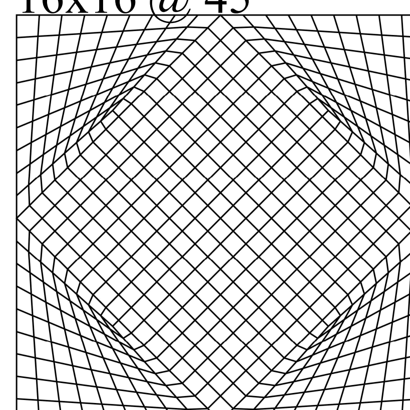
17x17 ~ 648+88 dofs



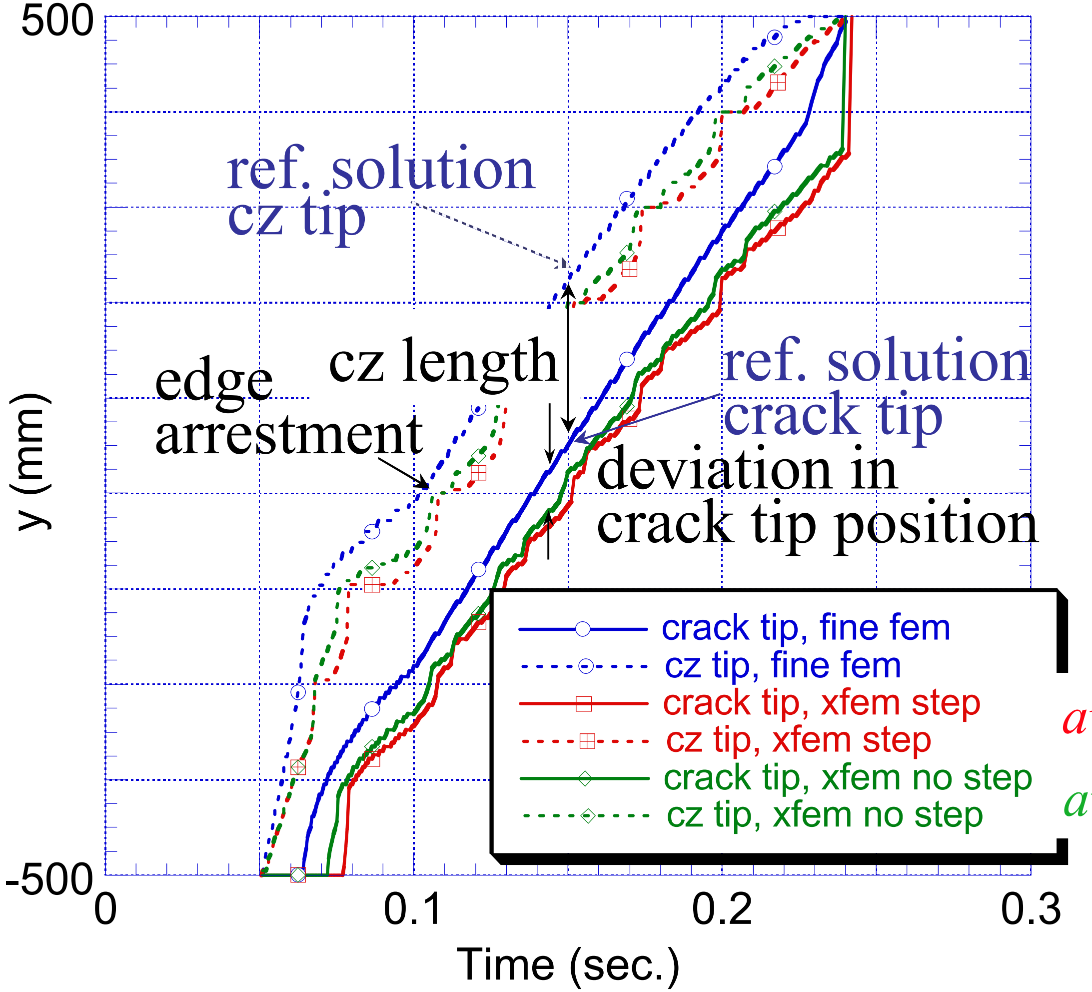
8x8 @ 45°



16x16 @ 45°



Extremes Histories



ave. dev. ~ 62 mm

ave. dev. ~ 46 mm

9x9 mesh, c = 125 mm

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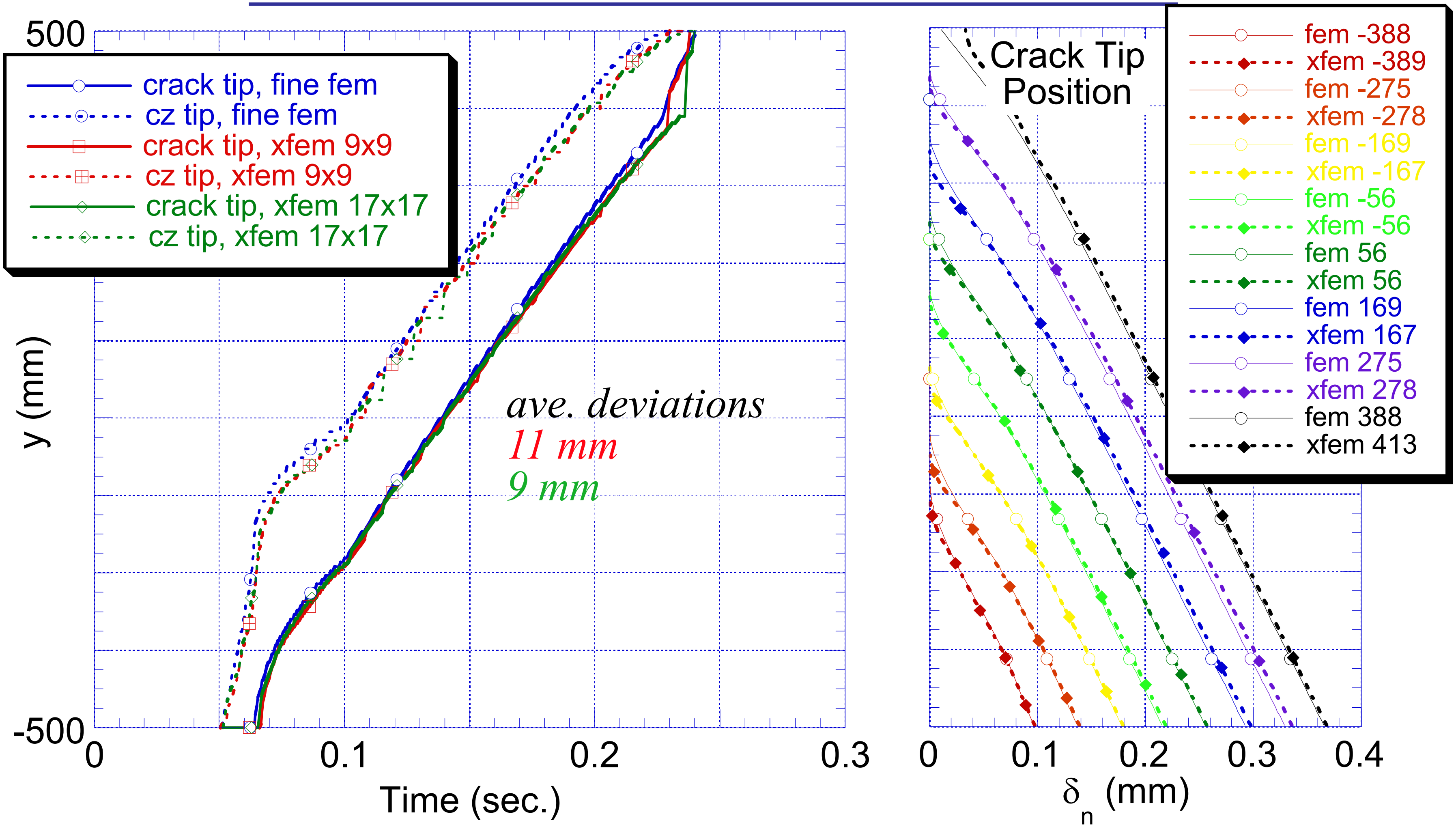
Extremes Histories and Crack Profiles



c = 50 mm

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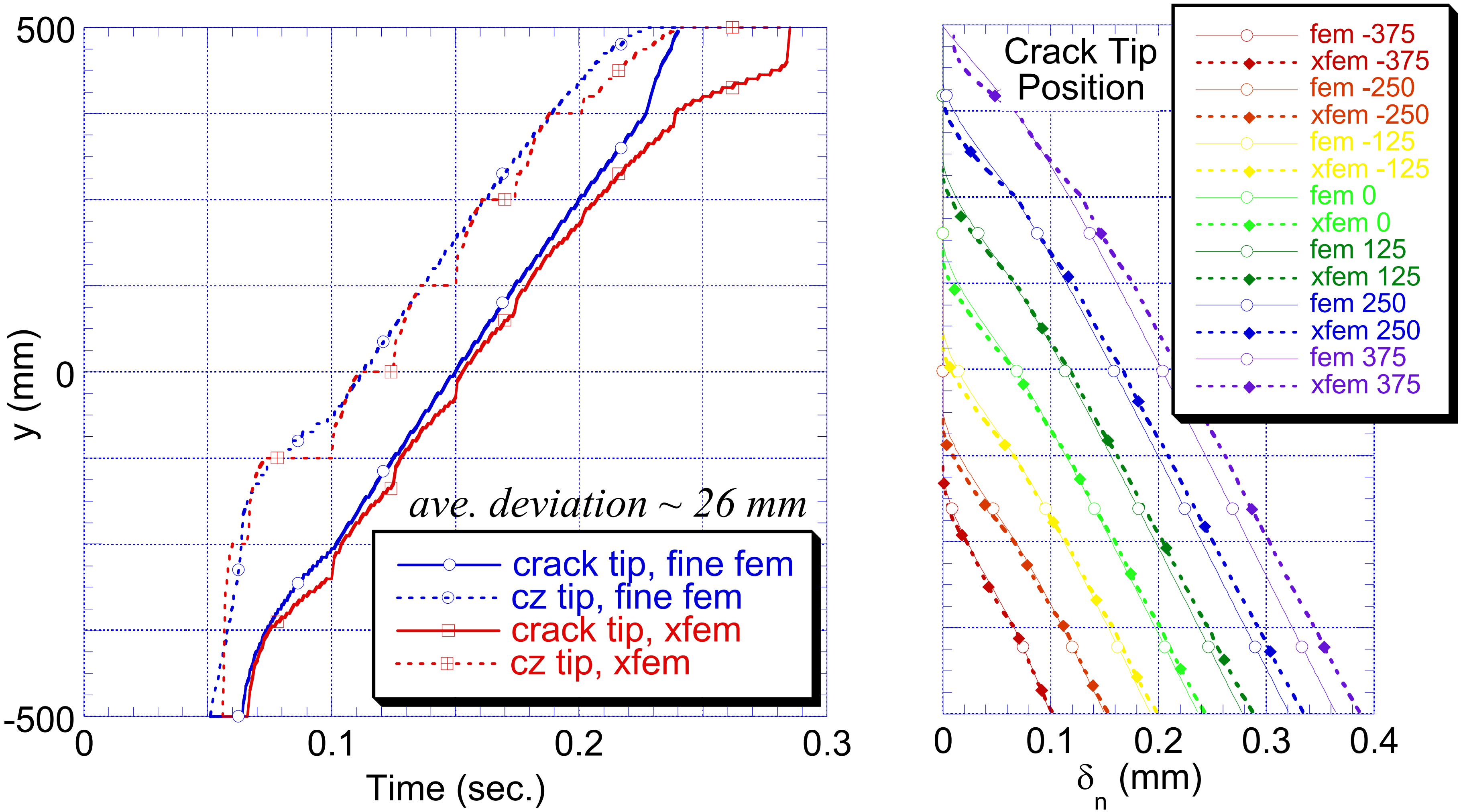
Extremes and Crack Profiles



8x8 mesh, c = 75 mm

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Summary of Average Deviations

Aligned Mesh	Ave. Deviation
9x9	11
17x17	9
33x33	6

Skewed Mesh	Ave. Deviation
8x8	18
16x16	13
32x32	6

Crack Propagation and Direction Calculations

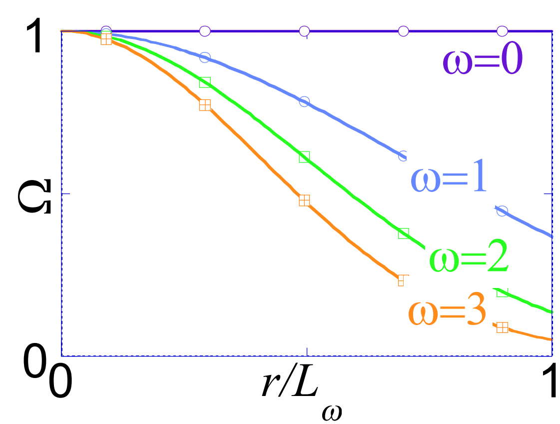
"Stress smoothing" used when a crack enters a new element

- Deviation between a polynomial approximation and the FEM approximation

$$d(\mathrm{x}) = \sigma^{p}(\mathrm{x}) - \sigma^{fem}(\mathrm{x}) = c_0 + c_1 x + c_2 y + \ldots - \sigma^{fem}(\mathrm{x})$$

- Weighting function

$$\Omega(\mathrm{x}) = \exp\left[-\omega\left(\frac{r}{L_\omega}\right)^2\right]$$

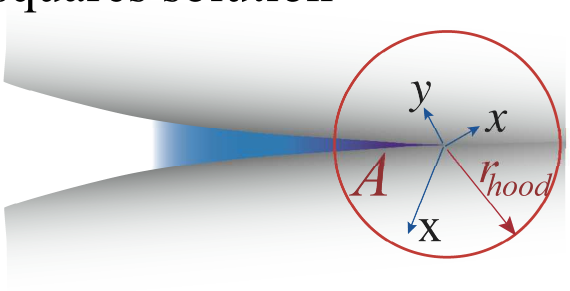


- Residual measure

$$R^2 = \int_A [\Omega(\mathrm{x}) d(\mathrm{x})]^2 \, dA$$

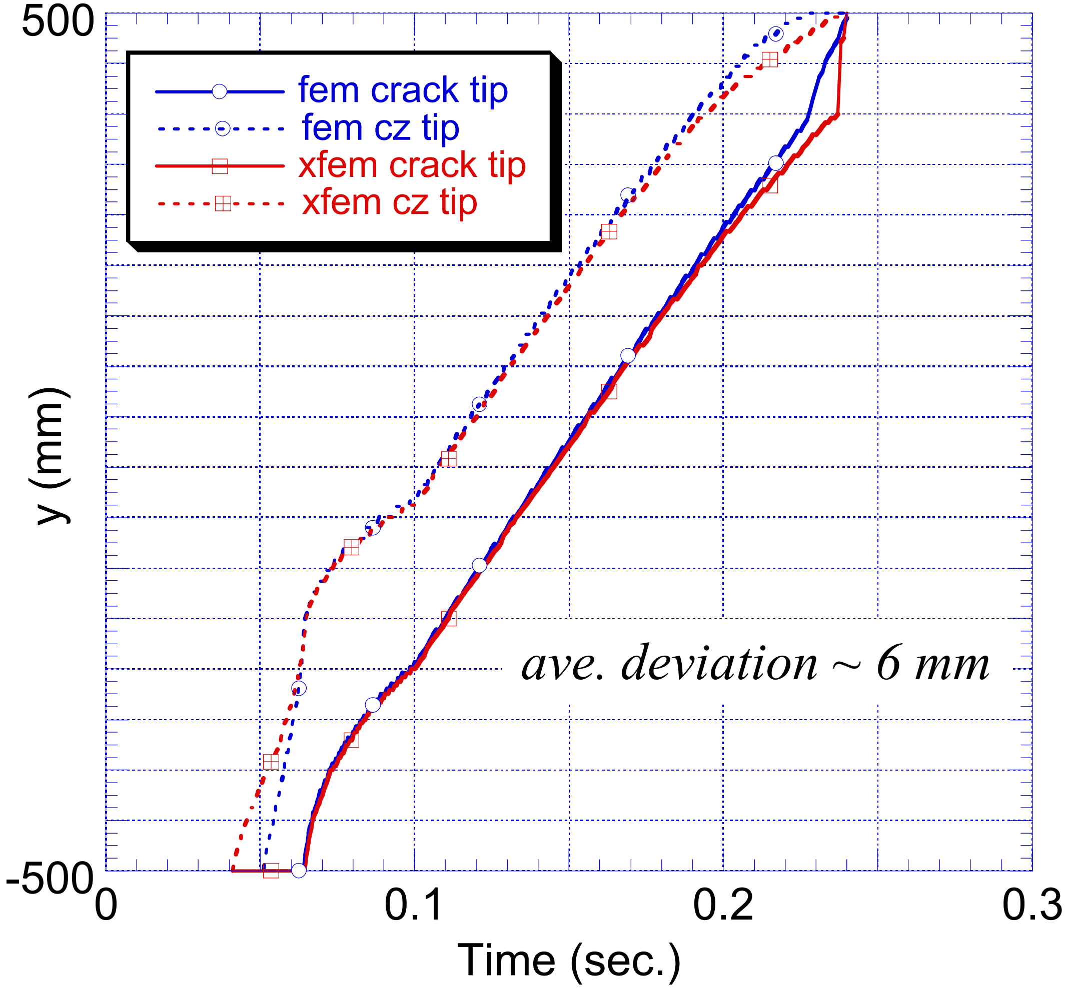
- Using the Gauss point values → weighted least squares solution

$$R^2 \approx \sum_{i=1}^{n} [\Omega(\mathrm{x}_i) d(\mathrm{x}_i)]^2 |J_i| w_i^{gauss}$$



Model Problem with Stress Smoothing

17x17 Aligned mesh



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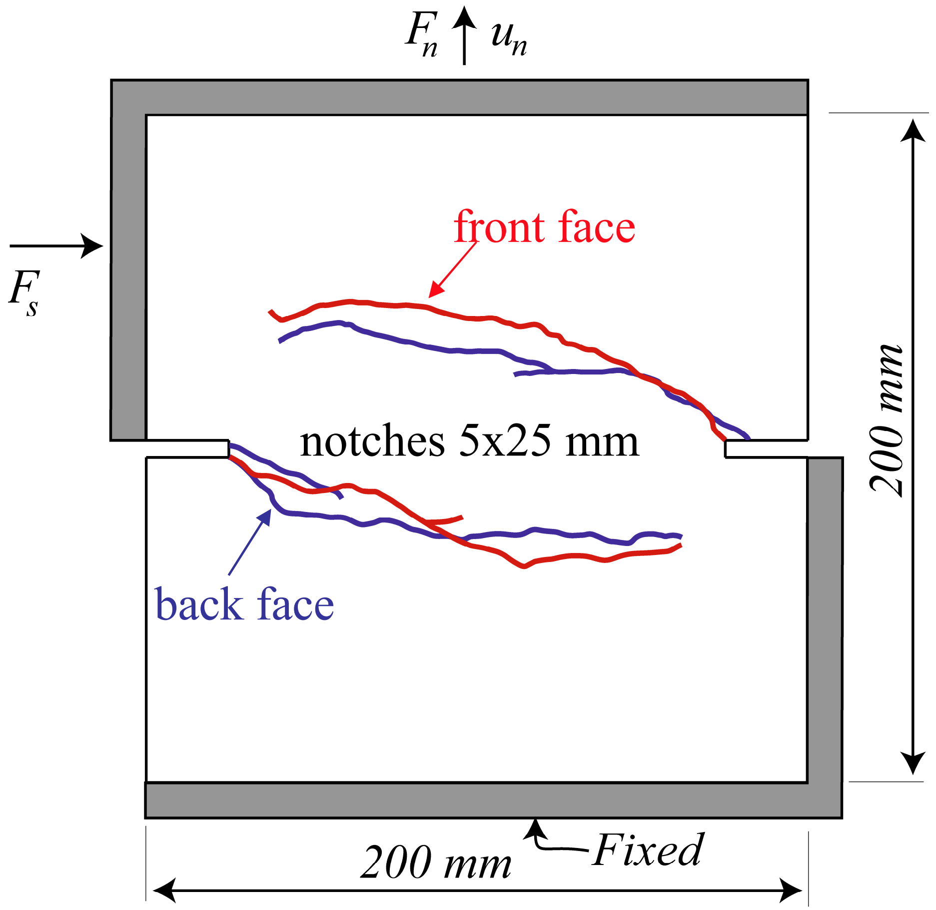
Mixed Mode Fracture Problem

Double edge-notched specimen (Nooru-Mohamed 1992) and experimental crack paths

Concrete square, 50 mm thick

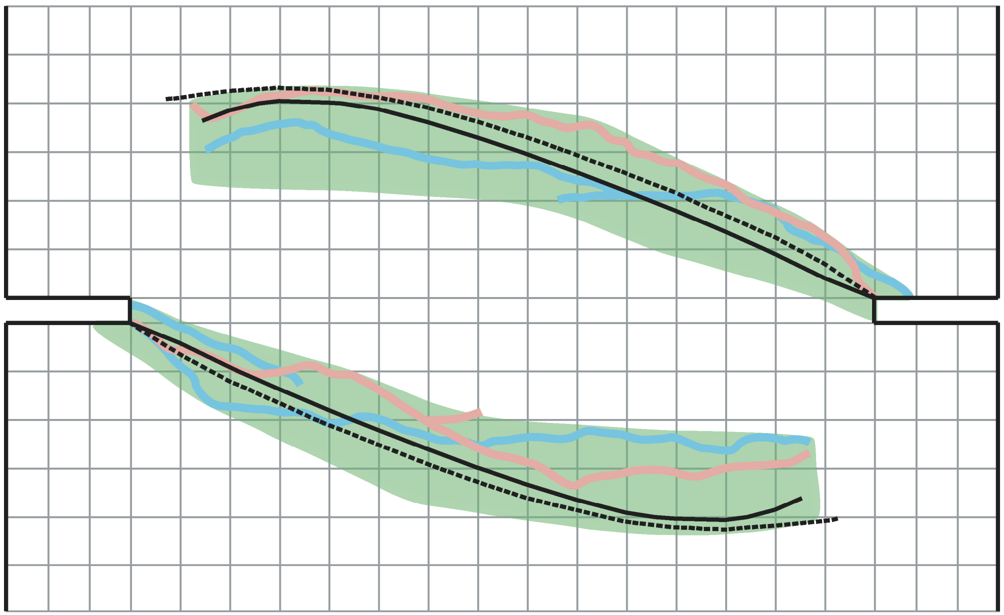
Load Path 4b:

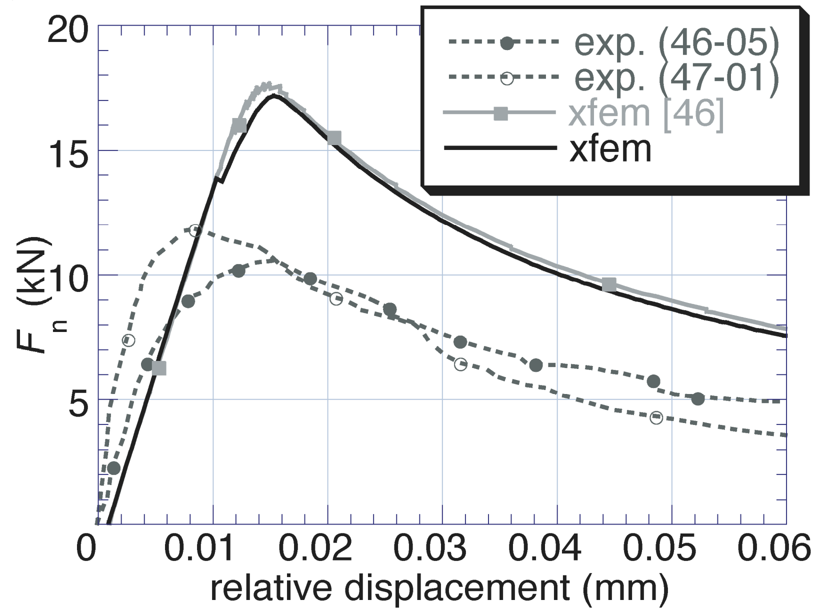
$F_s = 10$ kN



Mixed Mode Fracture Problem

XFEM simulation results for test give crack paths within the experimental scatter.





$r_\sigma = 20$ mm ~ $2h$, n=3, ω=0, full-disk, $\sigma_{cz}/\sigma_t = 0.8$

[46] Meschke & Dumstorff (2007)

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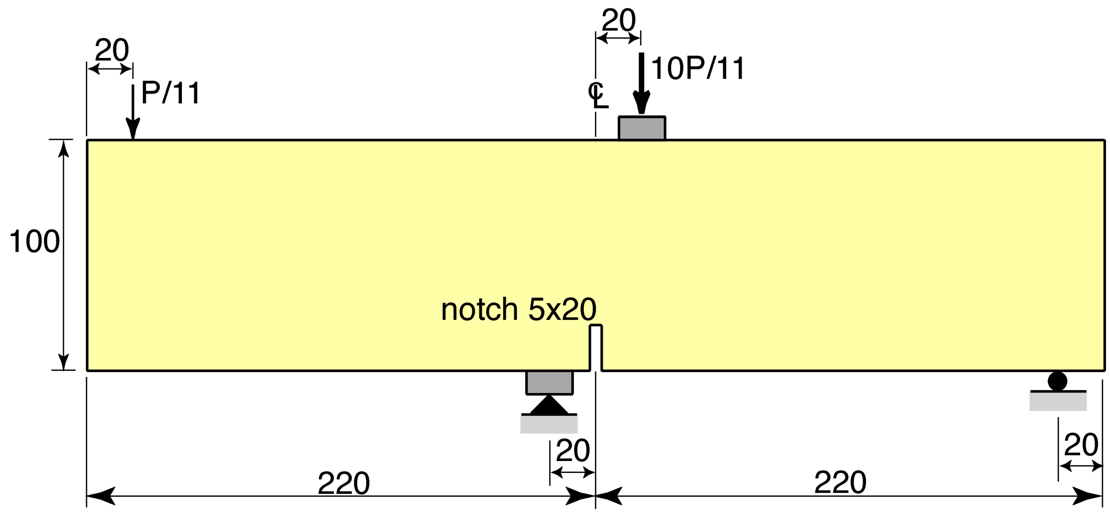
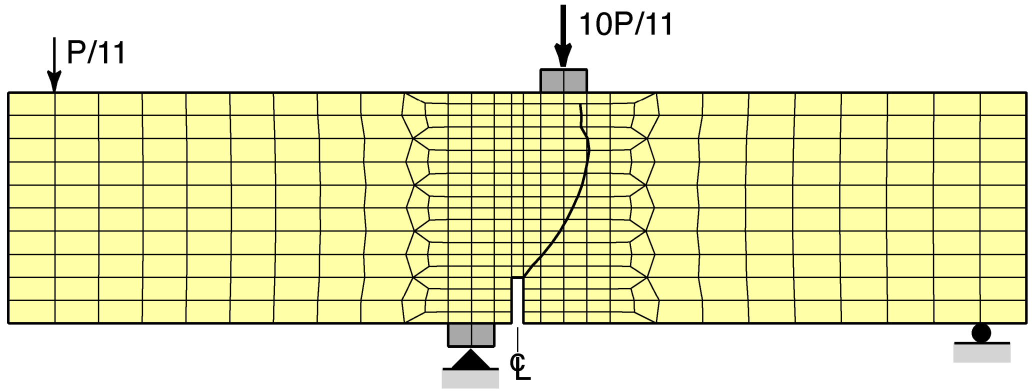
Single Edge-Notched Beam Specimen

Experimental work of Schlangen (1993)

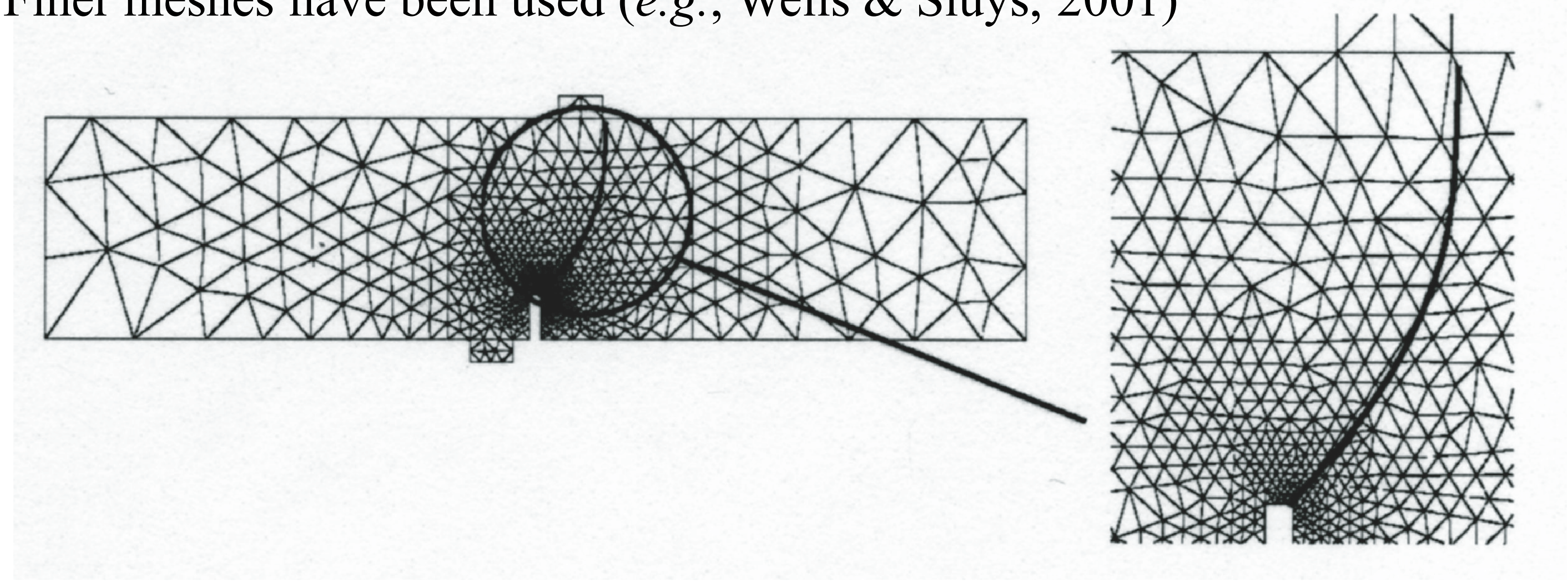
units ~ mm, thickness = 100 mm



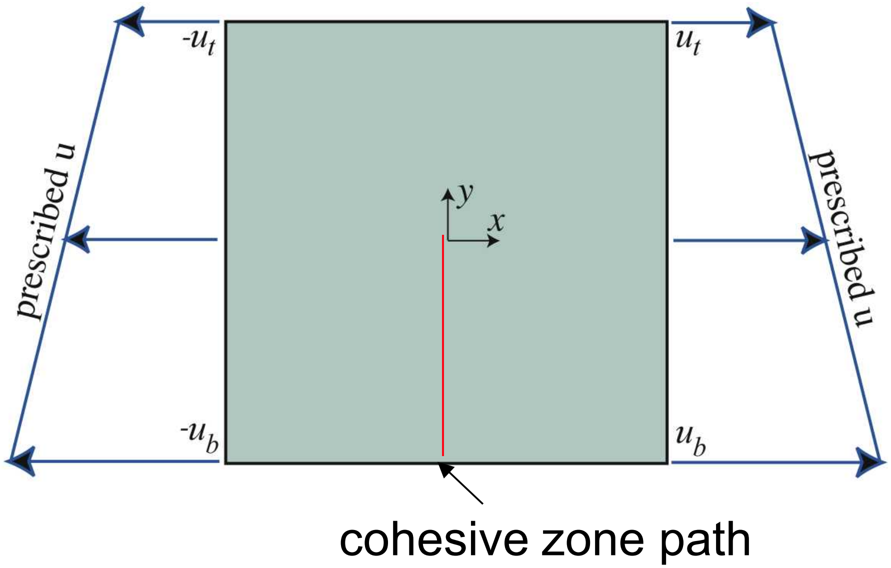
Single Edge-Notched Beam Specimen



Finer meshes have been used (*e.g.*, Wells & Sluys, 2001)



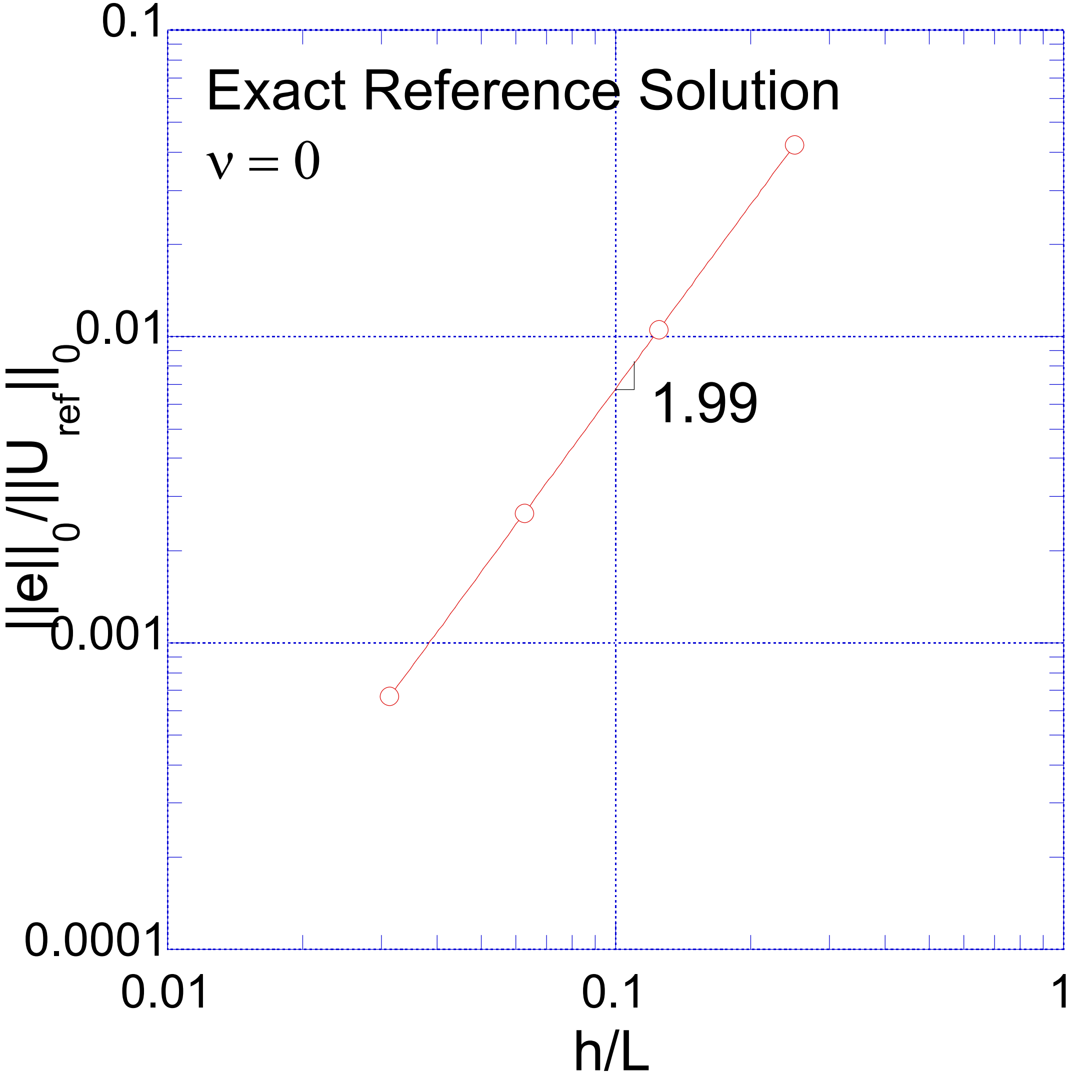
Initial Simple Test Problem



Initial tests exclude crack -- to establish limitations of approximate reference solution

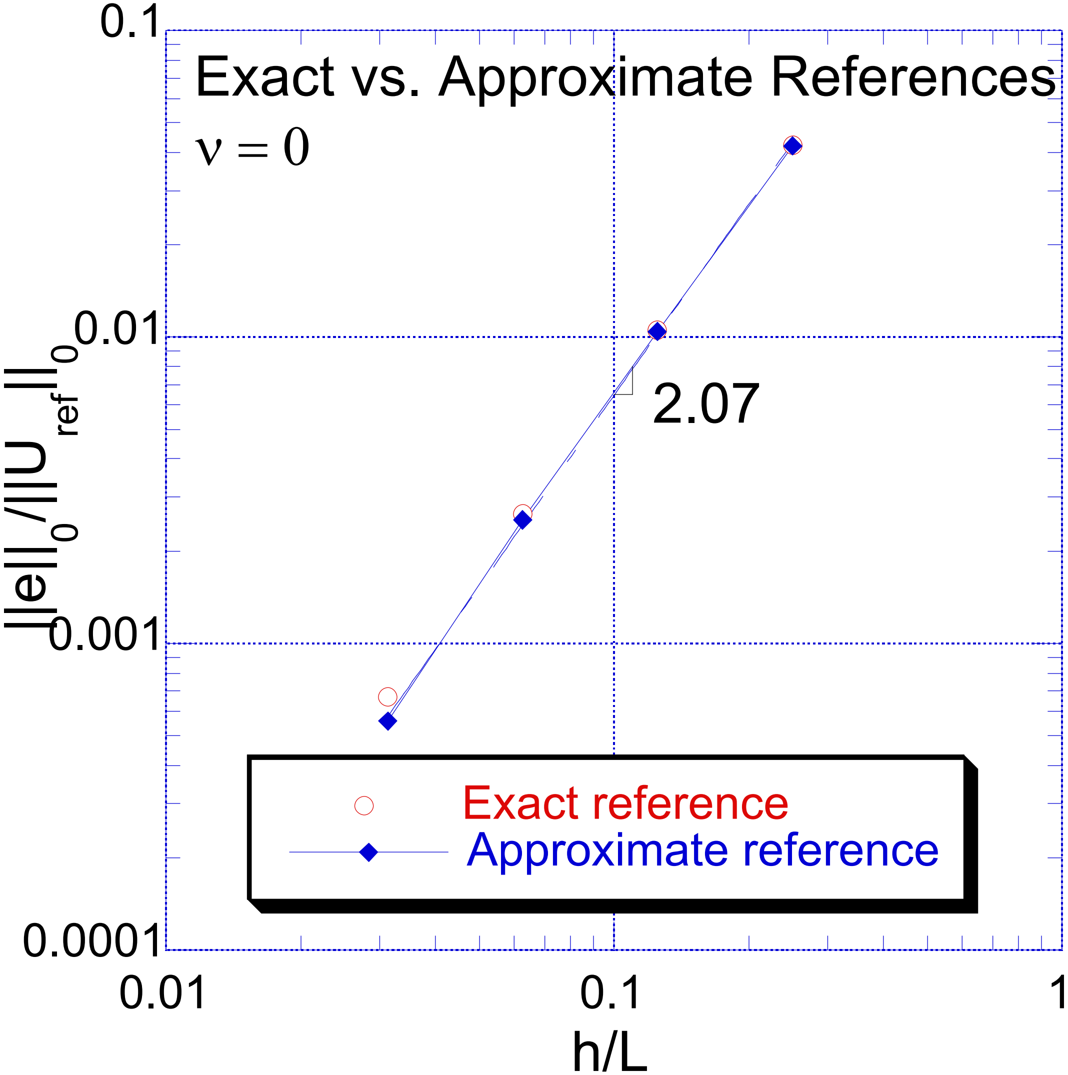
- Exact reference solution (ν=0)
- Approximate reference solution (ν=0)
- Approximate reference solution (ν=0.17)

Displacement Field Accuracy



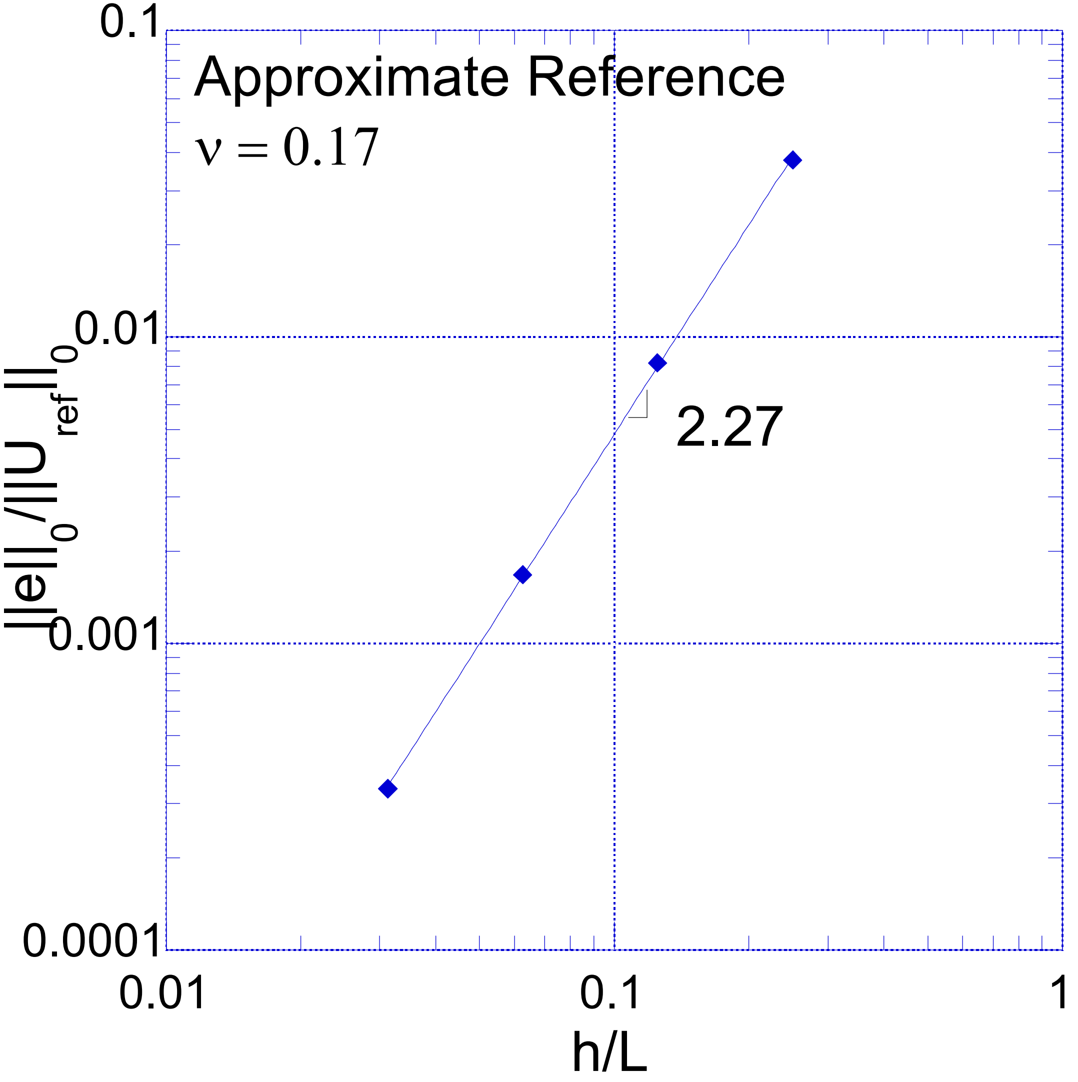
Meshes: 4x4, 8x8, 16x16, 32x32

Displacement Field Accuracy



Approximate reference: 81x80

Displacement Field Accuracy



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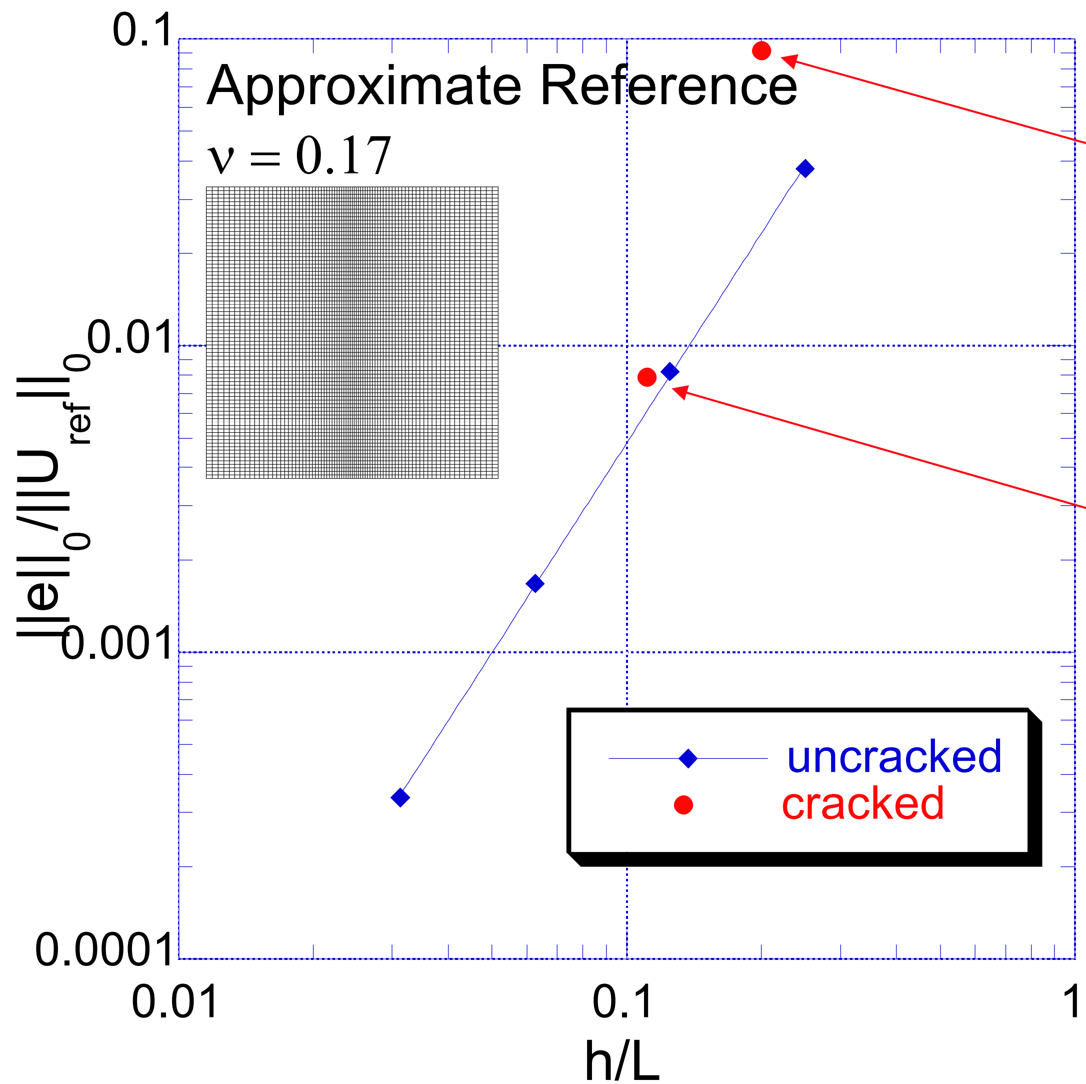

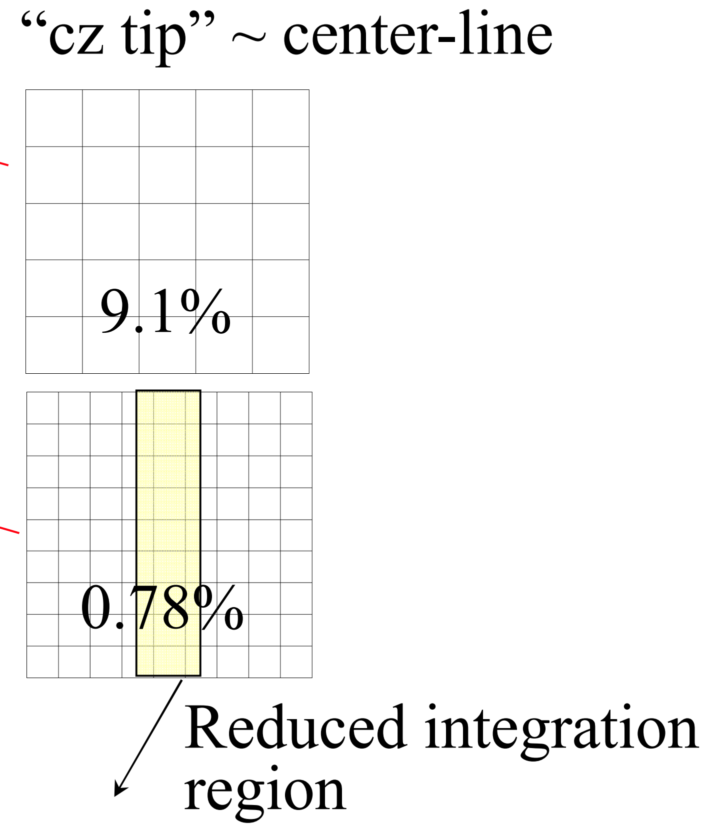

0.78% → 1.1%

Observations & Conclusions

- No free-lunch -- algorithm complexity ↑ with analytical enrichment
- Analytically enriched XFEM for cohesive zone modeling of localization has potential.
- Not the best approach for every application
- Several open issues, *e.g.*:
  - Value of c and its possible adjustment
  - Can the accuracy be improved?
  - How useful is analytical enrichment for materials that are:
    - anisotropic?
    - Inhomogeneous?
    - inelastic?
    - amenable to finite deformations?

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- Current funding is from the *Engineering Science Research Foundation* and *Lab Directed Research and Development*, Sandia National Laboratories.

For Sandia report e-mail *jvcox@sandia.gov*
Also ref. Cox [2008] IJNME.

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Questions?

Questions?

Eliminated slides

Some Recent Related Studies

XFEM/PUFEM-Cohesive Zone Studies

- Wells and Sluys (2001)
- Moes and Belytschko (2002)
- Zi and Belytschko (2003) -- tip function addresses tip position but not the field
- Xiao and Karihaloo (2006) -- asymptotic fields
- …

GFEM

- Strouboulis, Copps, Zhang, and Babuska (2000, 2001, 2003) numerical enrichment functions -- handbook functions

Fracture Models

- *Process zone* -- region where inelastic processes are occurring that lead to a new surface.

  Example processes in a quasibrittle material -- initiation, growth and coalescence of micro-cracks.

- *Linear Elastic Fracture Mechanics* -- assumes the process zone can be idealized as a curve (*i.e.*, a point in a 2D representation). Appropriateness depends upon:
  - material
  - scale (of structure relative to the "micro-structure")

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- *Linear Elastic Fracture Mechanics* -- assumes the process zone can be idealized as a curve (*i.e.*, a point in a 2D representation).
  Appropriateness depends upon:
  - material -- linear elastic, brittle
  - scale (of structure relative to the "micro-structure")

PUFEM Displacement Field Enrichment

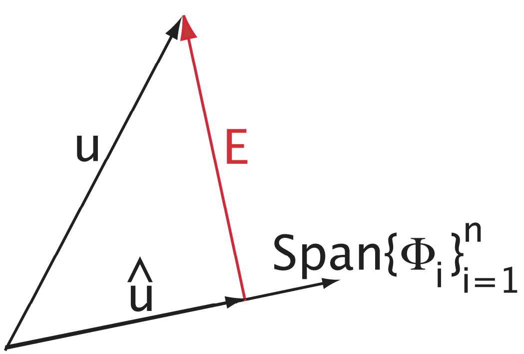
- Standard FEM
- PUFEM/XFEM

Global displacement approximations

$$u(x) = \sum_{i=1}^{N_\Phi} \Phi_i(x) u_i$$

$$u(x) = \sum_{i=1}^{N_\Phi} \Phi_i(x) u_i + \sum_{j=1}^{N_\Lambda} \sum_{i=1}^{N_\Phi} \Lambda_j(x) \Phi_i(x) \alpha_{ij}$$

FEM approximation



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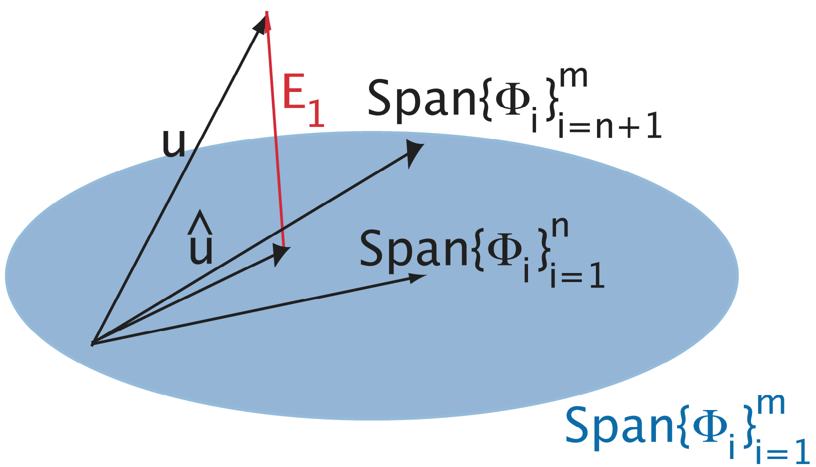
FEM approximation
with mesh refinement



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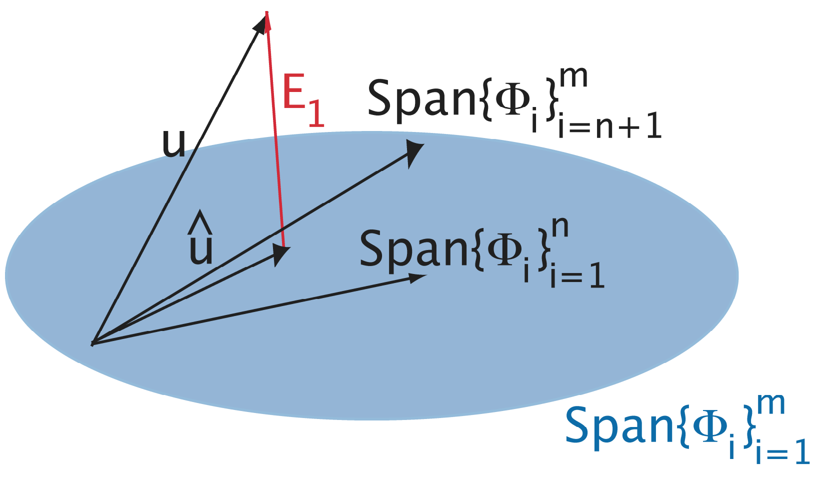
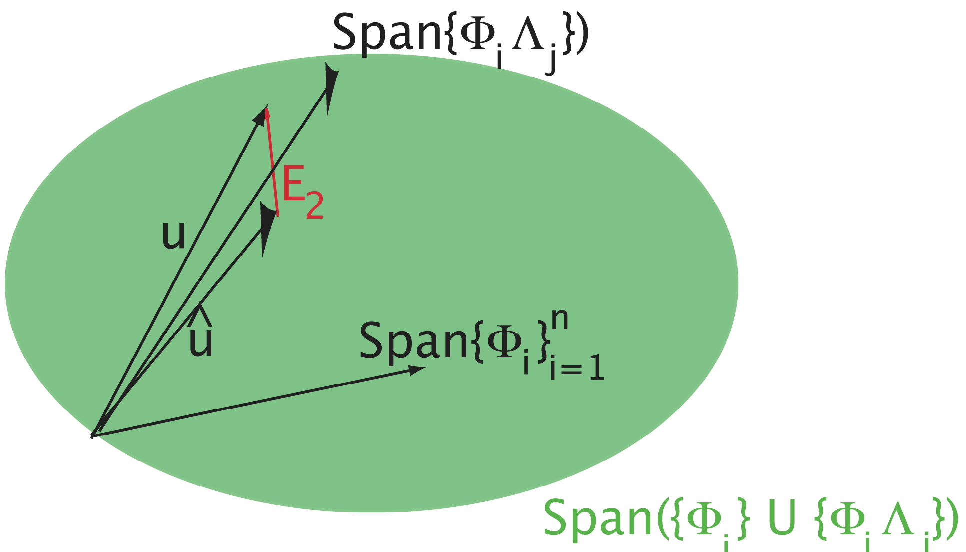
FEM approximation with mesh refinement

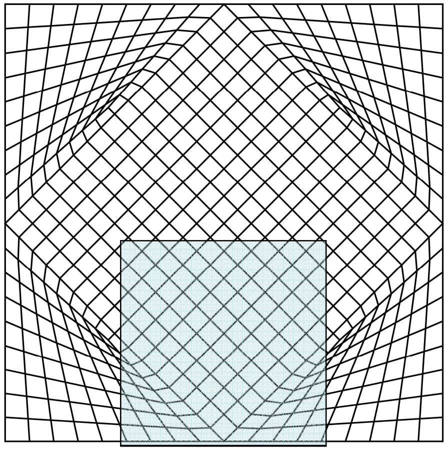


PUFEM approximation



Neighborhood Enrichment

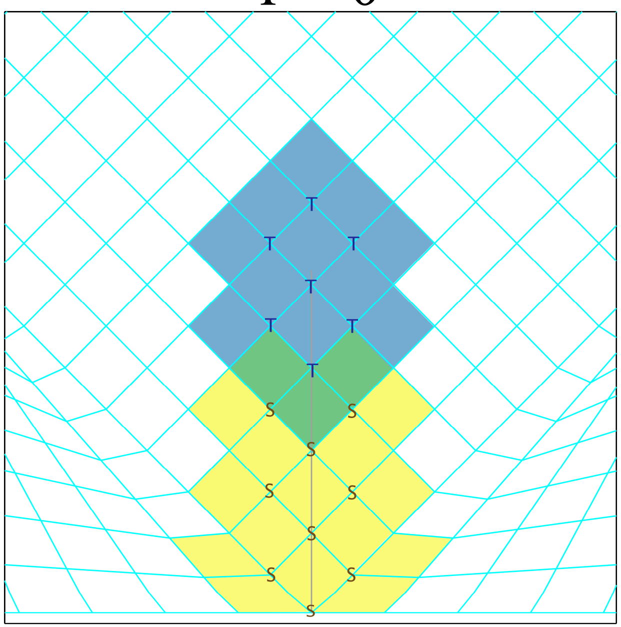
Aka the Mr. Roger's modification



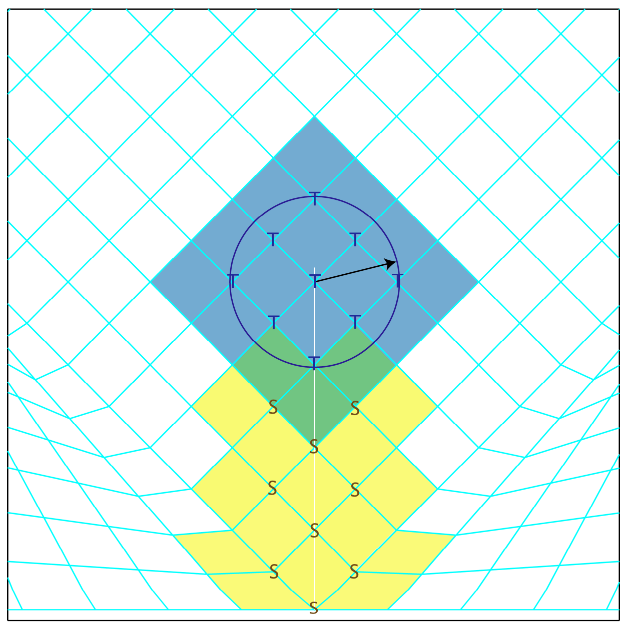
Enriches additional nodes within a user-defined neighborhood of the tip.

Done each time the tip enters a new element.

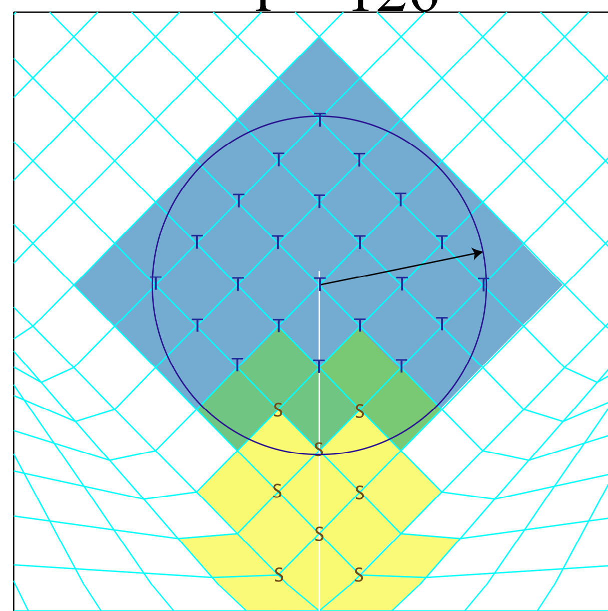
r = 0



r = 63

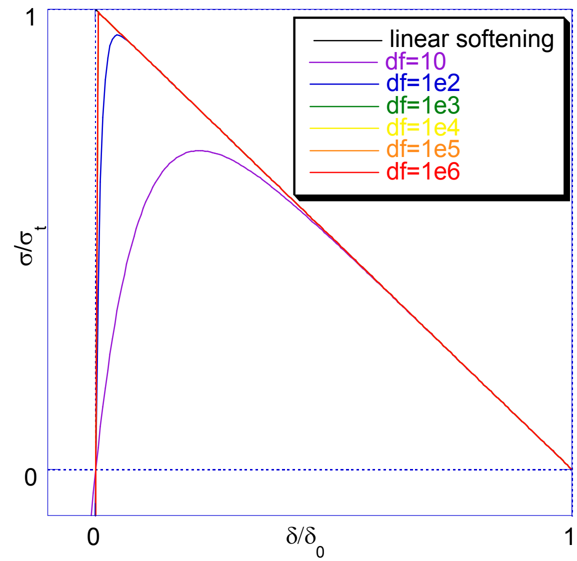


r = 126

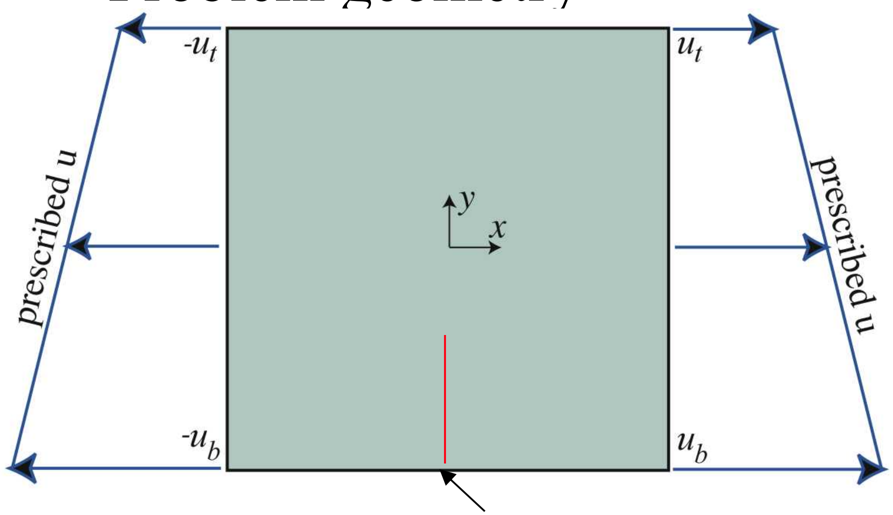


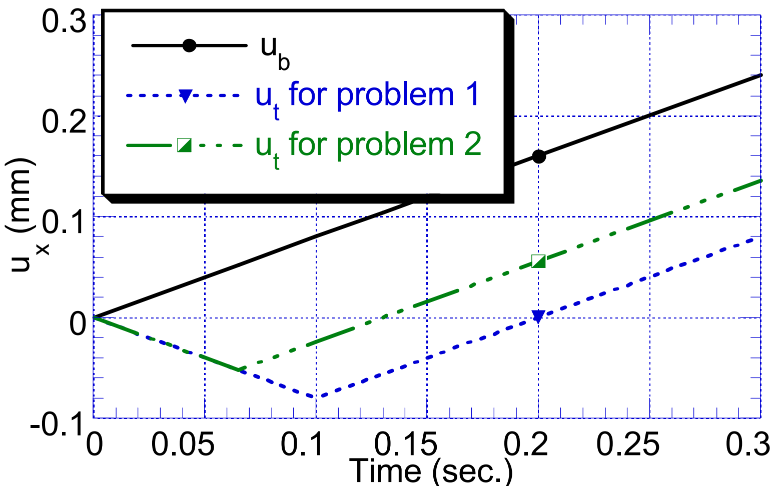
Initial Simple Test Problems

- Concrete test problems
  - relevant to HDBT
  - domain 1 m x 1 m
  - process-zone size ~ O(250 mm)
  - representative concrete tensile properties (except for simplified linear softening)
  - mode I quasistatic crack propagation



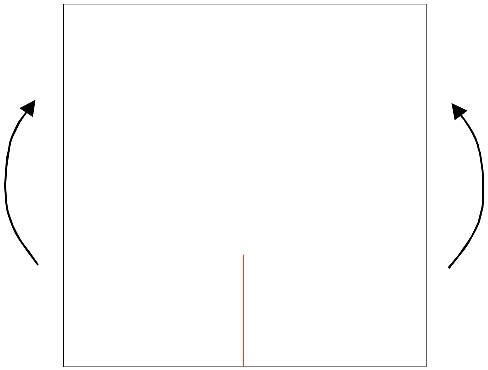
Problem geometry

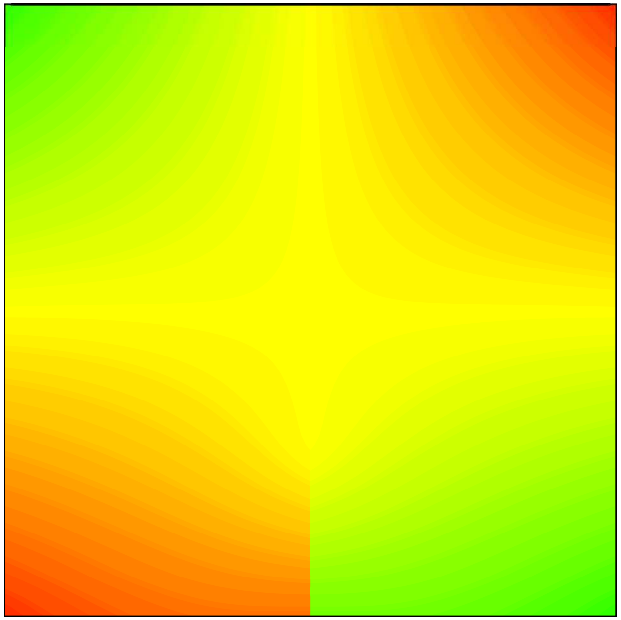




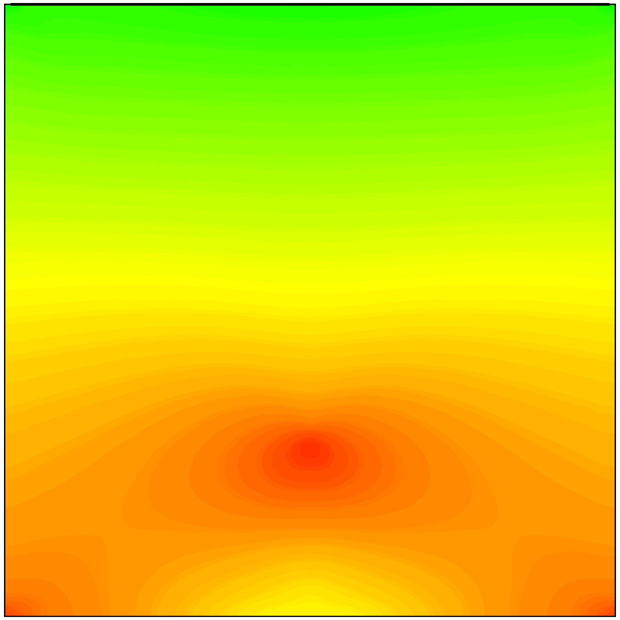
XFEM Displacement Field Enrichment

Example Problem:
concrete 1 m x 1 m
in bending





fine-scale FEM solution: u_x



fine-scale FEM solution: σ_{xx}

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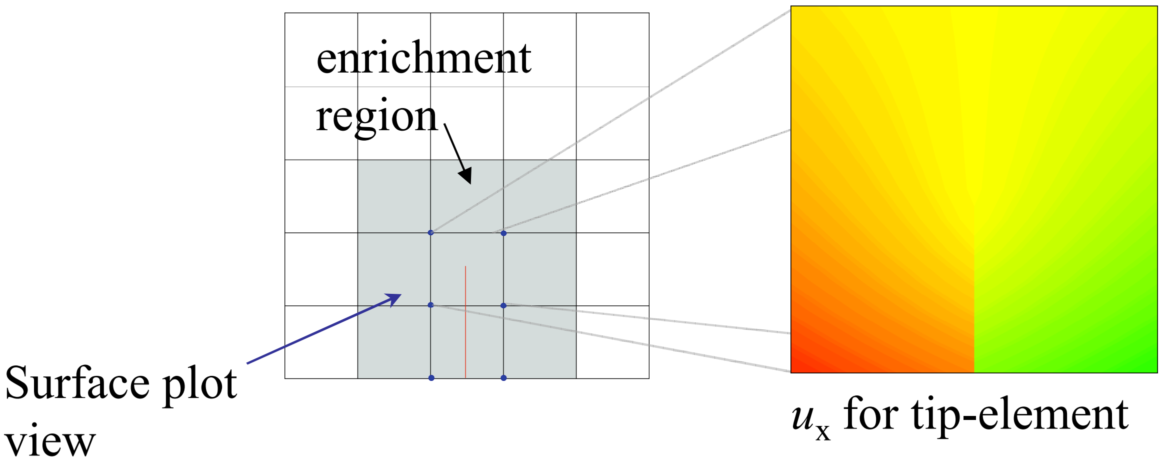
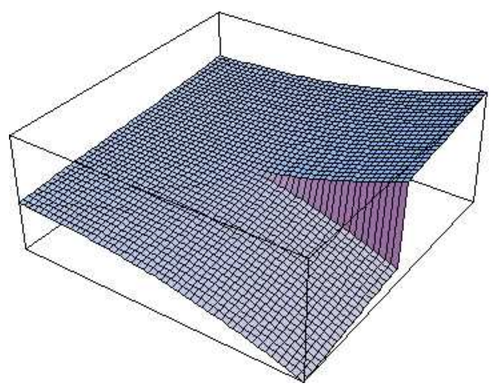
Example response in the "tip-element"

Fine-scale



u_x for tip-element

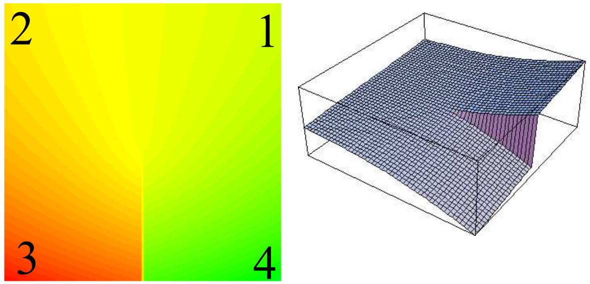
XFEM



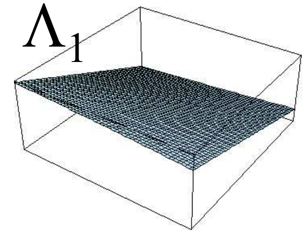
u_x for tip-element

Example enrichment in the “tip element”

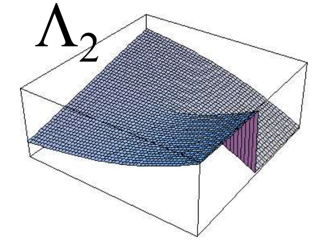
$u(x) =$



Λ_1



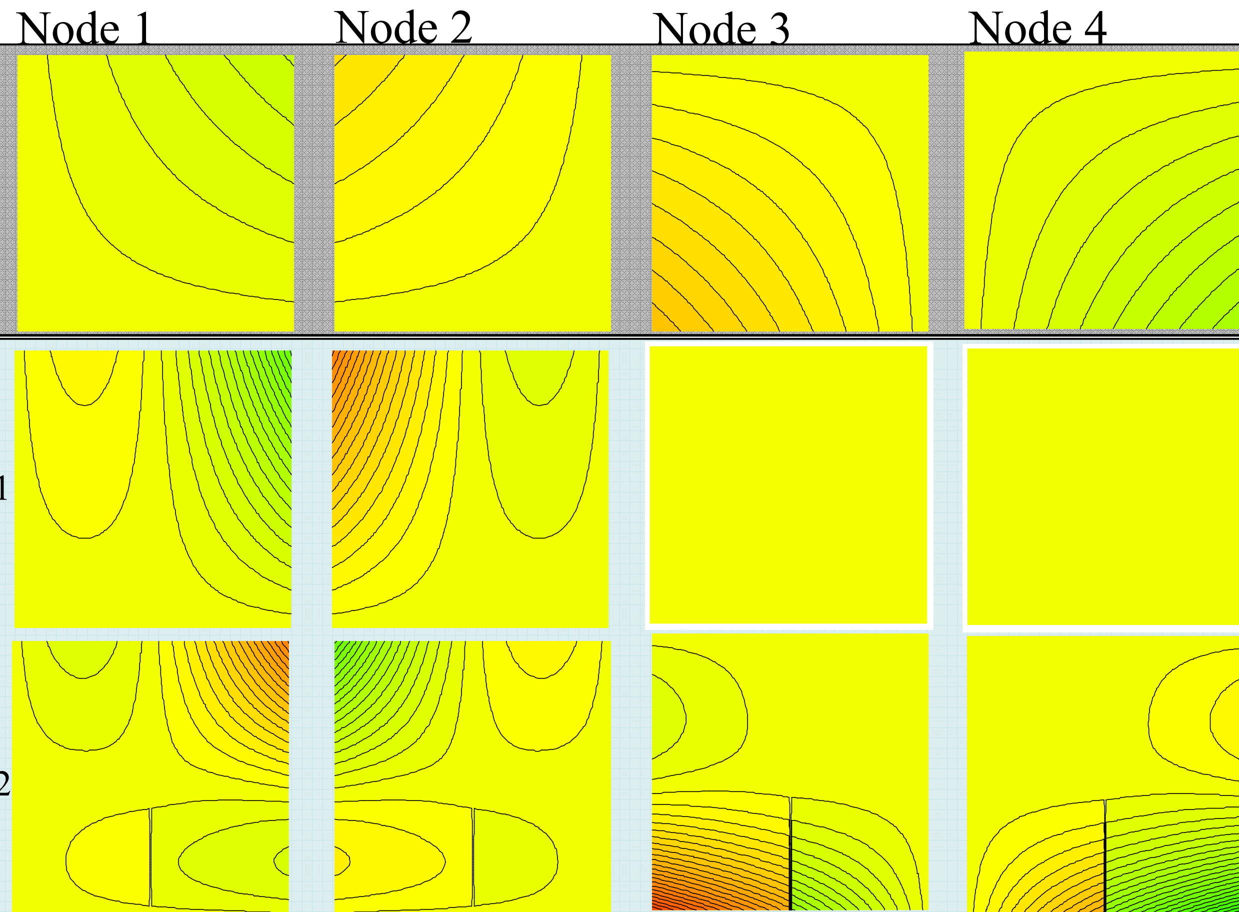
Λ_2



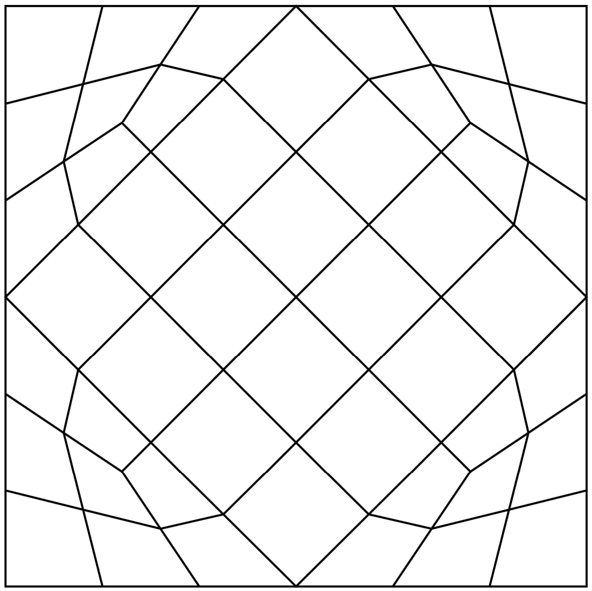
$$\sum_{i=1}^{N_N} N_i(x) u_i$$

$$\sum_{i=1}^{N_N} \Lambda_1(x) N_i(x) \alpha_{i1}$$

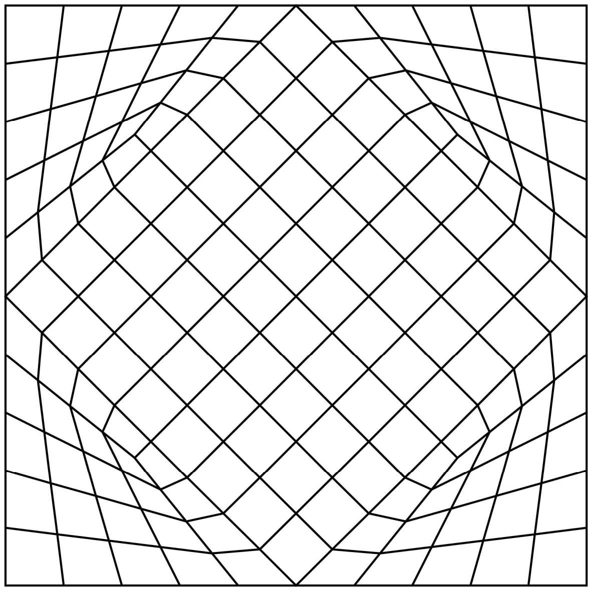
$$\sum_{i=1}^{N_N} \Lambda_2(x) N_i(x) \alpha_{i2}$$



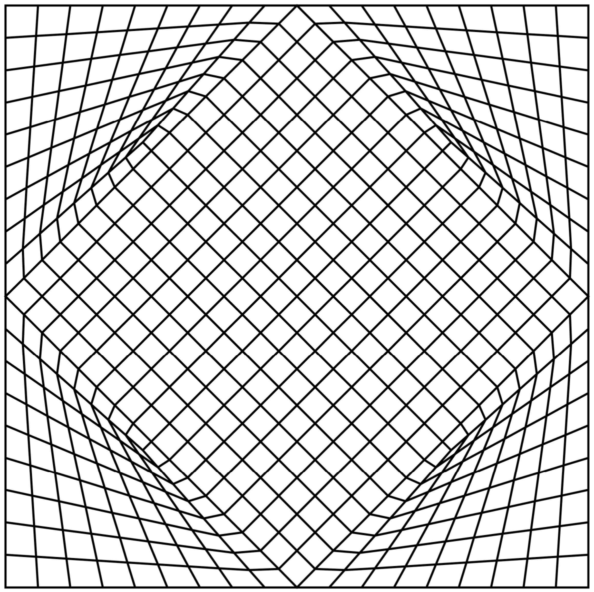
XFEM Skewed Mesh Tests



4x4 @ 45°

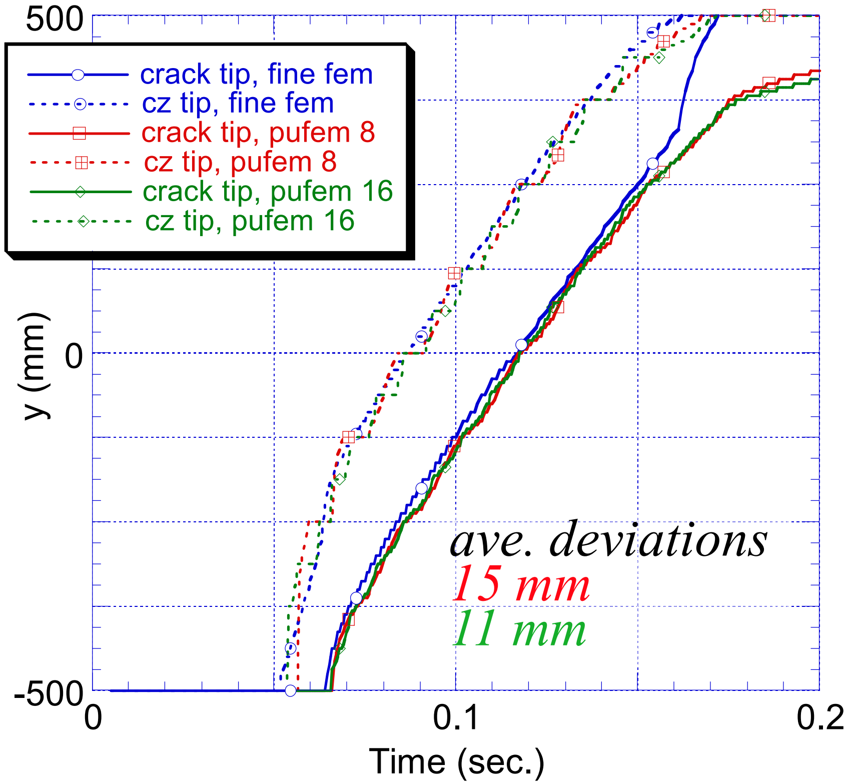


8x8 @ 45°



16x16 @ 45°

Extremes Histories

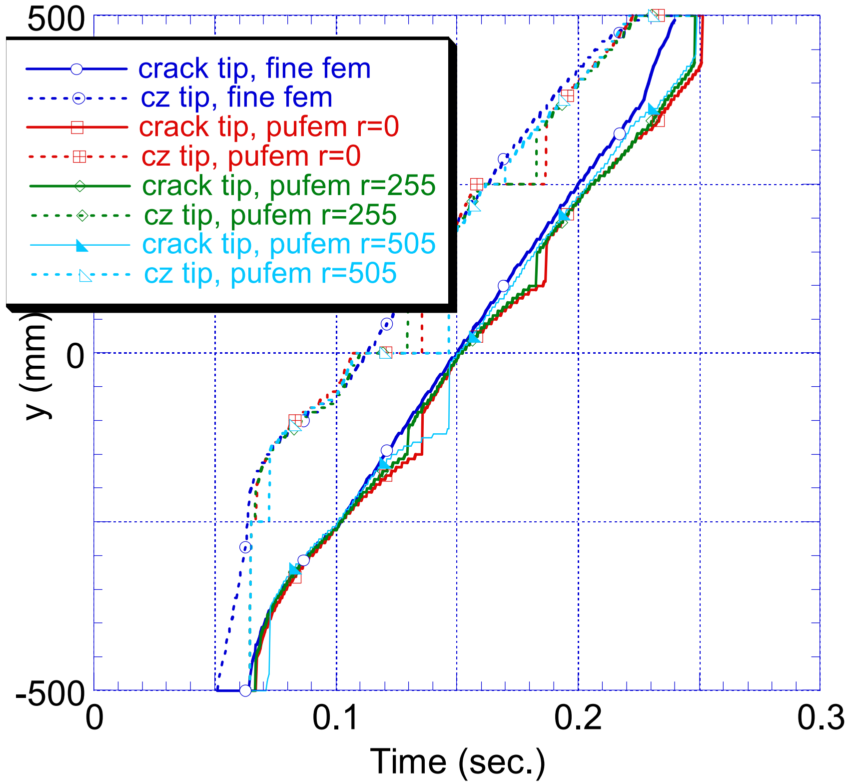


Problem 2

c = 75 mm

Extremes Histories

skewed meshes with the λ enrichment functions



4x4 mesh, c = 75 mm

Neighborhood enrichment

Average deviations: *29 mm for r=0*

20 mm for r=255 mm

19 mm for r=505 mm

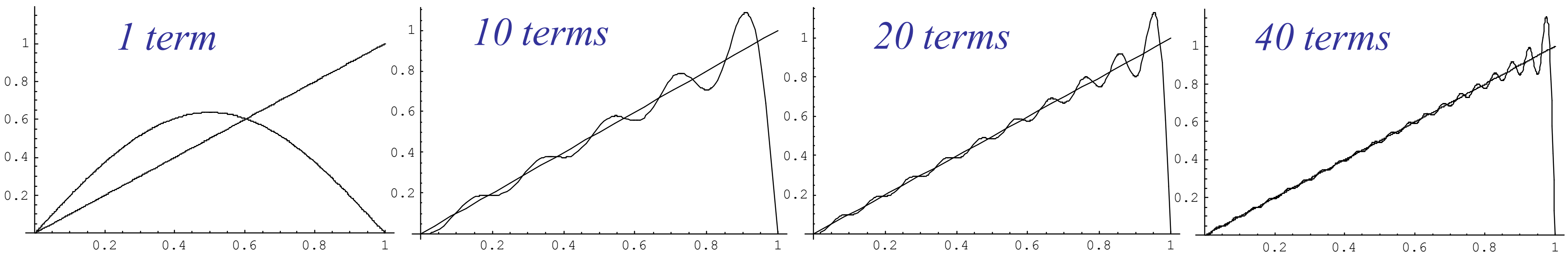
Value of c for 1 series term

Analogy to illustrate the point that if only one term of the series is used, adjusting another parameter of the single basis function can improve the solution.

Consider the approximation of the function $y=x$, on the interval [0,a].
In this case let the basis be an orthonormal sine series of the form:

$$\left\{\sqrt{\frac{2}{a}}Sin\left(\frac{i\pi x}{a}\right)\right\}_{i=1}^{\infty}$$

Approximation of y/a *vs*. x/a when keeping a finite number of terms:



Having a basis for the function we can approximate it as closely as desired in the sense of the L_2-norm, but 1 term is not very accurate.

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Value of c for 1 series term

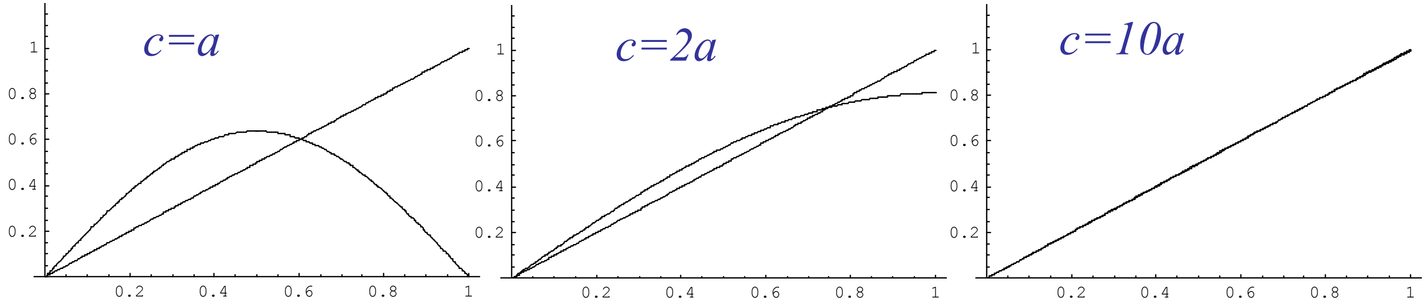
Analogy to illustrate the point that if only one term of the series is used, adjusting another parameter of the single basis function can improve the solution.

Consider the approximation of the function *y*=*x*, on the interval [0,a].
If we can only keep one term of the series, consider changing *a* to *c* and treating it as a parameter that can be adjusted. Our approximate solution then takes the form:

$$\hat{y} = b \sin\left(\frac{\pi x}{c}\right)$$

b ~ of the nodal unknown in the FEA. Here it is determined by a least squares fit. *c* ~ *c* of the cz analytical solution

Approximation of *y/a vs. x/a* when adjusting *c*:



When only 1 term is retained, increasing *c* improves the accuracy in the sense of the L_2-norm -- obvious from a Taylor series point of view.

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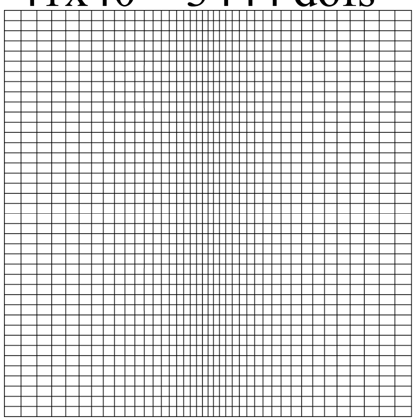
Preview

- Introduction
  - Cohesive crack model
  - Objective, goals, and approach
  - Background
  - Motivating problems

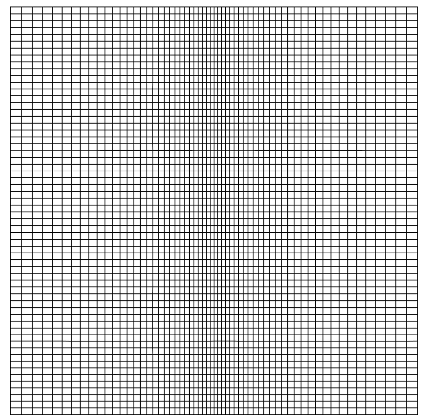
Spacial Discretizations

- Fine FEM meshes – accurate reference solution

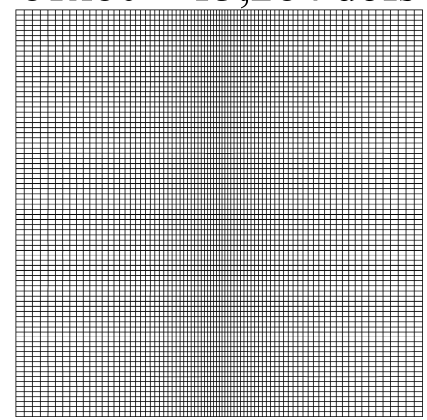
41x40 ~ 3444 dofs



61x60 ~ 7564 dofs

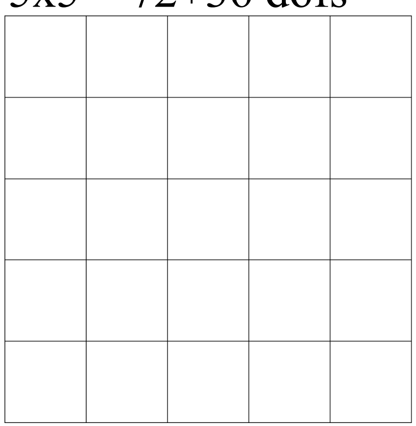


81x80 ~ 13,284 dofs

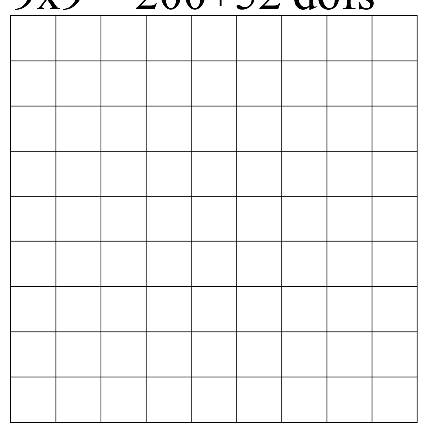


- XFEM – Aligned Meshes

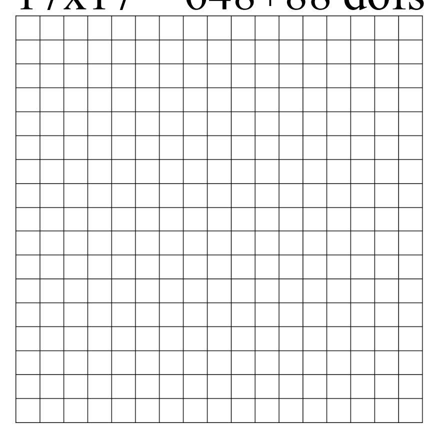
5x5 ~ 72+36 dofs



9x9 ~ 200+52 dofs



17x17 ~ 648+88 dofs



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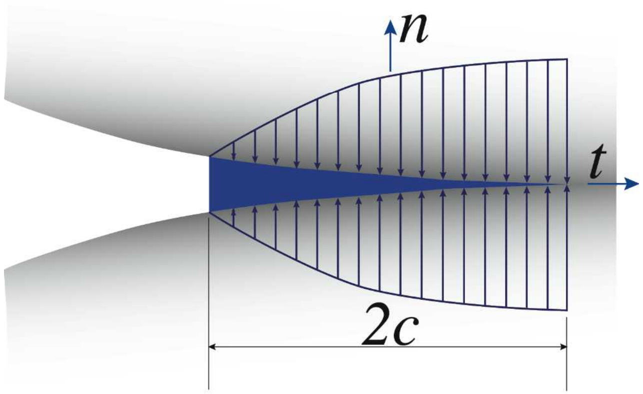
Cohesive Zone Insertion

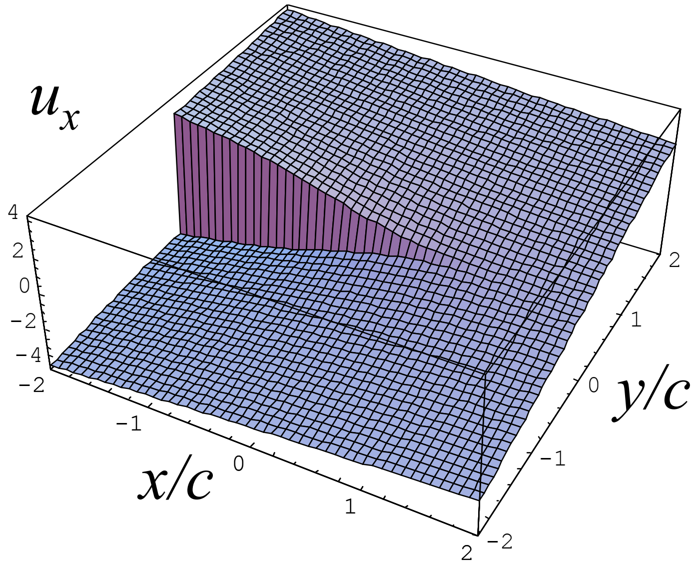
- In theory insertion occurs when $\sigma_{max} > \sigma_t$
- Issue: residual error between continuum and cohesive zone

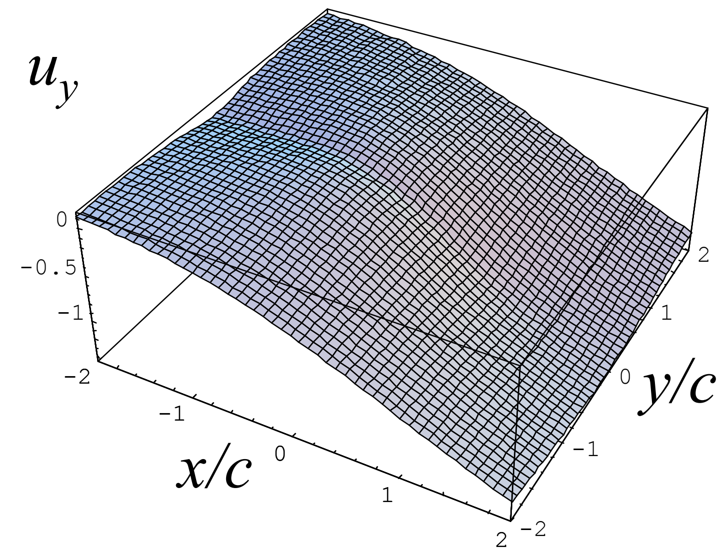


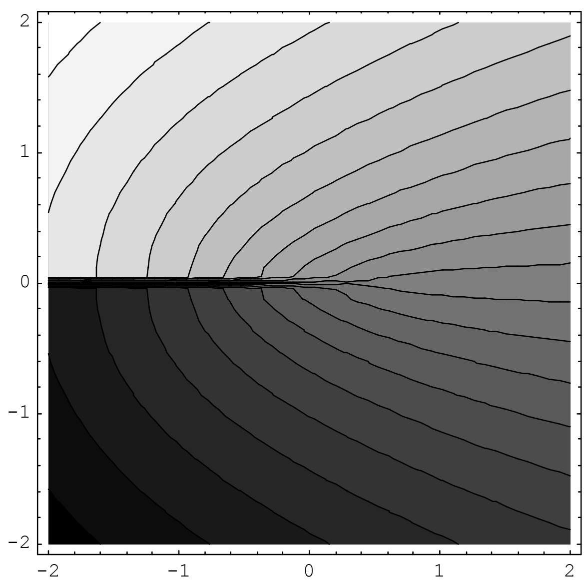
- Numerical criterion: $\sigma_t > \sigma_{max} > \sigma_{cz}$

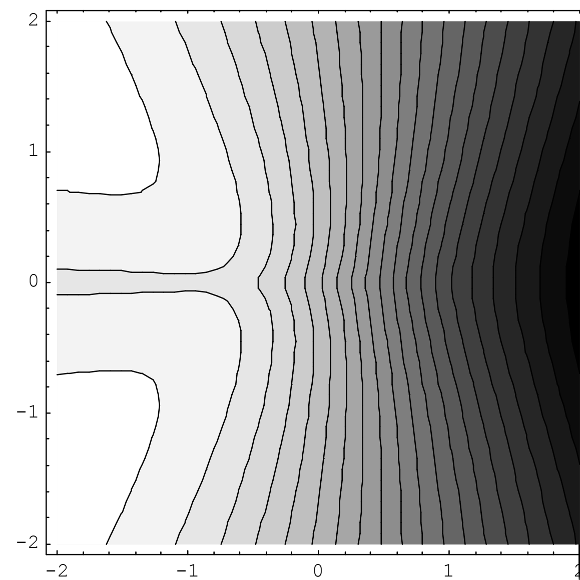
Mode II Enrichment Functions

Based on Zhang & Deng (2007)

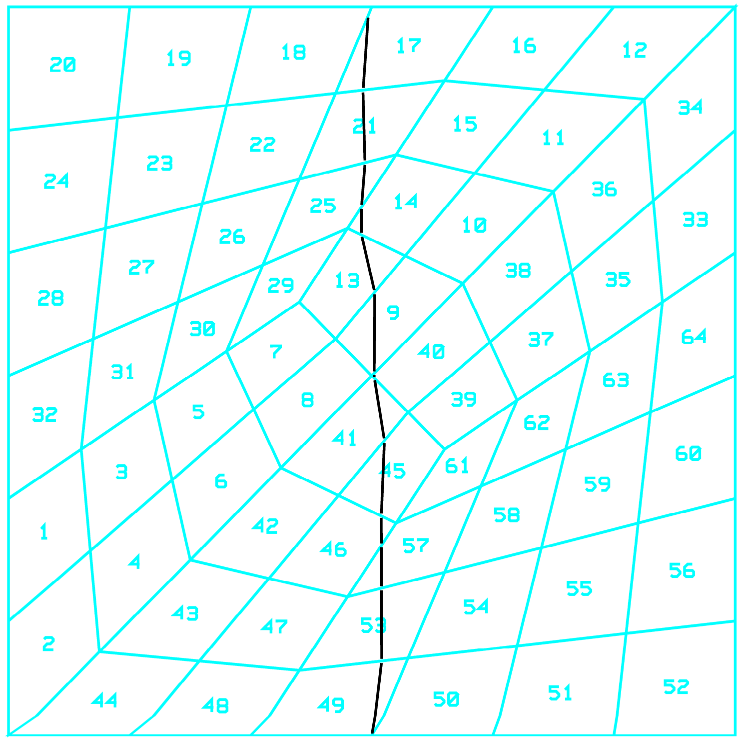






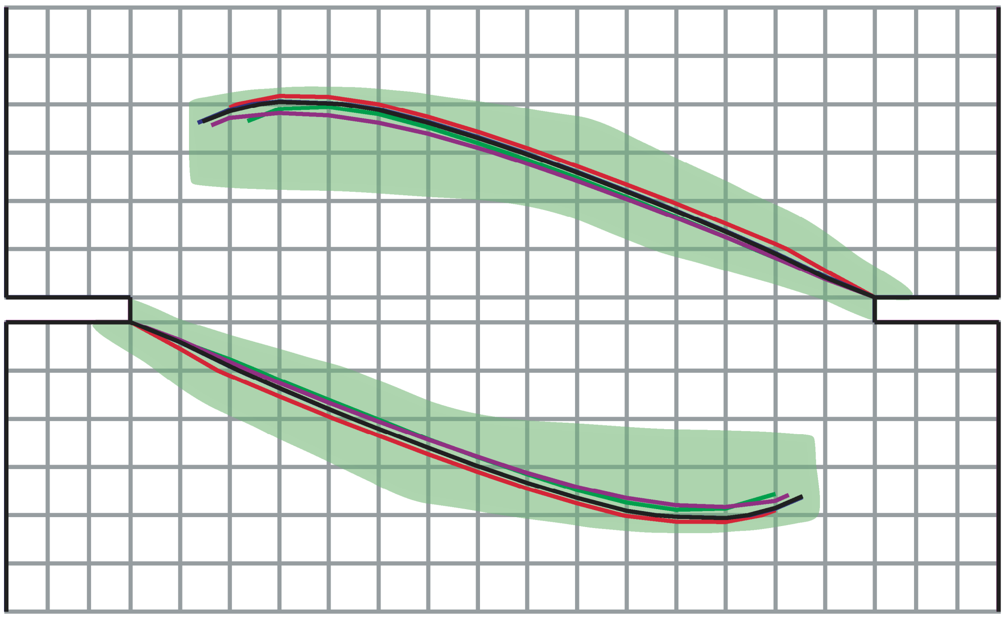


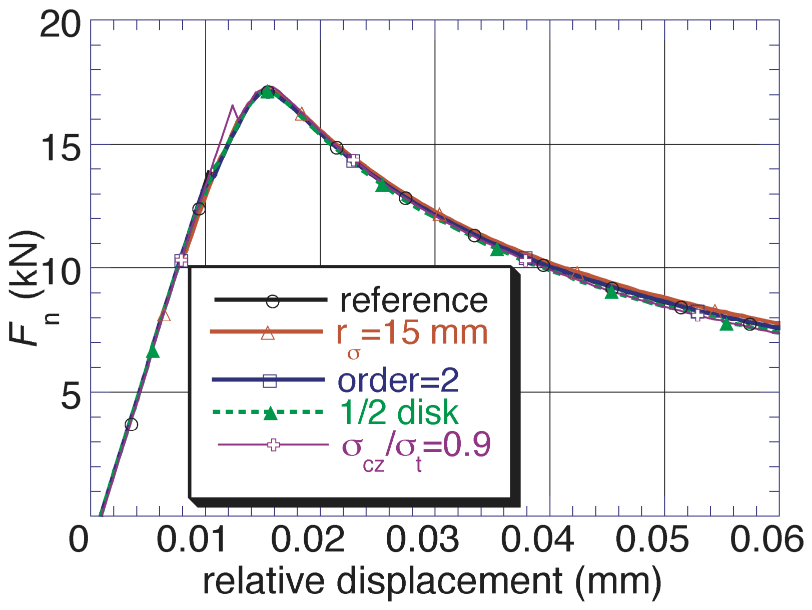
Model Problem with Arbitrary Intersects



Mixed Mode Fracture Problem

XFEM simulation results varying several parameters.





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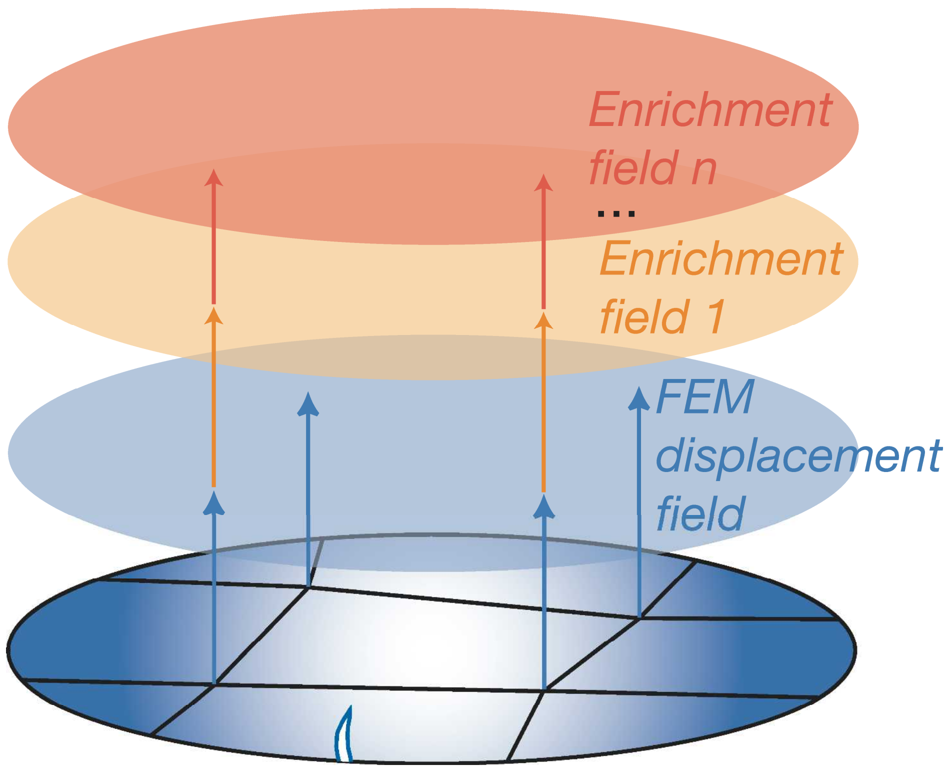
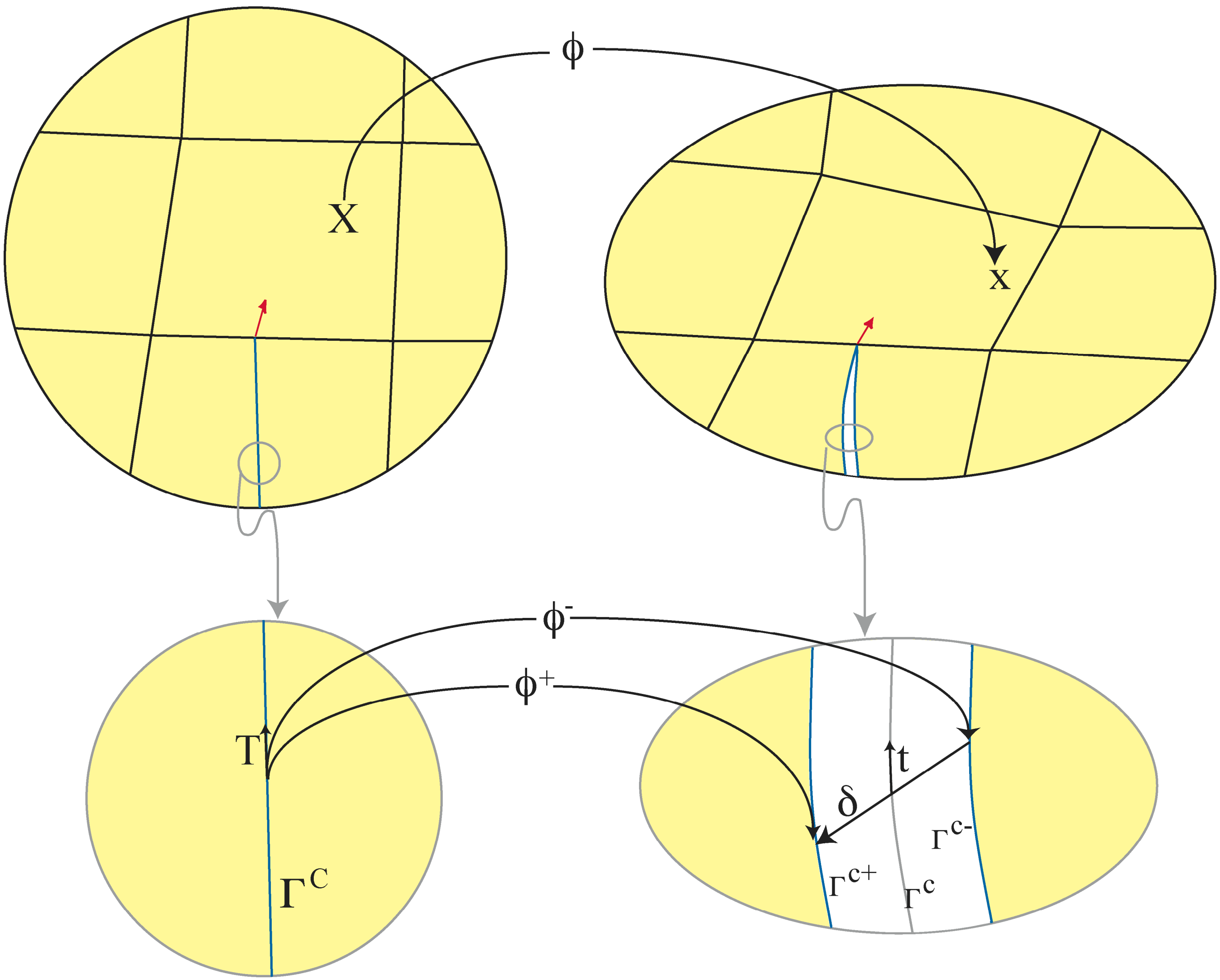
A “Global Issue” for the Tahoe Implementation

Management of enrichment DOFs



Extensions for Finite Deformations

Example: Mapping of crack direction and geometry



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