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A Model to Estimate Turbulent Wall Shear and B.L. Thickness Over Hydro-dynamically Rough Surfaces by Perturbing Known Smooth Results

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[Abstract] It is not uncommon when utilizing hydro-dynamically smooth experimental or computational results that one would like to estimate the additional effect of wall roughness without performing either new experiments or computations. Here we consider a simple analytical model based on inner law methods which extends smooth wall skin friction and boundary layer thickness to be valid for rough wall flows. The approach described here uses a formal perturbation model of the compressible (adiabatic) skin friction, the rough wall equivalent (Van-Driest) log-law boundary layer thickness. Though perturbation-based approaches provide a correction expression, they are not valid for many physically interesting problems where the roughness is more significant. For these flows we solve the appropriate roughness expressions using an approximate procedure which is valid for a wider range of roughness. Though useful to demonstrate behaviors associated with roughness and providing a connection to smooth expressions, this approximate method is of less value when one requires compressibility corrections, whereby it is perhaps more appropriate to dispense with approximation and simple solve the full expression numerically. We also note that the empirically based expression described by Fang et. al. (2003)[1] is in excellent agreement with the inner law based model described here.

Nomenclature

A	=	turbulent spot wave amplitude
B	=	inner law turbulent profile coefficient, B=5.0
C_f	=	skin friction coefficient
C_p	=	pressure coefficient
k	=	roughness height, wave number (L)
M	=	Mach number
p	=	pressure ($ML^{-1}t^{-2}$)
R	=	ideal gas constant ($L^2t^{-2}T^{-1}$)
Re	=	Reynolds number
ℓ_{mix}	=	turbulent mixing length (L)
r	=	recovery coefficient
s	=	distance along cone surface (L)
T	=	temperature (T)
u	=	streamwise velocity (Lt^{-1})
v	=	cross-stream velocity (Lt^{-1})
v^*	=	friction velocity (Lt^{-1})
x	=	stream wise spatial coordinate (L)
y	=	cross-stream spatial dimension (L)
z	=	friction variable (see equ. (3))
δ	=	wall boundary layer thickness (L)
ν	=	kinematic viscosity (L^2t^{-1})

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γ = ratio of specific heats
κ = inner law turbulent profile coefficient, κ=0.41
μ = absolute viscosity ($ML^{-1}t^{-1}$)
ω = power-law viscosity constant
ρ = density (ML^{-3})
τ = shear stress ($ML^{-1}t^{-2}$)

Subscripts/superscripts
( )' = turbulent fluctuating quantity
$\overline{()}$ = (over bar) mean flow turbulent quantity
rms = root means squarew = wall quantity
∞ = outer flow (free-stream) quantity
+ = inner law scaling (see White (1991)[2])
aw = adiabatic wall
w = wall

I. Introduction

THE goal of this research is to provide an engineering level predictive capability to estimate the additional effect of wall roughness without performing either new experiments or computations. Models that provide rough and smooth (as a degenerate case) information for simple flat plate flows are well known[2,3]. These models, however, do not allow us to take general skin friction/thickness input information and extend it to rough flows, but are inherently limited to a particular flow with a particular skin friction/B.L. thickness estimate. The model derived here, can use any wall layer information and increase of data base for that flow to include rough behavior. The family of models described by Fang et. al.[1] and references provide a simple (and we believe accurate) connection between smooth and rough behavior, but are empirical correlations. We attempt to derive such a model based on theoretical/inner law arguments.

The approach described here uses a formal perturbation expansion in terms of the "small" variable $k^+ = \dfrac{k_{rough} v^*}{v_w}$, of the compressible (adiabatic) skin friction, and a rough wall equivalent (Van-Driest) log-law boundary layer thickness. As noted by White[2], Van-Driest's expression can be seen as a compressibility transformation valid for either smooth or rough flow, our focus is more properly placed upon incompressible smooth and rough flows and here is where we focus our perturbation expansion analysis. Though this type of expansion is instructive and intuitively satisfying, we will find that for many problems, k^+ is NOT small, whereby the justification for the expansion is rather tenuous. For these flows we solve the appropriate incompressible roughness expressions using an approximate "single pass" iteration procedure which is valid for a wide range of k^+ values. Finally we solve the complete inner law expressions numerically, which we believe is sensible especially when including compressibility corrections.

II. Governing Equations

Here we summarize the analytical development of the models. As discussed, we start by delineating the relevant equations within the context of a perturbation method. Subsequently we consider an approximate solution that is more broadly applicable and then a numerical implementation.

A. Perturbation Expansion-Based Solution

The components required to perform this analysis include:

1. Access to hydro-dynamically smooth skin friction and boundary layer thickness results from measurement or simulation, say, C_{f0} and δ_0. These are the conditions that we "perturb" about.
2. The governing equations are an incompressible rough wall skin friction model, an equivalent compressible rough-wall (Van Driest) inner law skin friction expression and a boundary layer thickness result. We examine several alternatives for the B.L. thickness expressions.

Notice that while the smooth skin friction and boundary layer thickness may result from any flow (flat plate or otherwise), we are implicitly assuming that the near-wall behavior is adequately described by a flat plate flow a good approximation for flows where body curvature is a small effect. More complex "outer" flows transmit this information to the near wall via the pressure gradient, however, as shown by Bond and Blottner[4] neglecting convective and pressure gradient effects is often an excellent approximation.

Thus, if we define two perturbation series as:

$$c_f = c_{f0} + k^+ c_{f1} + \ldots + O(k^{+2}) \tag{1}$$

and

$$\delta = \delta_0 + k^+ \delta_1 + \ldots + O(k^{+2}) \tag{2}$$

where $k^+ = \dfrac{k_{rough} v^*}{v_w} = \mathrm{Re}_x \left(\dfrac{k}{x}\right) \lambda^{-1} << 1$ and $\lambda \equiv \left(\dfrac{2}{c_f}\right)^{1/2}$ (here we require that k/x<<<1).

The skin friction is computed utilizing an incompressible (with rough wall effects) inner-law skin friction model. The incompressible (both) smooth/rough skin friction model[1] is written as:

$$\mathrm{Re}_x = 1.73(1+0.3k^+)e^z(z^2 - 4z + 6 - \frac{0.3k^+}{1+0.3k^+})(z-1)$$

$$z \equiv \kappa\lambda \quad ; \quad \lambda \equiv \left(\frac{2}{c_f}\right)^{1/2} \quad ; \quad k^+ = \mathrm{Re}_x \left(\frac{k}{x}\right)\lambda^{-1} \tag{3}$$

Equation (3) is implicit in terms of the unknown c_f and as posed, Equation (3) is valid for incompressible flow only. However, Van Driest has shown that incompressible skin friction can be extended to be valid for compressible flow by a stretching transformation. This transformation takes the canonical form:

$$c_f = \frac{1}{F_c} c_{f,inc}(F_{\mathrm{Re}x}\,\mathrm{Re}_x) \equiv \frac{1}{F_c} c_{f,inc}(\widetilde{\mathrm{Re}}_x)$$

$$F_c = \frac{\frac{\gamma-1}{2}M_\infty^2}{\arcsin(a)} \quad ; \quad F_{rex} = \frac{\mu_\infty}{\mu_w} F_c^{-1} \tag{4}$$

where: $a^2 = \left[\frac{(\gamma-1)}{2}M_\infty^2\right]\left[1+\frac{(\gamma-1)}{2}M_\infty^2\right]^{-1}$; $y^+ = \frac{yv^*}{\nu_w}$; $k^+ = \frac{kv^*}{\nu_w}$. We close the viscosity ratio through the simple power law relationship: $\frac{\mu_\infty}{\mu_w} = \left(\frac{T_\infty}{T_w}\right)^\omega$; $\omega \approx 0.67$. The obvious benefit of this procedure is that one can analyze the incompressible expression only, then include appropriate stretching terms to map to a compressible expression, i.e. $c_{f\_comp} = \frac{1}{F_c}\left(c_{f0}(\widetilde{\mathrm{Re}}) + k^+ c_{f1}(\widetilde{\mathrm{Re}}) + \ldots + O(k^{+2})\right)$

The appearance of the Mach number terms is via the Crocco-Buseman law[1] $\frac{T}{T_w} = 1 - r\frac{(\gamma-1)}{2}M_\infty^2\left(\frac{u}{u_\infty}\right)^2\left(\frac{T_\infty}{T_w}\right) \Rightarrow \frac{T_w}{T_\infty} = 1 + r\frac{(\gamma-1)}{2}M_\infty^2$ (an approximate energy integral for Pr≈1 that is a reasonable approximation for air).

Equation (3) also permits estimation of a boundary layer thickness, since for $\frac{u}{u_\infty} \to 1 \Rightarrow y^+ \to \delta^+$. Hence we can solve equation (3) for δ^+ as:

$$\delta^+ = \frac{\delta v^*}{\nu_w} = \exp\left(\frac{\kappa}{a}\frac{u_\infty}{v^*}\arcsin(a) - \kappa B + \ln\left(1 + 0.3k^+\right)\right) \tag{5}$$

Equations (4) and (5) is based solely on inner law analysis. As written, equation (5) is expressed in terms of inner variables. Let's express it a more useful form as: $\mathrm{Re}_\delta = \frac{\delta u_\infty}{\nu_\infty}$.

$$\mathrm{Re}_\delta = \left(\frac{u_\infty}{v^*}\right)\left(1 + \frac{\gamma-1}{2}M_\infty^2\right)^{1+\omega} \exp\left(\frac{\kappa}{a}\frac{u_\infty}{v^*}\arcsin(a) - \kappa B + \ln\left(1 + 0.3k^+\right)\right) \tag{6}$$

with: $\frac{u_\infty}{v^*} = \frac{1}{\sqrt{2c_f}}\left(1 + \frac{\gamma-1}{2}M_\infty^2\right)^{-1/2}$.

Unfortunately, inversion of the inner law expression to obtain the boundary layer thickness tends to underestimate the actual boundary layer thickness since, which by definition, this expressions is only valid for the inner law. A more direct streamwise integral is preferable.

Following Tennekes and Lumley[5], one can write a streamwise momentum integral (integrated to remove cross stream dependence) for flat plate flow expression as:

$$\frac{d}{dx}\left(\Delta v^* u_\infty\right) = v^* u_\infty \frac{d\Delta}{dx} + \Delta u_\infty \frac{dv^*}{dx} = v^{*2} \tag{7}$$

where $v^*\Delta = \int_0^\infty (u_\infty - u)dy = u_\infty \delta^*$ and Δ is called "a normalized boundary layer thickness". Additional information is available by matching the inner-law to the outer law i.e. $\frac{u_\infty}{u^*} = \frac{1}{\kappa}\ln\left(\frac{\Delta v^*}{\nu_w}\right) + const.$. This expression can be differentiated as:

$$\left(1 + \frac{v^*}{\kappa u_\infty}\right)\frac{d}{dx}\left(\frac{u_\infty}{v^*}\right) = -\left(1 + \frac{v^*}{\kappa u_\infty}\right)\left(\frac{u_\infty}{v^{*2}}\right)\frac{dv^*}{dx} = \frac{1}{\kappa\Delta}\frac{d\Delta}{dx} \tag{8}$$

Using equations (7) and (8) we can eliminate $\frac{dv^*}{dx}$ and arrive at a governing equation for the normalized boundary layer thickness:

$$\frac{d\Delta}{dx} = \frac{1 + \frac{1}{\kappa}\left(\frac{v^*}{u_\infty}\right)}{\left(\frac{u_\infty}{v^*}\right) + 1 - \frac{1}{\kappa}} = \left(\frac{v^*}{u_\infty}\right)\frac{1 + 2.5\left(\frac{v^*}{u_\infty}\right)}{1 - 1.5\left(\frac{v^*}{u_\infty}\right)} \tag{9}$$

Since, $\frac{v^*}{u_\infty} = \sqrt{2c_f}\left(1 + \frac{\gamma - 1}{2}M_\infty^2\right)^{1/2} << 1$, it is traditional to simplify equation (9) to give: $\frac{d\Delta}{dx} \approx \left(\frac{v^*}{u_\infty}\right) \rightarrow \Delta = \left(\frac{v^*}{u_\infty}\right)x$, which finally gives:

$$\delta^* = \left(\frac{v^*}{u_\infty}\right)^2 x = 2c_f\left(1 + \frac{\gamma - 1}{2}M_\infty^2\right)x \tag{10}$$

Finally, using 1/7th power law profiles etc, one can infer that: $\delta \approx 16c_f\left(1 + \frac{\gamma - 1}{2}M_\infty^2\right)x$.

Thus, using equations (3) and (10) and the perturbation expansions, i.e. equations (1) and (2) it is possible to derive expressions for the first order terms, c_{f1} and δ_1 in equations (1) and (2). We examine the effect of the perturbation expansions on equations (3). It is convenient to introduce the intermediate variable: $z = z_0 + k^+ z_1 + \ldots + O(k^{+2})$ which permits us to expand equation (3) to arrive at the correction term z_1:

$$z_1 = -0.3\frac{\left(z_0^3 - 5z_0^2 + 9z_0 - 5\right)}{\left(z_0^3 - 2z_0^2 + 4\right)} \tag{11}$$

With the "z" correction term known we can use: $z = \kappa\lambda \equiv \kappa\left(\frac{2}{c_f}\right)^{1/2} = \kappa\sqrt{\frac{2}{c_{f0}}}\left(1 - \frac{1}{2c_{f0}}c_{f1}k^+ + \ldots\right)$ to derive an expression for c_{f1} as:

$$z = \kappa\lambda \equiv \kappa\left(\frac{2}{c_f}\right)^{1/2} = \kappa\sqrt{\frac{2}{c_{f0}}}\left(1 - \frac{1}{2c_{f0}}c_{f1}k^+ + \ldots\right)$$

$$c_{f1} = -\frac{\sqrt{2}}{\kappa}c_{f0}^{3/2}z_1 = 0.3\frac{\sqrt{2}}{\kappa}c_{f0}^{3/2}\frac{\left(z_0^3 - 5z_0^2 + 9z_0 - 5\right)}{\left(z_0^3 - 2z_0^2 + 4\right)} \quad ; \quad z_0 \equiv \kappa\left(\frac{2}{c_{f0}}\right)^{1/2} \tag{12}$$

Similarly, one can readily combine the boundary layer thickness expression: $\delta \approx 16c_f\left(1 + \frac{\gamma - 1}{2}M_\infty^2\right)x$ with equation (2) $\delta = \delta_0 + k^+\delta_1 + \ldots + O(k^{+2})$ so that:

$$\delta_1 = 16c_{f1}\left(1 + \frac{\gamma - 1}{2}M_\infty^2\right)x \tag{13}$$

Suggesting that $\delta = (1 + \frac{c_{f1}}{c_{f0}}k^+)\delta_0$ where c_{f1} is known from equation (12) and $k^+ = \mathrm{Re}_x\left(\frac{k}{x}\right)\sqrt{\frac{c_{f0}}{2}}$

Thus, equations (12) and (13) provide the unknown terms, c_{f1} and δ_1 in our expansions:

$$c_f = c_{f0} + k^+c_{f1} + \ldots + O(k^{+2})$$

$$\delta = \delta_0 + k^+\delta_1 + \ldots + O(k^{+2}) \tag{14}$$

These two expressions are, of course, intended to "extend" smooth results to be valid for rough flows.

B. Application of the Expansion

It is useful to consider a classical low speed M_∞<<1 problem. Under classical low speed conditions with Re_x=O(5E6), one finds that c_{f0} (smooth value) is approximately: c_{f0}=0.005 (say). If k/x is on the order of 1E-3 we find that the roughness k^+ is then: $k^+ = \mathrm{Re}_x\left(\frac{k}{x}\right)\sqrt{\frac{c_{f\_rough}}{2}}$, wherby k^+=5E6(1E-3)(1/2*5E-3)$^{1/2}$=180=O(100) (obviously this not formally a small value, however we can proceed to check behavior) This is sufficient information to estimate the roughness behavior as (using equation (12) we find that c_{f1}=O(2E-5):

$$c_f = c_{f0} + k^+c_{f1} = 0.005 + 2x10^{-5}(180) = 0.0086 \tag{15}$$

which corresponds to $c_f = 1.72 c_{f0}$ Examination of classical rough wall behavior suggests that C_f rough[1] is about 0.0055. The classical sand grain fully rough skin friction[2] (Reynolds number independent is about 0.0065) suggesting that our prediction is on the high side. The boundary layer thickness extension is: $\delta = \delta_0 (1 + k^+ \frac{c_{f1}}{c_{f0}}) = \delta_0 (1 + (180)(\frac{2x10^{-5}}{5x10^{-3}})) = 1.72\delta_0$. The skin friction and the boundary layer thickness have increased by approximately 72%.

Let's consider a second problem with skin friction and boundary layer thickness of (say) 0.0036 and 28 mm. The roughness is on the order of 2-6 mils (we actually use 4 mil) corresponding to $k^+ = \mathrm{Re}_x \left(\frac{k}{x}\right)\sqrt{\frac{c_{f0}}{2}} = (5x10^6)(0.0000254)(4)\sqrt{\frac{0.0036}{2}} = 21.6$ Under these conditions we expect that the skin friction is: $c_f = c_{f0} + k^+ c_{f1} = 0.0036 + 0.00011(21.6) = 0.0063$, implying that $c_f = 1.75 c_{f0}$ Following the empirical correlation of Fang et. al.[1] we shall see that this solution is too high by a factor of 3 (!) which is unacceptable.

C. "One Pass Iterative" Approximate Solution for k^+>1 to k^+>>1

The preceding analysis was based upon an elementary perturbation analysis which involves introduction (assumption) of a series solution form as: $c_f = c_{f0} + k^+ c_{f1} + ... + O(k^{+2})$. Obviously, this type of expansion is only (formally) useful for k^+<<1. Unfortunately, the definition of k^+: $k^+ = \mathrm{Re}_x \left(\frac{k}{x}\right)\sqrt{\frac{c_{f0}}{2}}$ suggests that unless k<<<1 that k^+ can easily be larger than 1 (as large as 100). Hence, the assumption inherent to the perturbation series (and moreover the simplifications utilized) cannot be truly correct. Moreover results derived using the perturbation approach yield solutions that are in poor agreement with data. Here, we consider an alternative solution procedure that can more formally accept larger values of the dimensionless roughness.

Our approach is to introduce an approximate, iteration scheme that is not dependent upon the size of the rough term. The procedure is based upon a simplification of equation (3) re-written here for convenience:

$$\mathrm{Re}_x = 1.73(1 + 0.3k^+)e^z (z^2 - 4z + 6 - \frac{0.3k^+}{1 + 0.3k^+}(z - 1)) \tag{16}$$

From the definition of z: $z \equiv \kappa\lambda$; $\lambda \equiv \left(\frac{2}{c_f}\right)^{1/2}$ we recognize that z>1 (typically O(10)) implying that the dominant terms in equation (A.1) would be:

$$\mathrm{Re}_x = 1.73(1 + 0.3k^+)e^z z^2 + ... + O(z) \tag{17}$$

The hydro-dynamically smooth solution form is: $\mathrm{Re}_x = 1.73 e^z z^2 + ... + O(z^2)$. This reduced expression is of value as a way to estimate the Reynolds number Re_x. It is also of value as a way to test the value of the approximation inherent to equation (17).

Obviously, to use equation (17) we need to be able to solve for z. Let's consider a basic canonical transcendental equation of this form, say: $ax^2 \exp(x) - \lambda = 0$. The solution for this expression is:

$$x = 2LambertW\left[\left(\frac{\lambda}{4a}\right)^{1/2}\right] \tag{18}$$

Where LambertW(β) is the special function solution[6] of the transcendental expression $\alpha \exp(\alpha) = \beta$. Though the LambertW function has well known properties (and approximations) it is not a convenient solution. As such, we consider a specific (empirical) approximation for equation (A.3) which is found to be:

$$x = 2LambertW\left[\left(\frac{\lambda}{4a}\right)^{1/2}\right] \approx \ln\left[\left(\frac{\lambda}{4a}\right)^{5/6}\right] \tag{19}$$

Indeed over a large range of values for the grouping: $\left[\left(\frac{\lambda}{4a}\right)^{1/2}\right]$ we plot the expressions in equation (19):

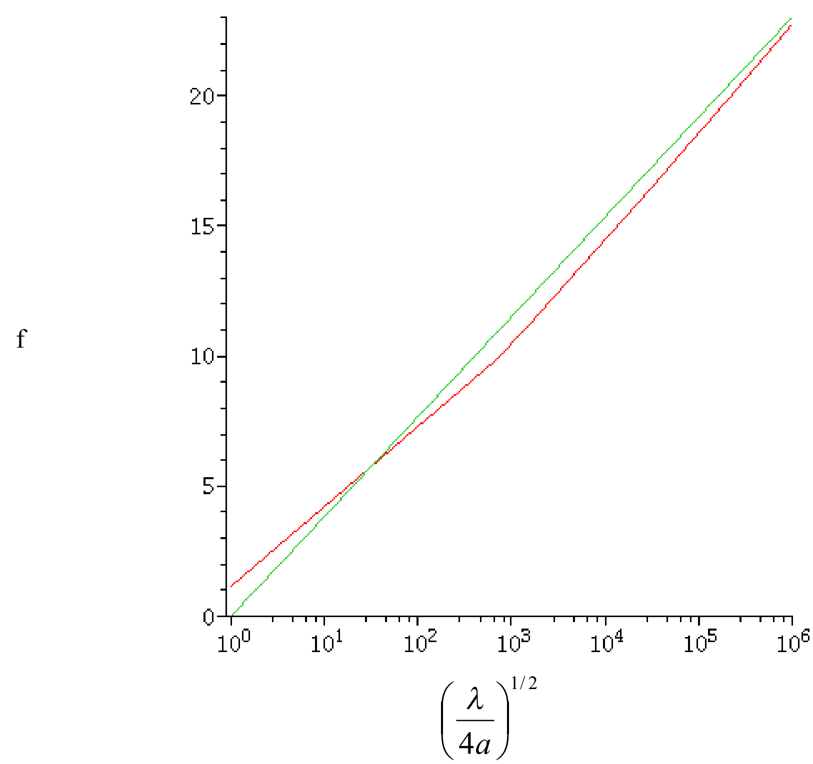


Figure 1 Comparison between $f = 2LambertW\left[\left(\frac{\lambda}{4a}\right)^{1/2}\right]$ and $f = \ln\left[\left(\frac{\lambda}{4a}\right)^{5/6}\right]$, note the good agreement over multiple decades of the independent variable.

Accepting the approximation in equation (19) it is possible to explicitly write the solution for $\mathrm{Re}_x = 1.73 e^z z^2$ as:

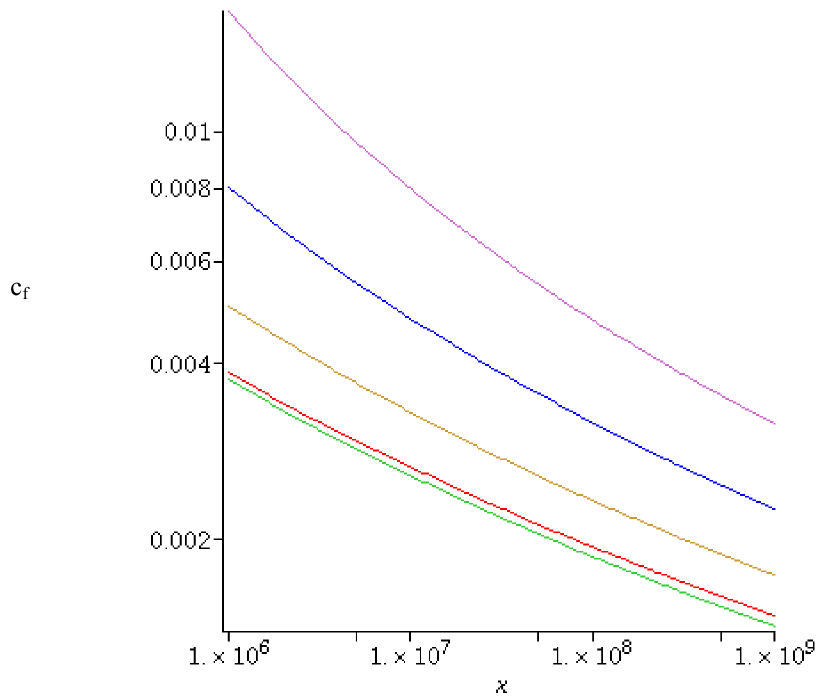
$z = \ln\left[\left(\frac{\mathrm{Re}_x}{4(1.73)}\right)^{5/6}\right]$ since z is defined in terms of c_f we can write:

$c_f = \frac{2\kappa^2}{z^2} = \frac{2\kappa^2}{\ln^2\left[\left(\frac{\mathrm{Re}_x}{4(1.73)}\right)^{5/6}\right]} \approx \frac{0.3362}{\ln^2\left[0.2\,\mathrm{Re}_x^{5/6}\right]}$. This expression is similar to White's[1] classical inner law model, i.e. $c_f = \frac{0.455}{\ln^2\left[0.06\,\mathrm{Re}_x\right]}$; suggesting that our expression for the Re_x using smooth skin friction results will be successful. Indeed, we suggest that it is appropriate to modify one of the constants in our expression to better to conform to White's result for via: $c_f \approx \frac{2\kappa^2}{\ln^2\left[\left(\frac{\mathrm{Re}_x}{8(1.73)}\right)^{5/6}\right]} \approx \frac{0.3362}{\ln^2\left[0.12\,\mathrm{Re}_x^{5/6}\right]}$

We examine the skin friction for rough flows and solve: $\mathrm{Re}_x = 1.73(1+0.3k^+)e^z z^2$ utilizing the preceding computations to write:

$$c_{f\_rough} = \frac{2\kappa^2}{\ln^2\left[\left(\frac{\mathrm{Re}_x}{8(1.73(1+0.3k^+))}\right)^{5/6}\right]} \tag{20}$$

Where we evaluate $z \equiv \kappa\lambda$; $\lambda \equiv \left(\frac{2}{c_f}\right)^{1/2}$ using c_f=c_{f_smooth} a value known from our CFD computation.



Re_x

Figure (2) Rough skin friction results; equation (20). $k^+=0$ ($c_f \approx \dfrac{0.3362}{\ln^2\left[0.12\,\mathrm{Re}_x^{5/6}\right]}$), Red; $k^+=0$ (Whites[2] solution $c_f = \dfrac{0.455}{\ln^2\left[0.06\,\mathrm{Re}_x\right]}$), Green; $k^+=10$, Orange; $k^+=100$, Blue; $k^+=1000$, Violet

As a useful asymptotic test, we consider equation (20) where $k^+ \gg 1$ whereby the solution becomes independent of Re_x and we have a "fully rough" solution. Under these conditions with $k^+ = \mathrm{Re}_x\left(\dfrac{k}{x}\right)\sqrt{\dfrac{c_{f\_rough}}{2}}$ equation (20) simplifies:

$$c_{f\_rough} = \frac{2\kappa^2}{\ln^2\left[\left(\dfrac{\mathrm{Re}_x}{8(1.73(1+0.3k^+))}\right)^{5/6}\right]} \approx \frac{2\kappa^2}{\ln^2\left[\left(\dfrac{\sqrt{2}}{8(1.73)(0.3)\left(\dfrac{k}{x}\right)c_{f\_rouf}^{1/2}}\right)\right]} \qquad (20)$$

and we can explicitly (in terms of LambertW or using the same type of "logarithmic squared" approximation $c_{f\_full\_rough} \approx \dfrac{0.55}{\ln^2\left[1.7\left(\dfrac{k}{x}\right)\right]}$) solve for c_{f_rough}. We plot this result for k/x in figure 3.

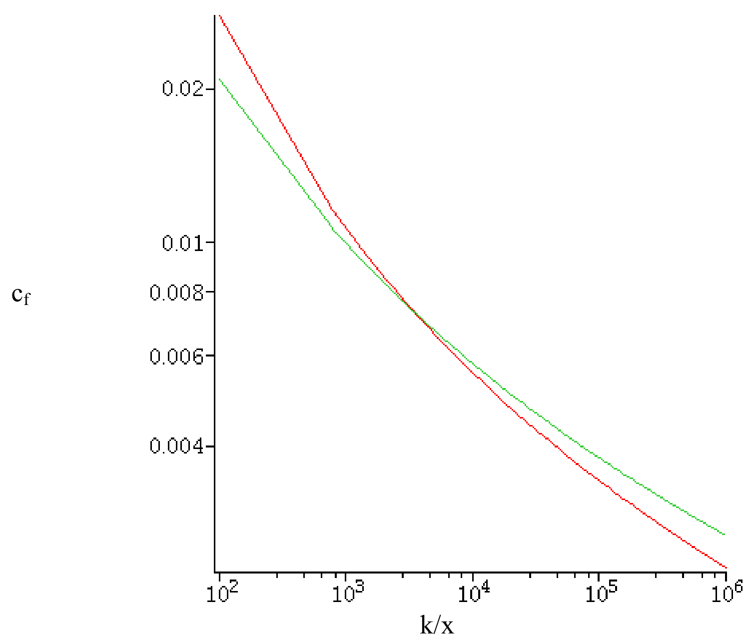


Figure (3) Fully rough (Re_x independent) skin friction results using: $c_{f\_full\_rough} \approx \dfrac{0.55}{\ln^2\left[1.7\left(\frac{k}{x}\right)\right]}$ **compared with correlation**[2] $c_{f\_full\_rough} \approx \left(1.4 + 3.7\log_{10}\left(\frac{x}{k}\right)\right)^{-2}$ **adequate agreement.**

Following section B, one can examine the small skin friction problem associated with high speed flows with corresponding skin friction and boundary layer thickness of (say) 0.0036 and 28 mm. The roughness is approximately (say) 4 mils.

Using the preceding equations we first estimate the *smooth* "z" value: $z \equiv \kappa\left(\frac{2}{c_f}\right)^{1/2} = 0.41\left(\frac{2}{0.0036}\right)^{1/2} = 9.66$ and Reynolds number as:

$$\mathrm{Re}_x = 1.73 e^z z^2 = 1.73\exp(9.66)(9.66)^2 = 2.52x10^6 \qquad (21)$$

Equation (21) is an estimate of the *smooth* Reynolds number.

We now must compute k^+ using the definition: $k^+ = \mathrm{Re}_x\left(\frac{k}{x}\right)\sqrt{\frac{c_{f0}}{2}} = \kappa\,\mathrm{Re}_x\left(\frac{k}{x}\right)z^{-1}$. Though, we have smooth estimates for Re_x and z, it is clear that these estimates cannot be correct since the problem (and this term in particular) are associated with rough flow. A simple approximation to provide a better estimate is to evaluate the z term in the k^+ definition using the fully rough approximation. This gives:

$k^+ = z^{-1}\,\mathrm{Re}_x\left(\frac{k}{x}\right)\sqrt{\frac{c_{f0}}{2}} = z_{fully\_rough}^{-1}\,\mathrm{Re}_x\left(\frac{k}{x}\right)\sqrt{\frac{c_{f0}}{2}}$. Where $z_{\text{fully\_rough}}$ was given as: $z_{full\_rough} \approx \kappa\sqrt{2}\left(\frac{0.55}{\ln^2\left[1.7\left(\frac{k}{x}\right)\right]}\right)^{-1/2}$.

The associated k^+ value for the problem of interest is then found to be:

$$k^+ = \kappa\,\mathrm{Re}_x\left(\frac{k}{x}\right) = (0.41)(2.52x10^6)(2.52x10^{-5})(4)(6.77)^{-1} = 15.56 \tag{22}$$

With the Reynolds number, Re_x and k^+ estimates available we use equation (20) to obtain the rough skin friction:

$$c_{f\_rough} = \frac{2(0.41)^2}{\ln^2\left[\left(\frac{2.52x10^6}{(4)(1.73)(1+0.3(15.56))}\right)^{5/6}\right]} = 0.0044 \tag{23}$$

Thus we find that: $c_f = \left(\frac{0.0045}{0.0036}\right)c_{f0} = 1.25_{f0}$ a much smaller increase in skin friction value confirming the concern that a classical perturbation method is of limited value.

Let's compare this value to the correlation[1]:

$$\frac{c_{f\_rough}}{c_{f0}} = 1 + 0.889(\log_{10}(k^+) - 1) \tag{23}$$

Which suggests that this value: $\frac{c_{f\_rough}}{c_{f0}} = 1 + 0.889(\log_{10}(15.56) - 1) = 1.17$ a lower, but similar value. In the next section, solution are performed for this problem by solving the equation (3) numerically rather than using the approximate method developed here. In this case we find that $\frac{c_{f\_rough}}{c_{f0}} = 1.17$ verifying the approach developed here. Though we develop a more complete description of compressibility in the next section, we note, that the expressions derived here can modified to included compressibility effects via equation (4). We suggest, however, that the added computational burden of the compressibility make the simple formulas less attractive and full numerical inversion more sensible.

D. Numerical Solutions and Compressibility Effects with Application to a Sharp Cone

A fully iterative solution was performed for the preceding example using a Newton-Raphson method to solve for the implicit variable, c_f, in Eq. (3). Examination of Eq. (3) shows that this computation is not dependent upon c_{f0}, except possibly as an initial guess to c_{f_rough}. The results of that computation are $c_{f\_rough} = 0.00414$, i.e. $c_{f\_rough} / c_{f0} = 1.15$, which is in excellent agreement with Fang's[1] correlation. Furthermore, k^+ is calculated from Eq. (3) as 23.10, a similar but higher result to those values calculated previously. Note that the larger value of k^+ would cause our approximation in section C to deviate even farther from the Fang's model and the numerical solution. We suggest, however, that the approximate solution section C has been optimized to use the fully rough expression and use of the numerical k^+ would not be consistent.

A particularly useful application is for friction on supersonic cones. The skin friction for cones at zero angle of attack can be obtained from the flat plate theory using the supersonic turbulent cone rule [1,2],

$$c_{f,cone}(Re_x) = c_{f,plate}(Re_x/2) \tag{24}$$

The 5° sharp cone data of Keel[7] provides an excellent test case for comparison of the theory to experiment. Keel measured skin friction in a wind tunnel for both smooth and rough walls on a 5° sharp cone at freestream Mach numbers of 2.5 and 5 and Reynolds numbers from $10^6 – 2.5{\cdot}10^7$. The measurement location was at x = 594 mm, and roughness heights of k = 0, 0.58, and 1.09 mm, were tested. Data were collected for both an adiabatic wall, and a cold-wall ($T_w = 0.35{\cdot}T_{aw}$) case, however only the adiabatic wall data are examined here. Smooth-wall skin friction measurements were ~ 0.001-0.003, resulting in k^+ ~ 15-1575. Since $k^+ >> 1$ and compressibility is certainly important for these cases, the perturbation approach is not valid and the approximation section C is less desirable, and a full iterative solution to Eq. (3) is appropriate.

A numerical iterative solution to Eq. (3) was performed using the Newton-Raphson method over the range of conditions provided in Table 1. Although Keel's paper does not provide enough boundary conditions to solve the problem, i.e. either T_0 or p_0, etc. are required in addition to Mach and Reynolds number, a range of wind tunnel operating pressures typically used for these M-Re combinations were tested and the final solution was found to be insensitive to pressure. The pressures reported below are averages of typical wind tunnel values. Since both the Mach 2.5 and 5 cases are in the compressible flow regime, it's necessary to apply the Van Driest compressible skin friction transformation of Eq. (4). To do this, the Reynolds number is first transformed by F_{Rex} (see equation (4)) and divided by 2 to account for the supersonic cone rule, then the transformed Reynolds number is inserted into Eq. (3), and finally, the resultant incompressible skin friction is transformed into a compressible value. The range of transformed Reynolds numbers, $\widetilde{Re}_s$,are provided in Table 1.

M_∞	**$Re_{x,edge}$ (10^6)**	**p_0 (kPa)**	**M_{edge}**	**$\widetilde{Re}_s$ (10^6)**	*(k/s)*
	1.0			0.45	
2.5	⋮	550	2.4	⋮	$9.8{\cdot}10^{-4}$, $1.8{\cdot}10^{-3}$
	20.0			8.5	
	1.0	414		0.19	
5.0	⋮	⋮	4.7	⋮	$9.8{\cdot}10^{-4}$, $1.8{\cdot}10^{-3}$
	20.0	1724		3.58	

Table 1. Boundary Conditions for comparison to Keel's skin friction data.

Figure 4 shows a comparison of the theoretically predicted compressible skin friction values with Keel's experimental data. Results are plotted as a function of the boundary layer edge Reynolds number. Data for M_{edge} = 2.4 are provided in the left figure, while those for M_{edge} = 4.7 are on the right. The theory shows fair agreement with the Mach 2.4 data, with particularly good agreement for k = 1.09 mm. Agreement is less favorable with the Mach 4.7 data, with the theory under-predicting by about 20-30%. Nevertheless, this example shows that the theory may be suitable for design purposes, particularly at lower Mach numbers.

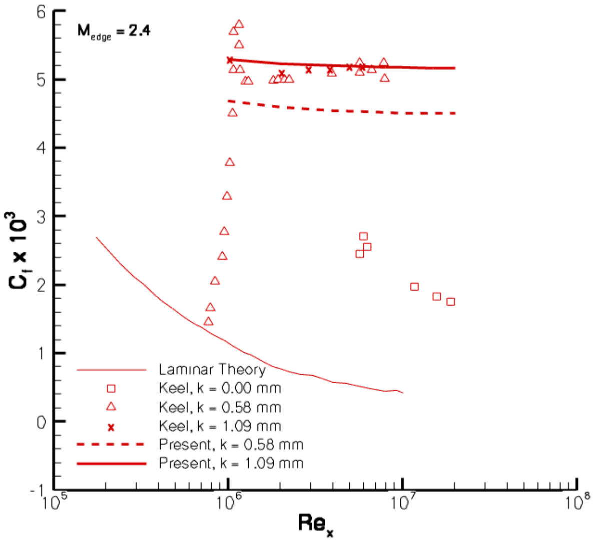


Figure 4. Comparison of theoretically-derived compressible, turbulent skin friction on a 5° sharp cone with experiment. Mach 2.4 data are shown on the left, while Mach 4.7 data are on the right.

III. Conclusion

We have developed a model to estimate for the increase in a known friction and boundary layer thickness due to roughness that may be utilized without performing either new experiments or computations. A simple perturbation method was used to motivate the problem, but for $k^+ = \frac{k_{rough}v^*}{\nu_w} > 1$ (the small parameter) the use of this series is usually not valid. For these flows we solved the appropriate incompressible roughness expressions directly using an approximate procedure which is valid for a wider range of k^+ values. Though useful to demonstrate behaviors associated with roughness and providing a connection to smooth expressions, the approximate method is of less value when one requires compressibility corrections, whereby it is perhaps more appropriate to dispense with approximation and simple solve the full expression numerically. We also note that the empirically based expression described by Fang et. al. is in excellent agreement with the inner law based model described here. More over their model is very convenient in application.

Further work must include examination of the procedure in a more comprehensive and systematic fashion. This effort should be both in terms of validating the method for simple problems, i.e. with knowledge of both smooth and rough behavior and more complex flows where the "local" flat plate assumption may not hold. More complex flow problems may require the explicit appearance of the pressure gradient. Geometric effect, e.g. axi-symmetric versus planar problems may also require a more formally explicit skin friction or boundary layer thickness model.

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