

Sensitive Detection: Cavity Ring-Down Spectroscopy and Crucial Intermediates Monitoring in Fuel Oxidation Chemistry

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\$\$\$

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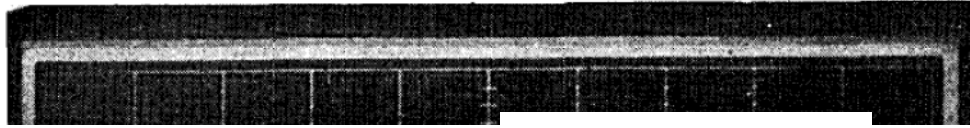


Outline

- **Cavity Ring-down Spectroscopy (CRDS)**
- **Challenges in CRDS Technique**
 - Higher order transverse mode interference
 - Linear birefringence and dichroism of supermirrors
 - Finite extinction ratio in CW-CRDS
- **Trace Methane (63 pptv) Detection by CW-CRDS**
- **Ultimate Sensitivity Limits of CRDS Method**
- **Alkane Oxidation: Probing OH and ROO (308 nm)**
- **OH Detection (2.9 μm) in QOOH Chemistry**
- **Conclusions**

Cavity Ring-Down Spectroscopy

In 1980s, dielectric coated “**supermirrors**” with **very high reflectivity (> 0.99)** measuring **lifetime of trapped photon** in a high finesse cavity



Cavity ring-down optical spectrometer for absorption measurements using pulsed laser sources

2544 Rev. Sci. Instrum. 59 (12), December 1988

Anthony O’Keefe and David A. G. Deacon

Deacon Research, 900 Welch Road, Palo Alto, California 94304

(Received 28 March 1988; accepted for publication 27 July 1988)

J. M. Herbelin, J. A. M

ancer, and D. J. Benard

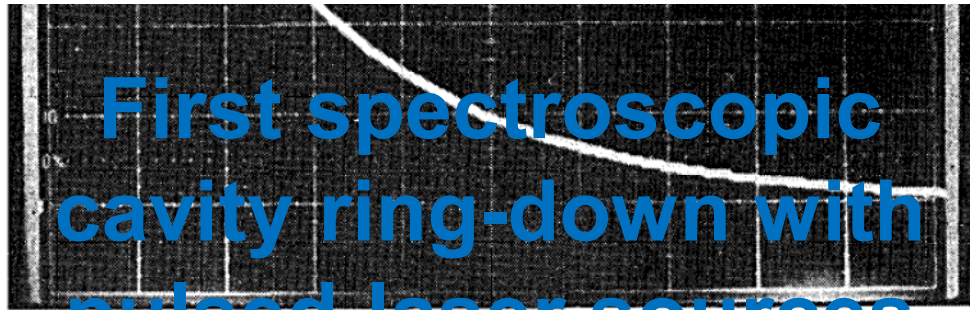


Fig. 1. Typical cavity decay curve from a **two-mirrored 10-m cavity** having a fall time of **23.2 μsec** starts on the left-hand side as the lower of the two curves. The second curve is the decay from a RC network having the same decay constant. The decays from the two events are perfectly merged showing that the cavity decay is indeed exponential.

CRDS has been *widely* employed for > 25 years.

- **Spectroscopy**

Cu₂, Cu₃ CPL **172**, 214 (1990) O'Keefe *et al.*

- **Chemical kinetics**

Phenyl JACS **115**, 4371 (1993) Lin *et al.*

- **Analytical chemistry**

¹³C/¹²C Anal. Chem. **74**, 2003 (2002) Crosson *et al.*

Trace H₂O Anal. Chem. **75**, 4599 (2003) Dudek *et al.*

- **Combustion diagnostics**

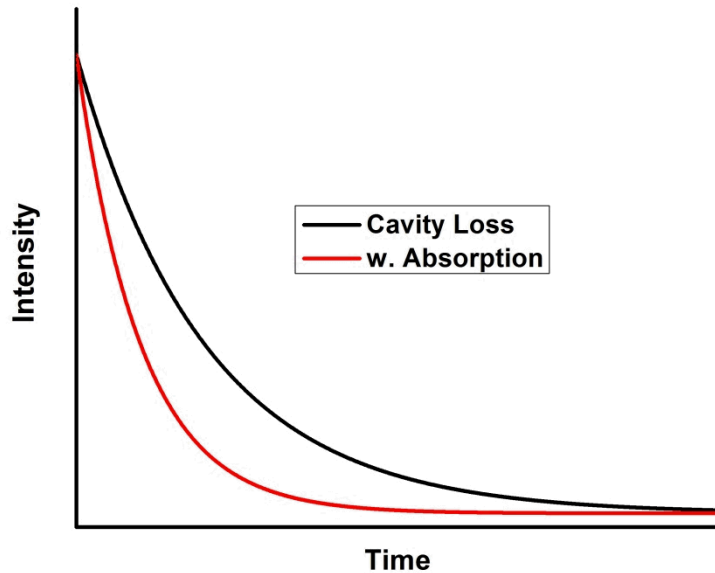
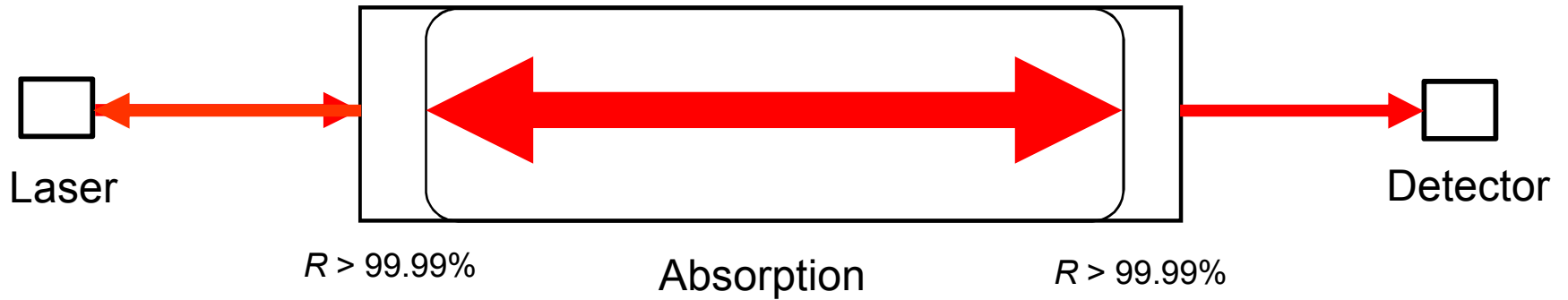
¹CH₂ CPL **296**, 151 (1998) McIlroy, A.

- **Medical gas diagnosis**

NO Appl. Opt. **43**, 2257 (2004) Bakhirkin *et al.*

- **and more**

CRDS: Basic Ideas



$$I(t) = Ae^{-kt} + B + \text{noise}$$

$$\tau = 1/k$$

$$\tau = \frac{L}{c(1 - R + \alpha L)}$$

Advantages of CRDS Method

- Much longer path lengths than traditional multi-pass cells

$$L_{\text{eff}} \sim 100 \mu\text{s} \cdot c = 30 \text{ km}$$

- Ring-down decay rate is **immune to the amplitude noise** of incident laser
- Very high sensitivity

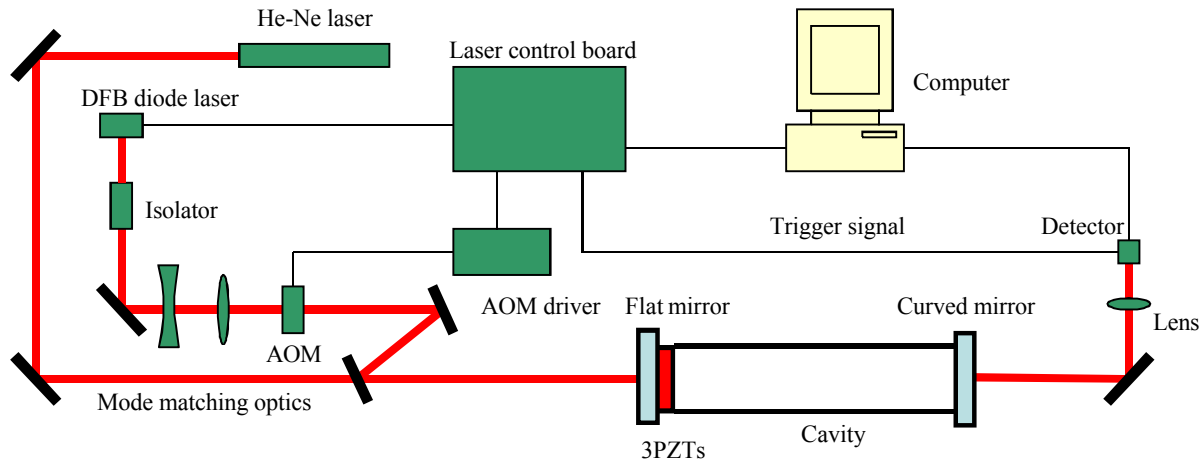
$$\alpha_{\text{min}} \sim 10^{-6} - 10^{-12} \text{ cm}^{-1}$$

$$\text{single pass} \quad \alpha L \sim 10^{-5} - 10^{-11} \text{ if } L = 30 \text{ cm}$$

- Method is calibration free if absorption cross section σ is known
- Cell is very compact; light contained in narrow spot of $\sim 1 \text{ mm}^2$

Sensitive detection methods such as CRDS are *essential* for probing very weak absorption features or detecting species of very low concentrations.

Typical CW-CRDS Setup



Key Parts:

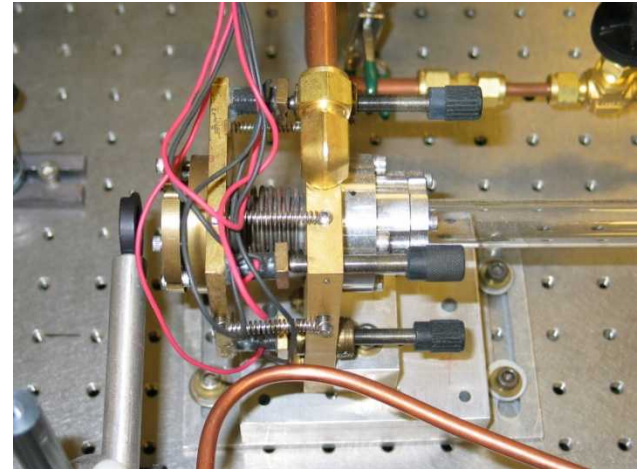
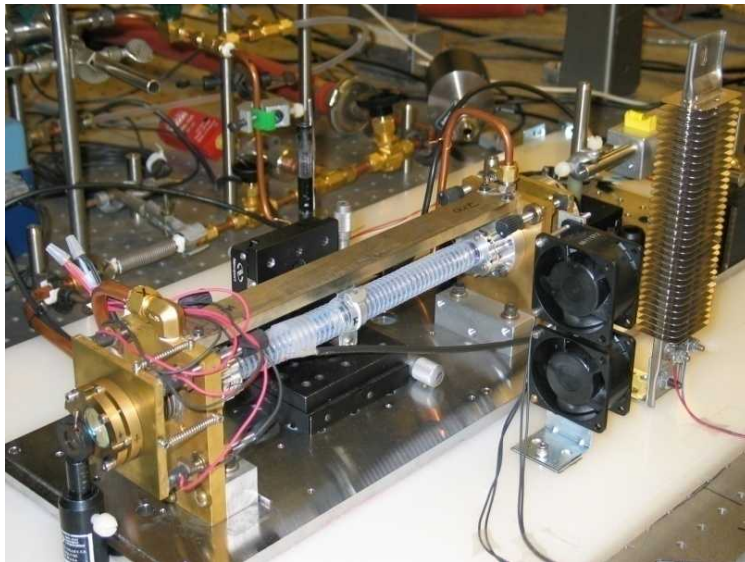
CW laser source

Stable ring-down cavity

Fast light switch

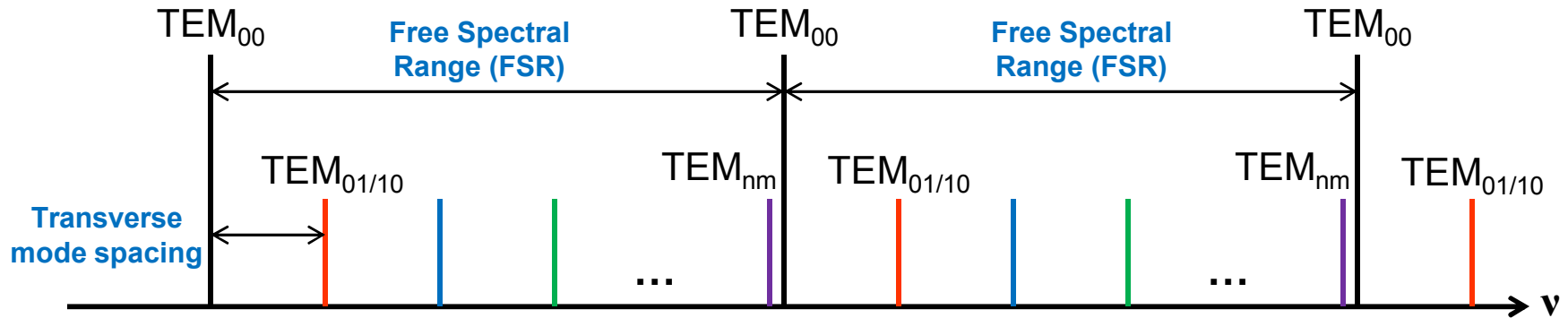
Fast detector

Piezo transducer (PZT)

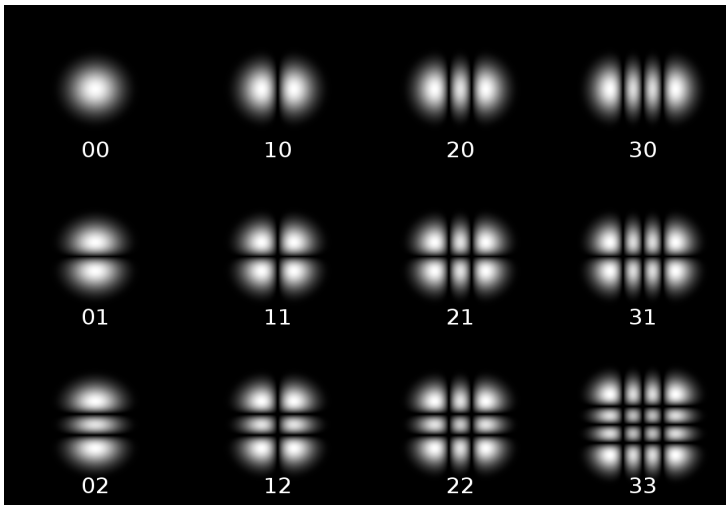


Eigenmodes in a Stable Optical Cavity

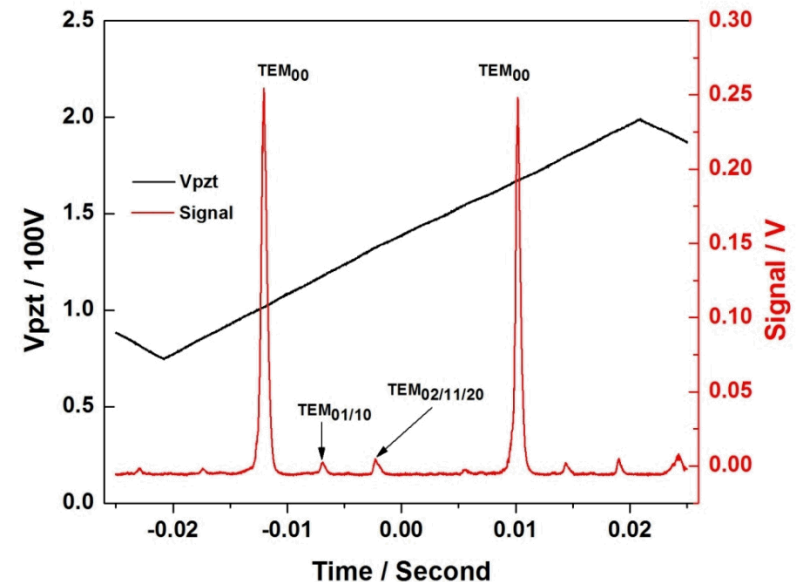
A. E. Siegman,
"Lasers", 1986



Single mode excitation produces exactly single exponential decay.
Multimode excitation can cause mode beating in decay signal.



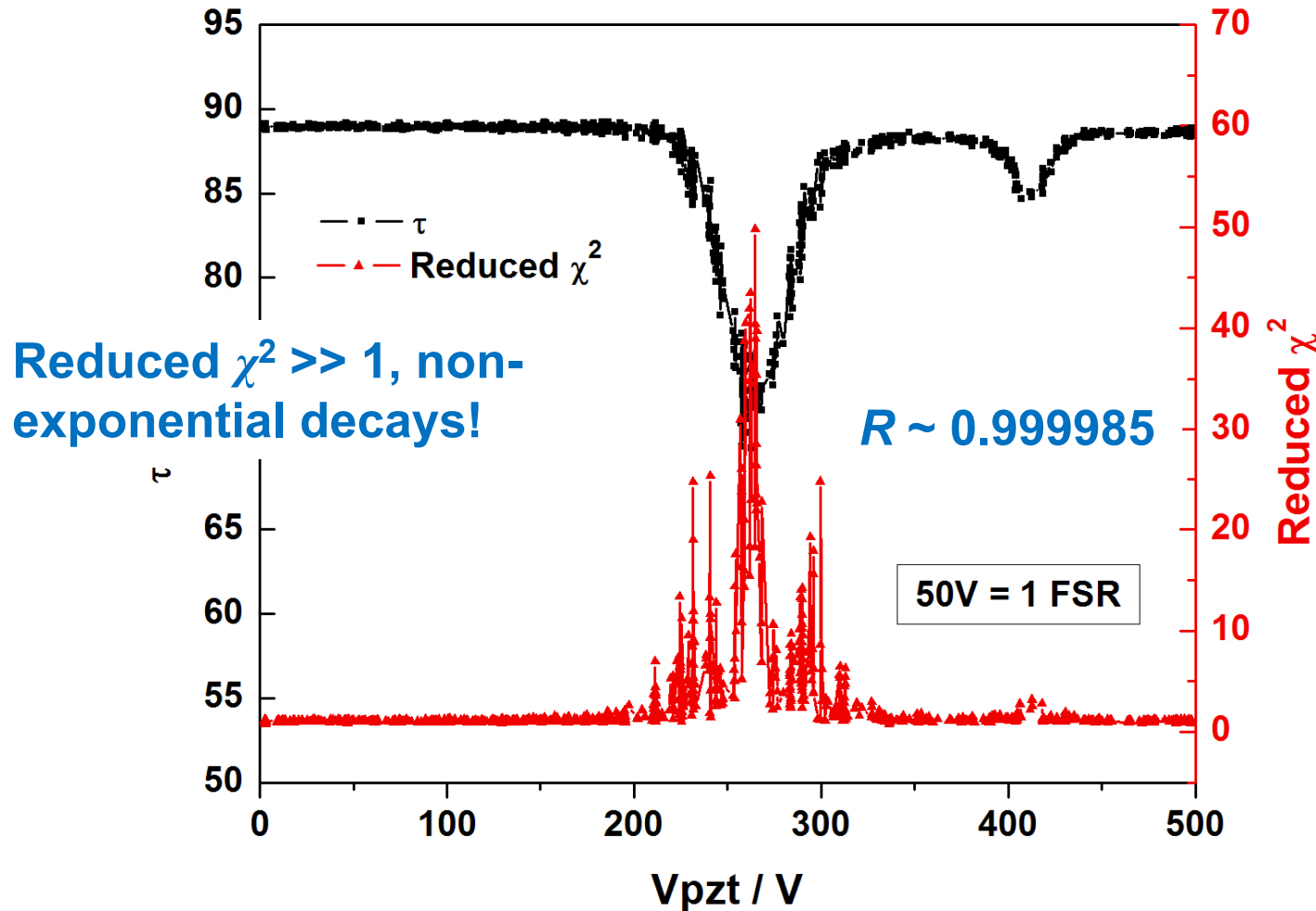
Wikipedia



Empty cavity decay time constant “decreased” a lot!

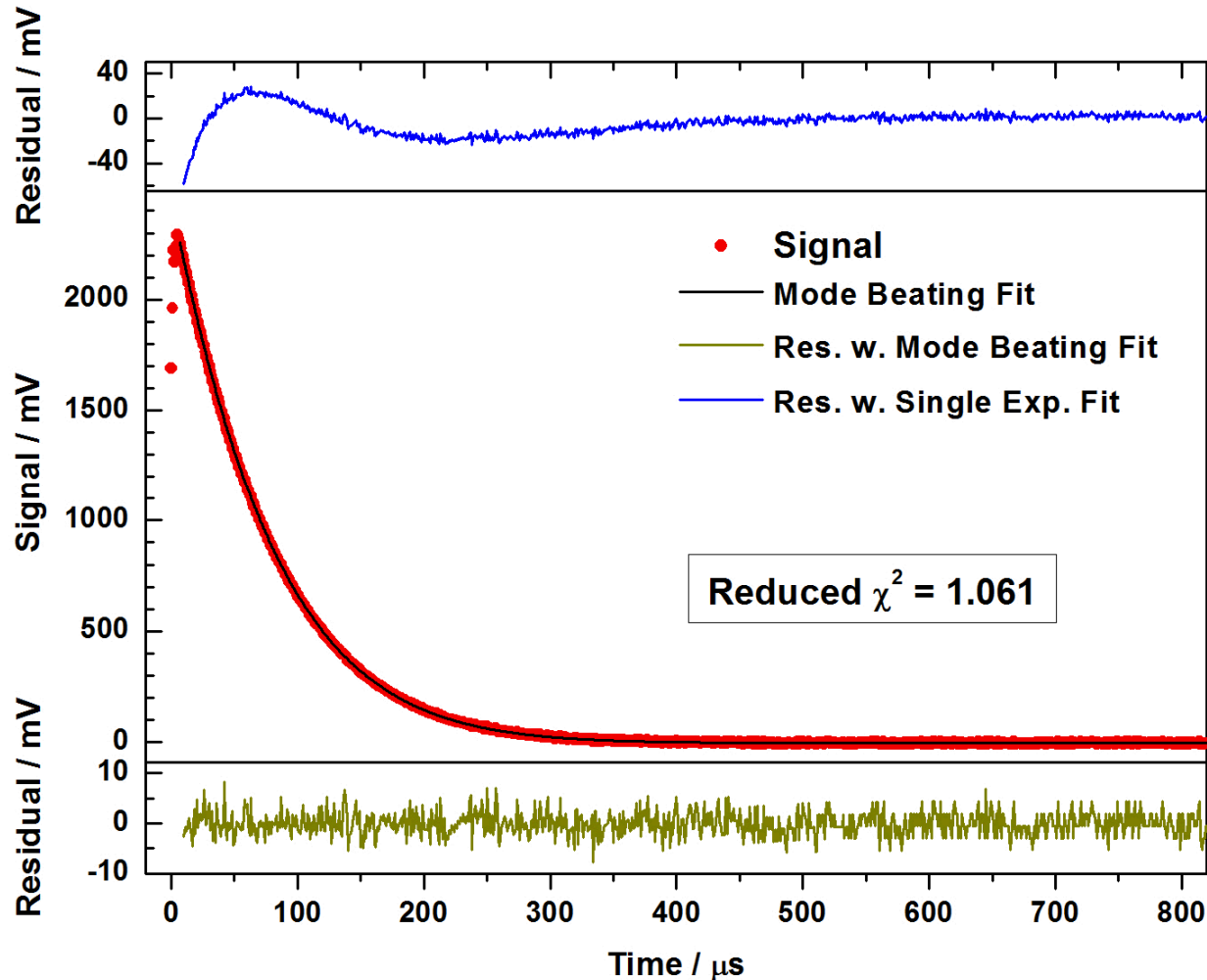
$$\chi^2 = \frac{1}{(N-3)\sigma^2} \sum_{i=0}^{N-1} [y_i - B - A \exp(-i\Delta t / \tau)]^2$$

Reduced χ^2 should be close to one.



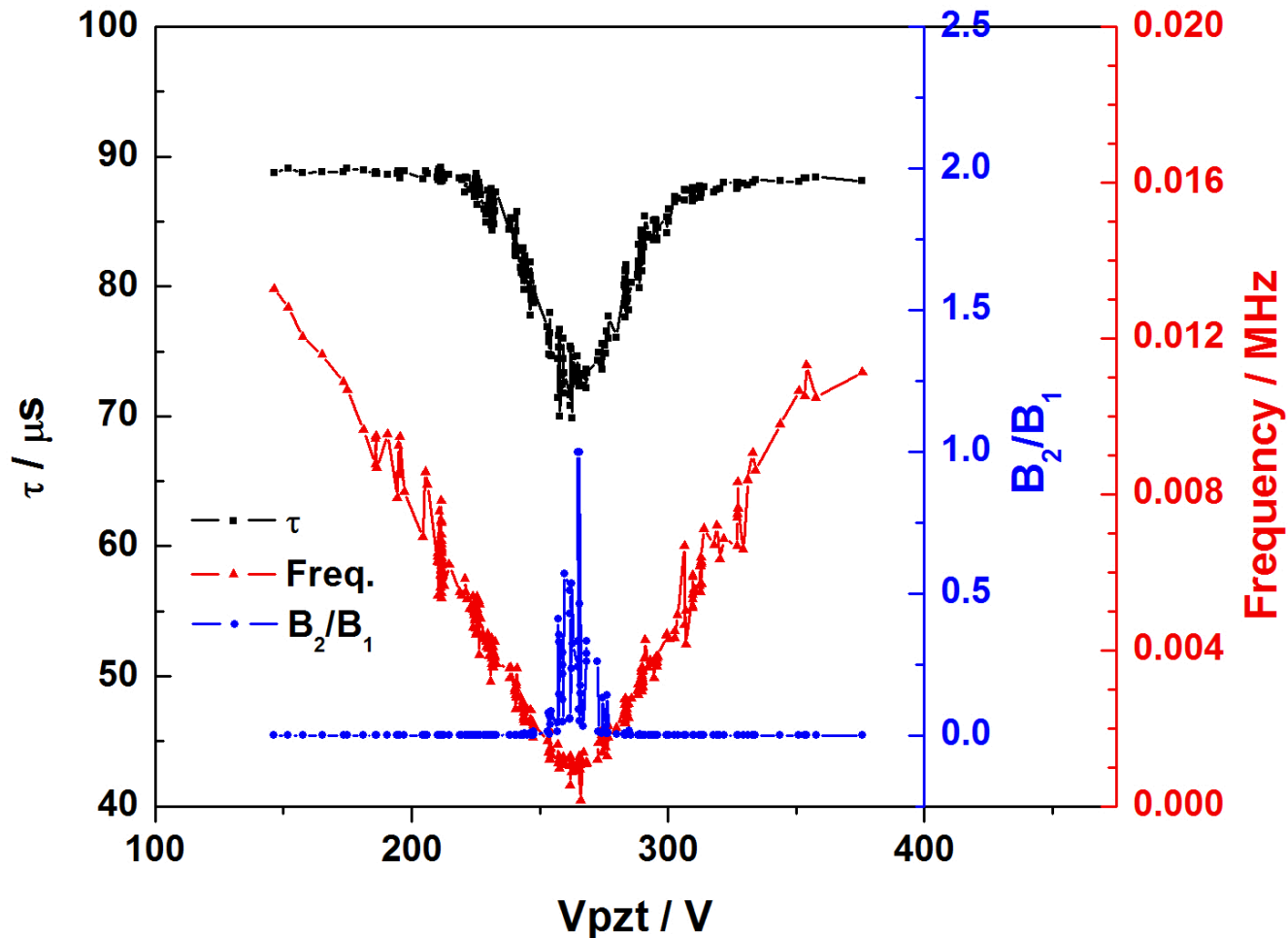
The noisy decay signal is described by a two-mode beating model perfectly.

$$y(t) = A + B_1 \exp(-k_1 t) + B_2 \exp(-k_2 t) + 2\sqrt{B_1 B_2} \exp(-\frac{k_1 t}{2}) \exp(-\frac{k_2 t}{2}) \cos(2\pi \cdot \Delta \nu \cdot t + \Delta \phi)$$

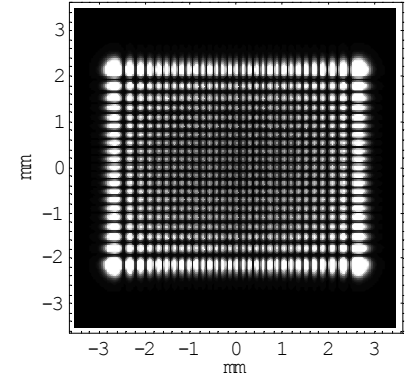
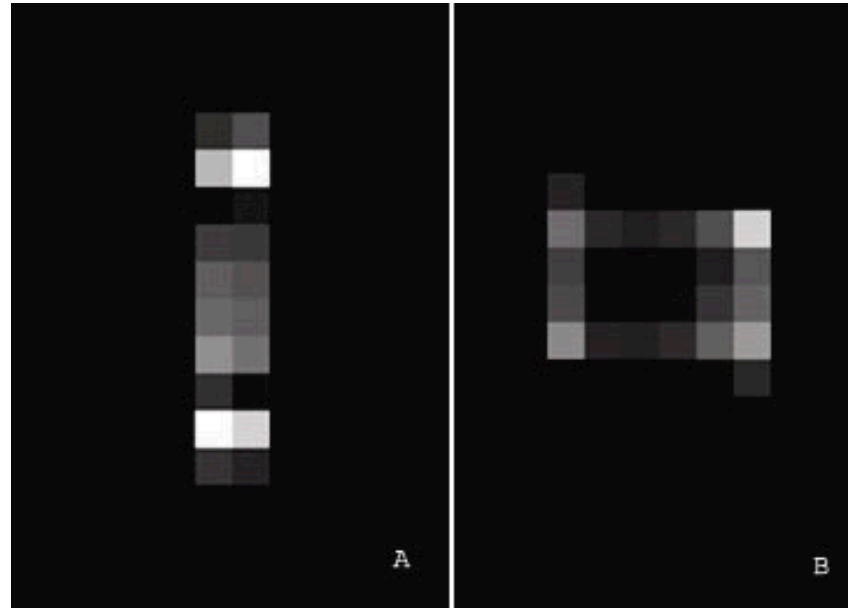
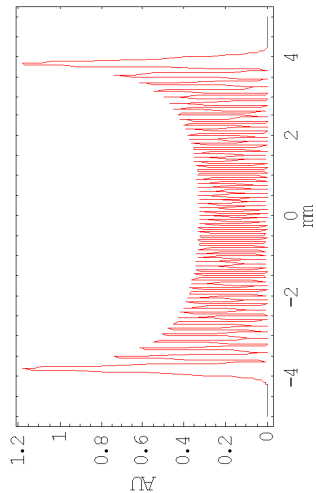


Data analysis shows signs of resonance by tuning.

- 50V = 1.0 FSR, laser $\lambda = 1.652 \mu\text{m}$
- $\Delta\nu$ tuning rate 120 Hz/V, or **7.22 kHz/ μm**



Images of very high order transverse mode have been captured by an IR camera.

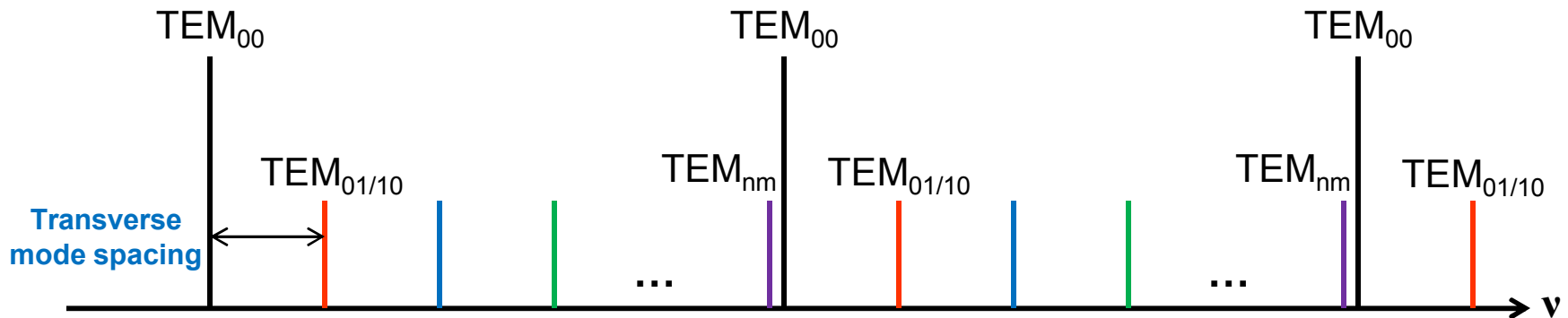


TEM_{nm} mode size:

$$\sqrt{n+m+1} \omega_{00}, \quad \omega_{00} = 0.507 \text{ mm}$$

- TEM₀₀ mode diameter ~ **1 pixel**
- Mode size of A and B: ~ **4mm** and **3.6mm**
- Index $n+m$: ~ **60** for A and ~ **50** for B

The excitation of very high order transverse modes is caused by **mode crossing** from cavity length tuning.



$$\frac{dv_{qnm}}{dL} = -\frac{v_{qnm}}{L} + \frac{c}{2n_0L} \frac{n+m+1}{2\pi\sqrt{L(R_c-L)}}$$

Relative tuning rate:

$$123(n+m) \text{ Hz}/\mu\text{m}$$

PZT scan:

7.22kHz/μm

$n+m \sim 59$

Higher order mode images

$n+m \sim 60, 50$

Cavity length 39.604 cm

$n+m \sim 60$

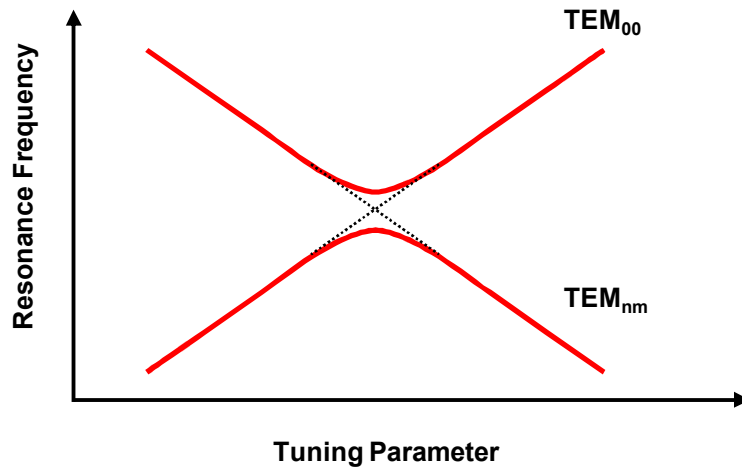
Cell inner radius 7.6 mm

$n+m < 135$

4 mm diameter aperture

$n+m < 14$

Those high order TEM_{nm} modes are excited through surface scattering coupling.



$$L_s = 1 - R - T \sim 10^{-6}$$

$$\Delta\Omega \sim \pi(\lambda/\pi\omega_{00})^2 = 3 \times 10^{-6}$$

$$L_s \Delta\Omega / 2\pi \sim 10^{-12}$$

$$\Delta\nu_{\min} = 2|S_1 + S_2|/t_r$$

$$\Delta\nu_{\min} \approx 1 \text{ kHz}$$

$$t_r = 2.63 \text{ ns}$$

$$|S_1|^2, |S_2|^2 \sim 10^{-12}$$

Relative Intensity of excited TEM_{nm} mode through coupling:

$$\sim (\Delta\nu_{\min} \tau_{nm})^2$$

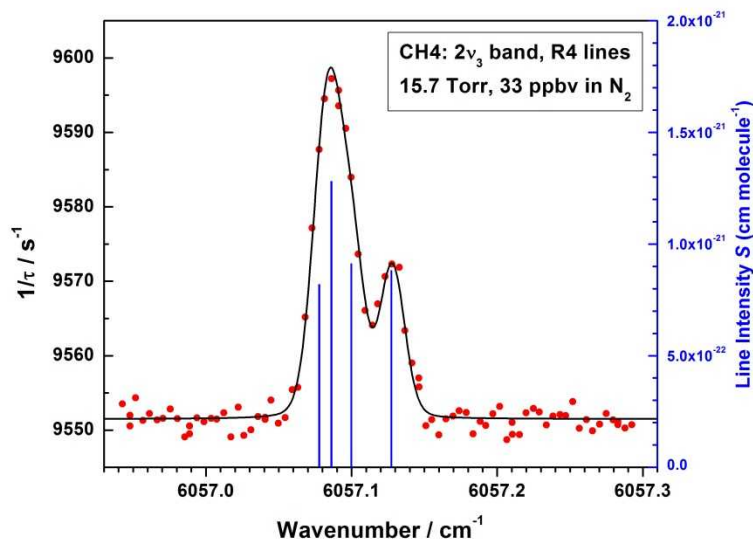
$$\tau_{nm} \approx 50 \mu\text{s} \quad \sim \frac{10^{-3}}{10 \text{ micron}}$$

$$\tau_{nm} \approx 100 t_r \quad \sim 10^{-10}$$

Not B_1/B_2 !

Sub-ppbv Methane Detection

Methane at ppbv level has been found in the atmosphere of Mars but the source of it is still under debate.

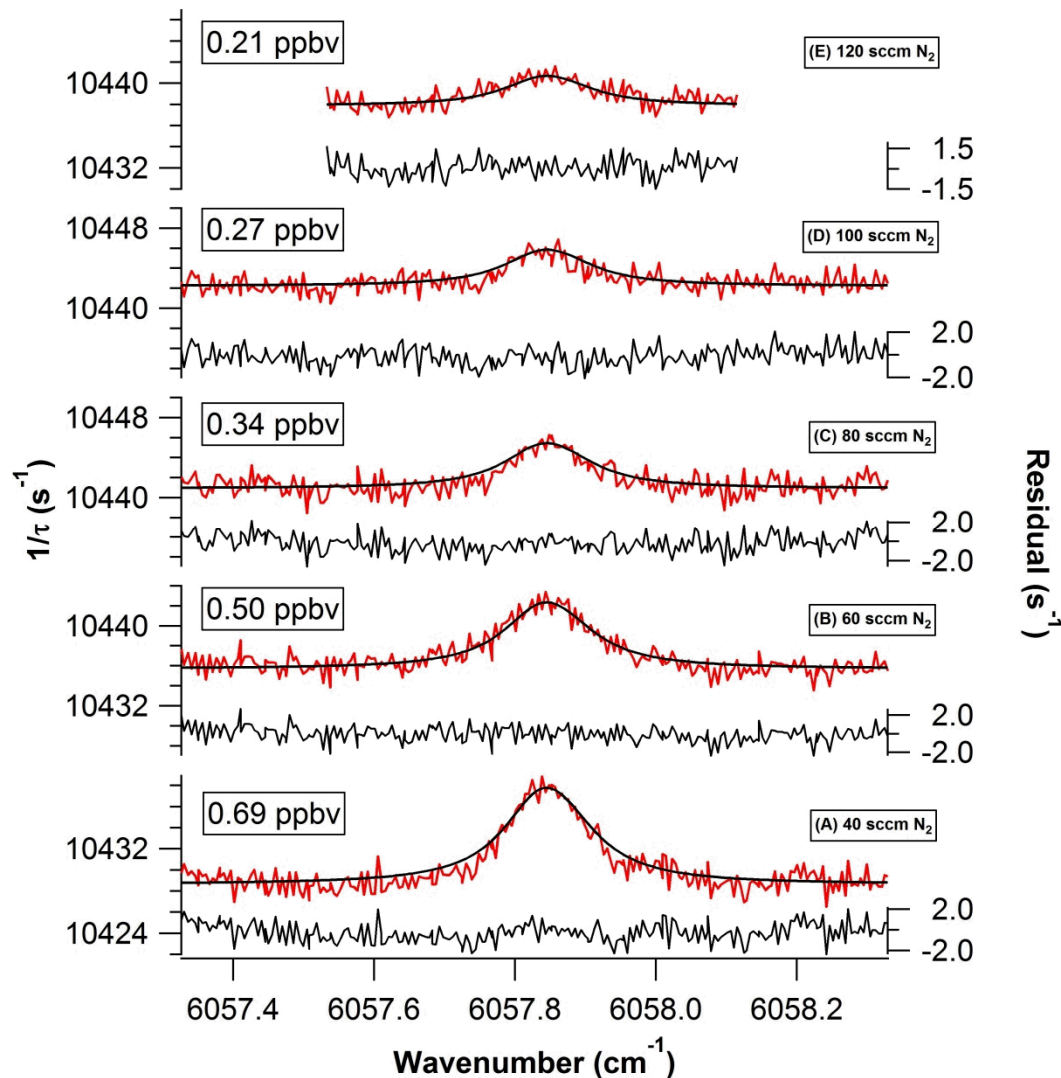


80 times spectrum averaging

1 σ detection limit:

63 pptv

760 Torr N_2 buffer gas



Single Shot Sensitivity of CRDS

What is the ultimate sensitivity of Cavity Ring-Down Spectroscopy?

Detector Noise + Shot Noise

$$\sigma^2(t) = \frac{P_N^2}{2} + \frac{h\nu}{Q} A e^{-kt}$$

Detector Noise Limit

$$\sigma_k^2 = 8k^3 \Delta t \left(\frac{\sigma}{A} \right)^2 = 4k^3 \left(\frac{P_N}{A} \right)^2$$

$$\tau = 100 \mu s \quad \Delta t = 1 \mu s \quad A/\sigma = 1000$$
$$\sigma_k = 2.8 s^{-1} \quad \alpha_{\min} = 9.4 \times 10^{-11} \text{ cm}^{-1}$$

Shot Noise Limit

$$\sigma_k^2 = k^3 \left(\frac{h\nu}{QA} \right)$$

$$\tau = 100 \mu s \quad Q = 0.64 \quad A = 1 \mu W \quad \lambda = 1.65 \mu m$$
$$\sigma_k = 0.43 s^{-1} \quad \alpha_{\min} = 1.4 \times 10^{-11} \text{ cm}^{-1}$$

Reached a sensitivity of $2.8 \times 10^{-11} \text{ cm}^{-1}$ in methane detection

What is the sensitivity of CRDS in unit bandwidth?

Larger triggering threshold and **longer fitting time window** reduce the error of measured **decay rate k** , but will decrease the **ring-down event rate** in a fixed time period.

This **tradeoff** between above factors gives the **highest sensitivity**.

Shot noise limited, $\Delta\nu_L \gg \Delta\nu_c$:

$$P_{\text{th}} = 2.47P_0 \quad \langle \Delta t_{\text{trig}} \rangle = 6.01\tau$$
$$N\Delta t = 5.67\tau \quad \text{RD rate} = 0.086k$$

$$\alpha_{\text{min}} = 2.31 \cdot \frac{1-R}{L} \sqrt{\frac{h\nu}{QP_0}}$$

$$P_0 = 1\mu\text{W} \quad \lambda = 1.65\mu\text{m} \quad Q = 0.64 \quad \tau = 100\mu\text{s}$$
$$0.086k = 860\text{Hz} \quad \alpha_{\text{min}} = 3.3 \times 10^{-13} \text{cm}^{-1} / \text{Hz}^{1/2}$$

We have realized a sensitivity of $7.9 \times 10^{-12} \text{cm}^{-1} \text{Hz}^{-1/2}$ by a cheap setup, only 24 times worse than the optimized shot noise limit.

NICE-OHMS

Noise-Immune Cavity-Enhanced Optical-Heterodyne Molecular Spectroscopy

Hall *et al.* combined **frequency modulation** with **high finesse cavity**

J. Ye, L-S Ma, J.L. Hall, **JOSA B 15**, 6 (1998)

L-S Ma, J. Ye, P. Dube, J.L. Hall, **JOSA B 16**, 2255 (1999)

$$\alpha_{\min} = \frac{1-R}{L} \sqrt{\frac{h\nu}{QP_0}} \frac{1}{J_0(\beta)J_1(\beta)} \xrightarrow{\beta=0.5} 4.4 \cdot \frac{1-R}{L} \sqrt{\frac{h\nu}{QP_0}}$$

Achieved a sensitivity of **$1.0 \times 10^{-14} \text{ cm}^{-1} / \text{Hz}^{1/2}$** , highest reported to date

This sensitivity was realized with a unique **intensity stabilized, sub-Hz linewidth** laser, with **$P_0 \sim \text{mW}$** .

Sensitive Detection: Crucial Intermediates Monitoring in Fuel Oxidation Chemistry

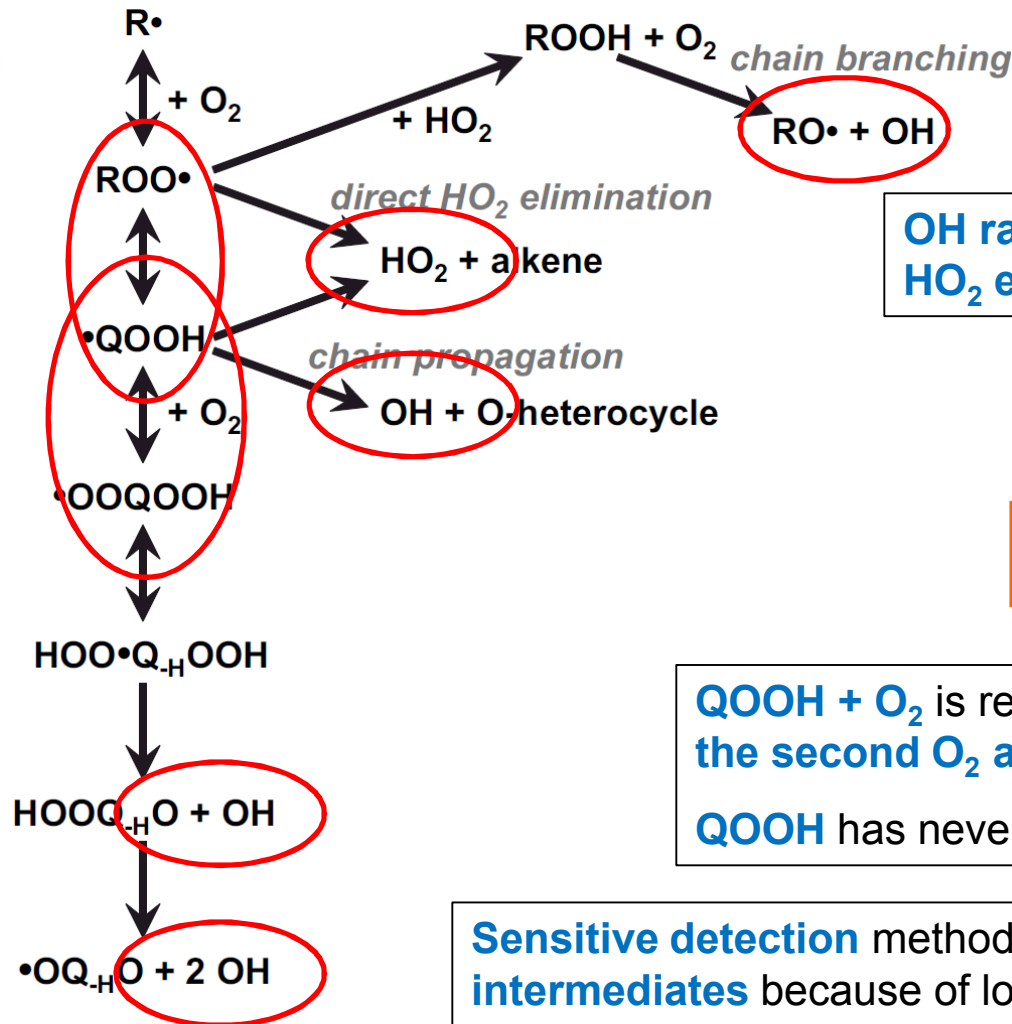
OH, RO₂, HO₂, QOOH, and CH₂O

ethane, propane, *n*-butane

ethanol, 1-butanol, diisopropyl ketone

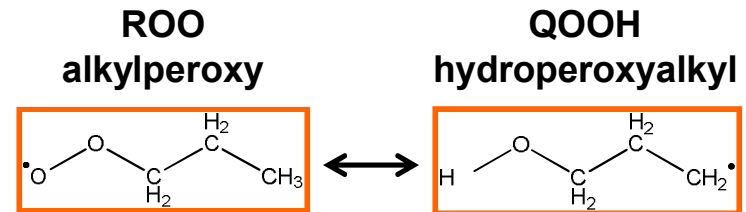
Hydrocarbon Autoignition Chemistry is fundamental to new clean efficient combustion strategies.

alkyl radical



The **ignition phasing** in an **HCCI engine**, in principle, is determined by **LT autoignition chemistry**.

OH radical is the major chain propagator and **HO_2 elimination** is a chain-terminating step.



$QOOH + O_2$ is responsible for **chain branching** through the **second O_2 addition**.

QOOH has never been detected **directly** by any means.

Sensitive detection methods are necessary in monitoring **reactive intermediates** because of low concentrations.

Herriott Cell Laminar Flow Reactor in Laser Chemistry Lab at CRF Sandia

Laser-photolysis **Cl-initiated**
oxidation of fuels in a multi-
pass reactor

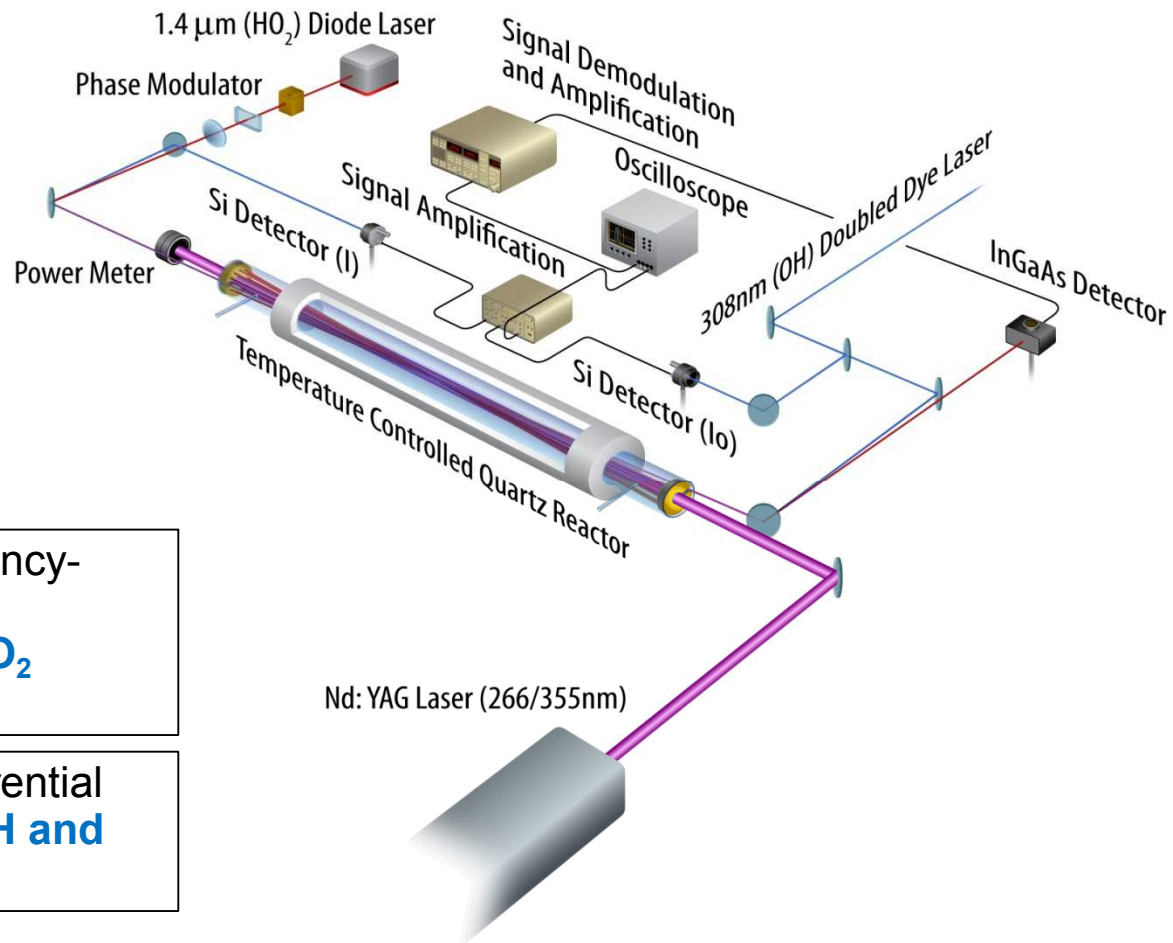
355nm Cl_2
266nm $(\text{COCl})_2$

Absorption length **10 ~ 20 m**

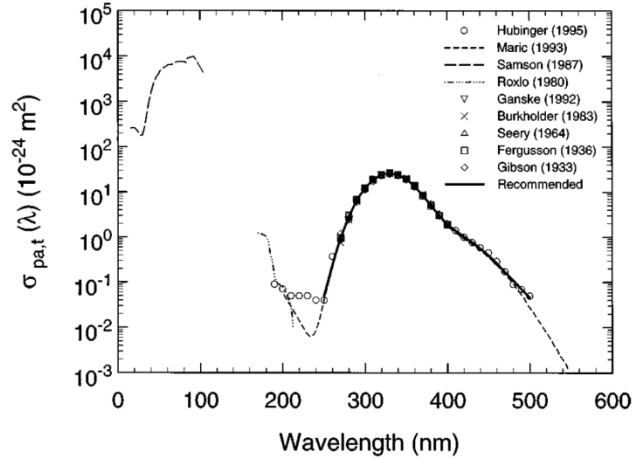
Time resolution **$\sim 1\mu\text{s}$**

Infrared (1509nm) frequency-
modulation or differential
absorption detection of **HO_2**
radicals

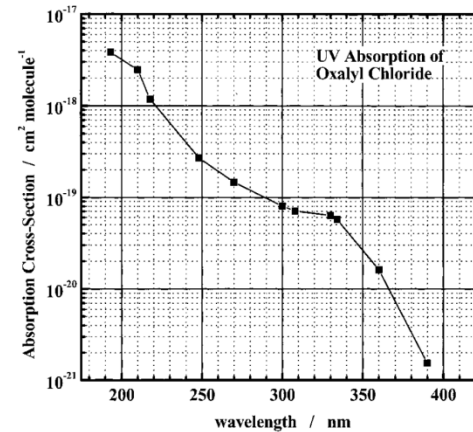
Ultraviolet (308nm) differential
absorption detection of **OH** and
ROO radicals



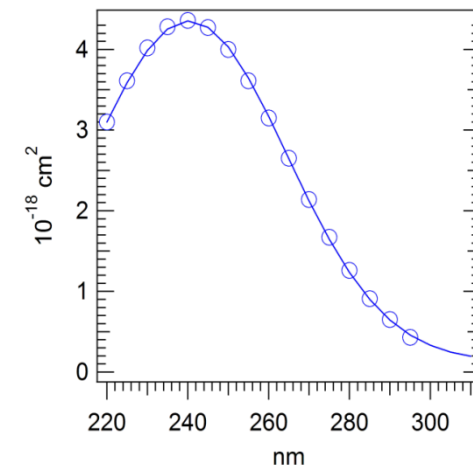
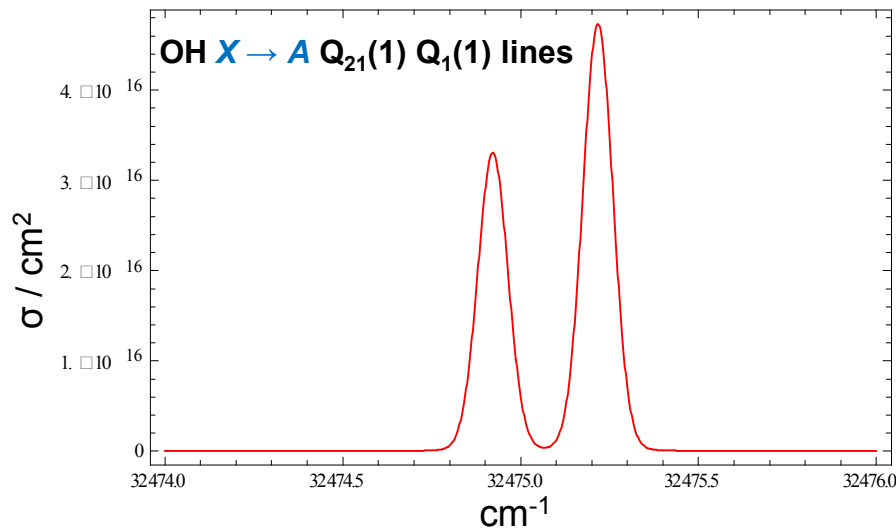
Absorption Features near 308nm



Cl_2 : $1.7 \times 10^{-19} \text{ cm}^2$ at 355nm



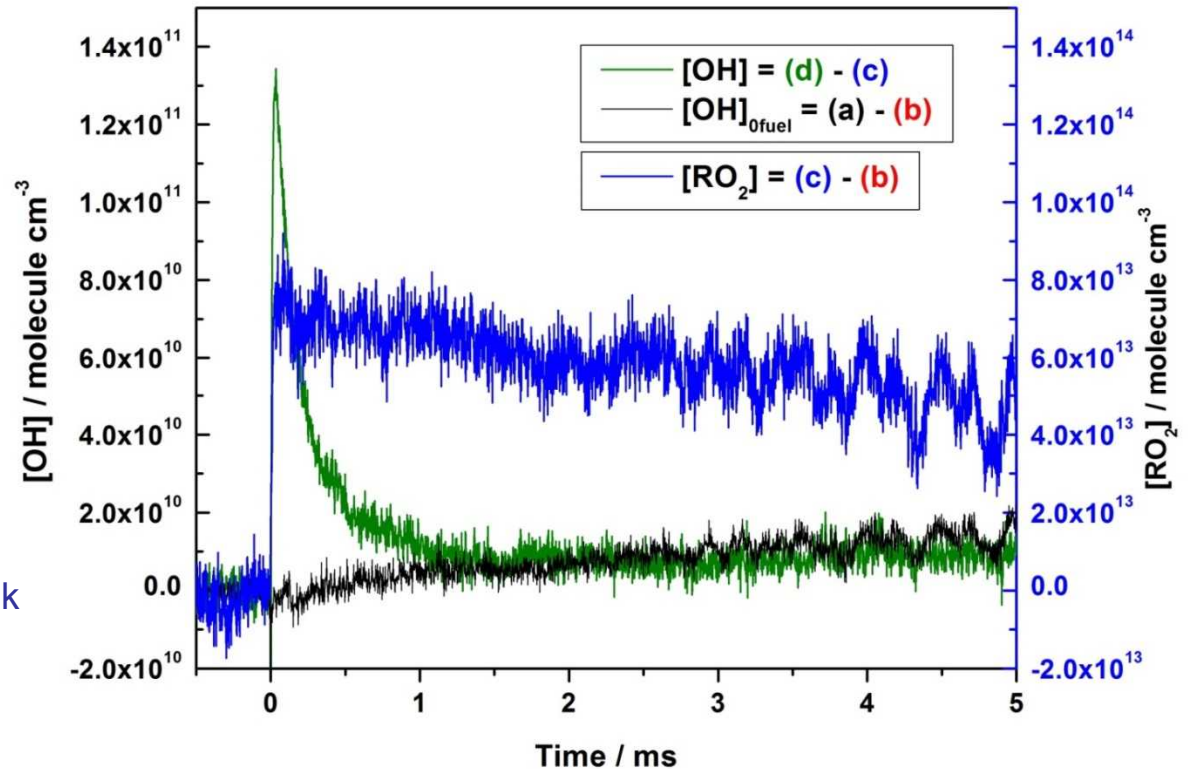
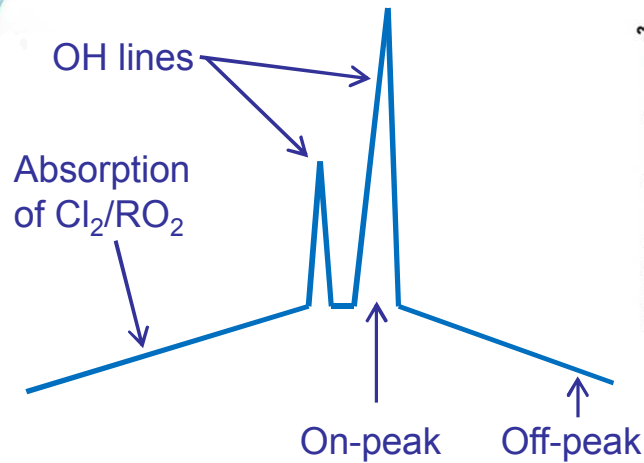
$(\text{COCl})_2$: $1.6 \times 10^{-19} \text{ cm}^2$ at 266nm



$\text{C}_2\text{H}_5\text{O}_2$: $2 \times 10^{-19} \text{ cm}^2$ at 308nm

JPCRD **28** (1999) 131; JPCA **105** (2001) 97; J Geophys Res **106** (2001) 12157; J Geophys Res **100** (1995) 7397; HITRAN Database 2000

We can detect both OH and ROO.

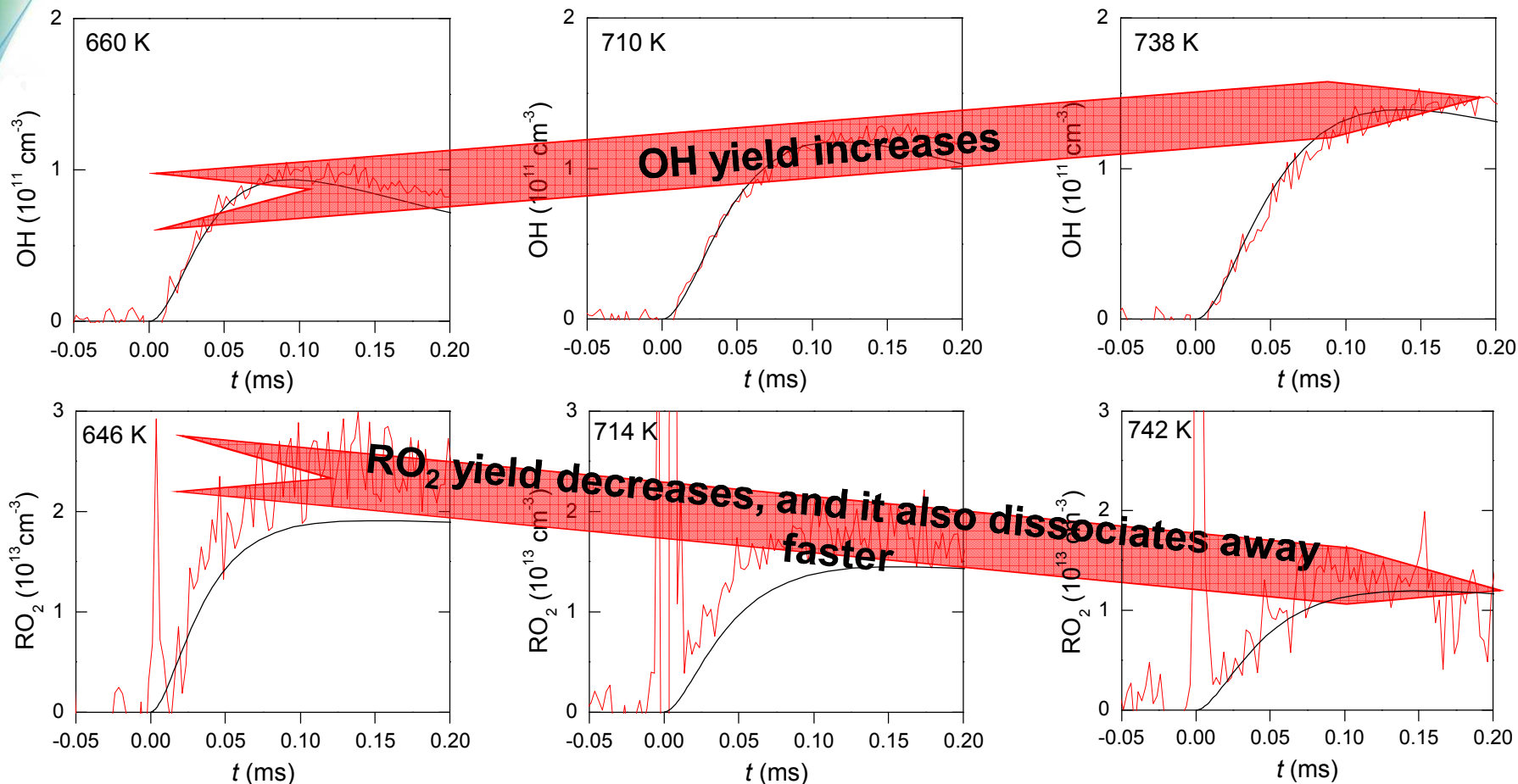


(a) no fuel, on-peak (b) no fuel, off-peak
(c) w. fuel, off-peak (d) w. fuel, on-peak

Thermal lensing noise is greatly removed by this subtraction.

$[\text{total}] \sim 10^{17} \text{ cm}^{-3}$, $[\text{Cl}]_0 \sim 10^{13} \text{ cm}^{-3}$, $[\text{OH}]$ detection limit $\sim 10^{10} \text{ cm}^{-3}$

OH and RO₂ in ethyl + O₂ system 660-740 K, 30 Torr, w. Cl₂

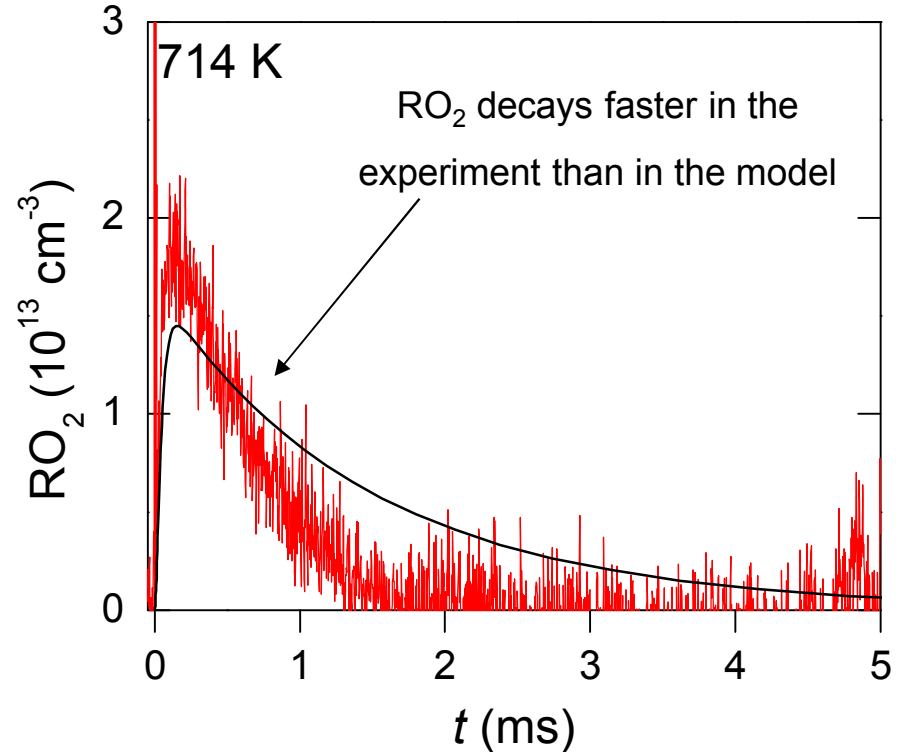
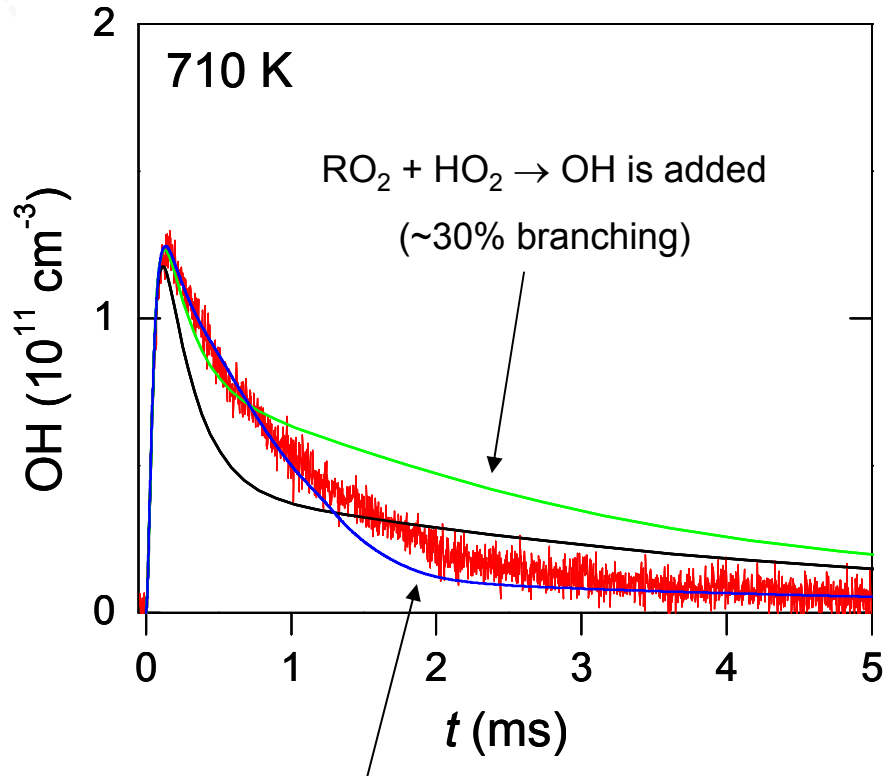


Measured and calculated **OH** absolute concentrations agree very well in a time window of **200 μs**.

Data unpublished

RO₂ detection clarifies effects of 2ndary chemistry.

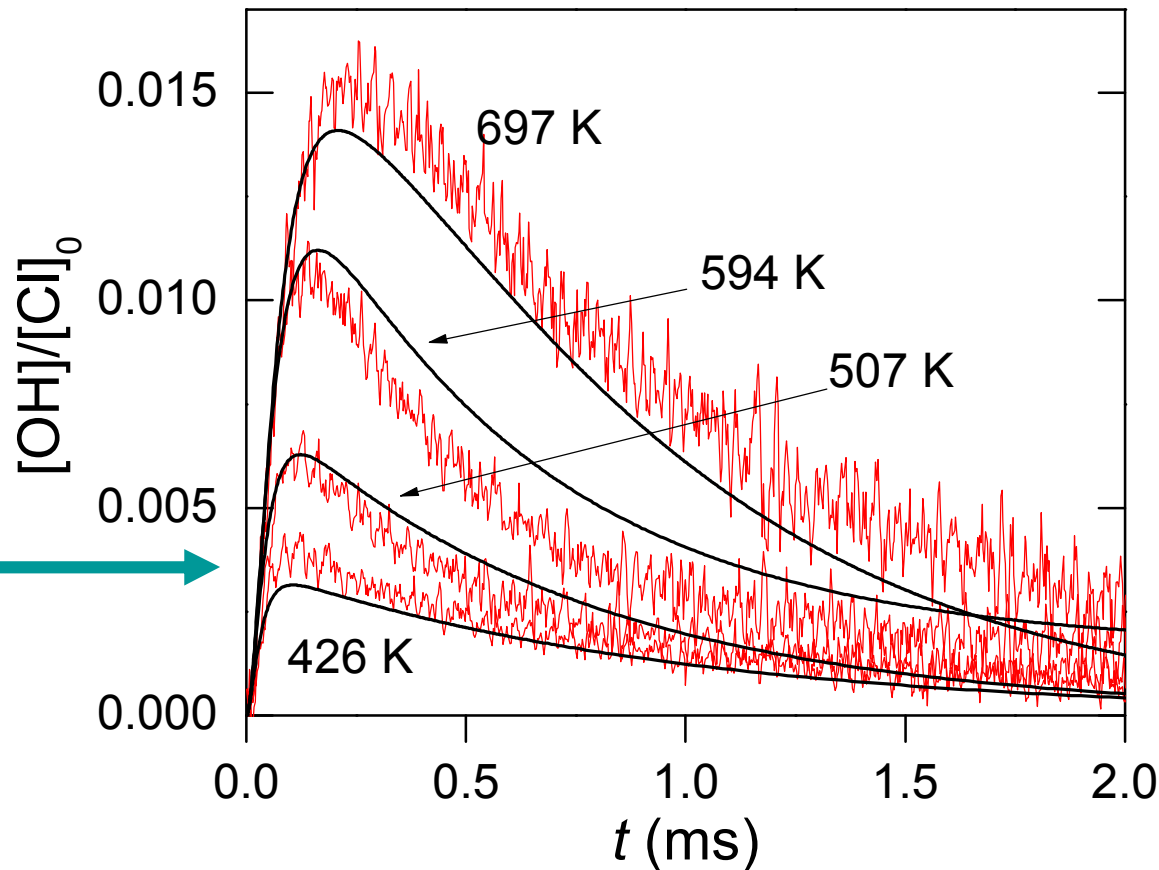
Constraining RO₂ to its measured values and adding RO₂ + HO₂ → OH eliminates almost all discrepancies on the whole 5 ms timescale.



Data unpublished

Results for the propyl + O₂ system 426-697 K, 10 Torr, w. (COCl)₂

~730K OH yield
from ethyl + O₂

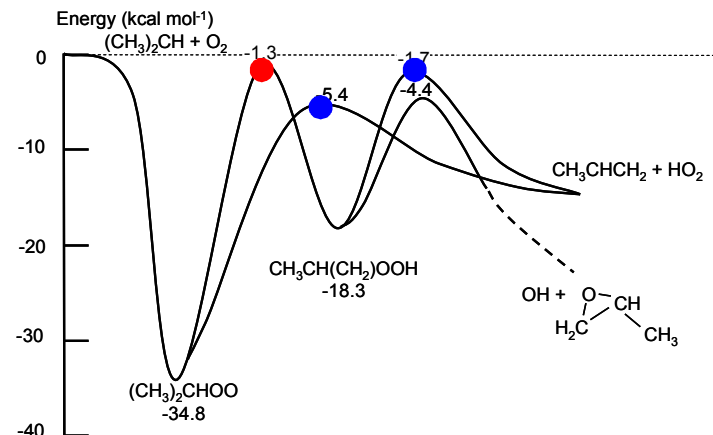
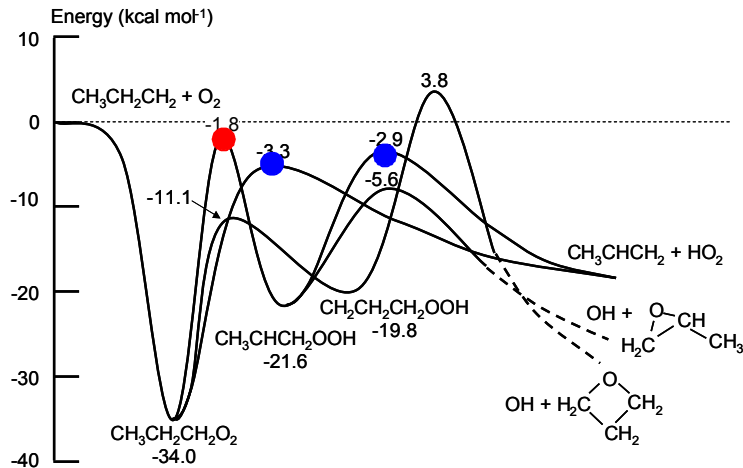
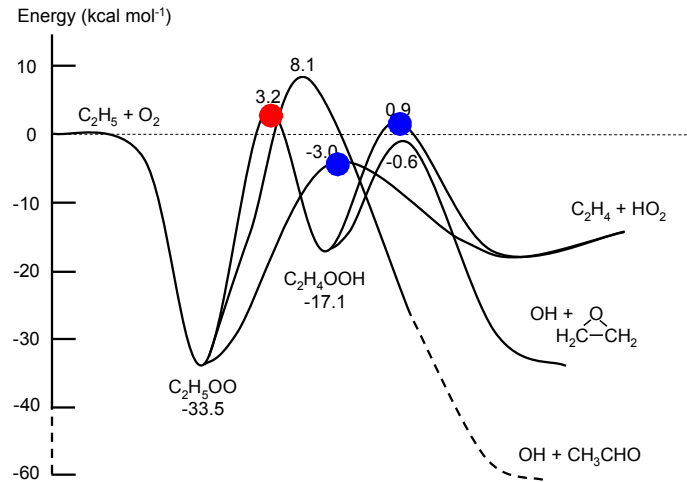


The propyl radical produces more OH, therefore, secondary chemistry is less important, even at longer times.

At 10 Torr, less RO₂ is stabilized.

Related barrier heights on the ethyl + O₂ and propyl + O₂ potential energy surfaces

- Previous HO₂ measurements (DeSain et al.)
- Current OH measurements



J. D. DeSain, S. J. Klippenstein, J. A. Miller, and C. A. Taatjes, *JPCA* **2003**, *107*, 4415-4427



Possible issues of OH detection at 308nm

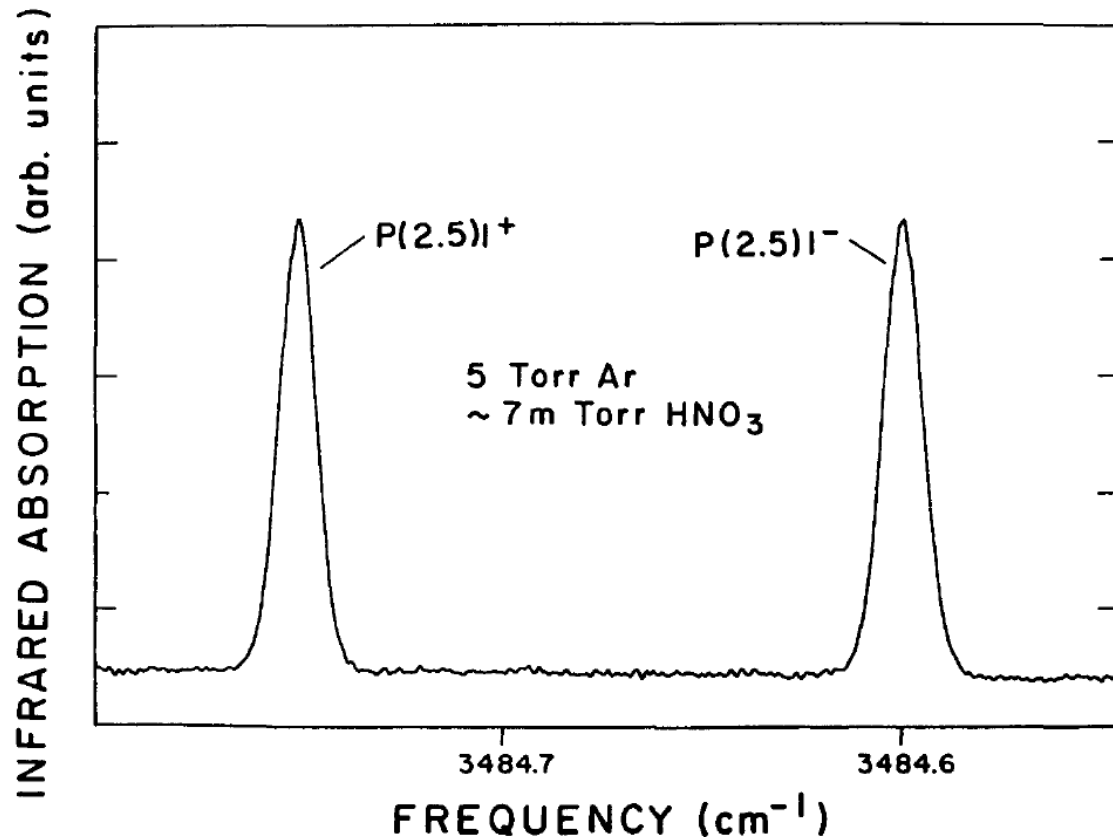
Measure **OH** with a cw **IR** laser

CW 308nm probe laser dissociates **Cl₂ / (COCl)₂** continuously, generating a background of reactions along the probe path before the **355nm / 266nm** photolysis pulse arrival.

308nm probe laser dissociates **RO₂** and **OOQOOH** radicals.

Some precursors, e.g. *tert*-butyl hydroperoxide (**TBHP**), absorb at **uv** region.

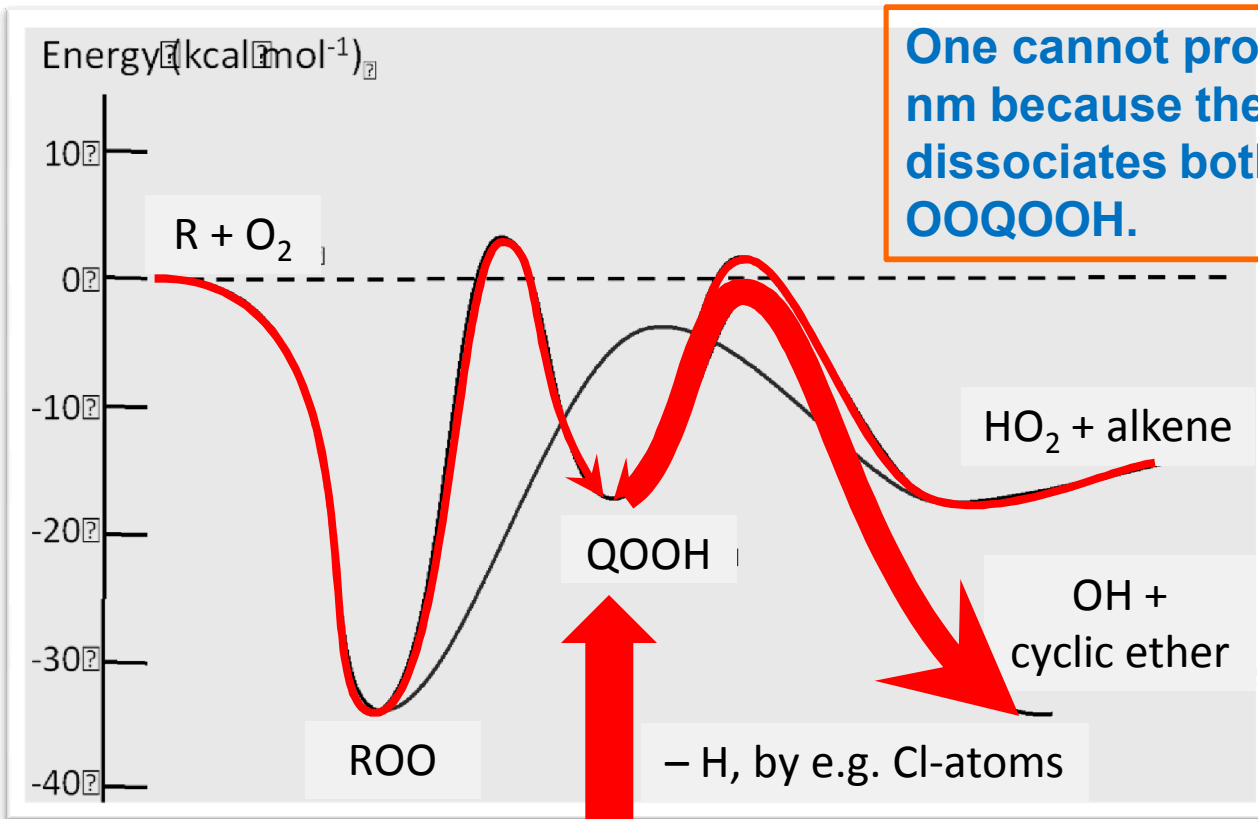
OH fundamental transition lines



Cross section $\sigma \sim 10^{-18} \text{ cm}^2$, [OH] detection limit $\sim 10^{11} \text{ cm}^{-3}$

Broad weak absorption features exist but can be subtracted.

Making enough QOOH radicals!

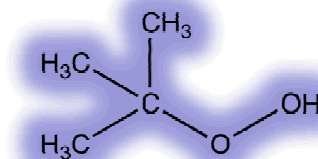


One cannot probe OH at 308 nm because the probe laser dissociates both TBHP and OOQOOH.

Recipes to make QOOH radicals:

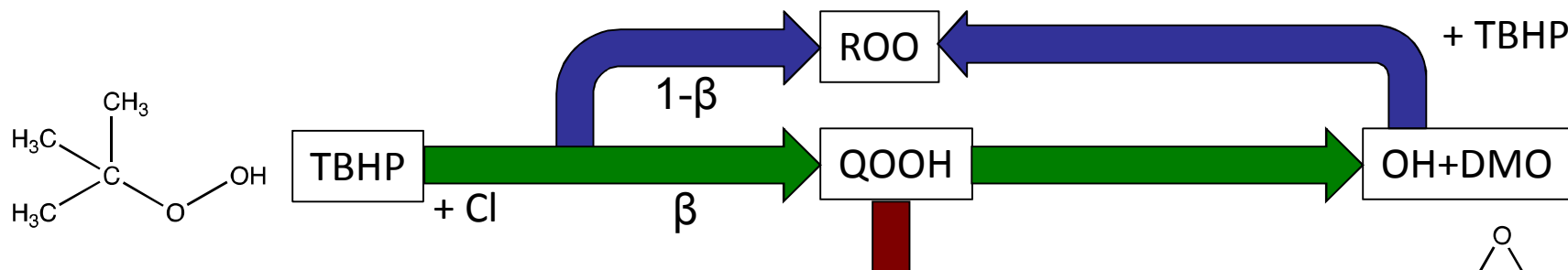
1. from $R + O_2$ or from ROO ⚡
2. from $HO_2 + \text{alkene}$ ⚡
3. ?

tert-butyl hydroperoxide
TBHP generates only one type QOOH.



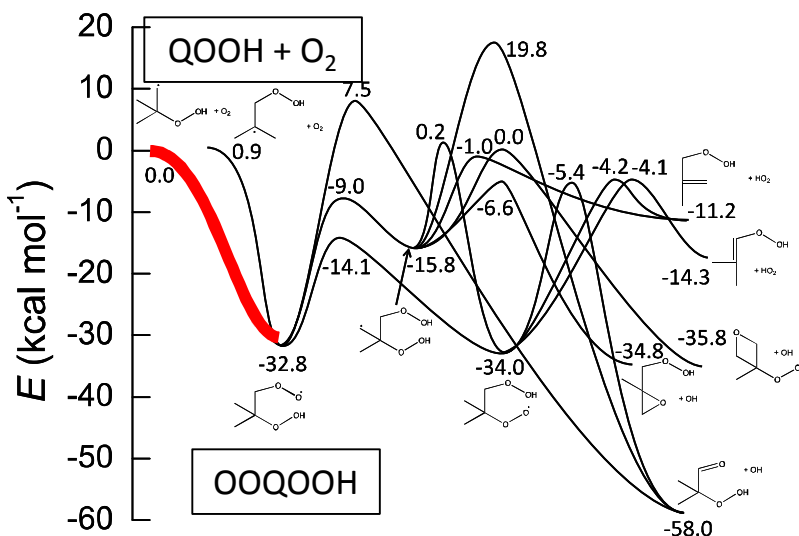
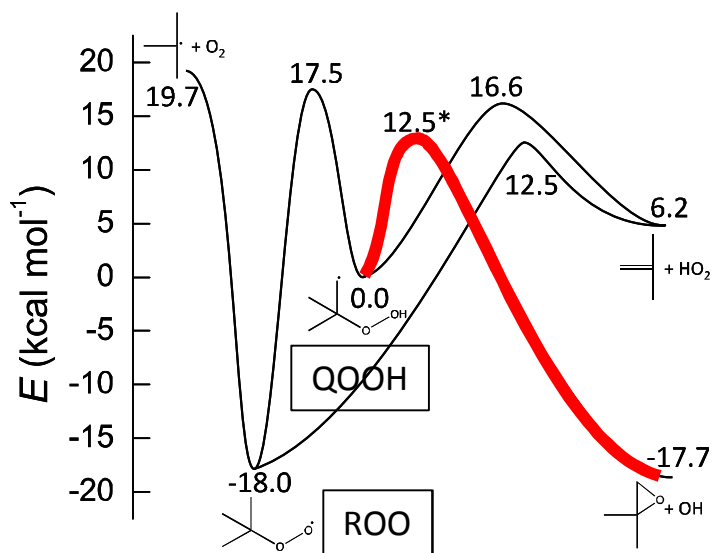
ROOH

With OxCl and at 300K the chemical mechanism has been greatly simplified.



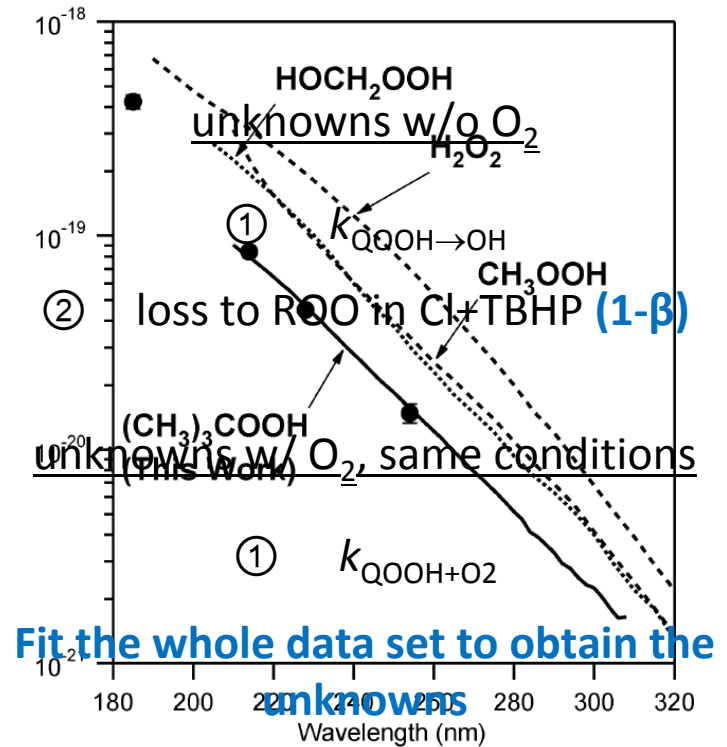
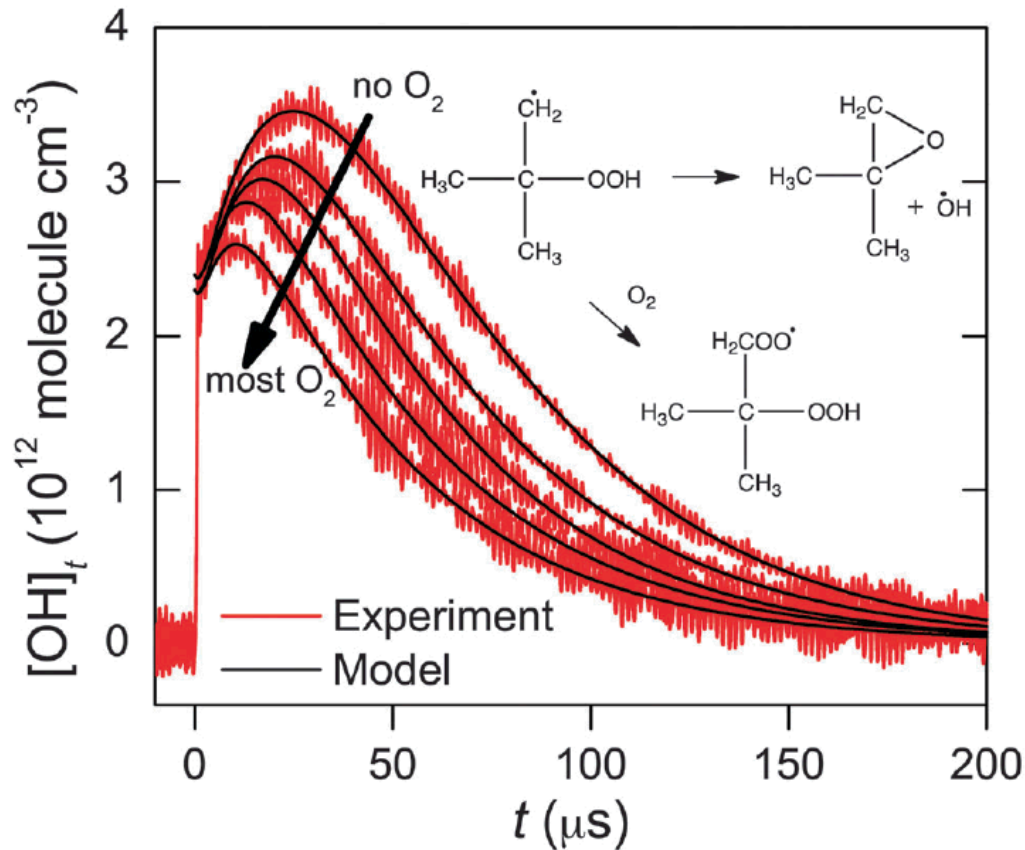
(CH₃)₃COOH (*tert*-butyl hydroperoxide): OH reaction rate coefficients between 206 and 375 K and the OH photolysis quantum yield at 248 nm[†]

Munkhbayar Baasandorj,^{ab} Dimitrios K. Papanastasiou,^{ab} Ranajit K. Talukdar,^{ab} Alam S. Hasson^c and James B. Burkholder^{*a}



Zador, Huang, Welz, Zetterberg, Osborn and Taatjes, *PCCP*, 2013, **15**, 10753--10760

Direct determination of QOOH decomposition and QOOH + O₂ rate coefficients



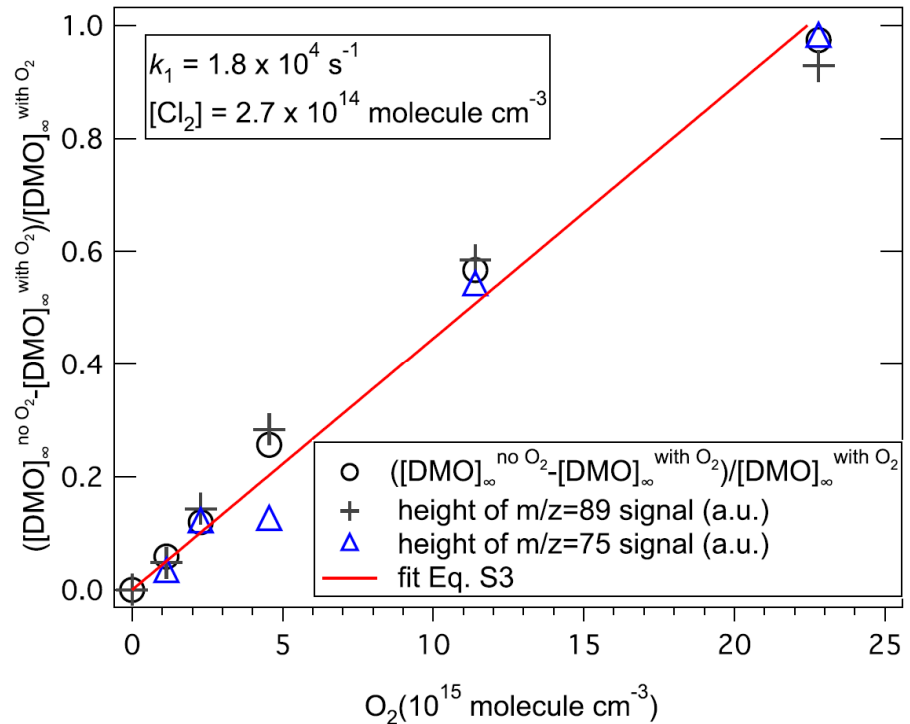
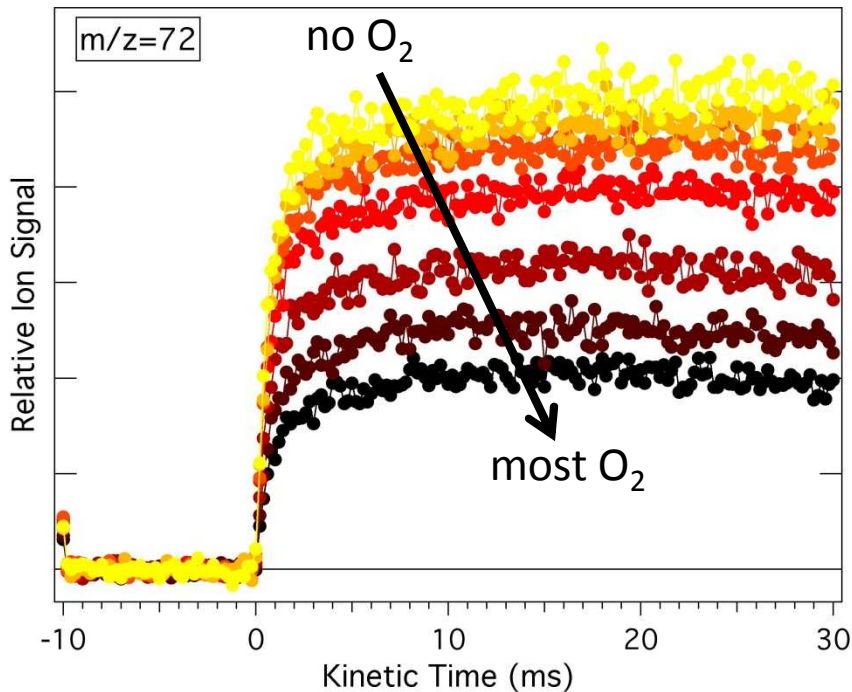
Baasandorj et al., PCCP, 2010, **12**, 12101–12111

Competition between **QOOH dissociation** and **addition of O₂**

Similar results using **Cl₂** except initial **OH** concentration and chain chlorination

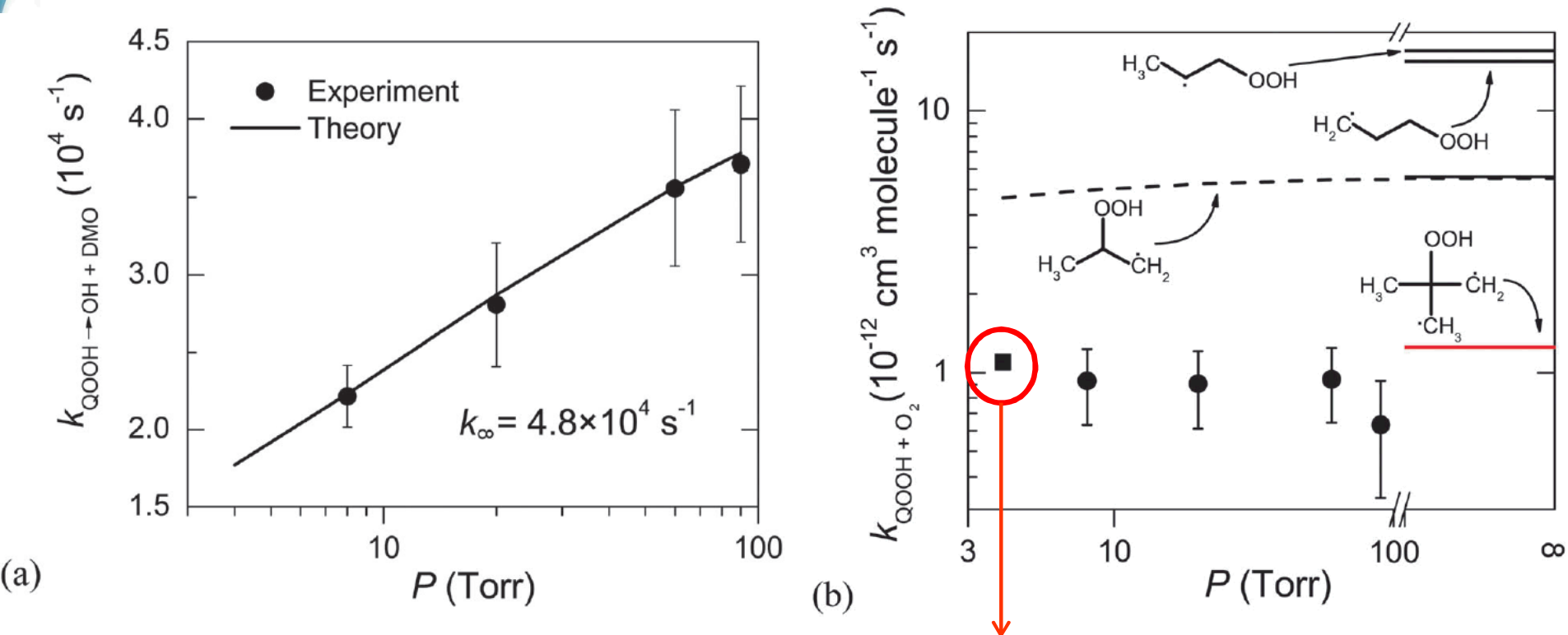
Yield of DMO ($m/z=72$) vs. O_2 gives k_{QOOH+O_2} at 4 Torr.

$$\frac{[DMO]_{\infty}^0 - [DMO]_{\infty}}{[DMO]_{\infty}} = \frac{k_{QOOH+O_2}}{k_{QOOH \rightarrow OH} + L_{QOOH}} [O_2]$$



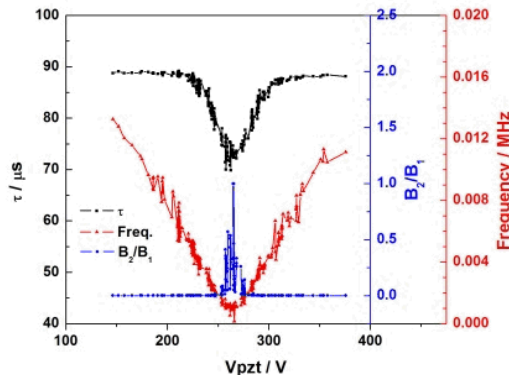
DMO: dimethyloxirane

OH measurement determines both decomposition and 2nd O₂ addition rate coefficients of QOOH.

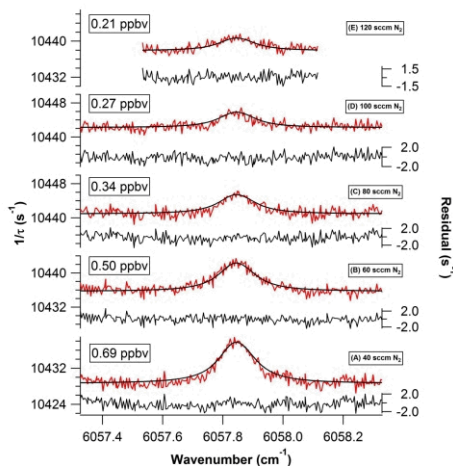


Multiplexed Photo-Ionization Mass Spectrometry (MPIMS) at ALS

Conclusions #1



High order transverse mode excitation can cause mode beating in ring-down signal, which deteriorates the sensitivity of CRDS severely.

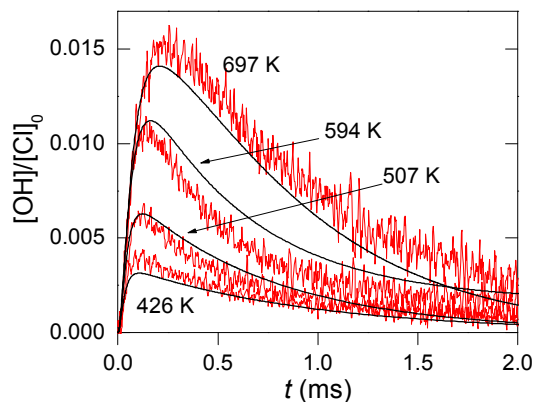


We have reached a detection limit of CH₄ of 63 pptv in N₂ at 760 Torr. Our short term sensitivity is $7.9 \times 10^{-12} \text{ cm}^{-1} \text{ Hz}^{-1/2}$, 24 times worse than the optimum shot noise limit under similar conditions, $3.3 \times 10^{-13} \text{ cm}^{-1} \text{ Hz}^{-1/2}$.

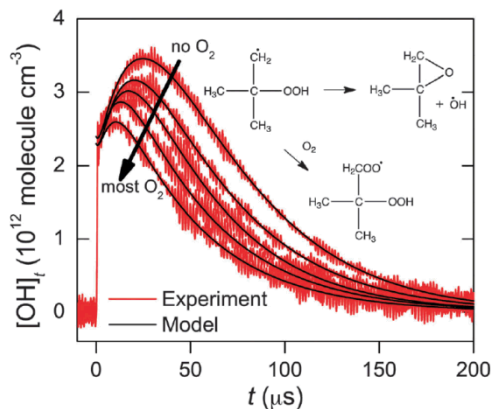
$$\alpha_{\min} = 2.31 \cdot \frac{1-R}{L} \sqrt{\frac{h\nu}{QP_0}}$$

Both CRDS method and NICE-OHMS are ultimately shot noise limited, with similar final sensitivities. But CRDS is a much simpler method.

Conclusions #2



OH and ROO detection at 308nm has been applied to study the chemical kinetics of ethyl / propyl + O₂ systems. More sensitive measurements, e.g. in the IR region are needed to reduce probe interference.



A new way to produce enough QOOH from alkyl hydroperoxide + Cl reaction

Successfully measured the decomposition and 2nd O₂ addition rates of one QOOH, supported by high level quantum chemistry calculations.

Next: direct observation of chain branching at higher temperatures

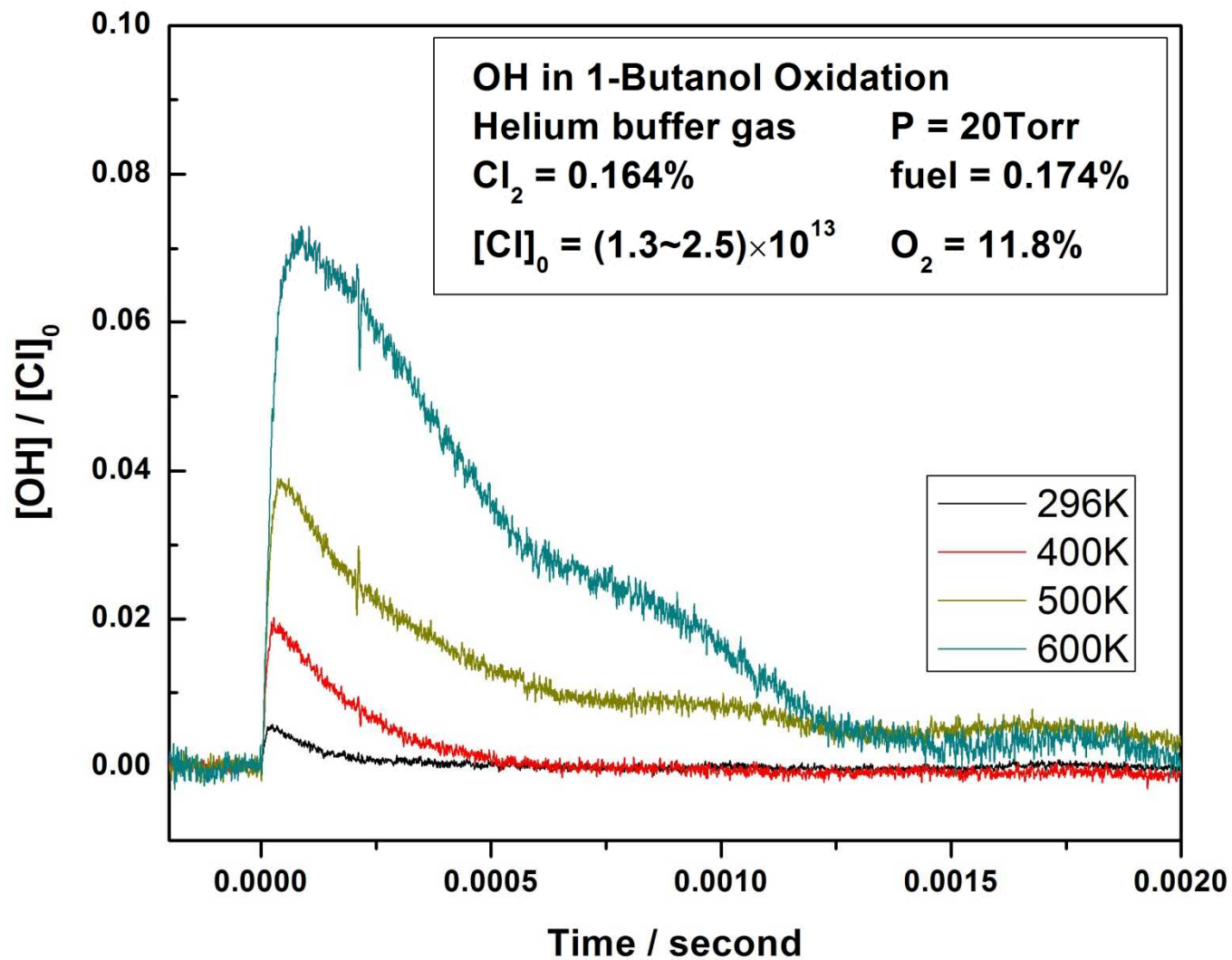


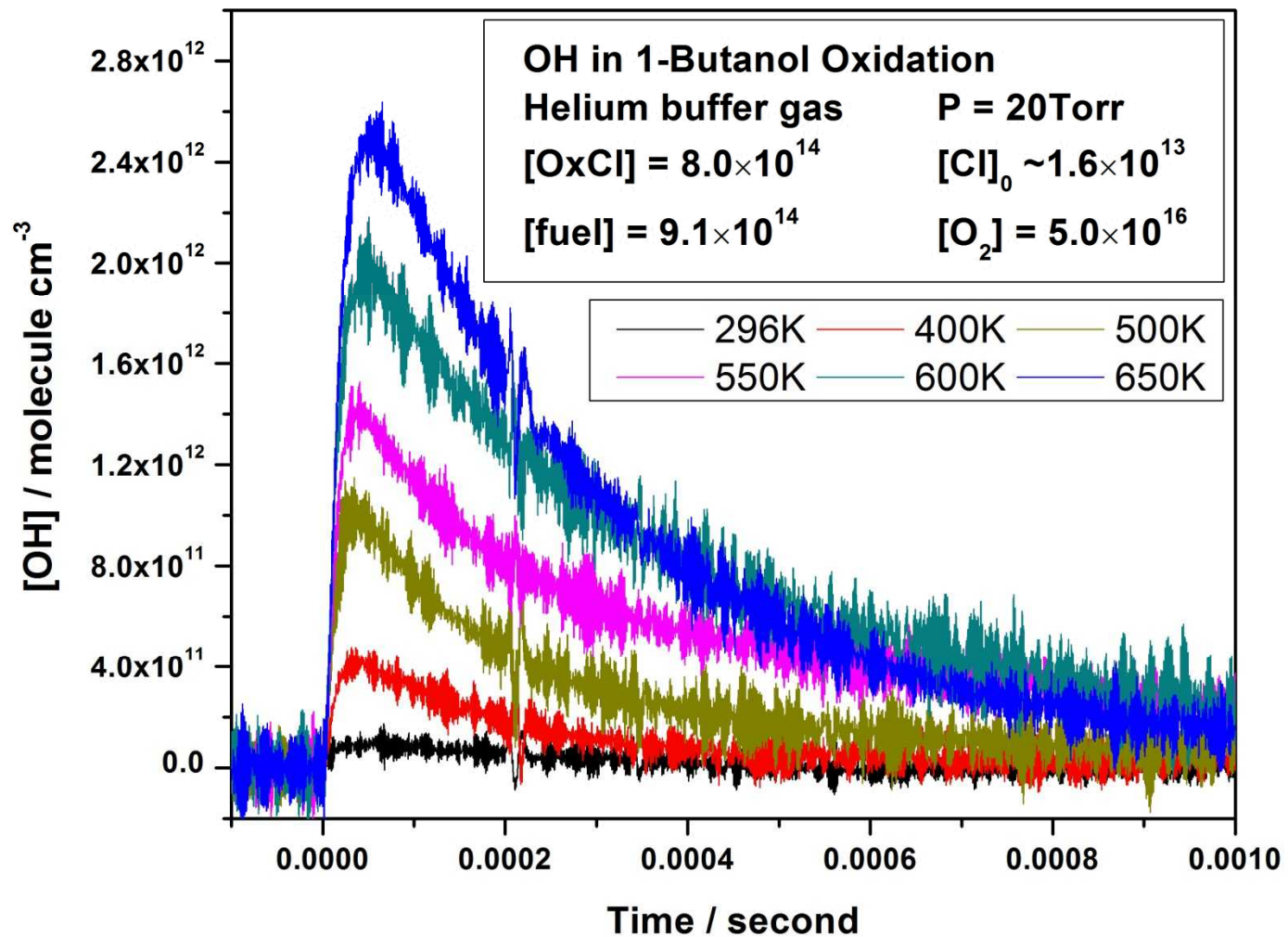
Thank you!

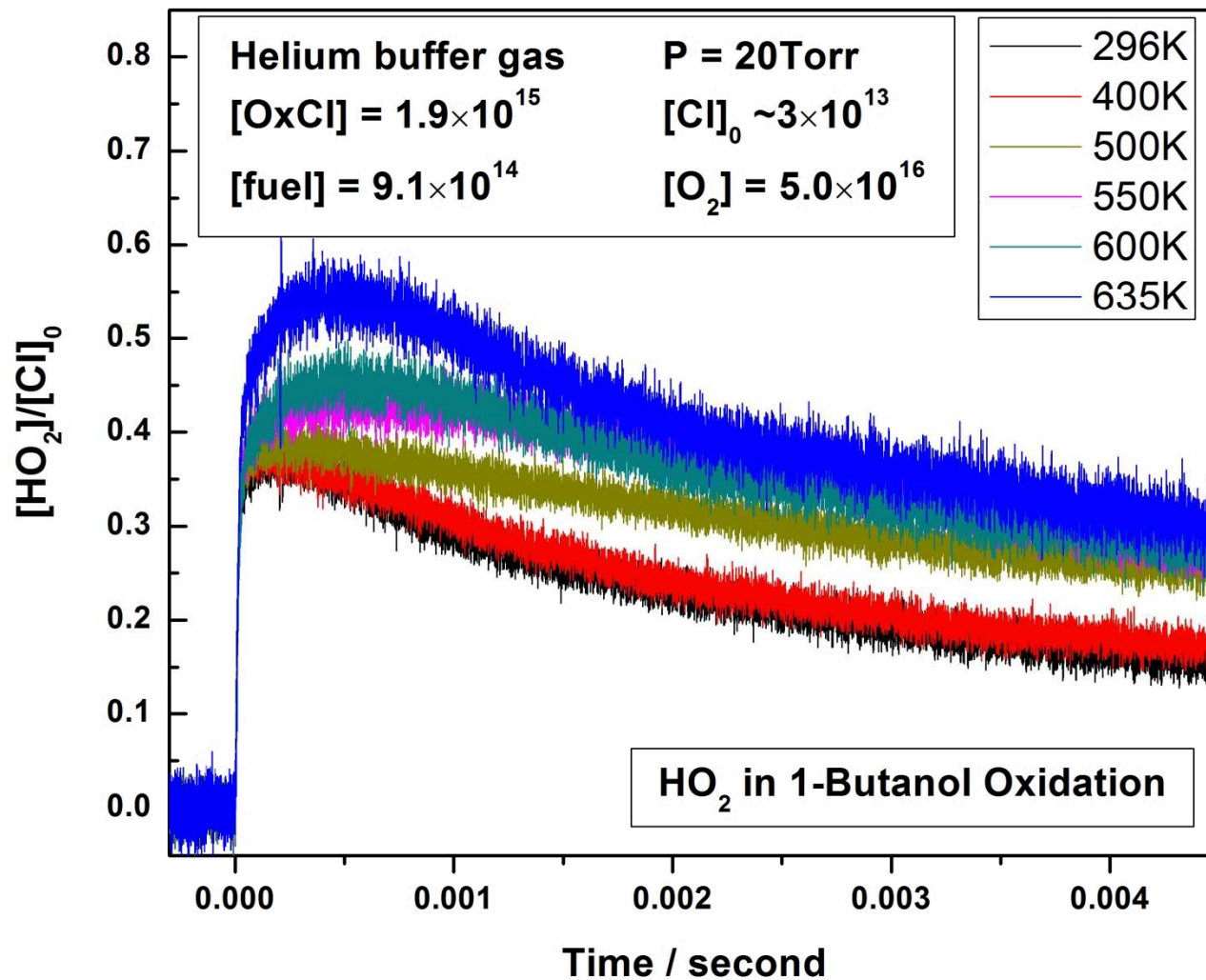


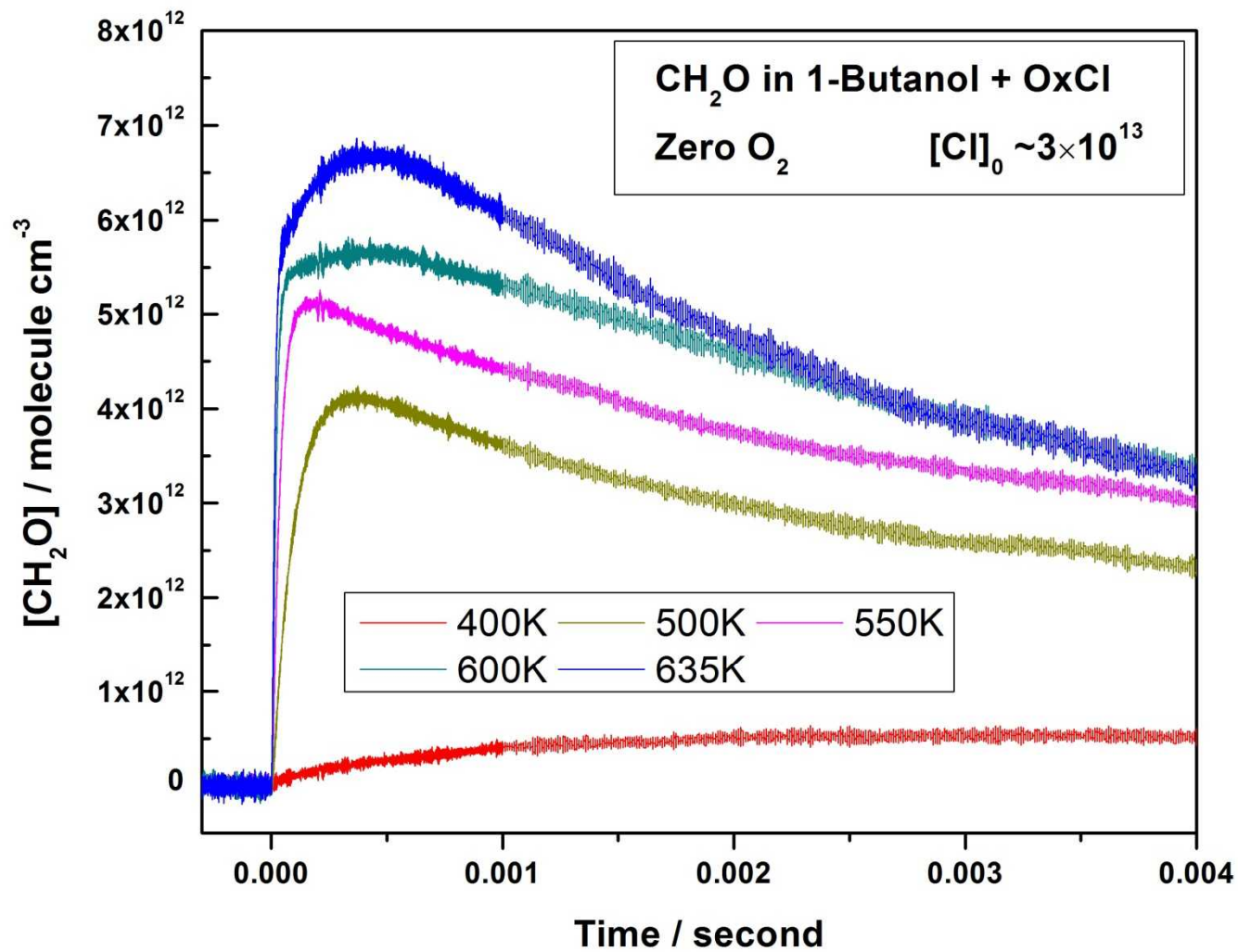
For Cl_2 , in **1 ms**, the background **[Cl]** by the probe laser is estimated to be less than **10%** of **[Cl]** by the photolysis pulse. For **(COCl)₂**, the same number is **5%**.

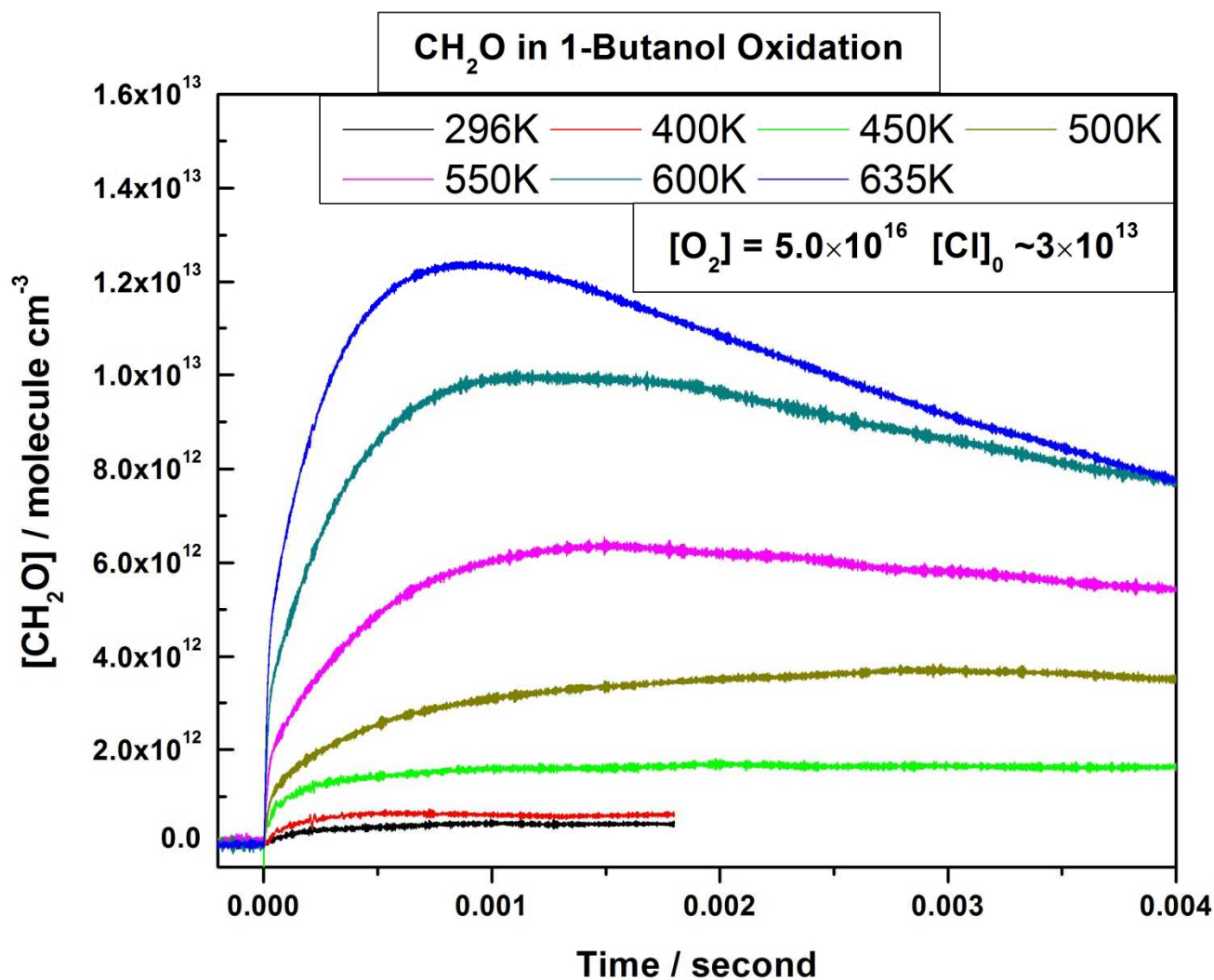
For ethylperoxy (**C₂H₅OO**), the dissociated **[RO₂]** by the 308nm probe laser in **1 ms** is less than **10%** of peak **[RO₂] ~ [Cl]₀**.

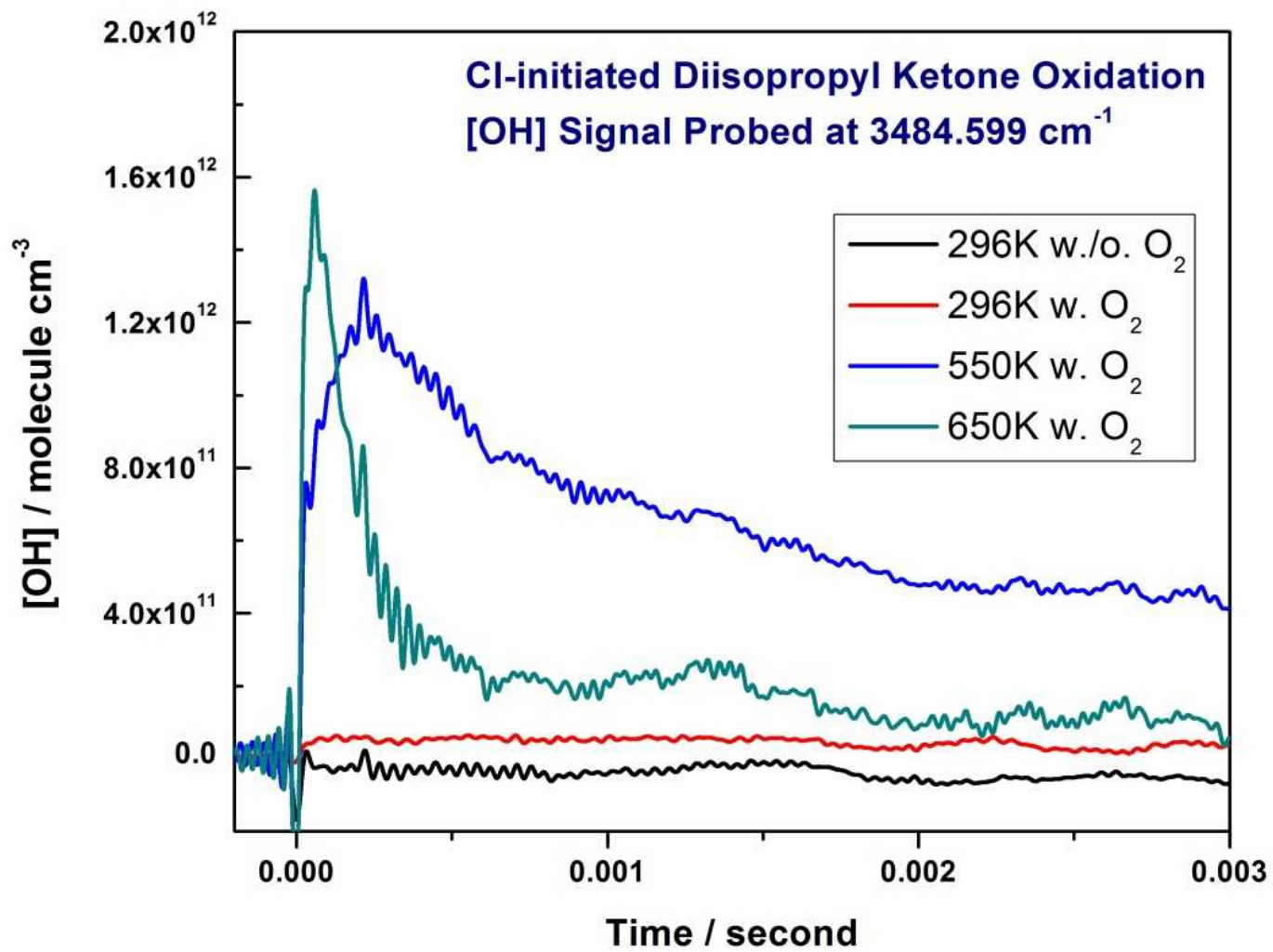




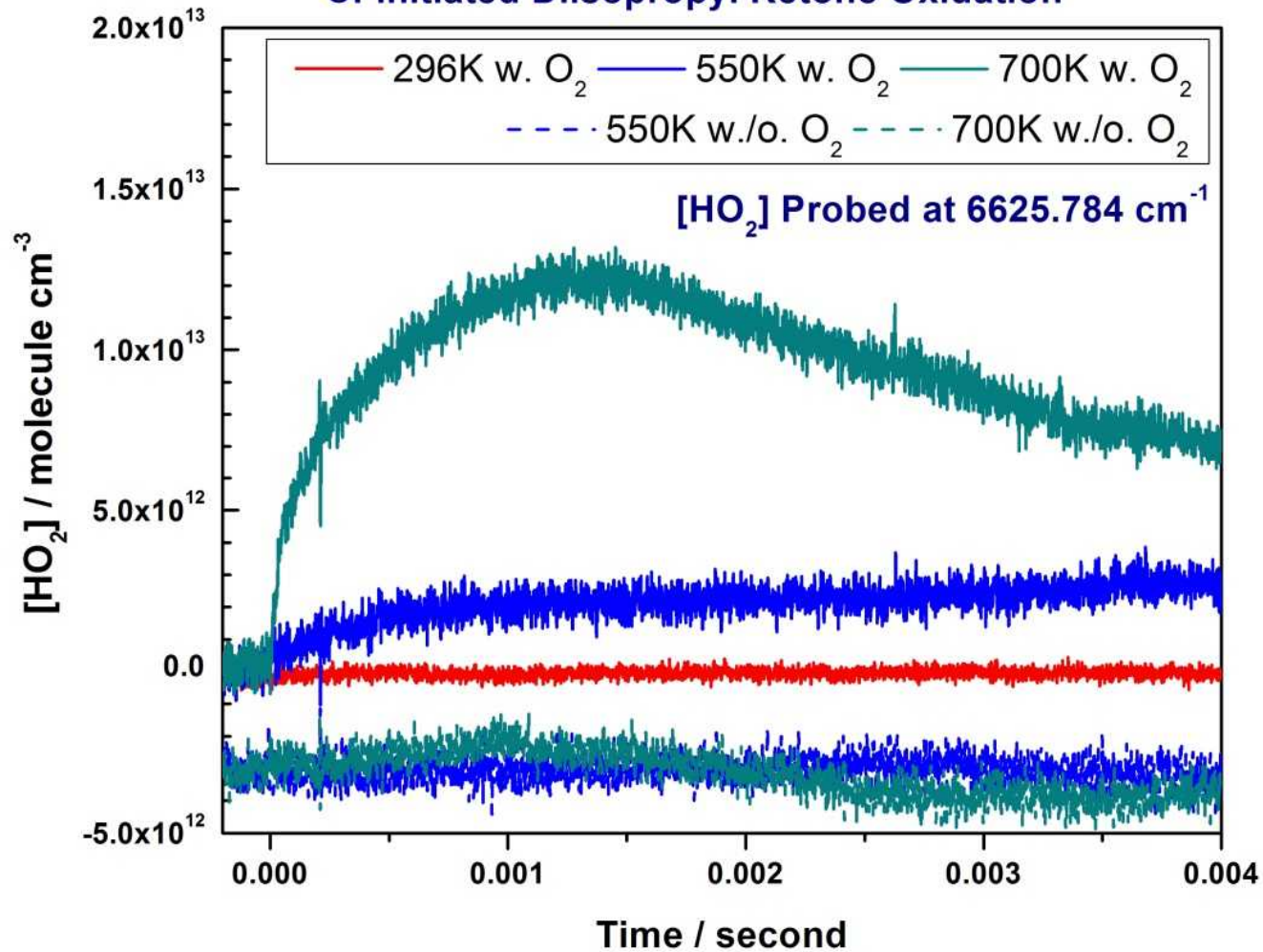


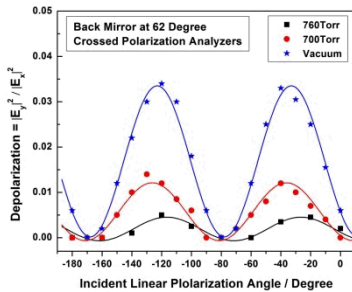






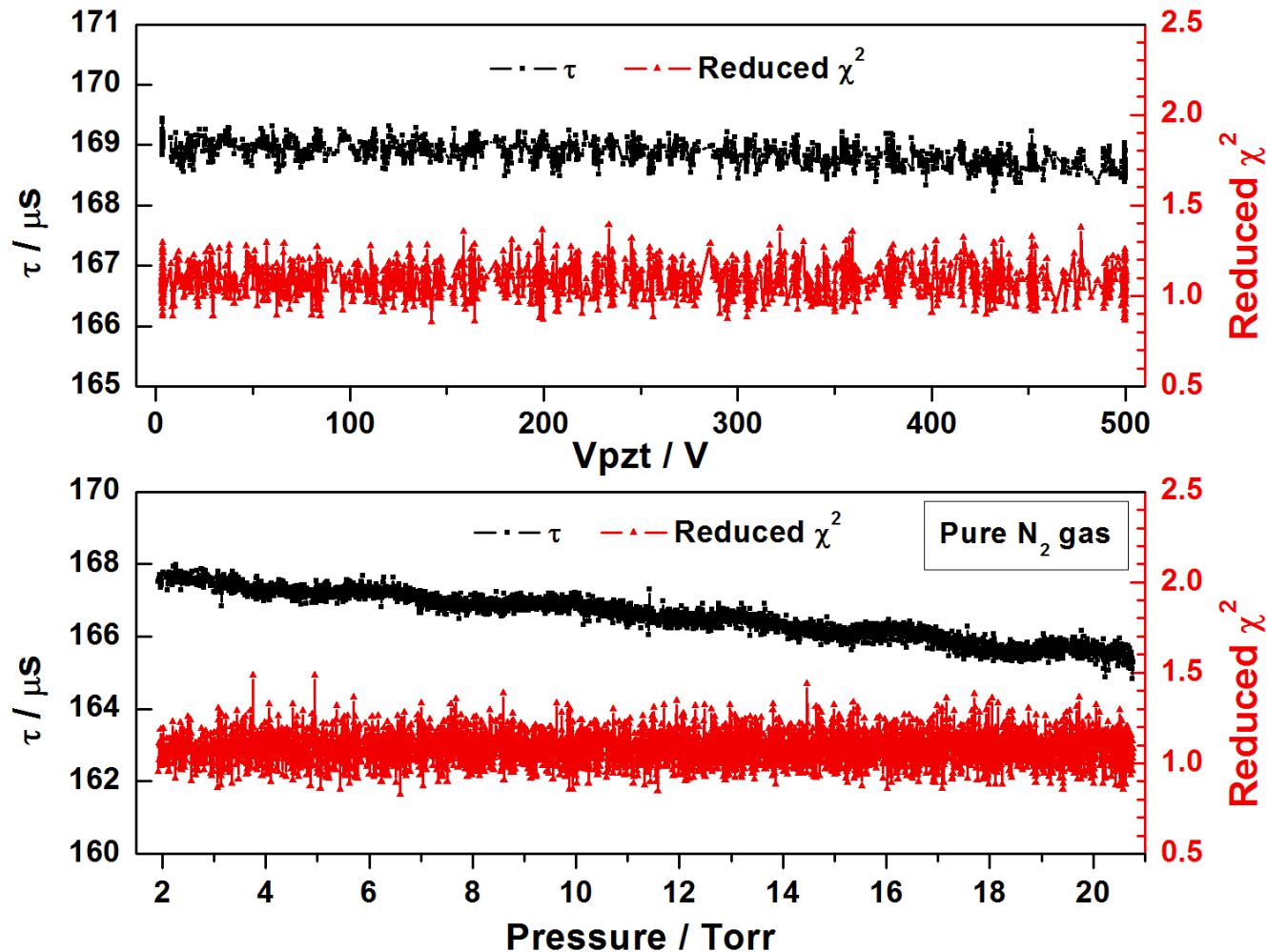
Cl-initiated Diisopropyl Ketone Oxidation





Supermirrors have both residual and strain induced linear birefringence, and polarization dependent loss (linear dichroism). Mismatched polarization analysis in detection can generate mode beating in ring-down signal, killing the sensitivity.

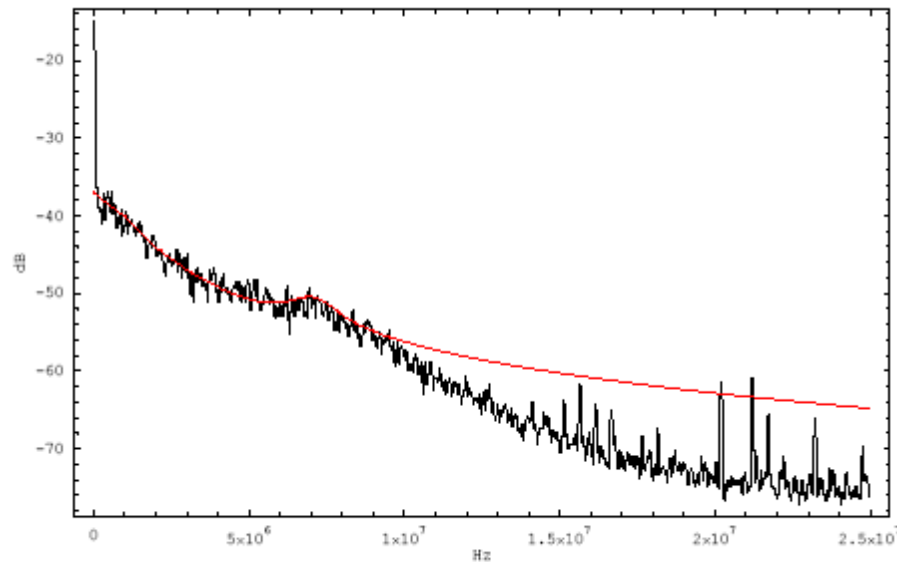
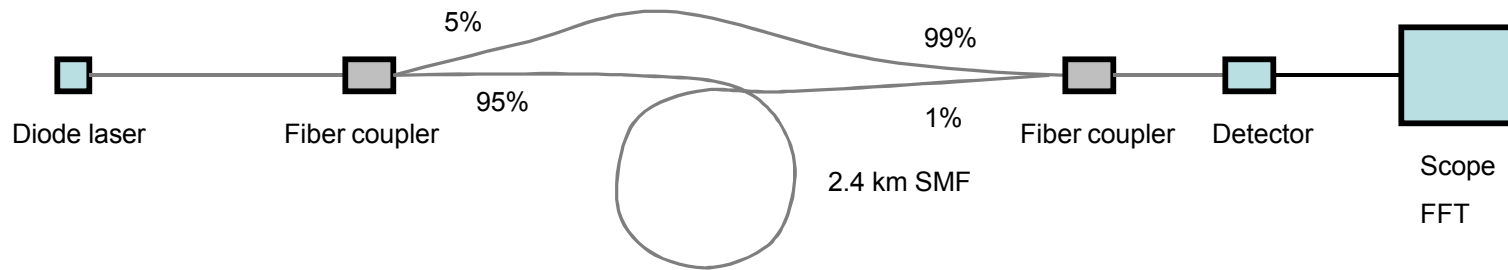
High order transverse mode excitation can be easily suppressed by a small size intracavity aperture.



TEM₀₀ mode size is **0.526mm** at the intracavity aperture of **4mm** diameter.

Linewidth of DFB Laser

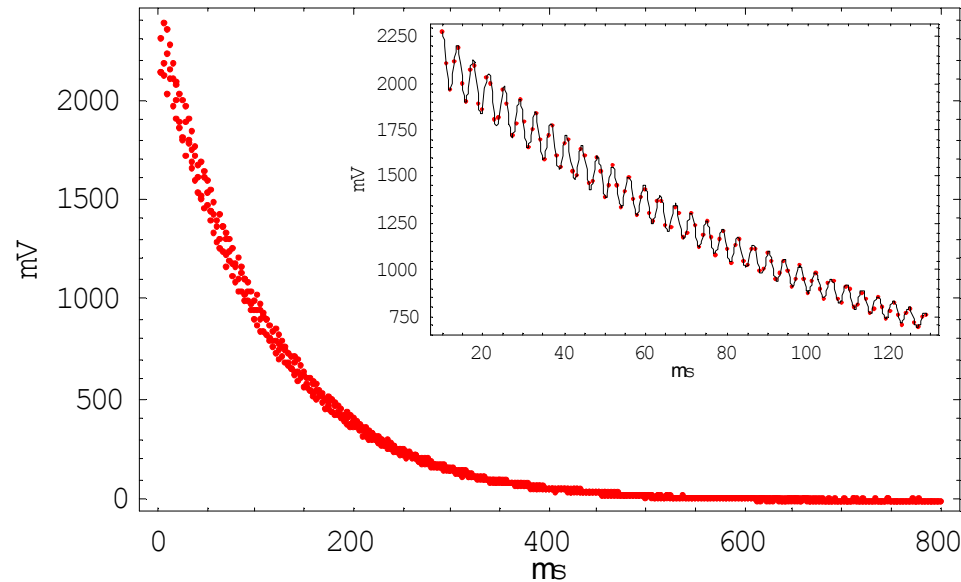
Self homodyne measurement: delay time $11.4\mu\text{s}$



$$\Delta\nu_L \approx 1\text{ MHz}$$

Mode Splitting of TEM_{10/01}

$$y(t) = A + B_1 \exp(-k_1 t) + B_2 \exp(-k_2 t) + B_3 \exp(-\frac{k_1 t}{2}) \exp(-\frac{k_2 t}{2}) \cos(2\pi \cdot \Delta\nu \cdot t + \Delta\varphi)$$



$$A = -2.60 \text{ mV}$$

$$B_1 = 1518.66 \text{ mV}$$

$$B_2 = 830.02 \text{ mV}$$

$$B_3 = 152.22 \text{ mV}$$

$$\tau_1 = 1/k_1 = 104.72 \text{ } \mu\text{s}$$

$$\tau_2 = 1/k_2 = 119.05 \text{ } \mu\text{s}$$

$$\Delta\nu = 261.25 \text{ kHz}$$

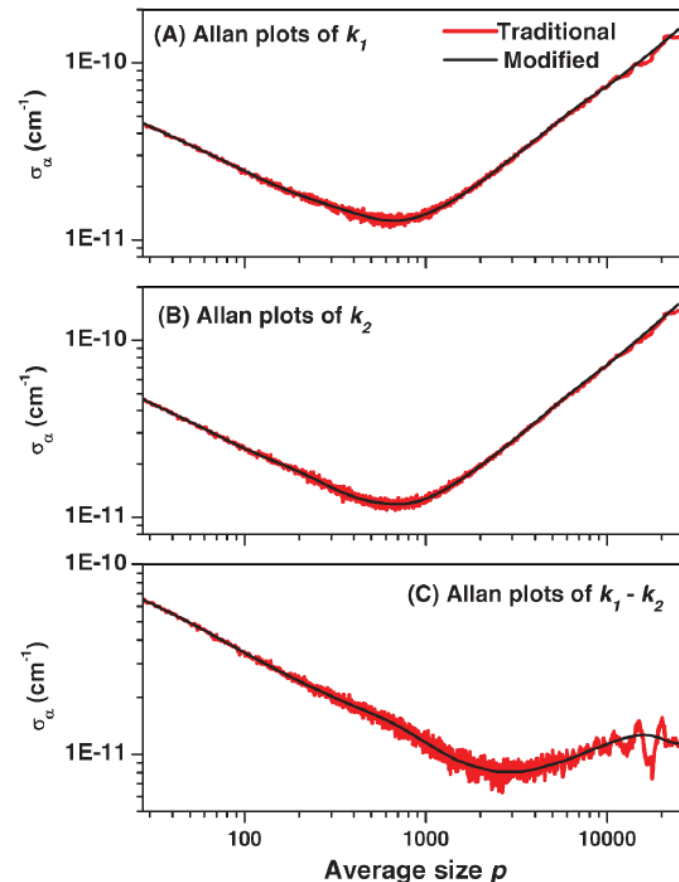
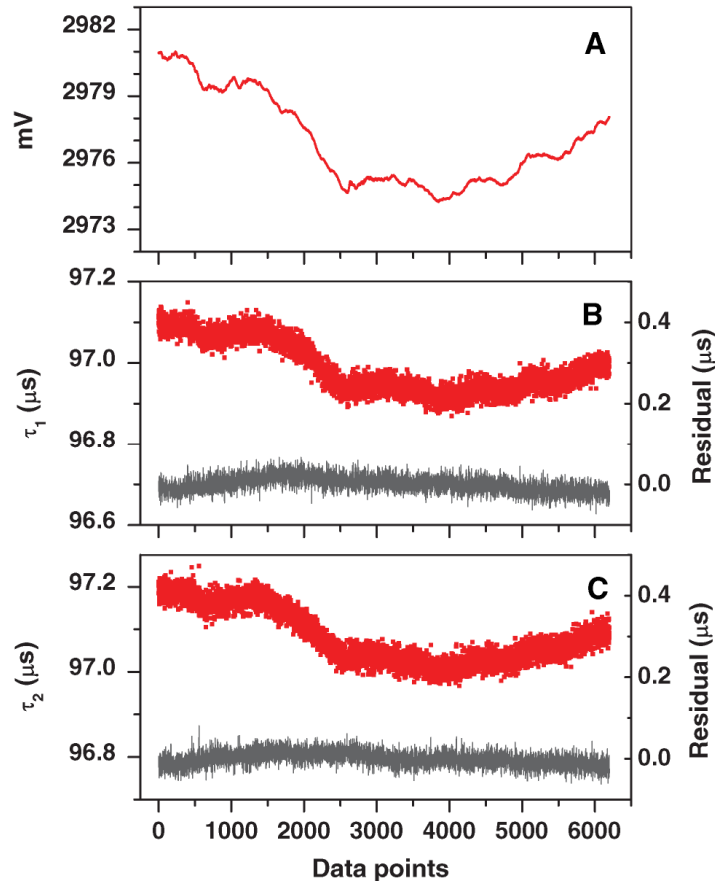
$$\Delta\varphi = 2.46$$

$$\chi^2 = 1.017$$

- The TEM₁₀ and TEM₀₁ modes are no longer orthogonal with each other ($B_3 > 0$). Possible reasons include spatial anisotropy of supermirrors or nonuniform quantum efficiency of the detector.
- A small difference of 0.26% between the radius of curvature of the mirror in x direction and that in y direction will generate this splitting.

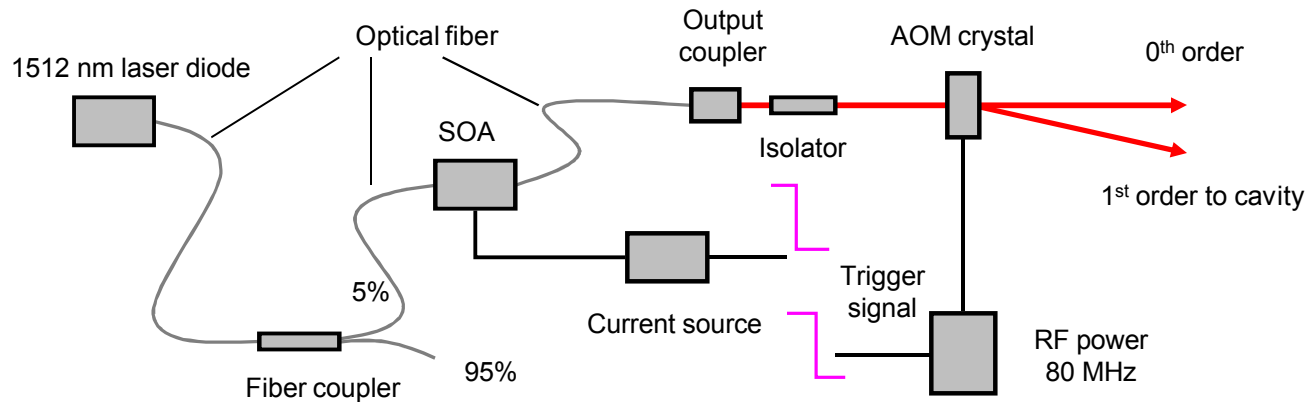
Effect of Ambient Pressure Change

- **A:** lab pressure change; **B:** τ by laser #1; **C:** τ by laser #2
- Differential measurement
- τ is strongly correlated with the change of pressure difference between inside and outside of the ring-down cell.

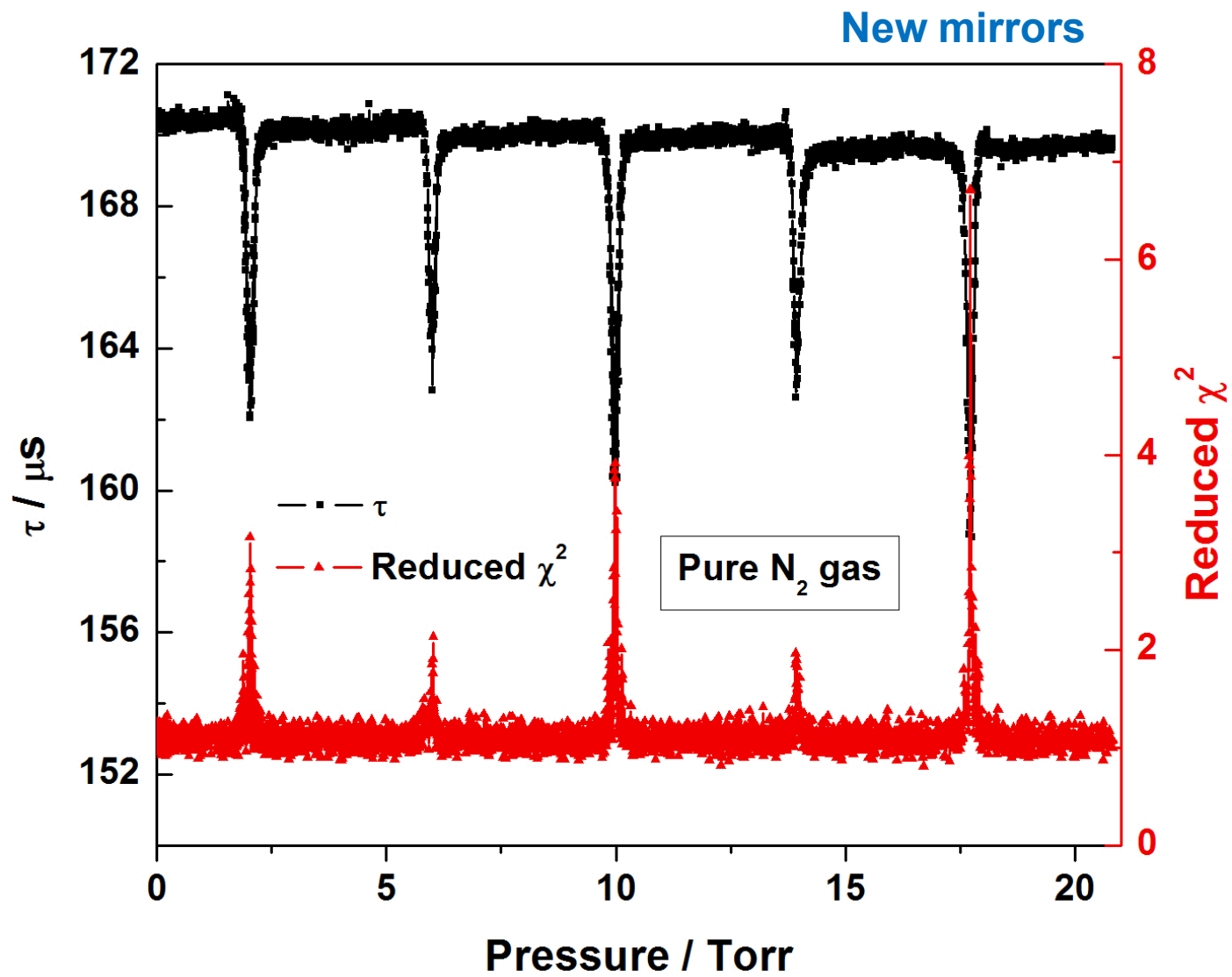


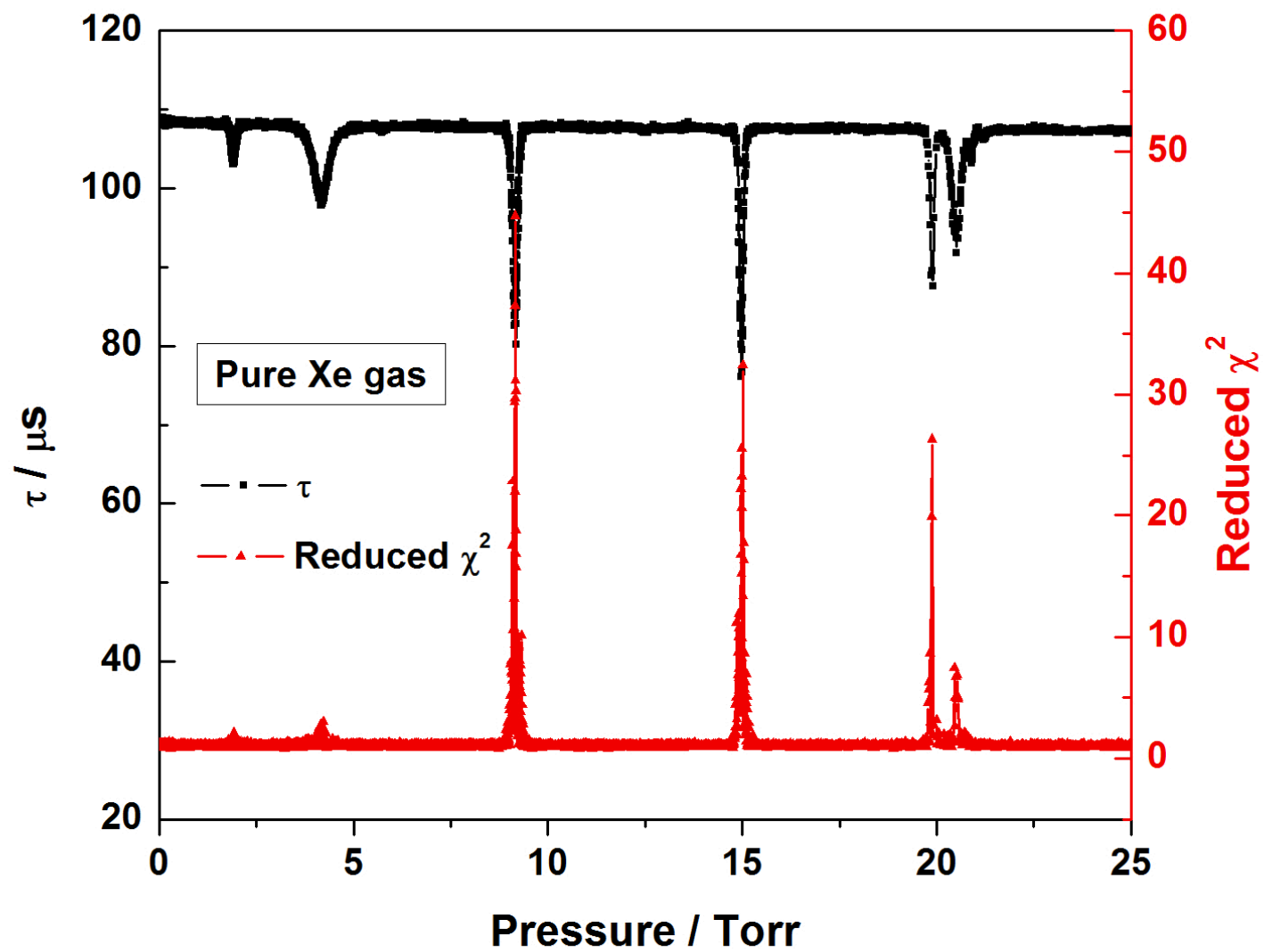
SOA as Light Modulator

Semiconductor Optical Amplifier (SOA):



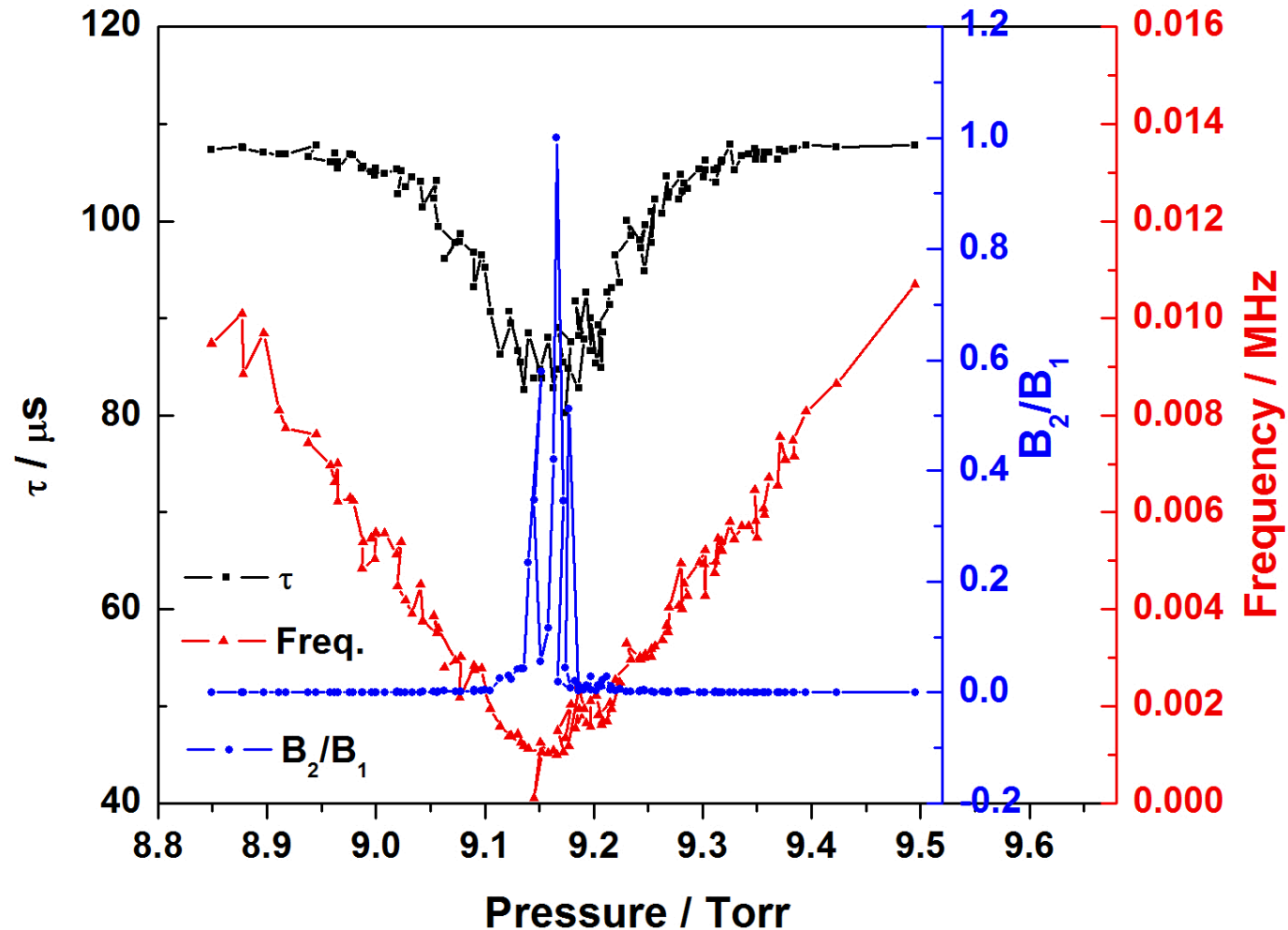
- Highest extinction ratio (> 80 dB) when used as light modulator
- Fast speed: ns or sub ns
- Broadband gain media: ~ 70 nm
- Optical fiber connected, no extra alignment needed when λ tuning





Resonance by tuning!

- Xenon: 81% scan from refractive index increase with pressure
- Cavity length change $\sim 0.1\mu\text{m}/\text{Torr}$, $\Delta\nu$ tuning rate **30.7 kHz/torr**



Optimal Least Squares Fit (LSF)

$$\vec{y} = \{y_i\} \quad i = 0, \dots, N-1 \quad \Delta t$$

$$y(t) = F(t) + \varepsilon(t) = Ae^{-kt} + B + \varepsilon(t) \quad \langle \varepsilon(t) \rangle = 0$$

$$\chi^2(A, B, k) = (\vec{y} - \vec{F})^T \cdot \tilde{W} \cdot (\vec{y} - \vec{F}) \quad \vec{F} = \{F_i\} = \{Ae^{-ki\Delta t} + B\}$$

$$(\vec{y} - \vec{F}(A, B, k))^T \cdot \tilde{W} \cdot \nabla F = \vec{0}^T \quad \nabla F_{ij} = \left(\frac{\partial F_i}{\partial p_j} \right) \quad \vec{p} = \{A, B, k\}^T$$

$$\tilde{\alpha} = \nabla F^T \cdot \tilde{W} \cdot \nabla F \quad \tilde{c} = \tilde{\alpha}^{-1}$$

$$\vec{\beta} = \nabla F^T \cdot \tilde{W} \cdot (\vec{y} - \vec{F}) \quad \delta \vec{p} = \tilde{c} \cdot \vec{\beta}$$

The diagonal elements of **covariance matrix c** are the variances of the fit parameters **A**, **B**, and **k**.

If exponential decay is sampled evenly in time, the **curvature matrix α** and the **vector β** can be written down in closed form expressions for either **detector noise limited (DNL)** or **shot noise limited (SNL)** signal.

α and β of DNL Case

$$\begin{aligned}
 \alpha_{B,B} &= \frac{N}{\sigma^2} \\
 \alpha_{B,A} &= \left(\frac{1 - a^N}{1 - a} \right) \frac{1}{\sigma^2} \\
 \alpha_{B,k} &= -A\Delta t \left(\frac{a(1 - a^N)}{(1 - a)^2} - \frac{Na^N}{1 - a} \right) \frac{1}{\sigma^2} \\
 \alpha_{A,A} &= \left(\frac{1 - a^{2N}}{1 - a^2} \right) \frac{1}{\sigma^2} \\
 \alpha_{A,k} &= -A\Delta t \left(\frac{a^2(1 - a^{2N})}{(1 - a^2)^2} - \frac{Na^{2N}}{1 - a^2} \right) \frac{1}{\sigma^2} \\
 \alpha_{k,k} &= (A\Delta t)^2 \left(\frac{2a^4(1 - a^{2N})}{(1 - a^2)^3} + \frac{a^2 - (2N + 1)a^{2N+2}}{(1 - a^2)^2} - \frac{N^2 a^{2N}}{1 - a^2} \right) \frac{1}{\sigma^2}
 \end{aligned} \tag{23}$$

$$a = \exp(-k\Delta t)$$

Detector noise σ

$$\begin{aligned}
 \beta_B &= \left(\sum_i y_i - A \left(\frac{1 - a^N}{1 - a} \right) - BN \right) \frac{1}{\sigma^2} \\
 \beta_A &= \left(\sum_i y_i a^i - A \left(\frac{1 - a^{2N}}{1 - a^2} \right) - B \left(\frac{1 - a^N}{1 - a} \right) \right) \frac{1}{\sigma^2} \\
 \beta_k &= -A\Delta t \left(\sum_i y_i i a^i - A \left(\frac{a^2(1 - a^{2N})}{(1 - a^2)^2} - \frac{Na^{2N}}{1 - a^2} \right) - B \left(\frac{a(1 - a^N)}{(1 - a)^2} - \frac{Na^N}{1 - a} \right) \right) \frac{1}{\sigma^2}.
 \end{aligned} \tag{24}$$

Each cycle of iteration:

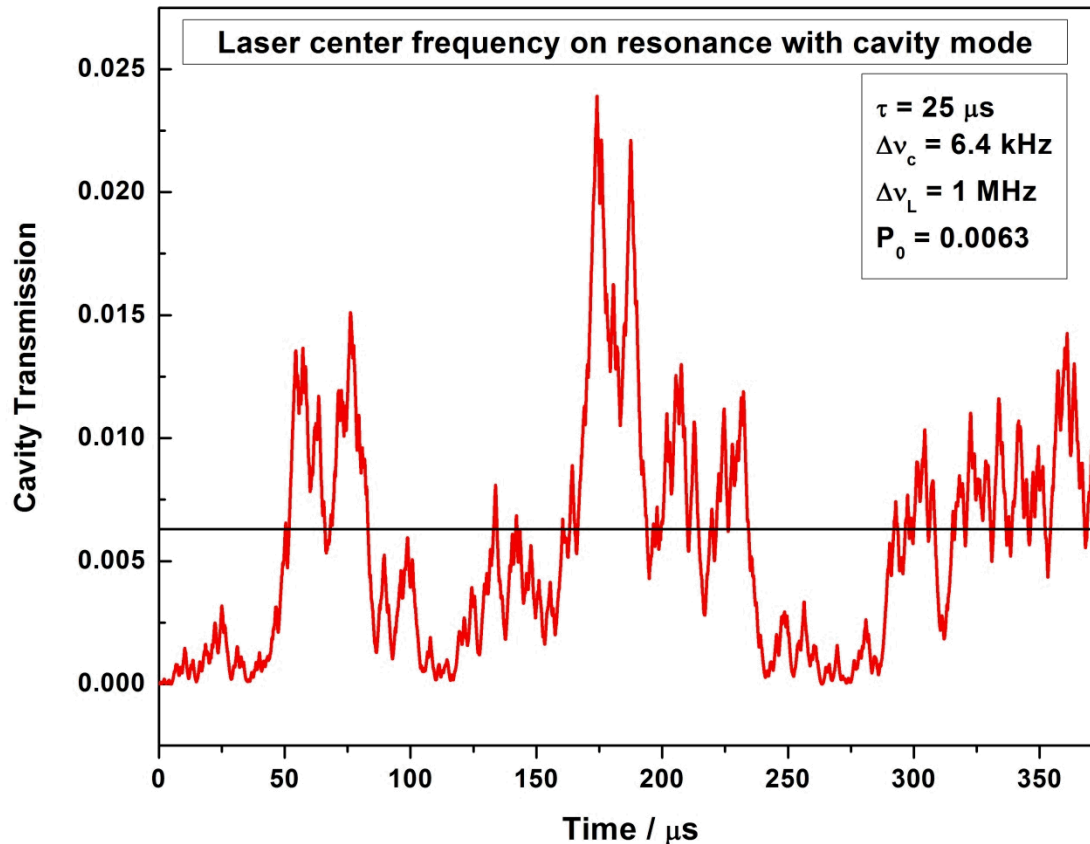
3N multiplications

3N additions

Substantially more efficient than a general purpose fitting package

Cavity Excited by a Noisy Laser

Lorentzian lineshape of a CW diode laser can be understood by a model of **random walk of its field phase**, with $D = \pi \Delta \nu_L$ the phase diffusion constant.

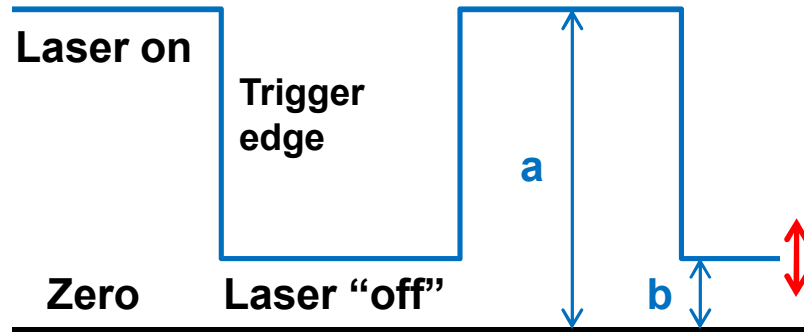


$$P_0 = \left(\frac{T}{1-R} \right)^2 \frac{\Delta \nu_c}{\Delta \nu_c + \Delta \nu_L} P_{\text{in}}$$

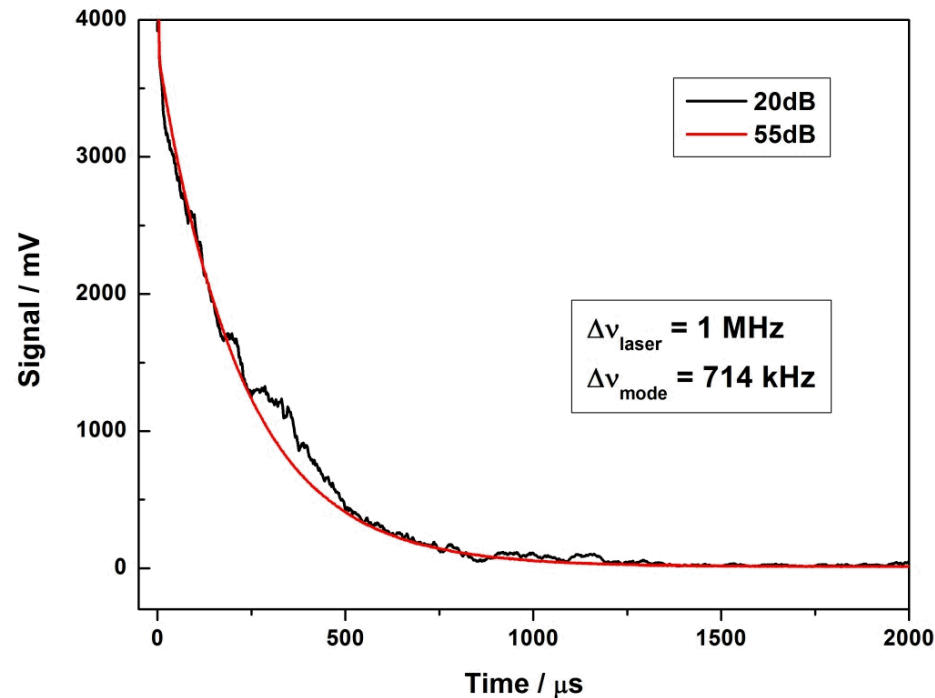
$$\Delta \nu_c = 1 / 2\pi\tau$$

$$P_0 \approx (10^{-4} \sim 10^{-3}) P_{\text{in}}$$

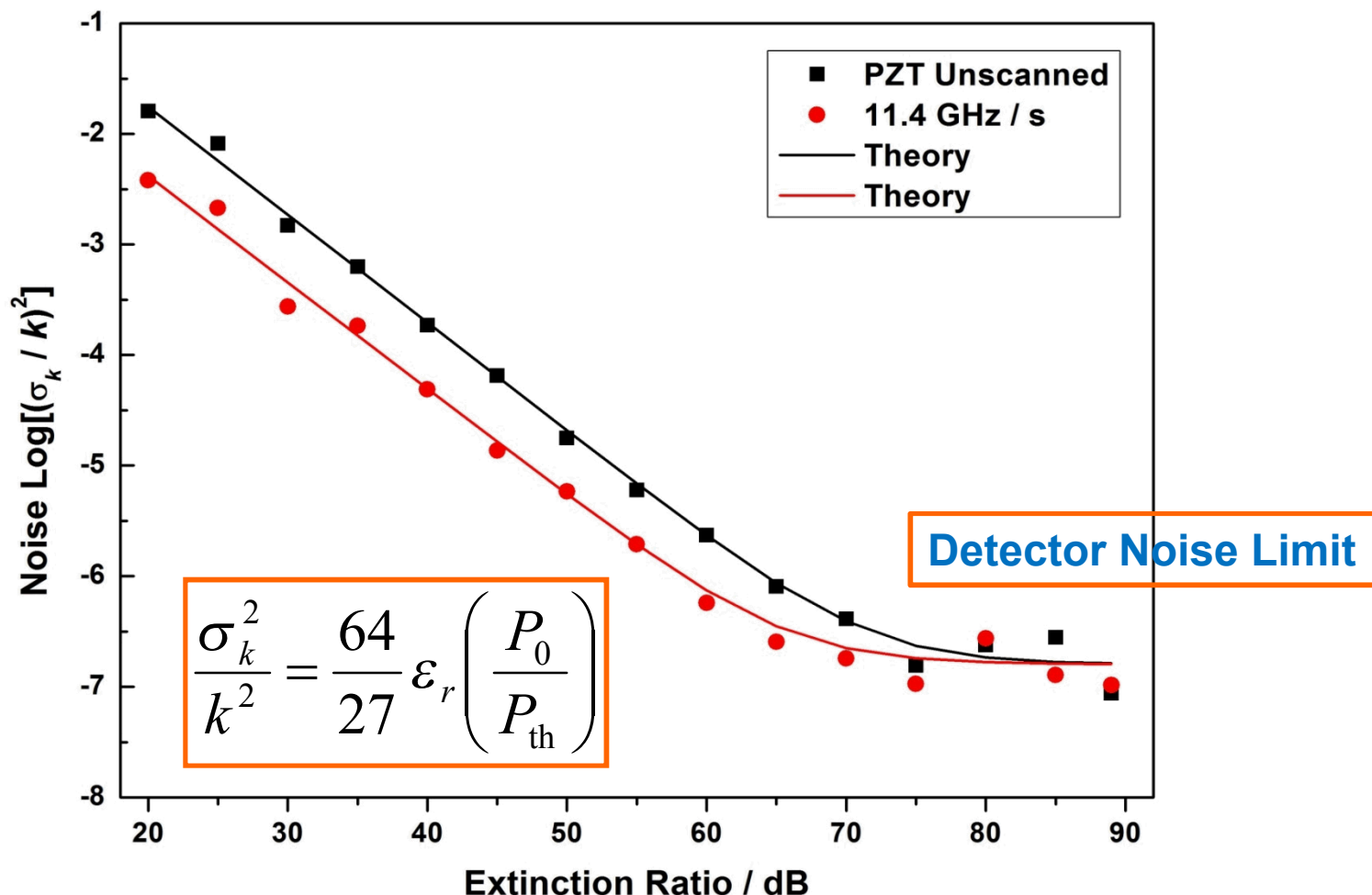
CW-CRDS: Finite Extinction



Extinction Ratio: $\epsilon_r = b / a$
or in dB = $10 \log(a / b)$



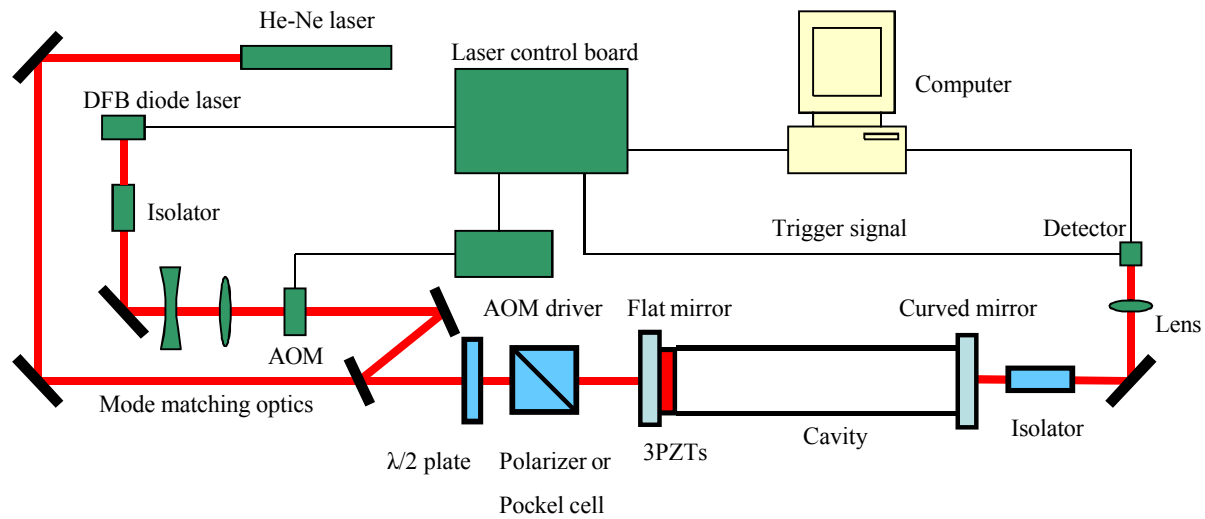
Noise linearly depends on the extinction ratio of light switch.



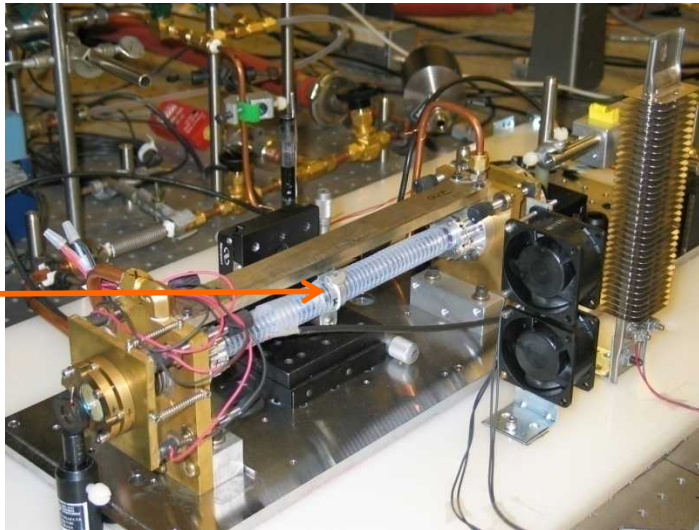
Semiconductor Optical Amplifier (SOA) is a perfect light switch for CW-CRDS.

Highest extinction ratio: > 80 dB, very fast speed: ~ ns, and broadband gain: ~ 70 nm

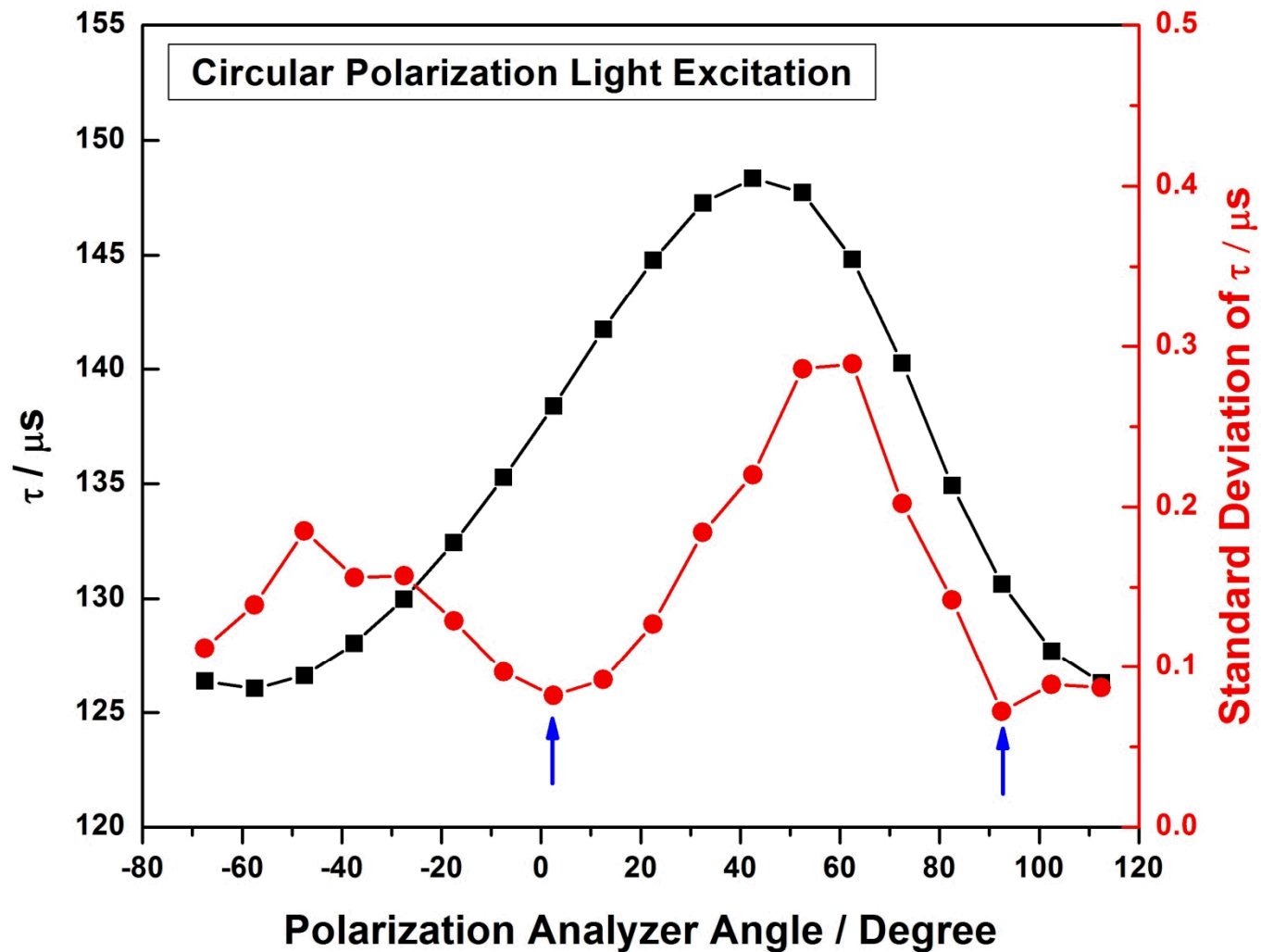
Setup w. Polarization Control



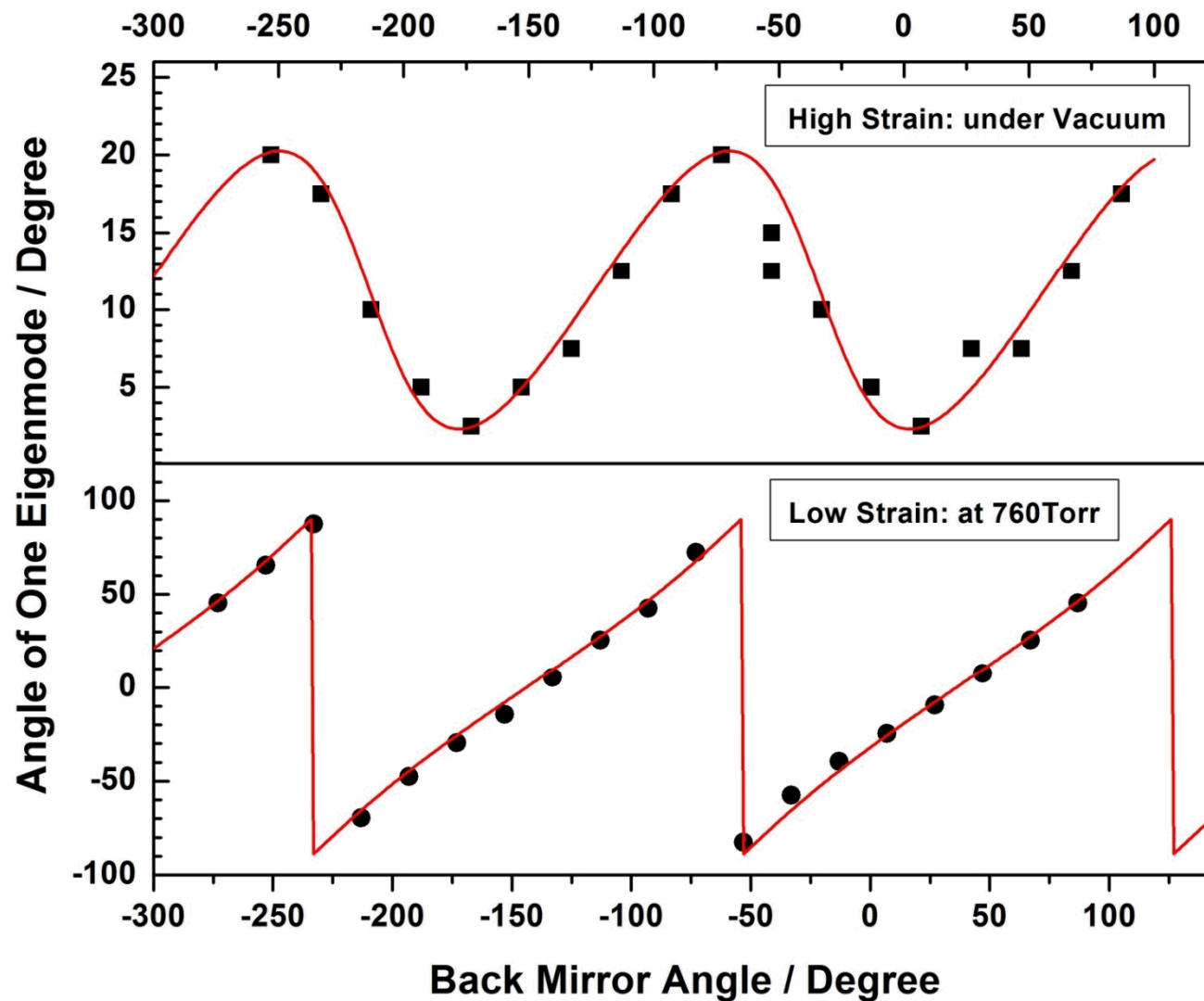
Intracavity
Aperture



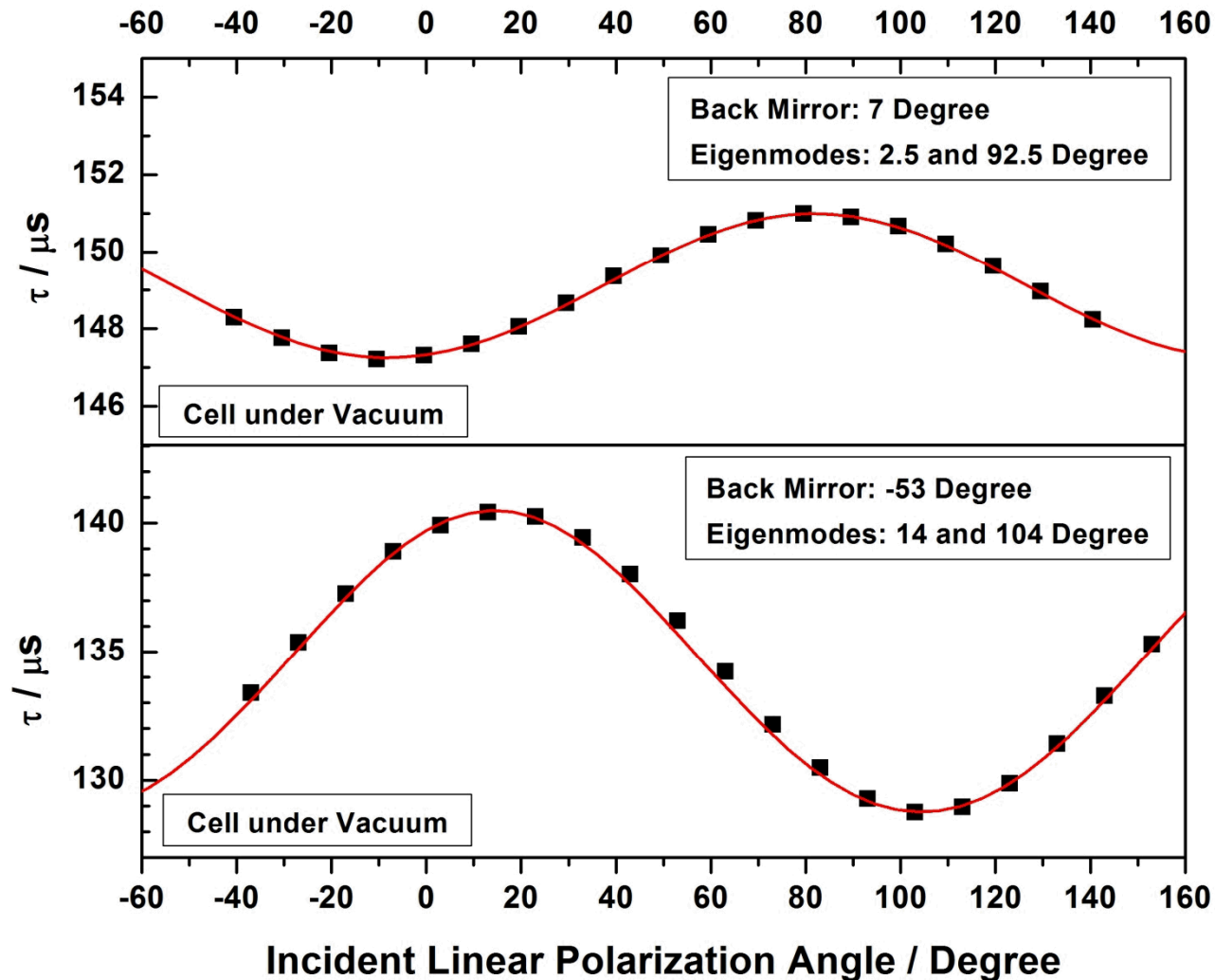
The two analyzer angles of lowest noise level of τ correspond to **two polarization eigenmodes** which are perpendicular with each other.

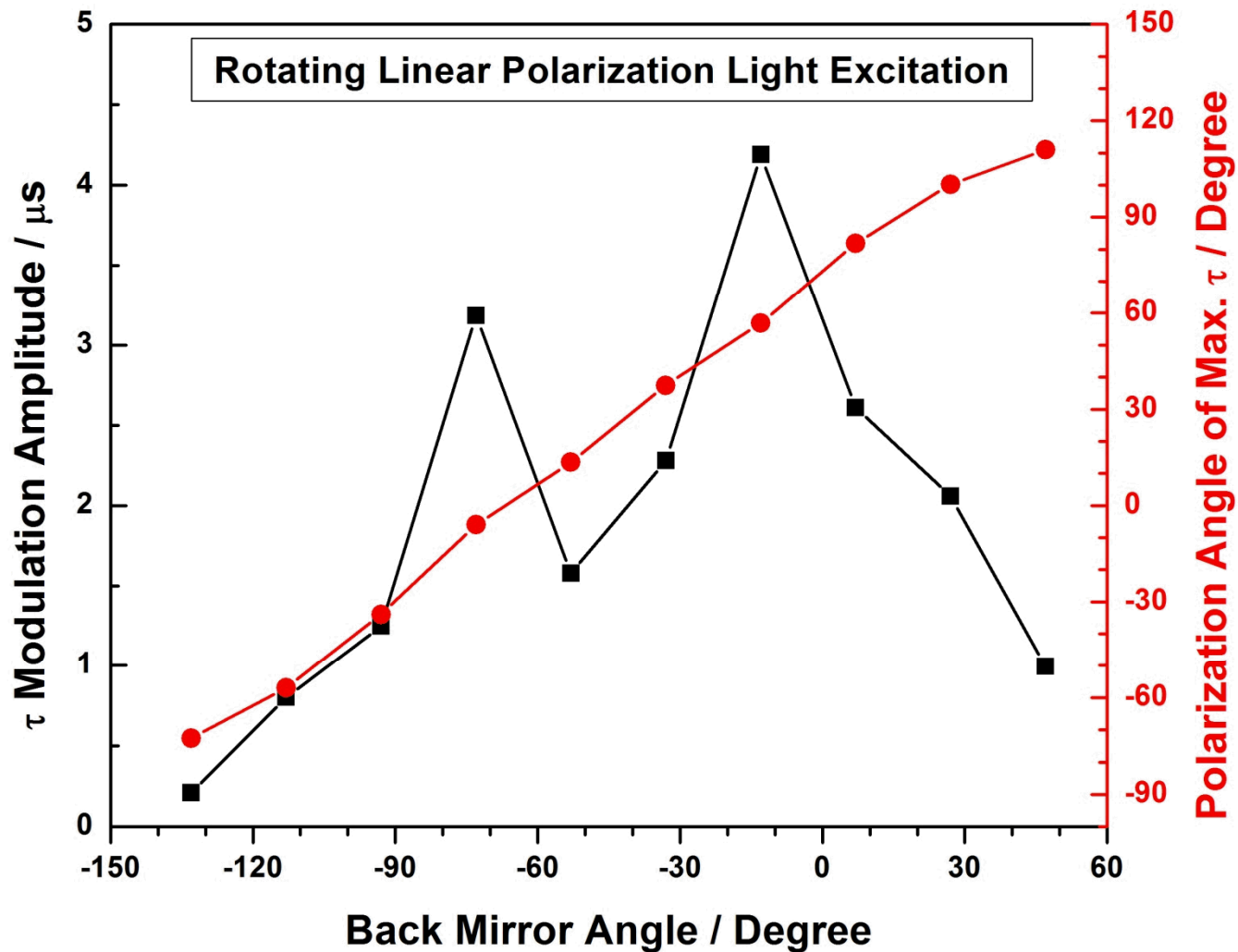


The two polarization eigenmodes rotate with back mirror orientation.



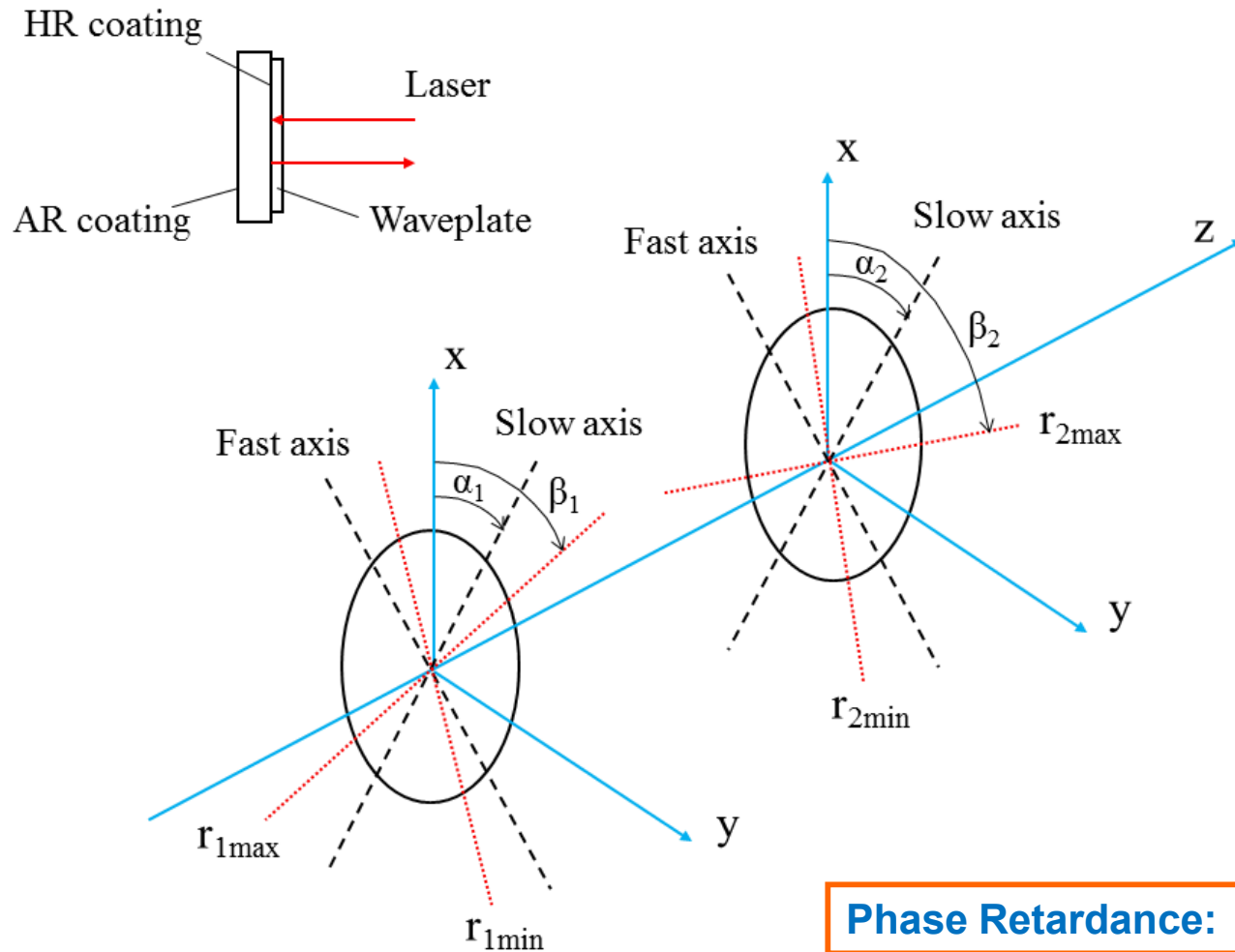
The reflectivity of cavity mirrors depends on the incident polarization angle (linear dichroism).





- τ strongly depends on local conditions (e.g. defects) of mirror surface.
- The incident laser polarization angle of **maximum τ** changes smoothly with **back mirror orientation angle**.

HR Coating as a Waveplate



The Model

Reflection
Jones matrix:

$$G_i = a_i R(-\beta_i) \cdot \begin{pmatrix} 1+b_i & 0 \\ 0 & 1-b_i \end{pmatrix} \cdot R(\beta_i), \quad \text{with } a_i = \frac{r_{i\max} + r_{i\min}}{2}, \quad b_i = \frac{r_{i\max} - r_{i\min}}{r_{i\max} + r_{i\min}}$$

Waveplate
transmission
Jones matrix:

$$F_i = R(-\alpha_i) \cdot \begin{pmatrix} \exp(j\varepsilon_i/2) & 0 \\ 0 & \exp(-j\varepsilon_i/2) \end{pmatrix} \cdot R(\alpha_i)$$

$$R(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

Round trip Jones
matrix with linear
approximation:

$$M_i = F_i \cdot G_i \cdot F_i \quad M = M_1 \cdot M_2$$

$$M = a_1 a_2 \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} b \cos(2\beta) + j\varepsilon \cos(2\alpha) & b \sin(2\beta) + j\varepsilon \sin(2\alpha) \\ b \sin(2\beta) + j\varepsilon \sin(2\alpha) & -b \cos(2\beta) - j\varepsilon \cos(2\alpha) \end{pmatrix} \right]$$

Round trip values
for linear dichroism
and birefringence:

$$b^2 = b_1^2 + b_2^2 + 2b_1 b_2 \cos(2(\beta_1 - \beta_2))$$

$$\beta = \frac{\beta_1 + \beta_2}{2} + \frac{1}{2} \tan^{-1} \left(\frac{b_1 - b_2}{b_1 + b_2} \tan(\beta_1 - \beta_2) \right)$$

$$\varepsilon^2 = \varepsilon_1^2 + \varepsilon_2^2 + 2\varepsilon_1 \varepsilon_2 \cos(2(\alpha_1 - \alpha_2))$$

$$\alpha = \frac{\alpha_1 + \alpha_2}{2} + \frac{1}{2} \tan^{-1} \left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \tan(\alpha_1 - \alpha_2) \right)$$

Round trip Jones
matrix ***M*** no longer
Hermitian

Two **almost linear**
polarization eigenmodes
but **no longer orthogonal**

$$\lambda_{1,2} = a_1 a_2 \left(1 \pm [b^2 - \varepsilon^2 + 2jb\varepsilon \cos(2(\beta - \alpha))]^{1/2} \right)$$

Frequency splitting
between two polarization
states of TEM₀₀ mode

$$\delta\nu = \frac{\arg(\lambda_1) - \arg(\lambda_2)}{2\pi} FSR \approx \frac{FSR}{\pi} \operatorname{Im}([b^2 - \varepsilon^2 + 2jb\varepsilon \cos(2(\beta - \alpha))]^{1/2})$$

$$\tau \approx 130\mu s, \tau_{\max} - \tau_{\min} = 5\mu s, \varepsilon_i \approx 10^{-6}, b_i \approx 10^{-8}, \delta\nu \approx 0.1kHz$$

Two eigenmodes have
different time constants.

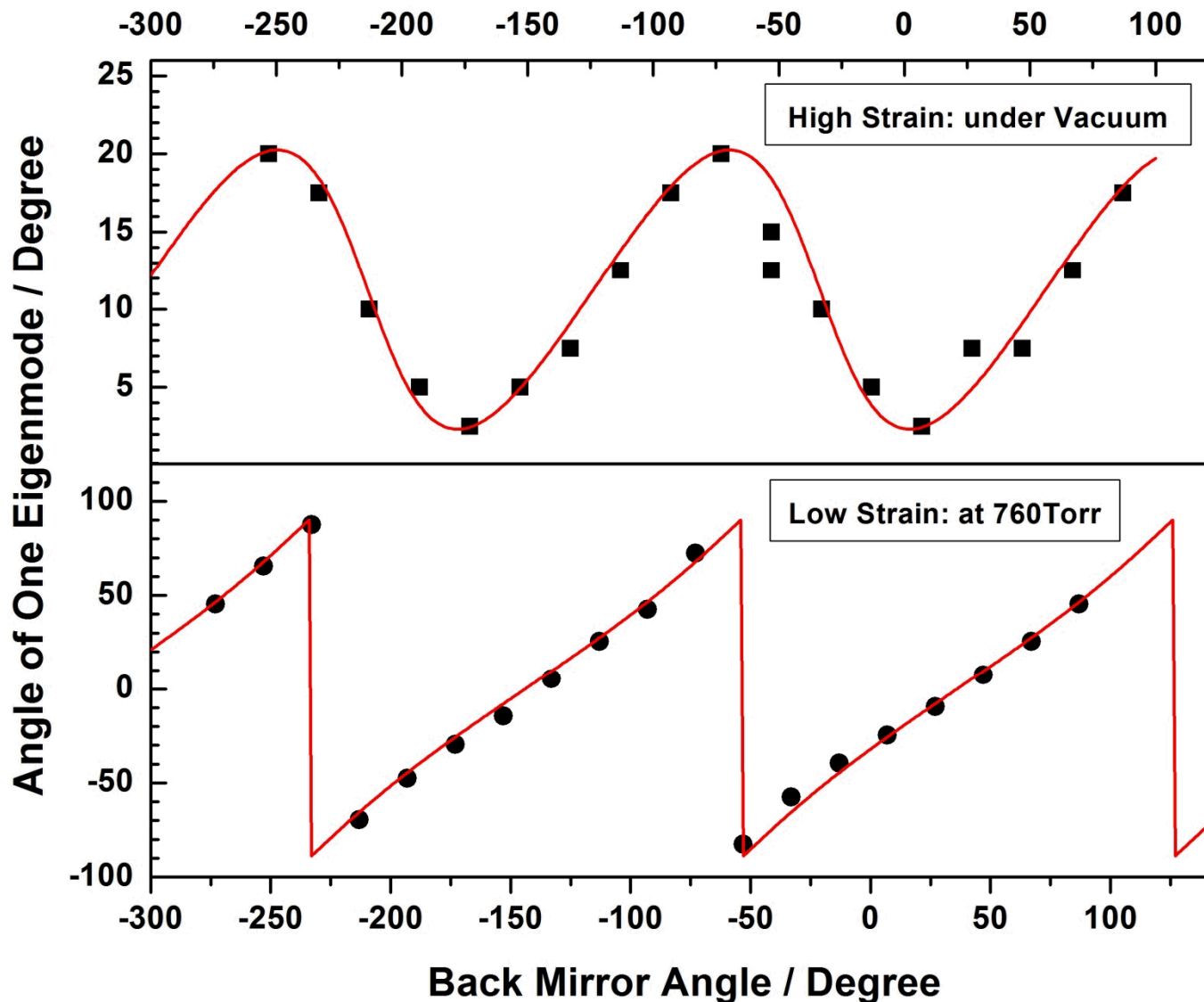
$$\tau_i = \frac{-t_r}{2 \ln(|\lambda_i|)}$$

If weak depolarization,
 τ is a **sine** of incident
polarization angle.

With $\varepsilon \cdot FSR \cdot \tau \ll 1$ and $b \cdot FSR \cdot \tau \ll 1$

$$u = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \quad u' = M \cdot u = M \cdot \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \quad \tau(\theta) \approx \frac{-t_r}{2 \ln(|u'|)}$$

The relative phase retardance determines the rotation behavior of two polarization eigenmodes.



Back mirror at **56** degree, both slow (fast) axes parallel

$$\varepsilon_1 \approx 3.3\varepsilon_2$$

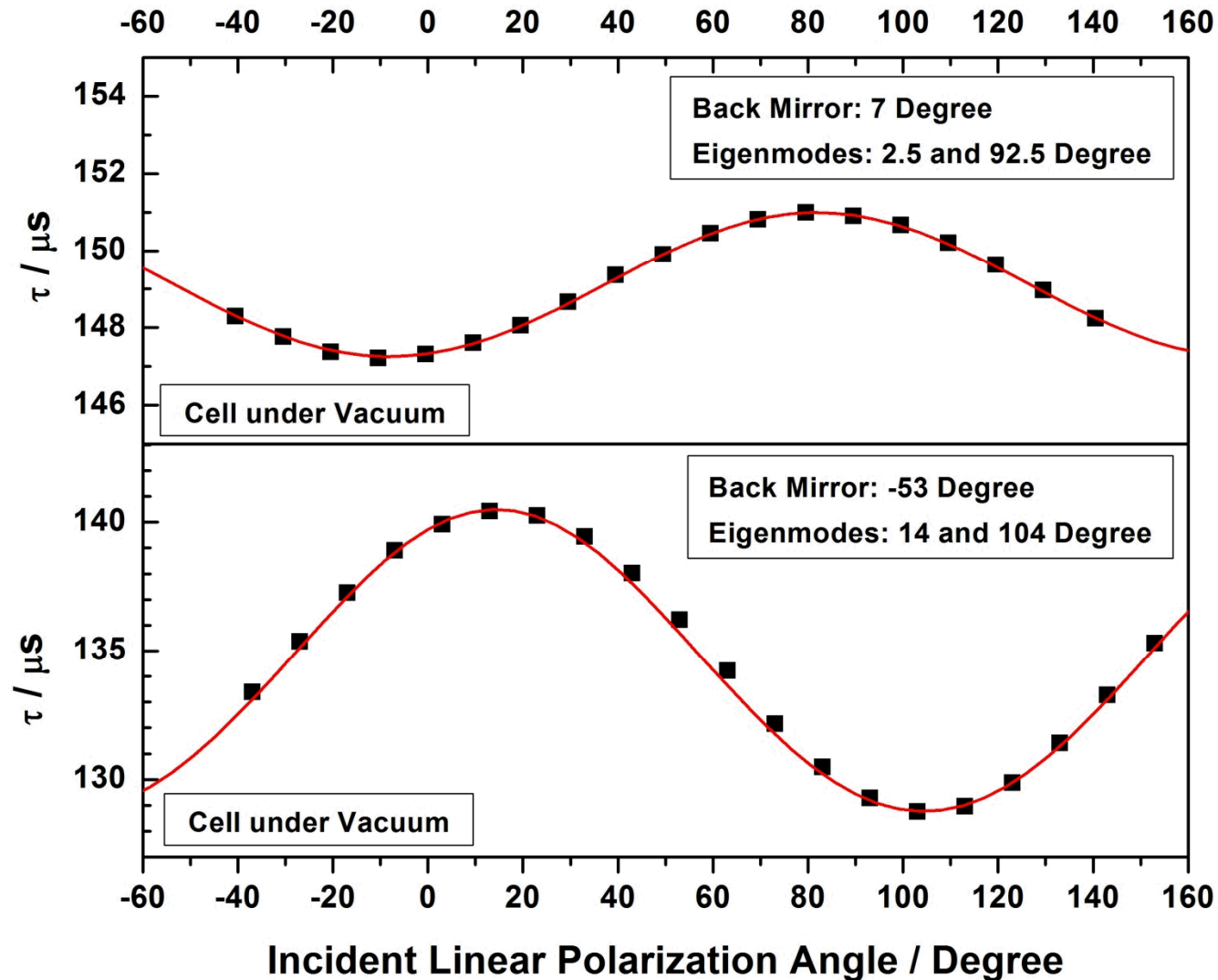
$$b \ll \varepsilon$$

Back mirror at **36** degree, its slow (fast) axis along **0** degree

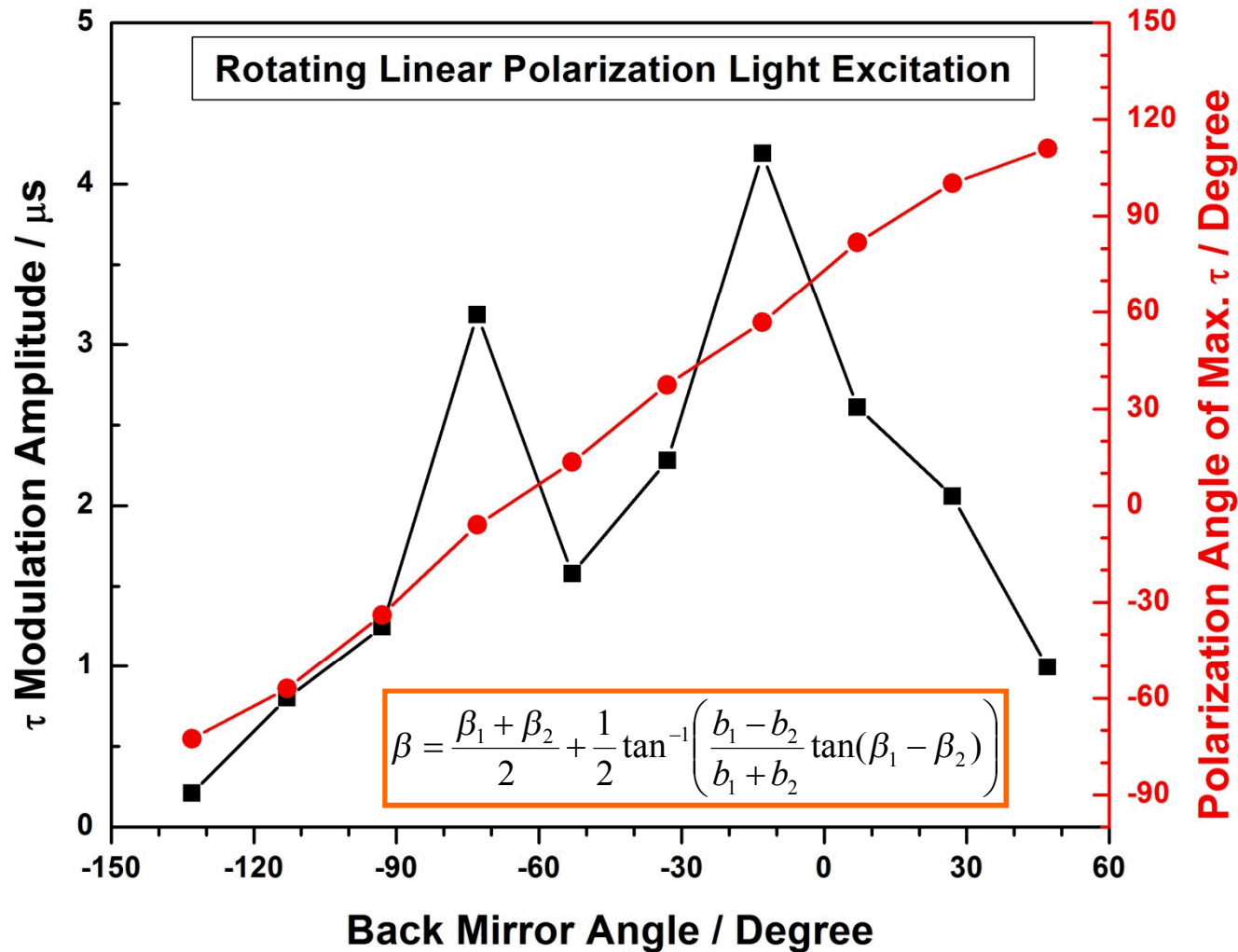
$$6\varepsilon_1 \approx \varepsilon_2$$

$$b \ll \varepsilon$$

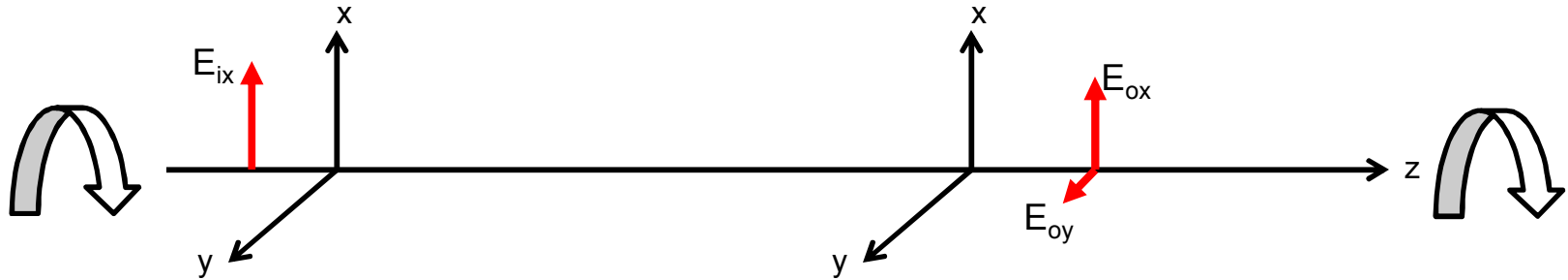
Under weak depolarization conditions, the decay time constant τ is a **sine** of incident polarization angle.



The polarization angle of maximum τ corresponds to the parameter β in the model.



Depolarization the cavity depend on mirror strain conditions.



α_1 : slow axis angle of mirror 1

$$_2), f = \varepsilon_1 / \varepsilon_2 \text{ and } \gamma = \alpha_2 - \alpha_1$$

High strain:

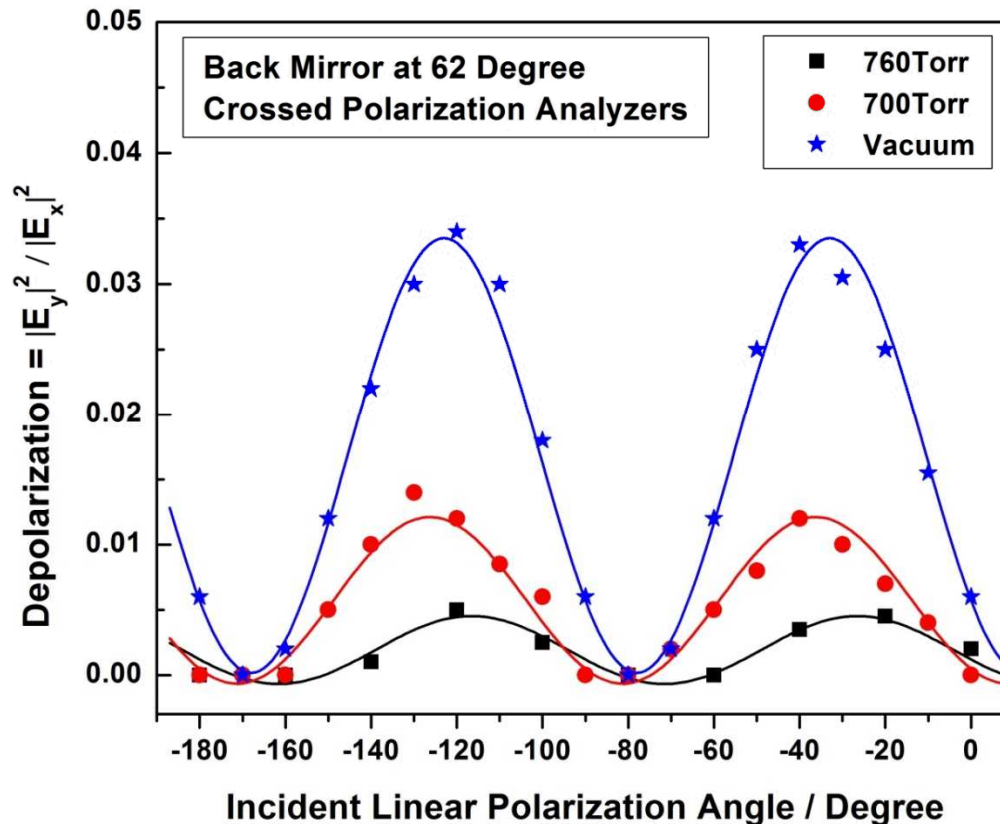
$$\varepsilon_1 \approx 1.0 \times 10^{-6} \text{ rad}$$

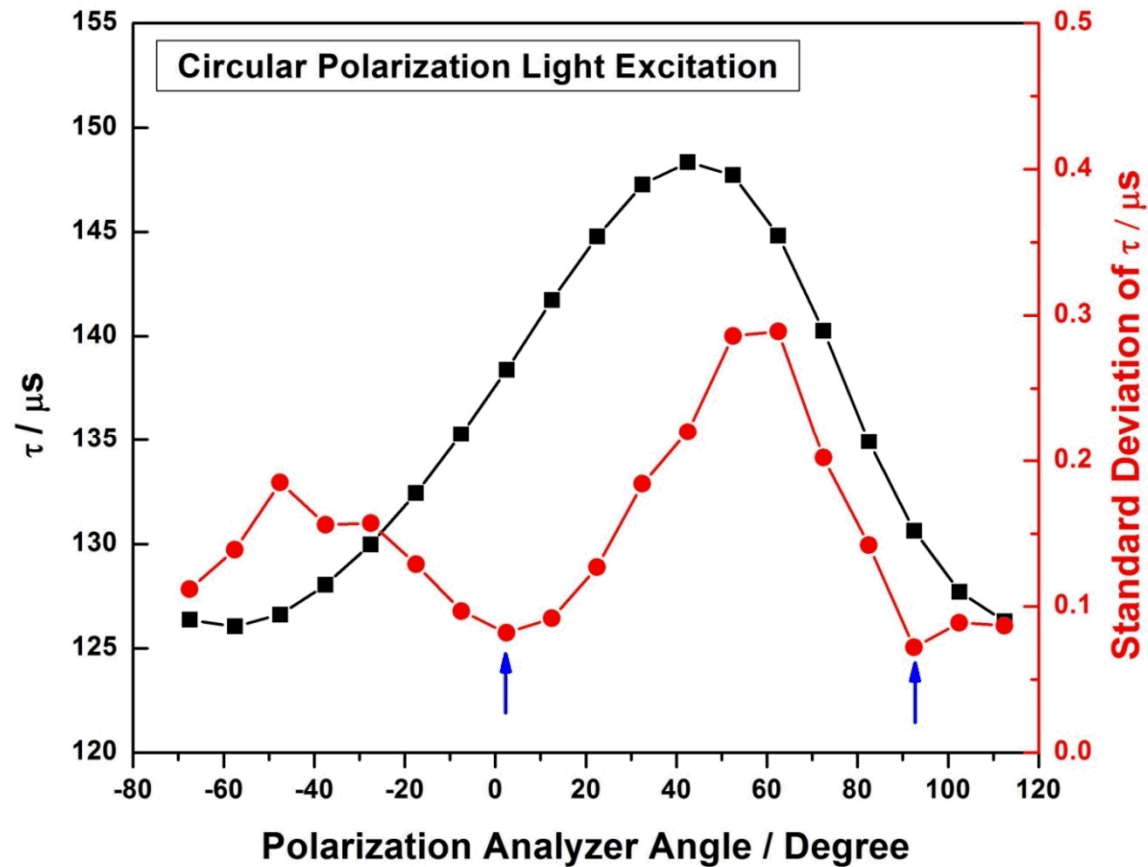
$$\varepsilon_2 \approx 3.1 \times 10^{-7} \text{ rad}$$

Low strain:

$$\varepsilon_1 \approx 8.3 \times 10^{-8} \text{ rad}$$

$$\varepsilon_2 \approx 5.0 \times 10^{-7} \text{ rad}$$





Because the splitting is only **0.1 kHz** and the mode overlap $b / \varepsilon \sim 10^{-2} - 10^{-1}$ is small, it will not cause an observable mode beating effect in ring-down signal.

Practically, linearly polarized incident laser at **two special angles** gives best signal.

However, mismatched polarization discrimination in the detection system will cause severe **mode beating** in ring-down signal, making **reduced $\chi^2 \gg 1$** .