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Label-invariant Mesh Quality Metrics

Patrick Knupp
Sandia National Laboratories

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Context

1. This paper isn't about using a priori quality metrics to detect 'bad' mesh elements.

2. Analysis performed to clarify certain aspects of the Target-matrix paradigm for mesh optimization.

3. We do not claim that we have a better way of measuring quality.

   We identify three approaches to measuring quality, whose utility depends on the context.

4. We are interested in measuring quality of quadratic finite elements, for use in mesh optimization.

5. Under what conditions can we make shape-quality of quadratic elements label-invariant?

6. The main results are mathematical and do not need numerical experiments to support them.

7. Only discuss triangle elements today. Quadrilateral and tetrahedral cases in report.

   There is no reason, based on the existing proofs of the results, to expect that the extension to quadratic hexahedra to be anything but straightforward.

8. Concepts such as label-invariance and symmetry metrics are introduced, that may be of interest outside the context of the Target-matrix paradigm, but those avenues are not explored here.

I. Measuring Quality Within Mesh Elements

Three ways to measure quality:
- Element-based,
- Mapping-based,
- Hybrid method

Hybrid Method within the context of the Target-matrix Paradigm

Element-based Quality Measurement

Unstructured meshing uses the element-based method of measuring mesh quality.

Element types include triangles, tetrahedra, quadrilaterals, and hexahedra.
Defined in terms of their connectivity in some canonical ordering scheme.

Elements have vertices, edges, faces, and volumes.

Particular mesh elements are defined in terms of their vertex coordinates.

The quality of a mesh element is a function of the vertex coordinates, via some expression involving edge-lengths, areas, angles, etc.

Example: the shape quality of a triangle is often measured as the scaled ratio of the maximum edge-length to the in-radius.

Nearly all element-based quality metrics apply only to linear mesh elements.

Mapping-based Quality Measurement

Structured meshing, based on a global mapping, uses the mapping-based method of measuring local mesh quality.

A global mapping from a logical block U to a physical block Ω is found and serves to define a discrete grid. The tangents to the mapping are used to define local quality.

$$\Xi = (\xi_1, \xi_2, \xi_3)$$

Logical Point

$$x = x(\Xi)$$

The Map

$$\frac{dx}{d\xi_1}, \frac{dx}{d\xi_2}, \frac{dx}{d\xi_3}$$

Tangents

$$g_{12} = \frac{dx}{d\xi_1} \bullet \frac{dx}{d\xi_2}$$

Orthogonal Quality

Hybrid Method of Quality Measurement

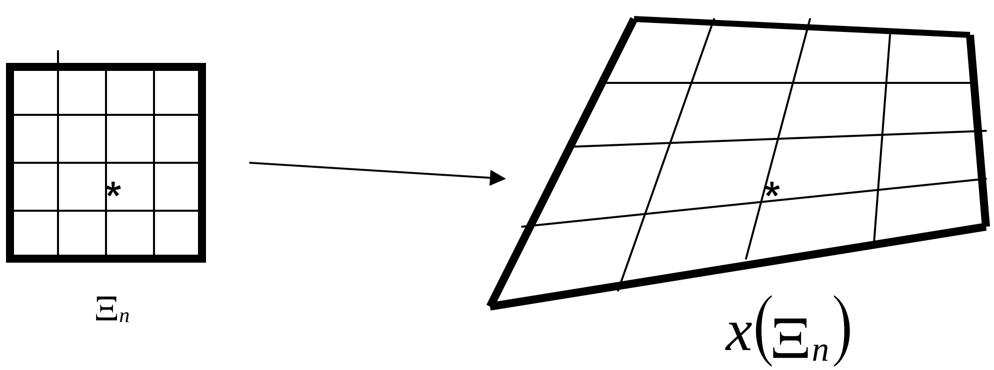
Measure Quality *within* Mesh Elements using local maps (as in finite elements).

Can measure local quality at element corners, center, Gauss points, mid-side nodes, mid-face, etc. Hybrid method applies to linear, quadratic, and higher-order elements. Local quality ultimately depends on vertex coordinates.

Hybrid method does not preclude the measurement of element quality:

$$q_\varepsilon = \max_n \left\{ \mu\left(x(\Xi_n) \right) \right\}$$

$$q_\varepsilon = \left(\frac{1}{N} \sum_{n=1}^{N} \left[\mu\left(x(\Xi_n) \right) \right]^p \right)^{1/p}$$



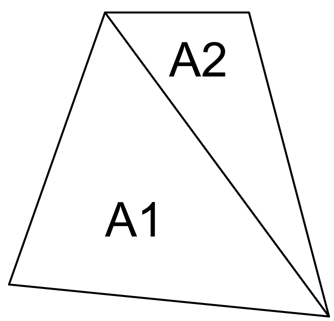
Example I: Area of a Quadrilateral

Element method: Area = A1 + A2

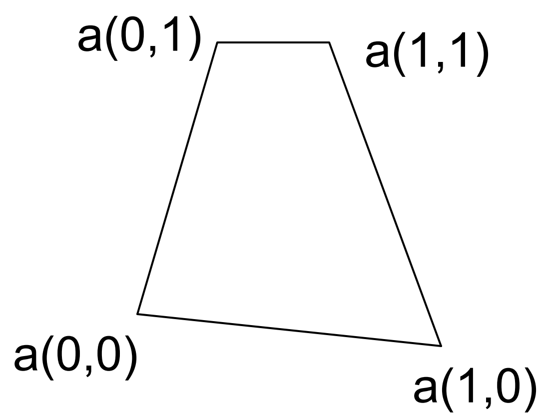
Hybrid Method:

Area = min(a(0,0), a(1,0), a(1,1), a(0,1))

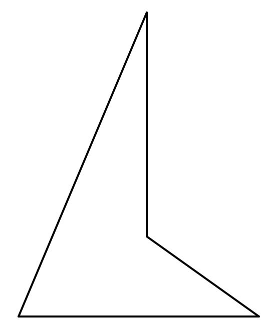
Element method can fail to detect Arrow-shaped quadrilaterals.



Element

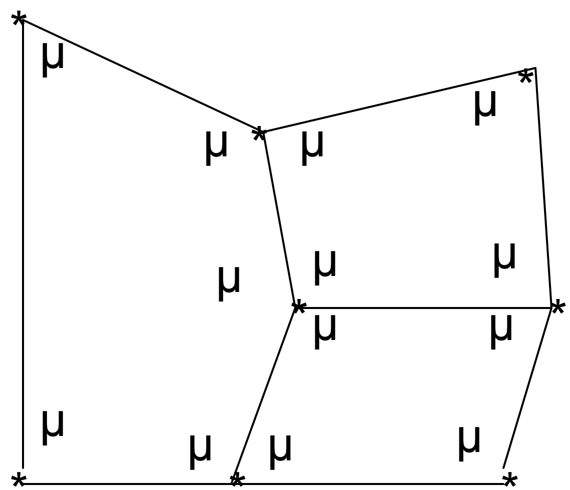


Hybrid



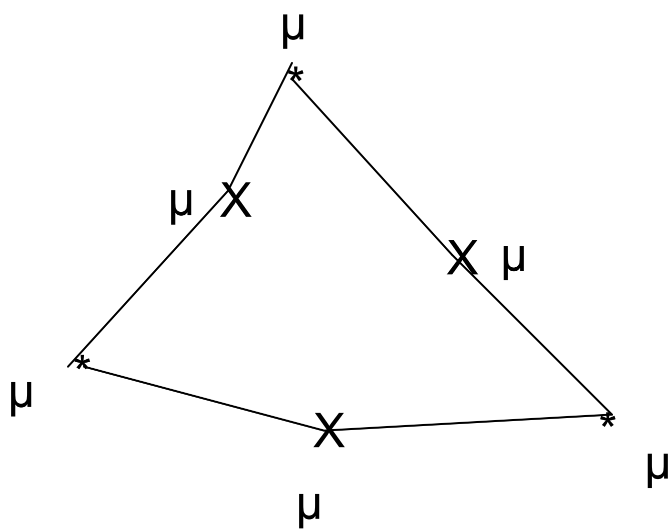
Arrow

Example II. Optimize a Mesh with Hanging Nodes



3 Quadrilaterals, 2 linear maps, 1 semi-quadratic map, 4+4+5 = 13 sample points.

Example III.   Optimize a mesh of Quadratic Triangles



Note: μ>0 at the six points does not guarantee that μ is positive everywhere in the interior of the element.

For our purpose, that's ok because we only are using these as 'monitor' points to control quality in mesh optimization, and not to rigorously measure the quality of isolated elements.

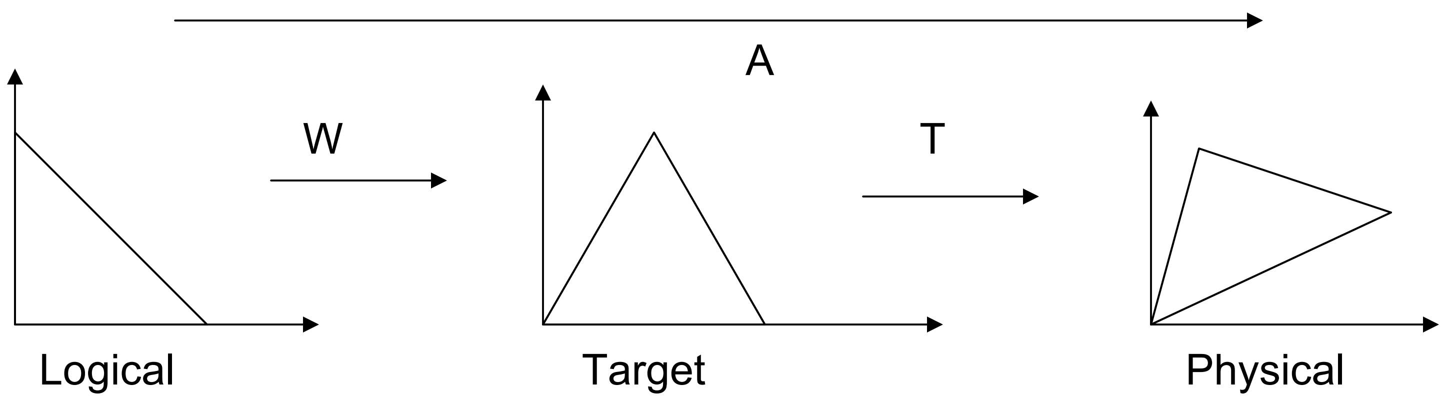
The Hybrid Method and the Target-matrix Paradigm

TMP is based on the Jacobian matrices A and W of local mappings from Logical to Physical and Logical to Target.

Local quality is measured on a set of sample points within each element.
For every sample point Ξ, A(Ξ) and W(Ξ) exist. Define $T = A W^{-1}$

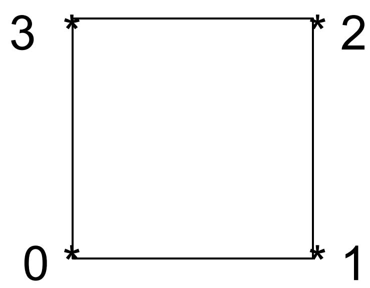
Let μ = μ(T) be the local quality at a sample point, relative to W. Example: $\mu(T) = \frac{\|T\|^2}{2\det(T)}$

In general, A depends on Ξ, so the quality varies as a function of Ξ.

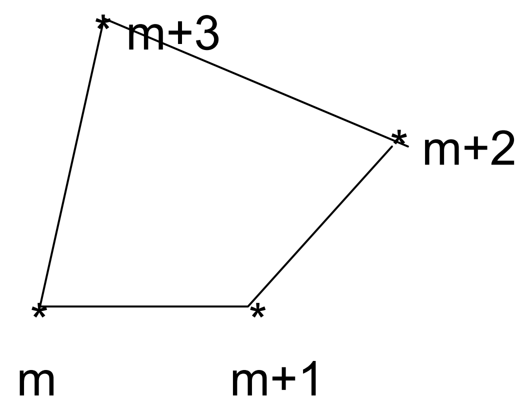


II. Label-invariance of Quality Metrics

Local (and sometimes element) quality depends, in part, on the labeling (or ordering) of the vertices in the element vertex list.



Master

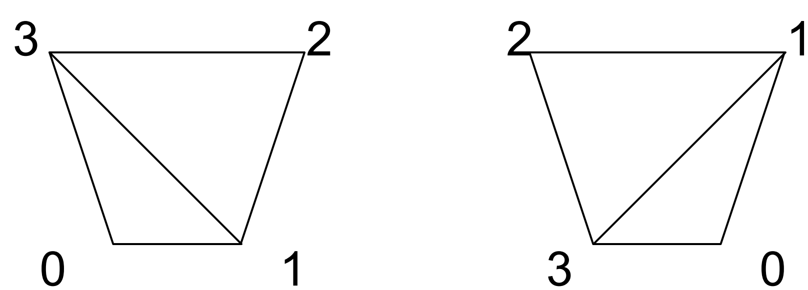


Mesh Element

Area is label-invariant, while Aspect Ratio is not.

$$Area(m+1) = Area(m)$$

$$AR(m+1) = 1/AR(m)$$



Can make AR label-invariant by symmetrizing the formula.

Definition 1. Label Invariance for <u>Element</u> Quality Metrics

An element quality metric is *label-invariant* if, for a fixed element type, its value is the same no matter which corner vertex is labeled zero.

Label-invariance is not the same concept as orientation-invariance.

Should quality metrics be label-invariant?
Not always, but metrics for area, volume, and shape should be label-invariant.

Recall the 3 approaches to measuring quality: element, mapping, hybrid.

Definition 1 is appropriate for the element approach.

Label-invariance is not an issue for the Global Mapping approach.

The hybrid approach requires a different definition of label-invariance.

Definition II. Label-invariance for <u>Local</u> Quality Metrics

Let $\mu_m(\Xi) = \mu(T_m(\Xi))$ be a local (target-matrix) quality metric. Let the reference element be fixed. Then the local quality metric is *label-invariant* at Ξ if, for an arbitrary physical element of the same type, $\mu_m(\Xi)$ is a constant for all m.

1. In the hybrid method, we talk about label-invariance at a *point*, not over an element.

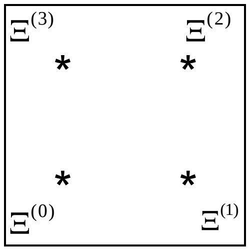
2. W is not a function of m. Only A is.

3. Label-invariance can depend on which point Ξ within the master element one chooses.

4. Label-invariance can depend on the choice of reference element.

Symmetry Metrics

Let $\left\{\Xi^{(0)},\ldots,\Xi^{(N)}\right\}$ be a collection of symmetry points in master element.



(The symmetry points themselves depend on Ξ).

Non-local Symmetry Metric: $\sigma_m(\Xi) = \max_n \left\{\mu_m\left(\Xi^{(n)}\right)\right\}$

Can also define in terms of the Min or Power Mean.
Note that this metric potentially depends on the labeling index m.

Definition III. Label-invariance of Symmetry Metrics.

Let σ_m be a symmetry metric derived from μ_m . Let the reference element be fixed. Then the symmetry metric is *label-invariant* at Ξ if, for any arbitrary physical element whose type is same as the reference, the value of the symmetry metric is constant for all m.

If the local metric is label-invariant at each of the symmetry points, then the symmetry metric is label-invariant.

As we shall see, symmetry metrics can be label-invariant even when the local metric is not.

III. Linear Planar Triangles

Label-invariance of the local quality metrics.

The Linear Map (depends on both Ξ and m).

$$x(\Xi, m) = x_m + (x_{m+1} - x_m)\xi + (x_{m+3} - x_m)\eta$$

The Jacobian Matrix (depends only on m).

$$A_m = [x_{m+1} - x_m, x_{m+3} - x_m]$$

It is clear that the Jacobian matrix is not label-invariant.

The three Jacobian Matrices are Related

$$A_m = A_0 P^m \qquad \text{with} \qquad P = \begin{pmatrix} -1 & -1 \\ +1 & 0 \end{pmatrix}$$

From this, it is easy to show that the determinant of the Jacobian matrix is label-invariant:

$$\det(A_m) = \det(A_0)$$

Let W be a fixed target-matrix, so that $T_m = A_m W^{-1}$

Then the determinant of T is label-invariant, for any W and any Ξ.

Let M be the set of matrices ρ R V, where ρ>0, R is a rotation, and

$V = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$ is derived from an equilateral reference element.

Remarkably, $V\ P^m V^{-1}$ is a rotation.

> Proposition 1.
>
> The local quality metric $|T_m|$ is label-invariant if and only if W is in M.
>
> Proof: Uses observation at top, plus the fact that the Frobenius norm is invariant to orthogonal matrices.

Note: $|T_m^t T_m|$ is also label-invariant, while the trace is not.

Corollary: Any local metric $\mu_m = \mu(T_m)$ which is a function of the determinant and/or norms of T, is label-invariant, provided W is in M.

Mean Ratio Shape metric is therefore label-invariant on the linear triangle.

IV. Quadratic Planar Triangles

Label-invariance of the local quality metrics
and of the symmetry metrics.

The Quadratic Map (depends on both Ξ and m).

$$x(\Xi, m) = c_{0,m} + c_{1,m}\xi + c_{2,m}\eta + c_{3,m}\xi^2 + c_{4,m}\xi\eta + c_{5,m}\eta^2$$

The Jacobian Matrix

$$A_m(\Xi) = \left[c_{1,m} + 2c_{3,m}\xi + c_{4,m}\eta, c_{2,m} + c_{4,m}\xi + 2c_{5,m}\eta\right]$$

The Jacobian Matrix for the Quadratic Element is not Label-invariant, nor is it a constant over the element (unless the sides are straight).

In the following analylsis, we allow the reference element to be curved, i.e.,

$$W = W(\Xi)$$

Derivation of the Symmetry Points for a Quadratic Triangle

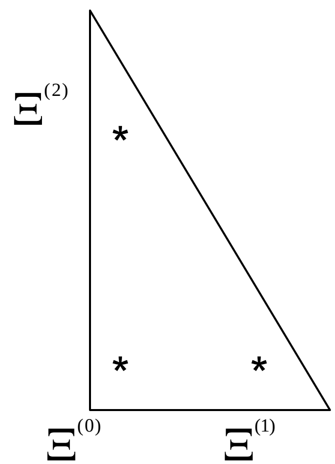
Let $\Xi^{(0)} = \Xi$ be the first symmetry point.

Then the other two are defined by

$$x\left(\Xi^{(1)}, m\right) = x\left(\Xi^{(0)}, m+1\right)$$
$$x\left(\Xi^{(2)}, m\right) = x\left(\Xi^{(0)}, m+2\right)$$

These can be solved to find:

$$\Xi^{(0)} = (\xi, \eta)$$
$$\Xi^{(1)} = (1-\xi-\eta, \xi)$$
$$\Xi^{(2)} = (\eta, 1-\xi-\eta)$$



Examples:

Ξ=(0,0) gives the three corners,
Ξ=(1/2, 0) gives the three mid-edge positions, and
Ξ=(1/3,1/3) gives the centroid (all three symmetry points are the same).

Symmetry Points Relate the Jacobian Matrices

Proposition 2. For k,m=0,1,2 $A_m\left(\Xi^{(k)}\right) = A_0\left(\Xi^{(k+m)}\right)P^m$

Proof: Differentiation of the symmetry point definitions with respect to the first symmetry point, gives the result.

Corollary: $\det\left(A_m\left(\Xi^{(k)}\right)\right) = \det\left(A_0\left(\Xi^{(k+m)}\right)\right)$

Therefore, *the area at a point in a quadratic triangle is* *<u>not</u>* *label-invariant.*

Neither are the <u>local</u> metrics det(T), |T|, nor trace(T).

Unless….

Proposition 3.

The local metrics det(T), |T|, and |T^t T| are label-invariant if $\Xi = \left(\frac{1}{3}, \frac{1}{3}\right)$

This is true for det(T) for any W, while it is true for the others only when W is in M.

Corollary: Local metrics that involve combinations of det(T), |T|, and/or |T^t T| are label-invariant provided they are evaluated at the centroid of the master element and W in M.

Unfortunately, monitoring quality only at the centroid of an element can be mis-leading. (e.g., Arrow-quad example)

Label-invariance of Symmetry Metrics on the Quadratic Map

Proposition 4. If, for a particular reference element, a local metric satisfies

$$\mu_{m+r}\left(\Xi^{(s)}\right) = \mu_{m}\left(\Xi^{(r+s)}\right)$$

then the derived symmetry metric is label-invariant.

Thus, symmetry metrics can be label-invariant even when the local metric is not.

Corollary: Symmetry metrics derived from combinations of det(T), |T|, and |T^t T| are label-invariant provided W in M.

V. Application to TMP

The Target-matrix paradigm of mesh optimization:

- uses a mapping from a master element to each element of the mesh.
- each master element contains a set of sample points at which one computes a Target-matrix and the Jacobian of the Active mesh.
- Local quality metrics for Size, Shape, and Orientation are functions of these matrices.
- TMP thus uses the Hybrid approach to measuring quality.

For metrics such as Size and Shape, label-invariance is highly desirable.

If the Jacobian is constant over the element, as in the linear triangle and linear tetrahedral cases, Label-invariance of the Shape metric is achieved by choosing the Target-matrix to correspond to an equilateral reference element.

In all other cases, there are two ways to achieve label-invariance in TMP.

- First, for local (non-symmetry) metrics, it is required that the sample point be located at the element centroid, and the Target correspond to equilateral.
- Second, one can use symmetry metrics in which the sample points are strategically placed, along with a Target based on the equilateral reference.

This analysis lays the foundation for improving the Shape quality of elements with quadratic maps via optimization.