Calculating Damping from Ring-Down Using Hilbert Transform and Curve Fitting

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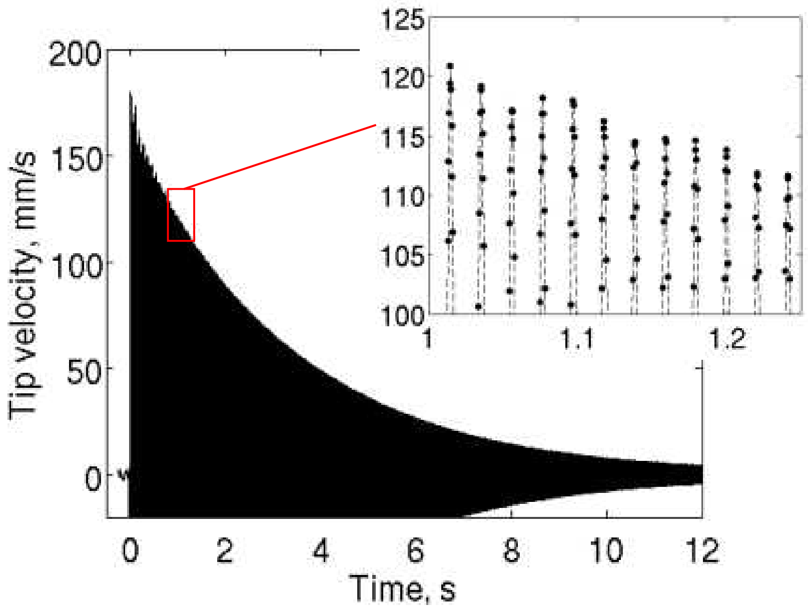
Introduction

- Damping plays an important role in structural vibration.
  - For example, damping determines how much energy is dissipated as heat when a PZT bimorph is used as a vibration energy harvester.
- The natural frequency and damping could be obtained by measuring the time lapse and the logarithmic decrement between consecutive peaks of the trace of free-decay time history .
- In practical response histories, however, the peaks may not be captured exactly. For example, a peak may appear higher than its preceding peak because of nonlinearity, sampling, noise, and various other conditions, resulting in erroneous negative damping.
- The method presented in this paper is based on the Hilbert transform. This method is more robust and considerably easier to apply than previous methods.
- The method allows modeling of nonlinearity.

Response to “plucking” is a cosine with time-decaying amplitude.

- For an underdamped linear SDOF oscillator, the free decaying velocity response to an initial condition is



$$v(t) = A(t)\cos(\omega_d t + \phi_0)$$

$$A(t) = A_0 \exp(-\zeta\omega_n t)$$

A_0 is the initial amplitude, ζ is the damping ratio and ω_n is the natural frequency.

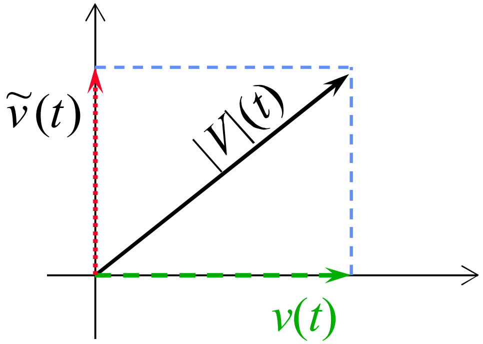
Hilbert transform gives the envelope.

The Hilbert transform of a signal $v(t)$ is $\tilde{v}(t) = \mathscr{H}\{v(t)\} = \dfrac{1}{\pi}\int_{-\infty}^{\infty} \dfrac{v(\tau)}{t-\tau}\, d\tau$

- $\tilde{v}(t)$ is an imaginary signal that is 90° lagging from phase from the real signal $v(t)$.

$$V(t) = v(t) + \mathrm{j}\tilde{v}(t)$$

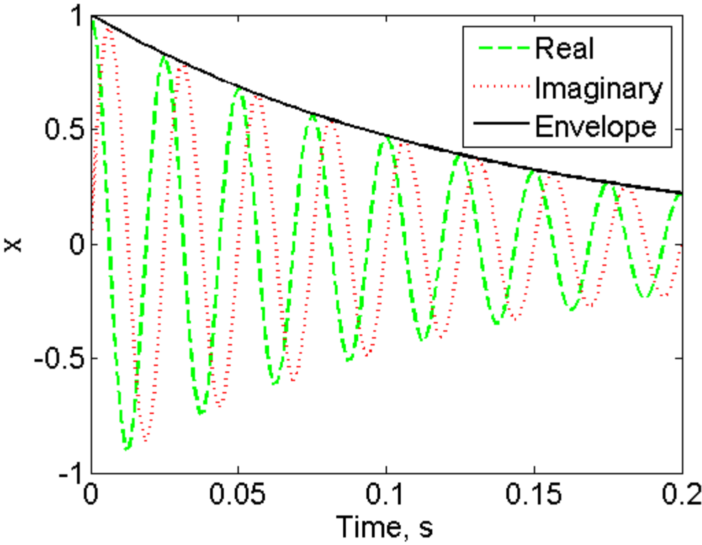
- The |vector sum| of the real signal and the imaginary signal is the amplitude



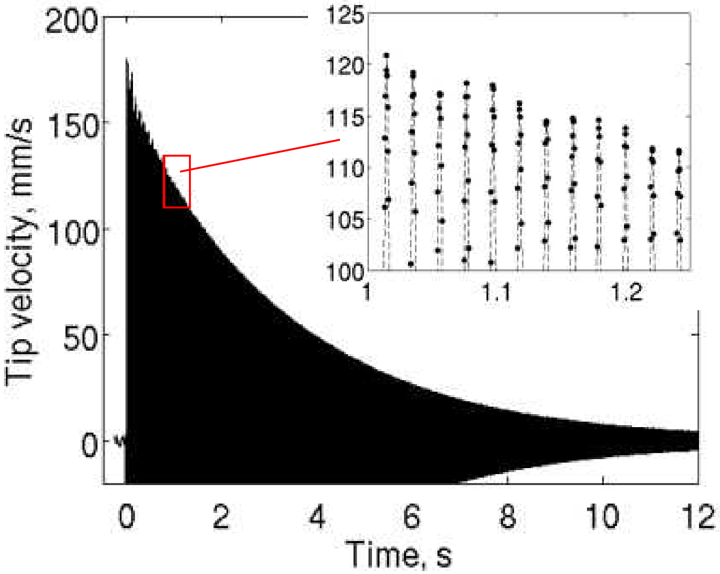
$$|V|(t) = \sqrt{v^2(t) + \tilde{v}^2(t)}$$

- The amplitude of a decaying sinusoid is the envelope

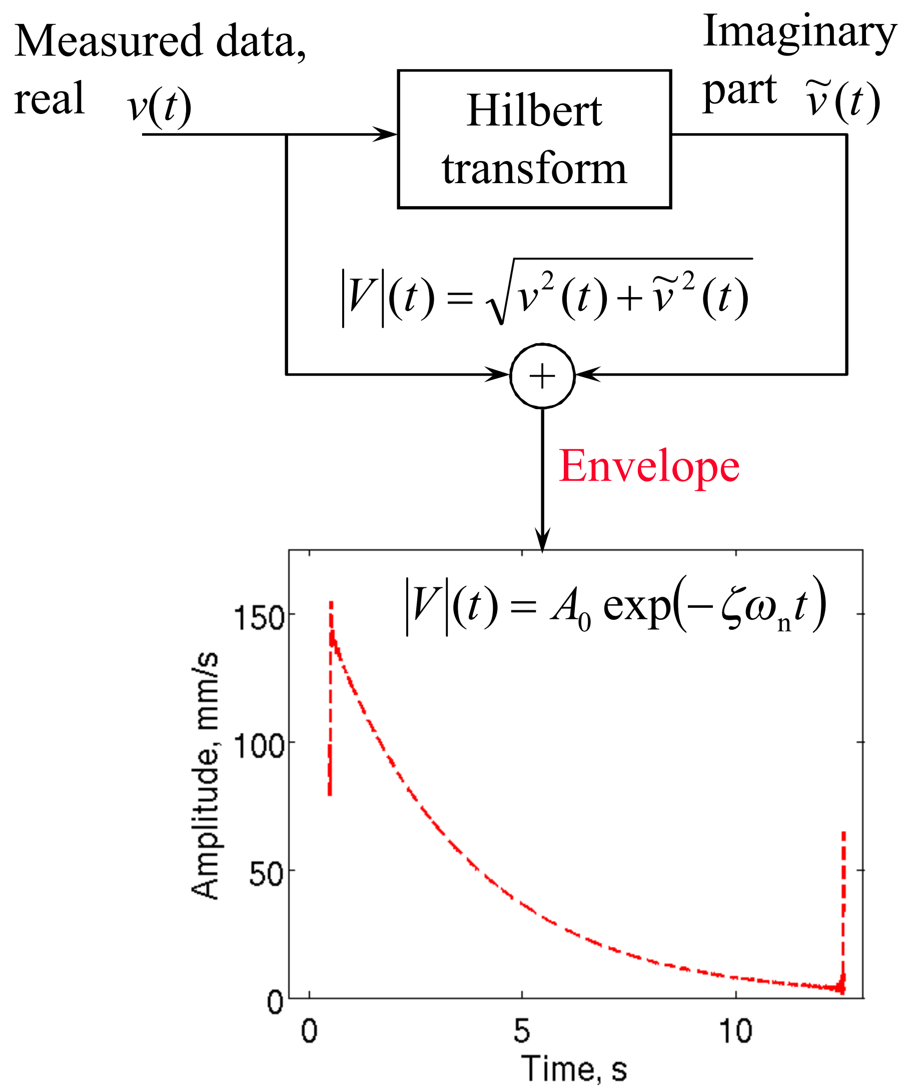
$$|V|(t) = A_0 \exp(-\zeta\omega_n t)$$



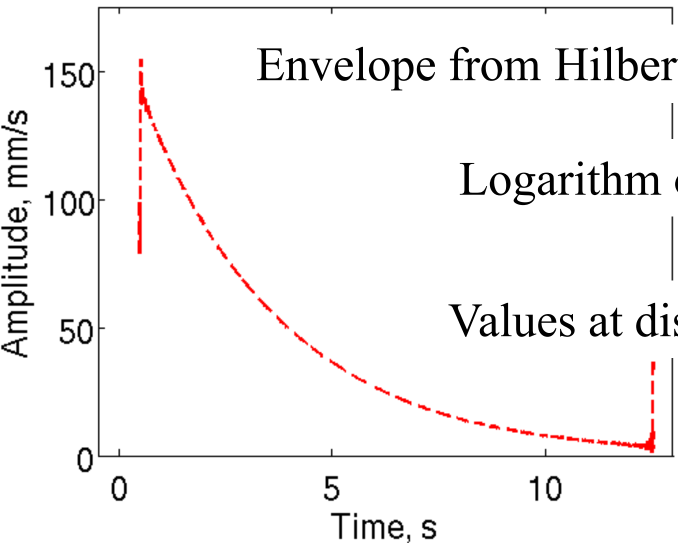
The envelope is a decaying exponential.



$$v(t) = A(t)\cos(\omega_d t + \phi_0)$$



Curve fitting gives initial amplitude and decay rate.



Envelope from Hilbert transform: $|V|(t) = A_0 \exp(-\zeta\omega_n t)$

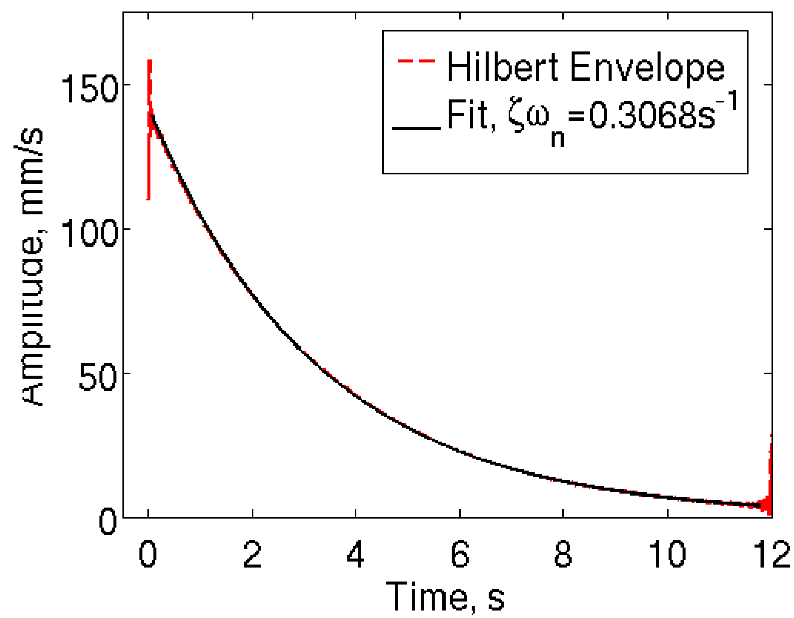
Logarithm of envelope: $\ln\left(|V|(t)\right) = \ln\left(A_0\right) - \zeta\omega_n t$

Values at discrete times: $\ln\left(|V|(t_0)\right), \ln\left(|V|(t1)\right), \ldots, \ln\left(|V|(t_{N-1})\right)$

Linear regression analysis gives the least-squares solution to

$$\begin{Bmatrix} \ln|V(t_0)| \\ \ln|V(t_1)| \\ \vdots \\ \ln|V(t_{N-1})| \end{Bmatrix} = \begin{bmatrix} t_0 & 1 \\ t_1 & 1 \\ \vdots & \vdots \\ t_{N-1} & 1 \end{bmatrix} \begin{Bmatrix} \zeta\omega_n \\ \ln(A_0) \end{Bmatrix}$$

- product of
  - linear damping
  - natural frequency
- initial amplitude

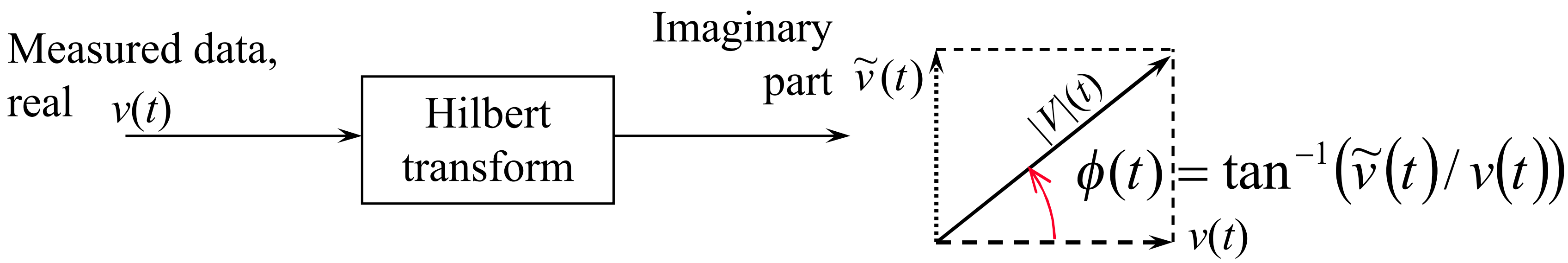


For later, define $\boldsymbol{\zeta\omega_n = -C}$.

Besides amplitude, Hilbert transform gives phase $\phi(t)$.

In any sinusoid $v(t) = A(t)\cos\boxed{\left(\omega_d t + \phi_0\right)}, \quad \omega_d t + \phi_0 = \phi(t)$

Signal phase can be obtained from the Hilbert transform

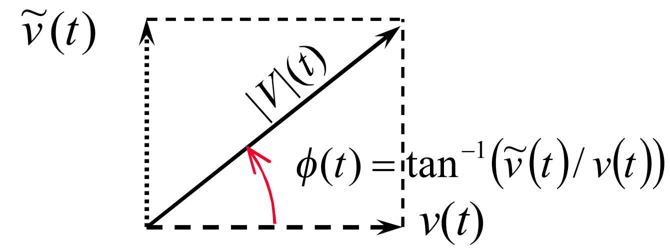


From the phase $\phi(t) = \omega_d t + \phi_0$

oscillation frequency is time derivative of signal phase: $\dfrac{d}{dt}\phi(t) = \omega_d$

Curve-fitting gives the oscillation frequency from the phase.

Signal phase is obtained from the Hilbert transform



$$\phi(t) = \omega_d t + \phi_0$$

Curve fitting of the signal phase gives

- oscillation frequency
- initial phase

$$\begin{Bmatrix} \phi(t_0) \\ \phi(t_1) \\ \vdots \\ \phi(t_{N-1}) \end{Bmatrix} = \begin{bmatrix} t_0 & 1 \\ t_1 & 1 \\ \vdots & \vdots \\ t_{N-1} & 1 \end{bmatrix} \begin{Bmatrix} \omega_\mathrm{d} \\ \phi_0 \end{Bmatrix}$$

Natural frequency is

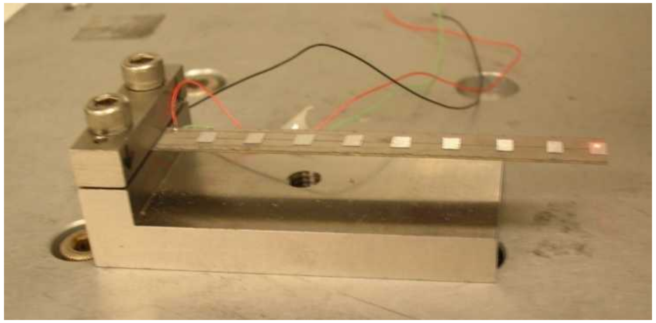
$\zeta\omega_n$ was obtained from envelope curve fitting: $\zeta\omega_\mathbf{n} = -\boldsymbol{C}$.

Using $\omega_\mathrm{d} = \omega_\mathrm{n}\sqrt{1-\zeta^2}$,

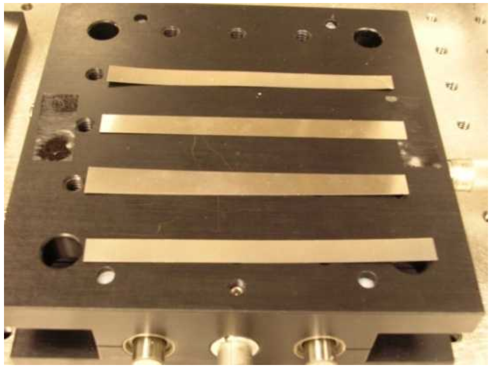
<u>the natural frequency ω_n can be obtained as</u> $\omega_\mathrm{n} = \sqrt{\omega_\mathrm{d}^{\,2} + C^2}$

<u>And damping as</u> $\zeta = -\boldsymbol{C}/\omega_\mathbf{n}$.

Cantilever was plucked.
Velocity was measured by LDV.



a)



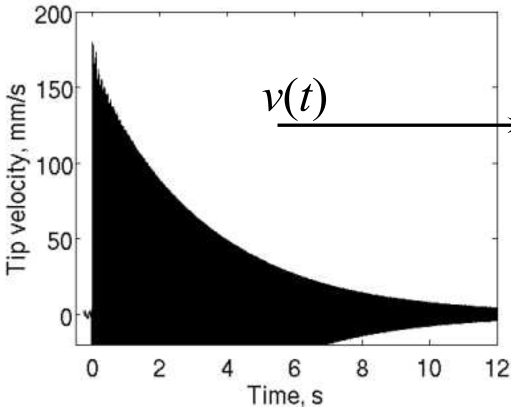
b)

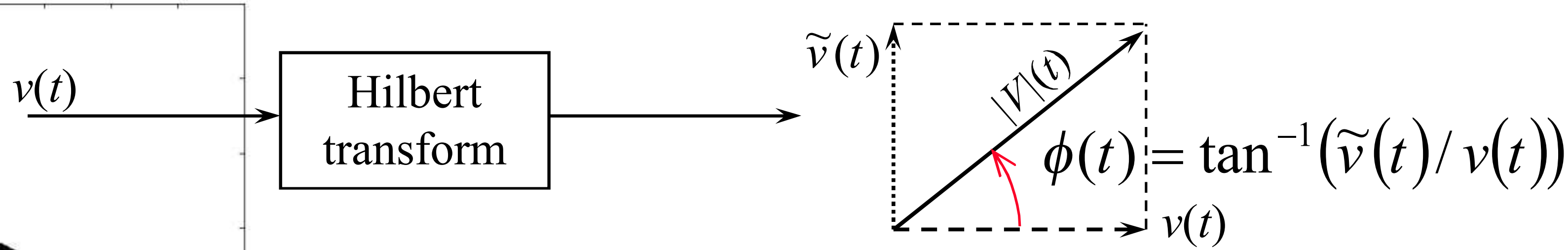
Figure 1 : a) Cantilever with LDV laser beam at the tip. b) Thin beams of four different metals.

- A laser Doppler vibrometer (LDV) measured the tip velocity $v(t)$.

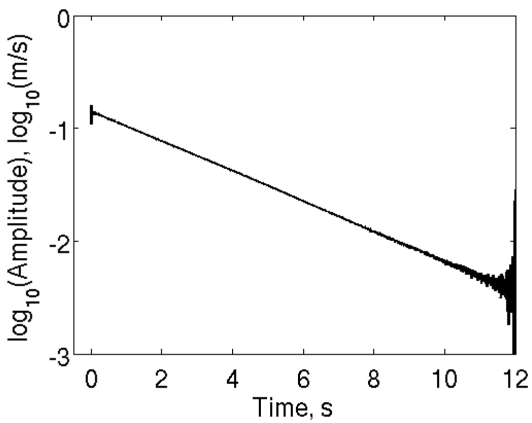
Monel cantilever gave a clean linear ringdown.

Time history

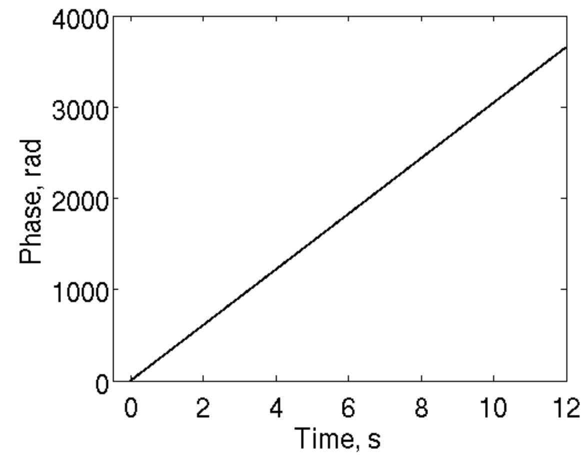




Log of envelop



Phase growth with time



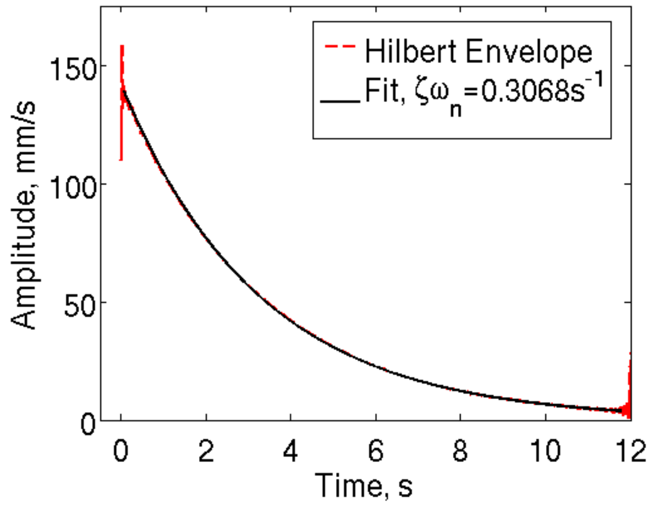
Monel cantilever decay rate and oscillation frequency were curve-fit linearly.

Curve fitting of the signal amplitude

$$\begin{Bmatrix} \ln|V(t_0)| \\ \ln|V(t_1)| \\ \vdots \\ \ln|V(t_{N-1})| \end{Bmatrix} = \begin{bmatrix} t_0 & 1 \\ t_1 & 1 \\ \vdots & \vdots \\ t_{N-1} & 1 \end{bmatrix} \begin{Bmatrix} \zeta\omega_n \\ \ln(A_0) \end{Bmatrix}$$

- product of
  - linear damping
  - natural frequency
- initial amplitude

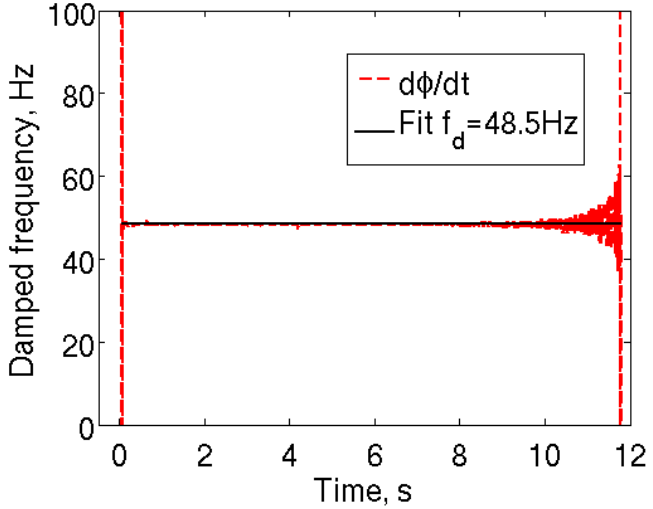
$\zeta\omega_n = -C.$



Curve fitting of the signal phase gives

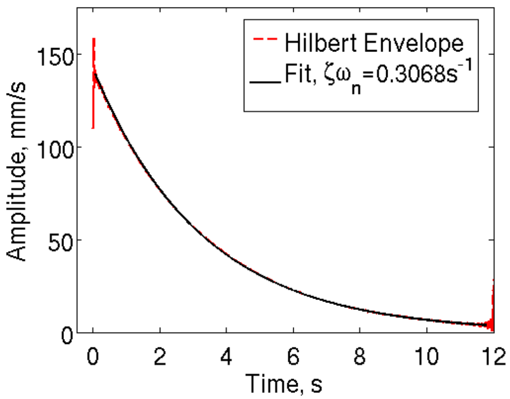
$$\begin{Bmatrix} \phi(t_0) \\ \phi(t_1) \\ \vdots \\ \phi(t_{N-1}) \end{Bmatrix} = \begin{bmatrix} t_0 & 1 \\ t_1 & 1 \\ \vdots & \vdots \\ t_{N-1} & 1 \end{bmatrix} \begin{Bmatrix} \omega_d \\ \phi_0 \end{Bmatrix}$$

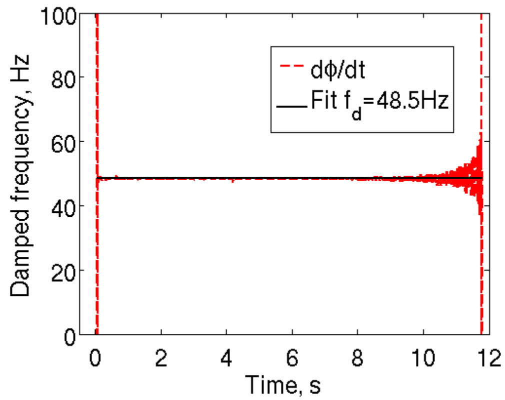
- oscillation frequency
- initial phase



Monel cantilever frequency and damping were easy to extract.

Linear regression analysis gave $-C=\zeta\omega_n$ and ω_d.

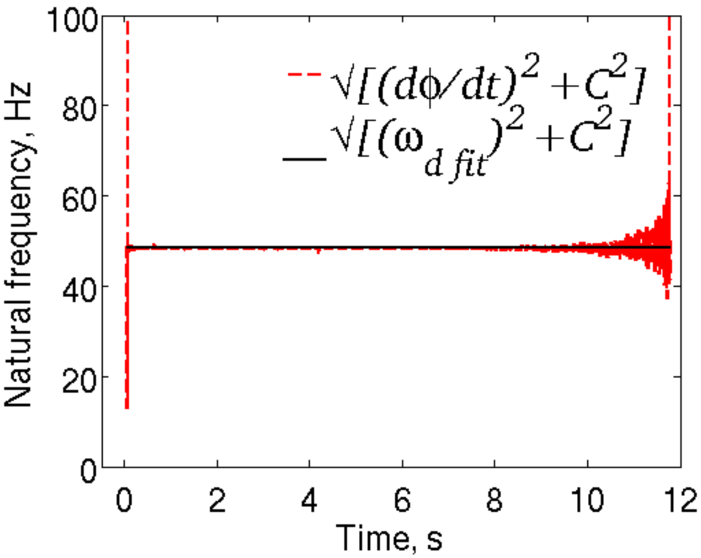


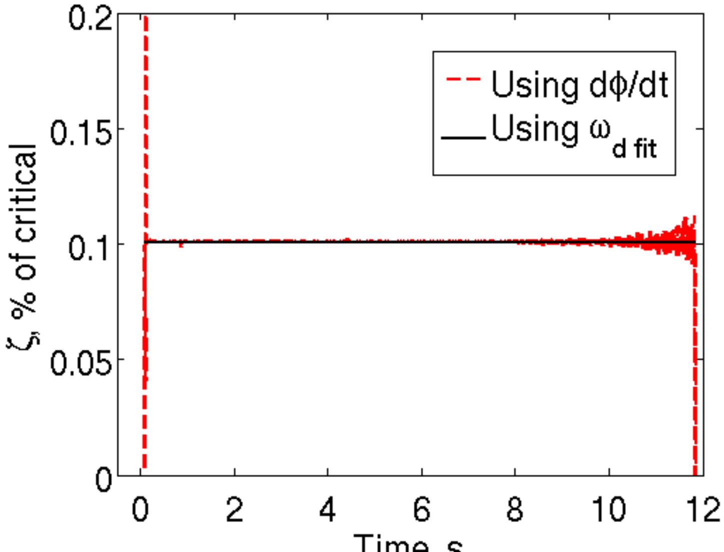


Transform them into ω_n and ζ as follows:

Natural freq is $\omega_n = \sqrt{\omega_d{}^2 + C^2}$

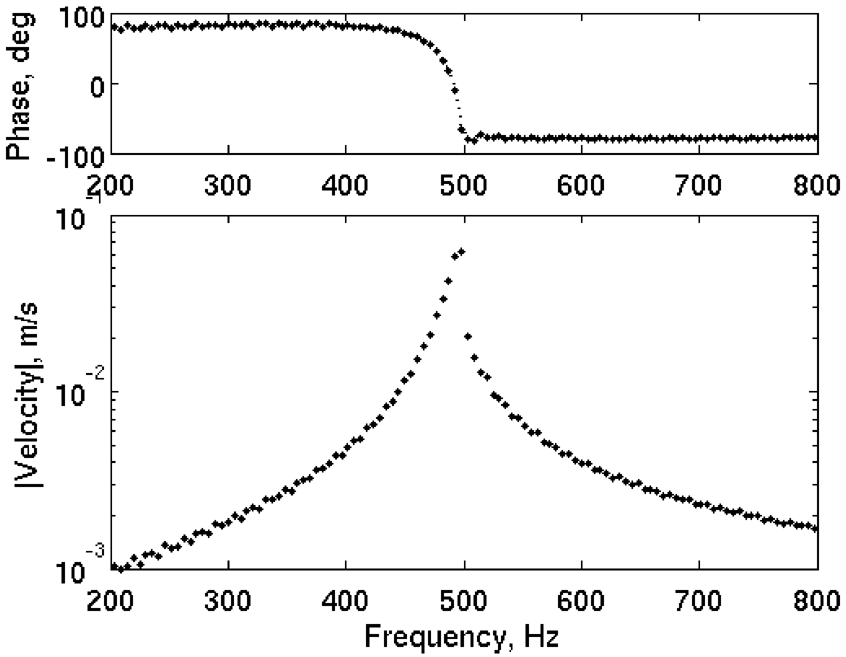
Damping is = $-C/\omega_n$.



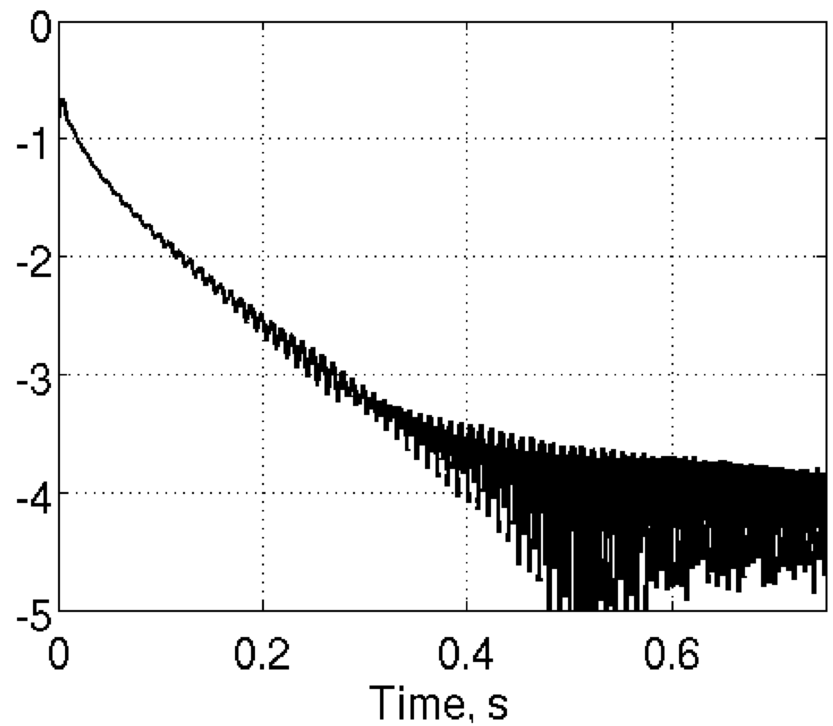


PZT bimorph cantilever showed nonlinearity and high damping.

- Response FFT tilts to the right.



- High damping reduces response to noise floor quickly.



- The above problems require nonlinear curve fitting.

Nonlinearity requires curve-fitting with higher degrees.

Linear, ω_d is constant:

$$\phi(t) = \omega_d t + \phi_0$$

$$\begin{Bmatrix} \phi(t_0) \\ \phi(t_1) \\ \vdots \\ \phi(t_{N-1}) \end{Bmatrix} = \begin{bmatrix} t_0 & 1 \\ t_1 & 1 \\ \vdots & \vdots \\ t_{N-1} & 1 \end{bmatrix} \begin{Bmatrix} \omega_\mathrm{d} \\ \phi_0 \end{Bmatrix}$$

$$\omega_\mathrm{d} = d\phi(t)/dt = \mathrm{const.}$$

Nonlinear, ω_d is a function of time:

$$\phi(t) = b_P t^P + b_{P-1} t^{P-1} + \ldots + b_1 t + b_0 \phi_0$$

$$\begin{Bmatrix} \hat{\phi}(t_0) \\ \hat{\phi}(t_1) \\ \vdots \\ \hat{\phi}(t_{N-1}) \end{Bmatrix} = \begin{bmatrix} t_0{}^P & \cdots & t_0 & 1 \\ t_1{}^P & \cdots & t_1 & 1 \\ \vdots & \cdots & \vdots & 1 \\ t_{N-1}{}^P & \cdots & t_{N-1} & 1 \end{bmatrix} \begin{Bmatrix} b_P \\ \vdots \\ b_1 \\ b_0 \end{Bmatrix}$$

$$\omega_\mathrm{d}(t) = d\phi(t)/dt$$

$$\hat{\omega}_d(t) = \frac{d\hat{\phi}(t)}{dt} = \begin{bmatrix} Pt_0{}^{P-1} & \cdots & 1 & 0 \\ Pt_1{}^{P-1} & \cdots & 1 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ Pt_{N-1}{}^{P-1} & \cdots & 1 & 0 \end{bmatrix} \begin{Bmatrix} b_P \\ \vdots \\ b_1 \\ b_0 \end{Bmatrix}$$

Curve fitting of amplitude is also expanded from linear to polynomial.

Linear, ζ is constant:

$$|V|(t) = A_0 \exp(-\zeta\omega_n t)$$

$$\ln(|V|(t)) = \ln(A_0) - \zeta\omega_n t$$

$$\begin{Bmatrix} \ln|V(t_0)| \\ \ln|V(t_1)| \\ \vdots \\ \ln|V(t_{N-1})| \end{Bmatrix} = \begin{bmatrix} t_0 & 1 \\ t_1 & 1 \\ \vdots & \vdots \\ t_{N-1} & 1 \end{bmatrix} \begin{Bmatrix} \zeta\omega_n \\ \ln(A_0) \end{Bmatrix}$$

Nonlinear, ζ is a function of time:

$$\begin{Bmatrix} \ln|V(t_0)| \\ \ln|V(t_1)| \\ \vdots \\ \ln|V(t_{N-1})| \end{Bmatrix} = \begin{bmatrix} t_0{}^3 & t_0{}^2 & t_0 & 1 \\ t_1{}^3 & t_1{}^2 & t_1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ t_{N-1}{}^3 & t_{N-1}{}^2 & t_{N-1} & 1 \end{bmatrix} \begin{Bmatrix} c_3 \\ c_2 \\ c_1 \\ \ln(A_0) \end{Bmatrix}$$

$$\hat{A}(t) = A_0 \exp(-c_1 t - c_2 t^2 - c_3 t^3)$$

$$\zeta(t)\omega_n(t) \equiv C(t) = (c_1 + c_2 t + c_3 t^2)$$

$$\omega_n(t) = \sqrt{(\omega_d(t))^2 + (C(t))^2}$$

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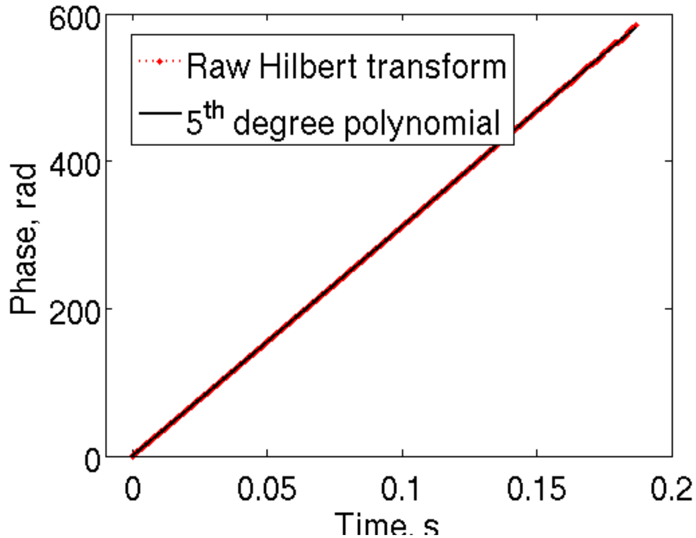
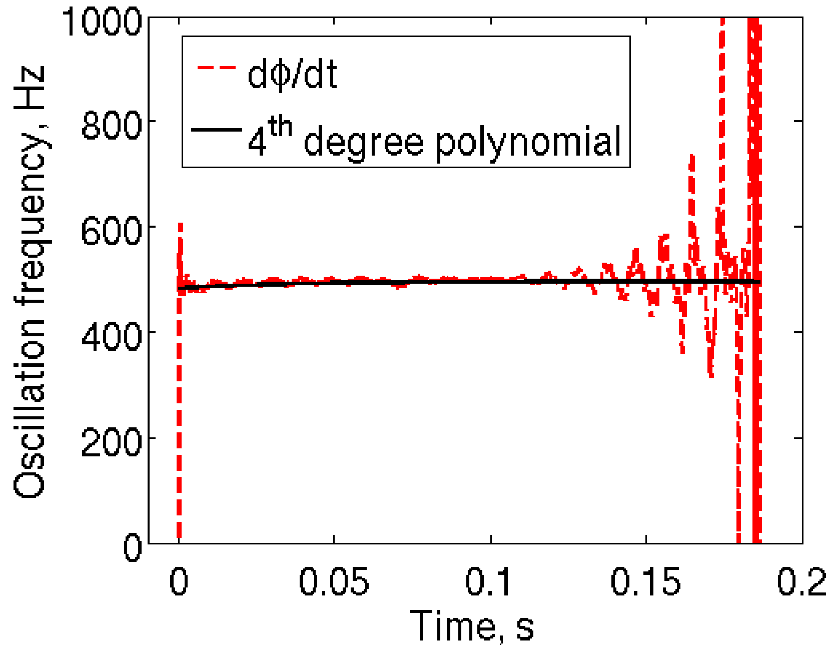
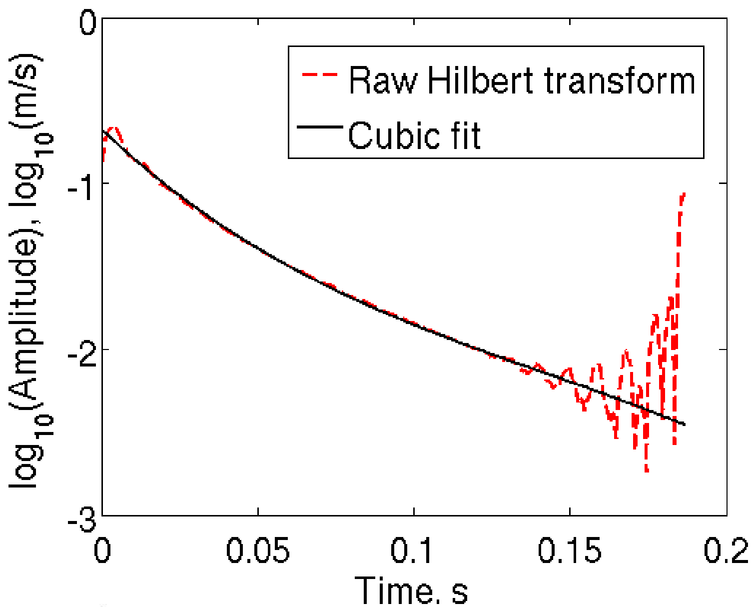
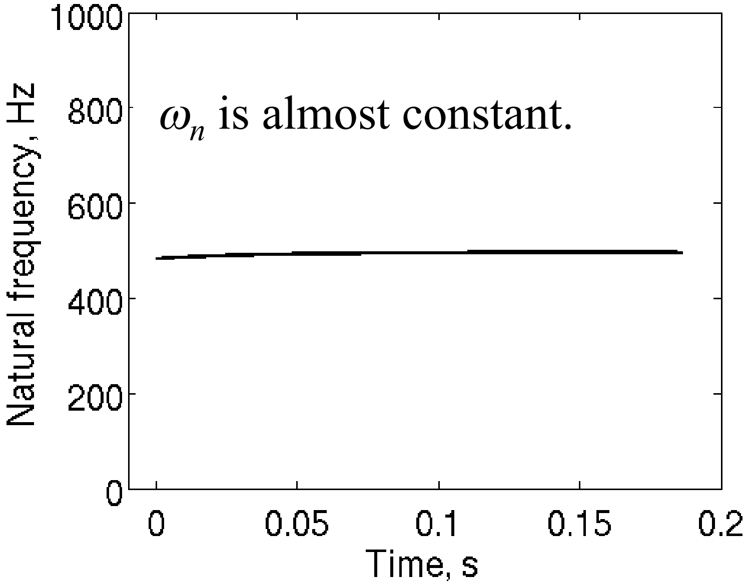
PZT Bimorph nonlinear ringdown is curve-fit.

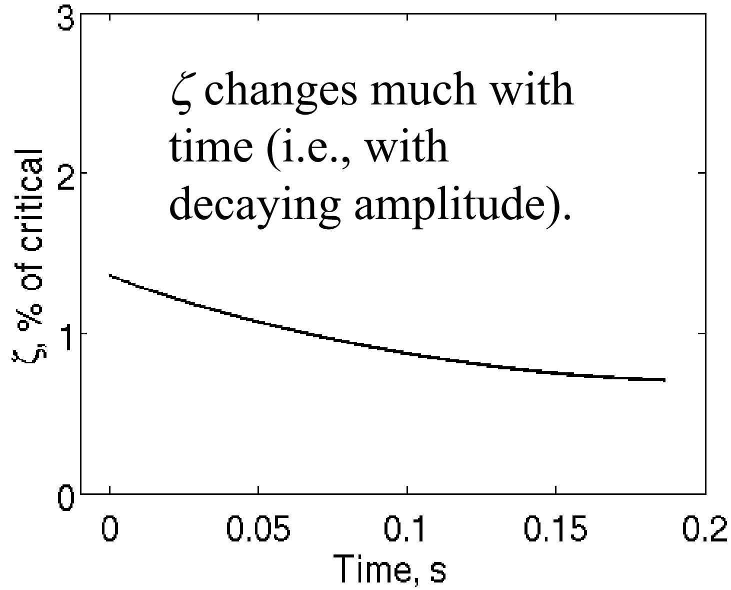




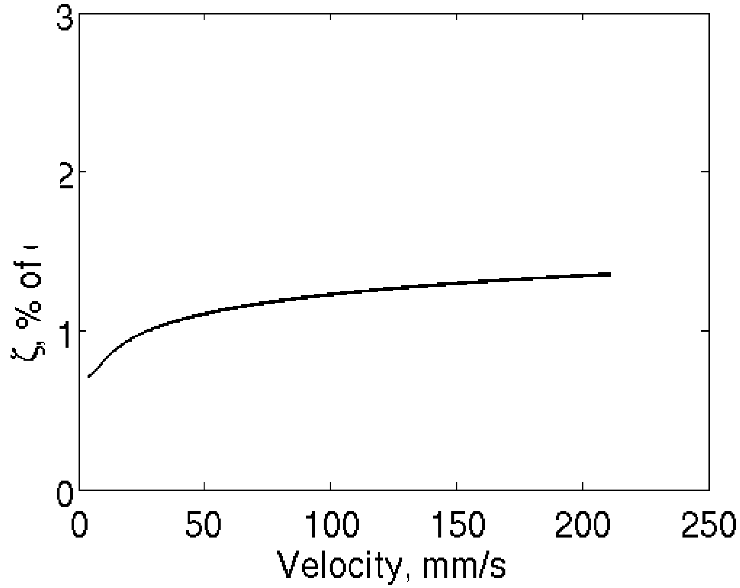


Natural frequency and damping are functions of time or amplitude.





ζ is a function of amplitude.



Test procedure gave damping ratios of various cantilevers.

Material	Test run #	$Q = 1/(2\zeta)$	f_n, Hz
Brass from sheet stock	1	580	36.3
Brass from commercial bimorph supplier	1	640	62.18
Brass from commercial bimorph supplier	2	640	62.18
Bronze	1	625	37.8
Bronze	2	624	37.8
Monel	1	498	48.4
Monel	2	492	48.5
Monel	3	495	48.4
PZT 402 sheet (in-house)	1	357	60.5
PZT 402 sheet (in-house)	2	351	60.5
PZT 5A bimorph (in-house)	1	80	377
PZT 5A bimorph (in-house)	2	47	378
PZT 5A bimorph from commercial supplier	1	96	499.8
PZT 5A bimorph from commercial supplier	2	96	500.7

Conclusions:

•The method presented here has enabled the computation of natural frequency and damping of an SDOF system from its free ring-down history.

•Noise is a major problem. Curve fitting helps suppress the noise from time-differentiation.

•The method produced very good results for the linear beams.

•The nonlinear bimorph beams with higher damping gave very noisy data.

•In that case, nonlinear curve fitting is very important.

•The test results give an idea of how much damping (hence energy dissipated as heat) is typical of PZT bimorphs used for vibration energy harvesters.

•They also help in selecting the material to be used as the shim metal in fabricating the bimorphs.

Acknowledgment

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Questions ??

Thank you!

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