# The Role of the Global Phase in Optimal Quantum Control to Implement Partial Isometries

Charles H. Baldwin[(1)], Matthew D. Grace[(2)], and Ivan H. Deutsch[(1)]
[(1)] Center for Quantum Information and Control, University of New Mexico, [(2)] Sandia National Labs

## Abstract

Controlling quantum systems is an important step towards the implementation of quantum information protocols. We consider "geometric control," whereby time-dependent waveforms modulate a set of Hamiltonians that are generators of the Lie algebra su(d) for a d-dimensional Hilbert space. In such a scenario, there is a "quantum speed limit," i.e., the minimum time that it is needed to produce a specified control task for a given set of time dependent Hamiltonians. This speed limit is typically studied for two tasks: state-to-state mappings and the implementation of a full unitary map on the Hilbert space. We study the range of intermediate cases -- partial isometries that map an under-complete set of orthonormal states to another under-complete set of orthonormal states. For full unitary control, it was recently shown that the global phase of the target unitary, restricted to root of unity phases, affects the quantum speed limit. We observe that, in the partial isometry case as well as state-to-state mappings, the idea of global phase is not restricted to root of unity phases but can take any value. This means that each control task has a range of speed limits that must be understood in order to implement the control.

## Notation Convention

*d*: dimension of Hilbert space
*n*: dimension of the partial isometry
*V*: Target of unitary control
*U(T)*: unitary created by control protocol
$A_n$: *n*-dimensional projector for partial isometry
*Δ*: Objective function to minimize for control search
*T*: Total time of applying the control field
$T_c$: Miniumum time control field can achieve objective function
$\varphi_i$: Control phases

### State-to-state

Mapping an initial state to a target state

$$|\psi_i\rangle \rightarrow |\psi_t\rangle$$

$$\Delta_1 = \frac{1}{2} - \frac{1}{2}\mathrm{Re}\{\langle\psi_i|\psi_t\rangle\}$$

### Coherent Evolution/Observable

Creating a unitary map on a system

$$1 \rightarrow U$$

$$\Delta_d = \frac{1}{2} - \frac{1}{2d}\mathrm{ReTr}\left(U^\dagger V\right)$$

### Partial Isometry Control

Mapping a set of *n* orthonormal states to another set of *n* orthonormal states

$$\{|\psi_i\rangle\} \rightarrow \{|\psi_t\rangle\}$$

Creating a modified evolution operator with projector to *n*-dimensional subspace $A_n = \sum_{i=1}^{n} |\psi_i\rangle\langle\psi_i|$

$$1 \rightarrow VA_n$$

$$\Delta_n = \frac{1}{2} - \frac{1}{2n}\sum_{i=1}^{n}\mathrm{Re}\{\langle\psi_i^u|\psi_i^v\rangle\} = \frac{1}{2} - \frac{1}{2n}\mathrm{Re}\{\langle UA_n, VA_n\rangle\}$$

Unitary control search for *d = 4* hyperfine-spin system

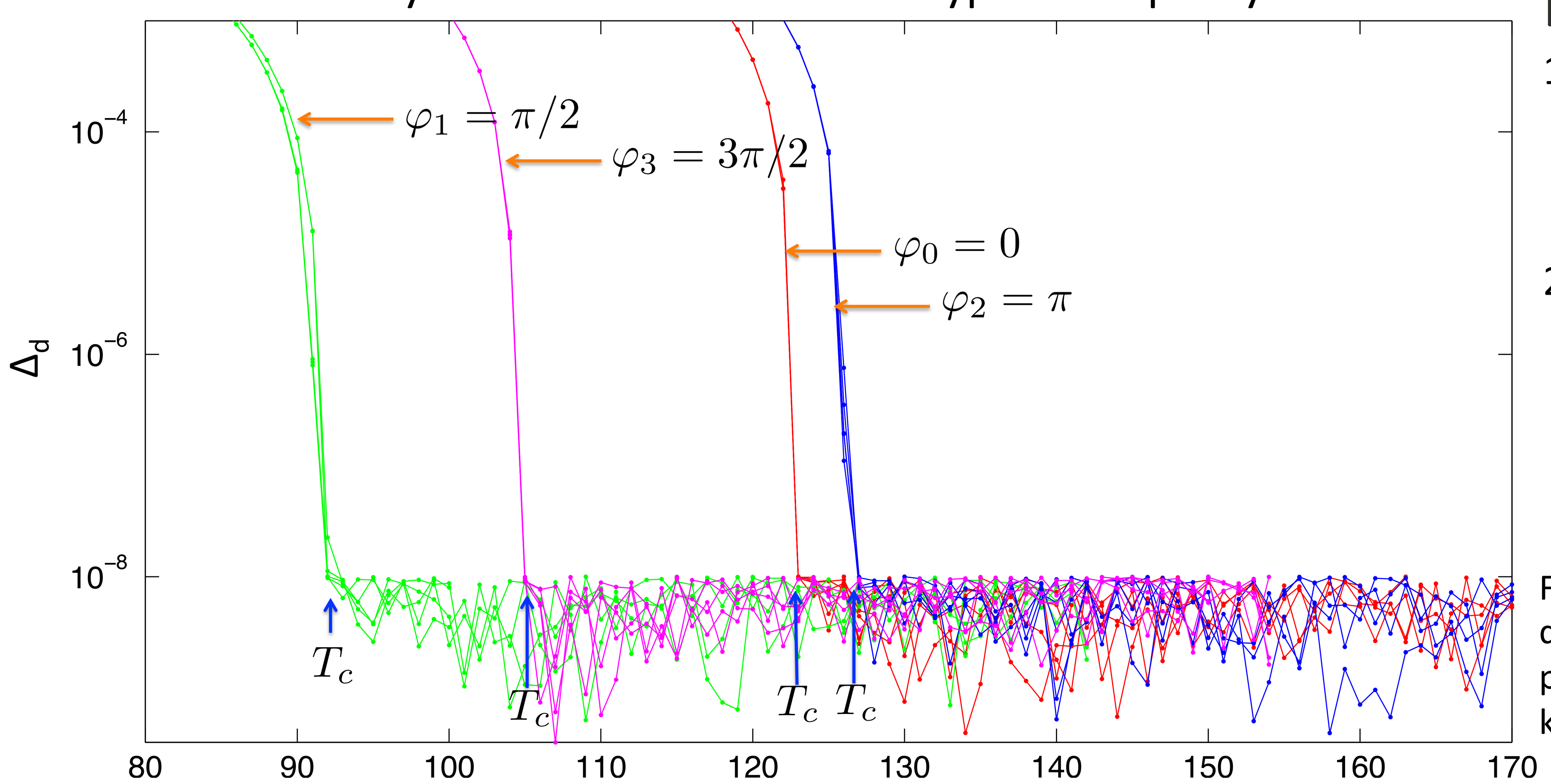
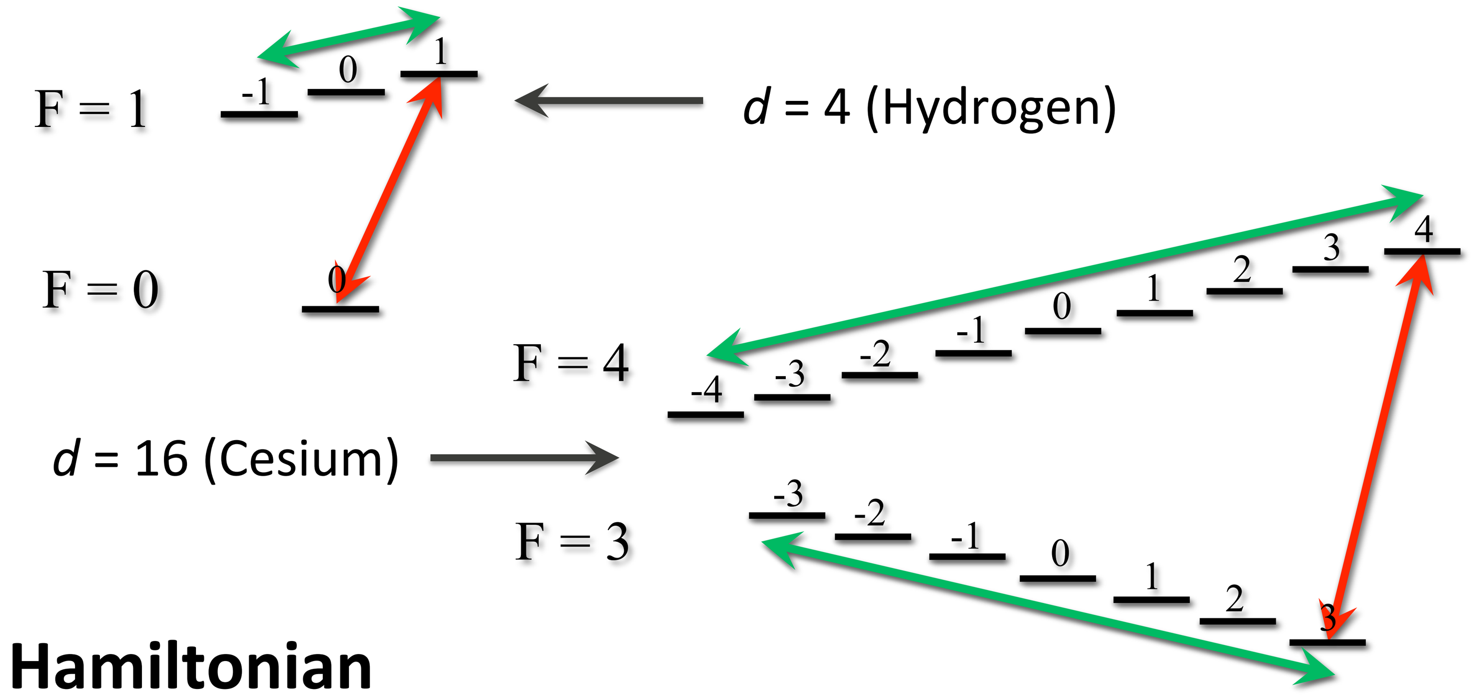



Fig. 1: Unitary control search for a *d* = 4 dimensional system with the four root-of-unity phases shown with $\Omega_x = \Omega_y$ = 1 kHz, $\Omega_{\mu W}$ = 25 kHz, and Δt = 10 μs.

## Numerical Procedures

1. Control Search: Minimize Δ using GRAPE (gradient ascent pulse engineering) for time *T* to obtain an optimal control field
2. Pareto Front Tracking: Apply control search at *T* to find *Δ* then decrease *N* by one and reapply control search at new total time

## Hyperfine-coupled Spin System




### Hamiltonian

$$H_{tot}(t) = H_0 + H_{RF}(t) + H_{\mu W}(t)$$

$$H_0 = \frac{\Delta_{\mu W}}{2}\left(P^{(+)} - P^{(-)}\right) + \Delta_{RF}\left(F_z^{(+)} - F_z^{(-)}\right)$$

$$H_{RF}(t) = \frac{\Omega_x}{2}\left[\cos(\phi_x(t))\left(F_x^{(+)} - F_x^{(-)}\right) + \sin(\phi_x(t))\left(F_y^{(+)} - F_y^{(-)}\right)\right] + \frac{\Omega_y}{2}\left[\cos(\phi_y(t))\left(F_y^{(+)} - F_y^{(-)}\right) + \sin(\phi_y(t))\left(F_x^{(+)} - F_x^{(-)}\right)\right]$$

$$H_{\mu W}(t) = \frac{\Omega_{\mu W}(t)}{2}\left[\cos(\phi_{\mu W}(t))\sigma_x + \sin(\phi_{\mu W})\sigma_y\right]$$

### Control Field

Divide total time (*T*) into *N* steps and piecewise define control phases for *j = x, y,* and *μW*

$$\phi_j(t) = \begin{cases} \phi_j^{(0)}, & 0 \le t < \Delta t \\ \phi_j^{(1)}, & \Delta t \le t < 2\Delta t \\ \vdots & \vdots \\ \phi_j^{(N)}, & T - \Delta t \le t < T \end{cases}$$

Then the evolution is described by a time independent Schrodinger equation for each time step leading to a total unitary of the system

$$U(T) = U_N \cdots U_2 U_1$$

# Role of Global Phase

**Unitary Control:** *d* root-of-unity phases lead to a family of *d* equivalent unitaries, up to global phase, for a given target

$$V_0 \in \mathrm{SU}(d) \rightarrow V_p = e^{i\varphi_p} V_0 \in \mathrm{SU}(d)$$

Where $\varphi_p$ is constrained by det(V) = 1 so that

$$\varphi_p = 2\pi p/d, \quad p = 0, 1, 2, ..., d-1$$

While each $V_p$ applies the same evolution, up to a global phase, the Hamiltonian to generate each evolution are not necessarily equivalent. Therefore, some may be harder (or easier) to produce leading to different critical times as seen in Fig. 1. We have found, through, numerical studies, that this effect can be reduced with different parameter regimes as shown in Fig. 2.

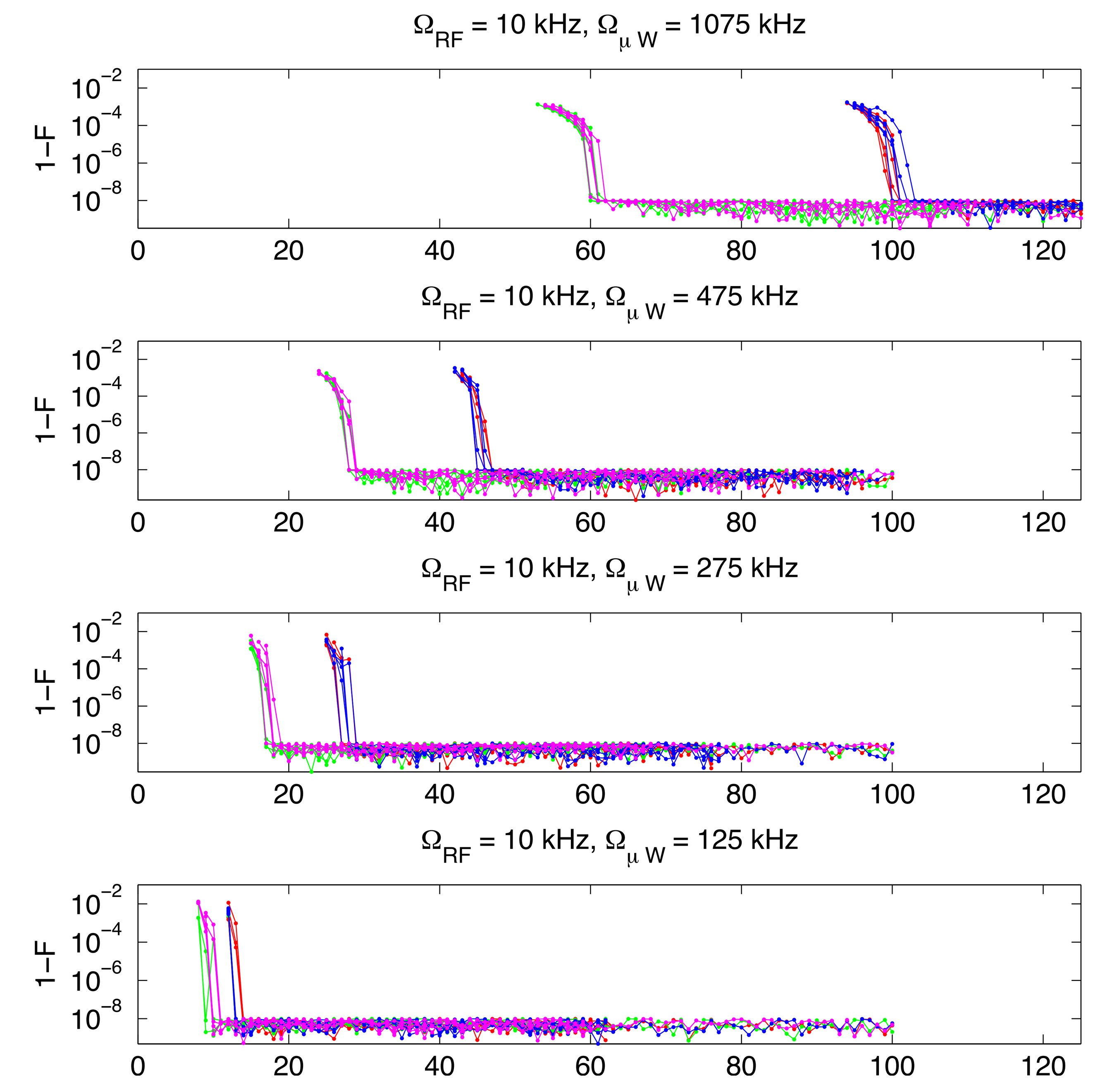



Fig. 2: Control serach results from a unitary control objective with decreasing the strength of the μW field. As the strength decreases the control times also decrease contrary to what may be expected.


Sandia National Laboratories
U.S. DEPARTMENT OF ENERGY

This work was supported by the NSF, and the Laboratory Directed Research and Development program at Sandia National Laboratories. Sandia National Laboratories is a multi-program laboratory managed and operated by Sandia Corporation, a wholly owned subsidiary of Lockheed Martin Corporation, for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000.


# *d* = 4 Numerical Studies

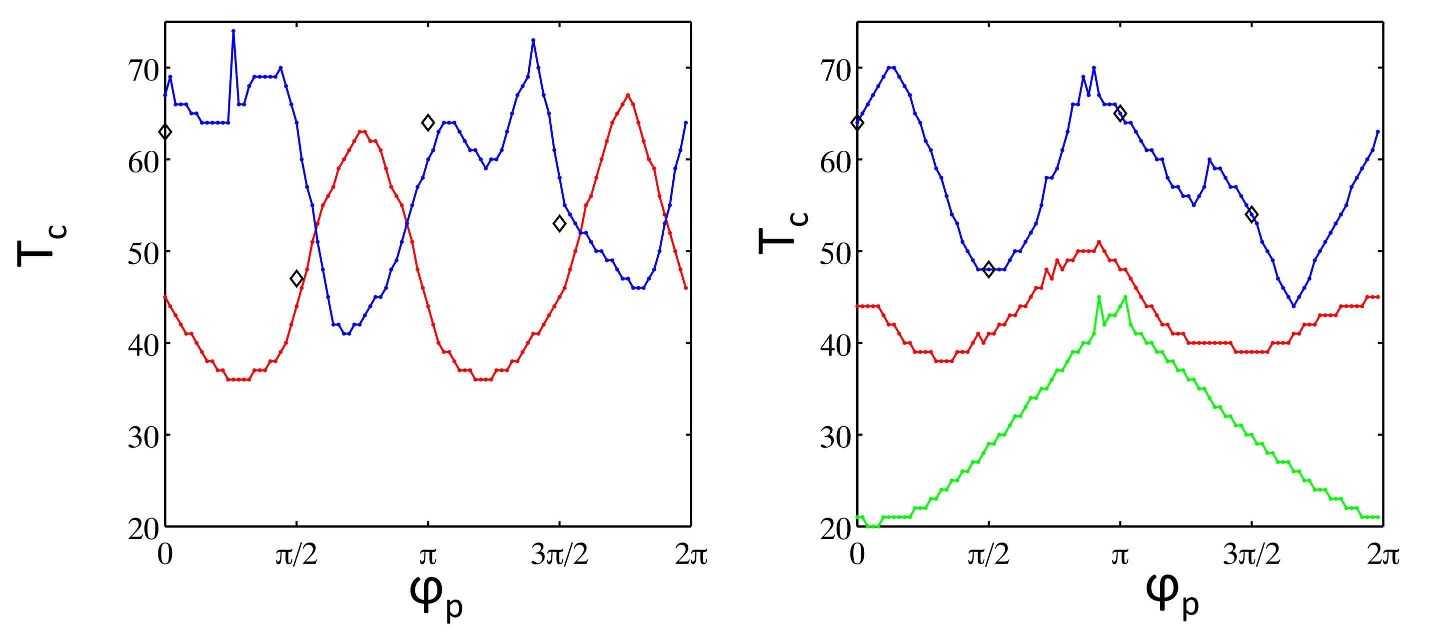



Fig. 3: Partial Isometry control of the *d* = 4 dimensional hyperfine system with $\Omega_x = \Omega_y$ = 1 kHz, $\Omega_{\mu W}$ = 25 kHz, and Δt = 10 μs. Black triangles correspond to the root-of-unity phases for full unitary control (*n = 4*.) The lines correspond to partial isometries for *n = 3* (blue), *n = 2* (red), and *n* = 1 (green) with 100 phases between 0 and 2π. (a) is for a partial isometry where *V* has a block diagonal structure and (b) is for a partial isometry for a normal unitary.

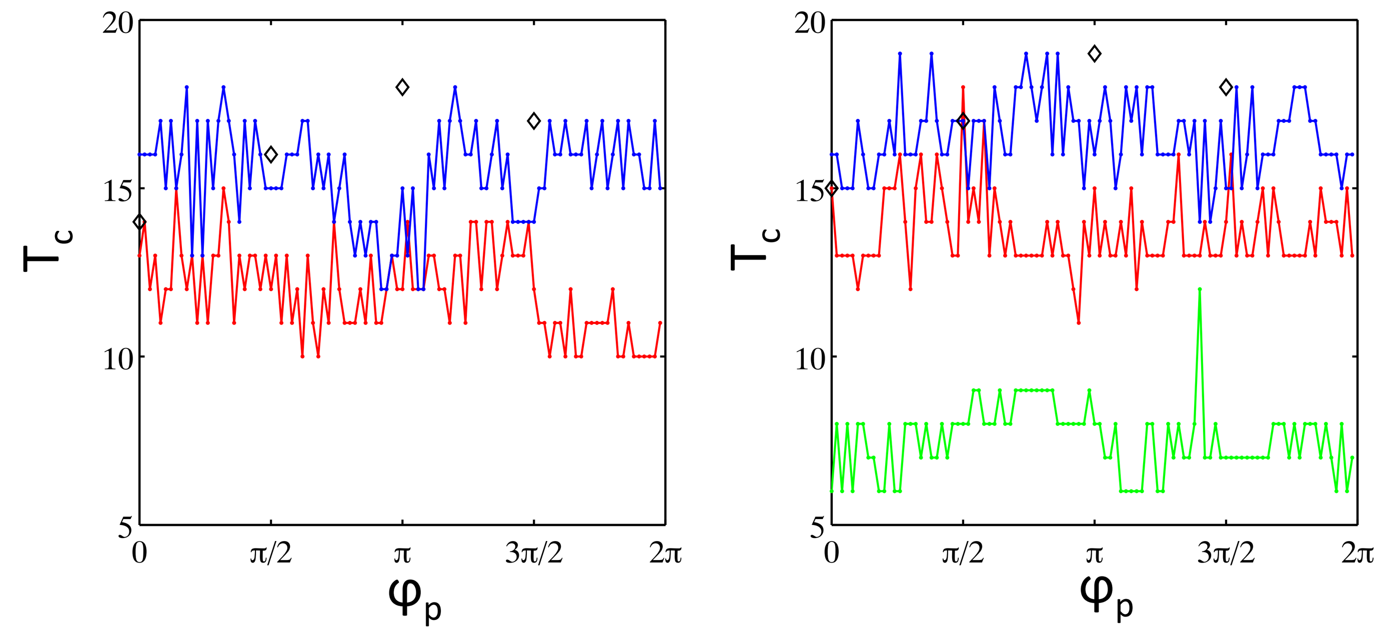



Fig. 4: Partial Isometry control of the *d* = 4 dimensional hyperfine system with $\Omega_x = \Omega_y = \Omega_{\mu W}$ =10 kHz and Δt = 10 μs. Black triangles correspond to the root-of-unity phases for full unitary control (*n = 4*.) The lines correspond to partial isometries for *n = 3* (blue), *n = 2* (red), and *n = 1* (green) with 100 phases between 0 and 2π. (a) is for an isometry where *V* has a block diagonal structure and (b) is for an isometry for a normal unitary.

# Analysis

Often times control objectives can be accomplished with partial isometries instead of full unitaries. In this case we have demonstrated numerically that these tasks can be accomplished in a shorter time which is advantageous in experimental applications. This reduction comes at a cost, however, since partial isometries of dimension *n* < *d* have a family of infinitely many equivalent maps which only vary in global phases. As was seen in unitary control the different phases correspond to different critical times since the Hamiltonian to generate each map may be easier (or harder) to create as seen in Fig. 3. In an experiment it is important to stay above the critical times for each phase, since global phase is irrelevant to observables. One way to deal with this issue is to probe the parameter regime to find an area where the control is saturated and the the critical time varies less with global phase as shown in Fig. 4 and also applied to the larger system in Figs. 5 and 6.

# *d* = 16 Numerical Studies

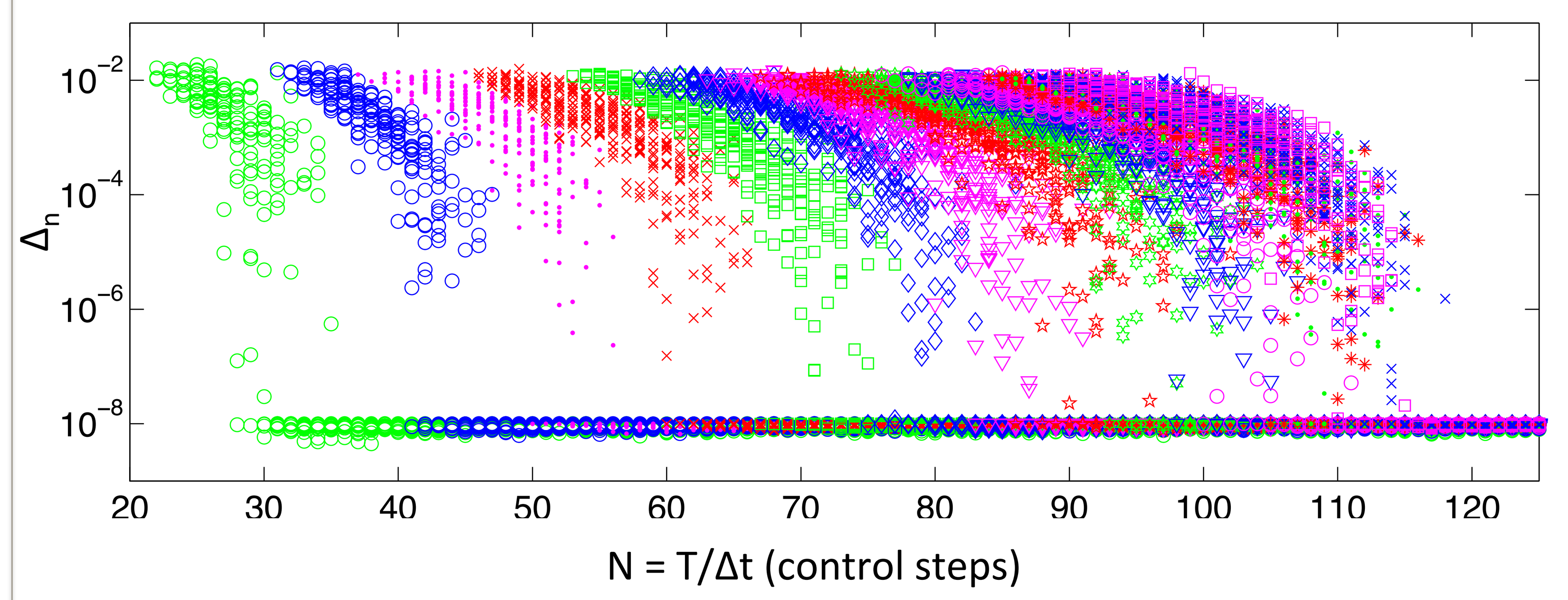



Fig. 5: Five partial isometry reconstructions for a *d=16* dimensional system for a target with a block diagonal structure for each *n* = 1,…,16. When *n=1,…,15* there are 21 phases between 0 and 2π and when *n* = 16 there are the 16 root-of-unity phases. The colors alternate for each dimension of the partial isometry. Instead of clean Pareto fronts as seen in Fig. 2 there is a tight range of critical times similar to Fig. 4.

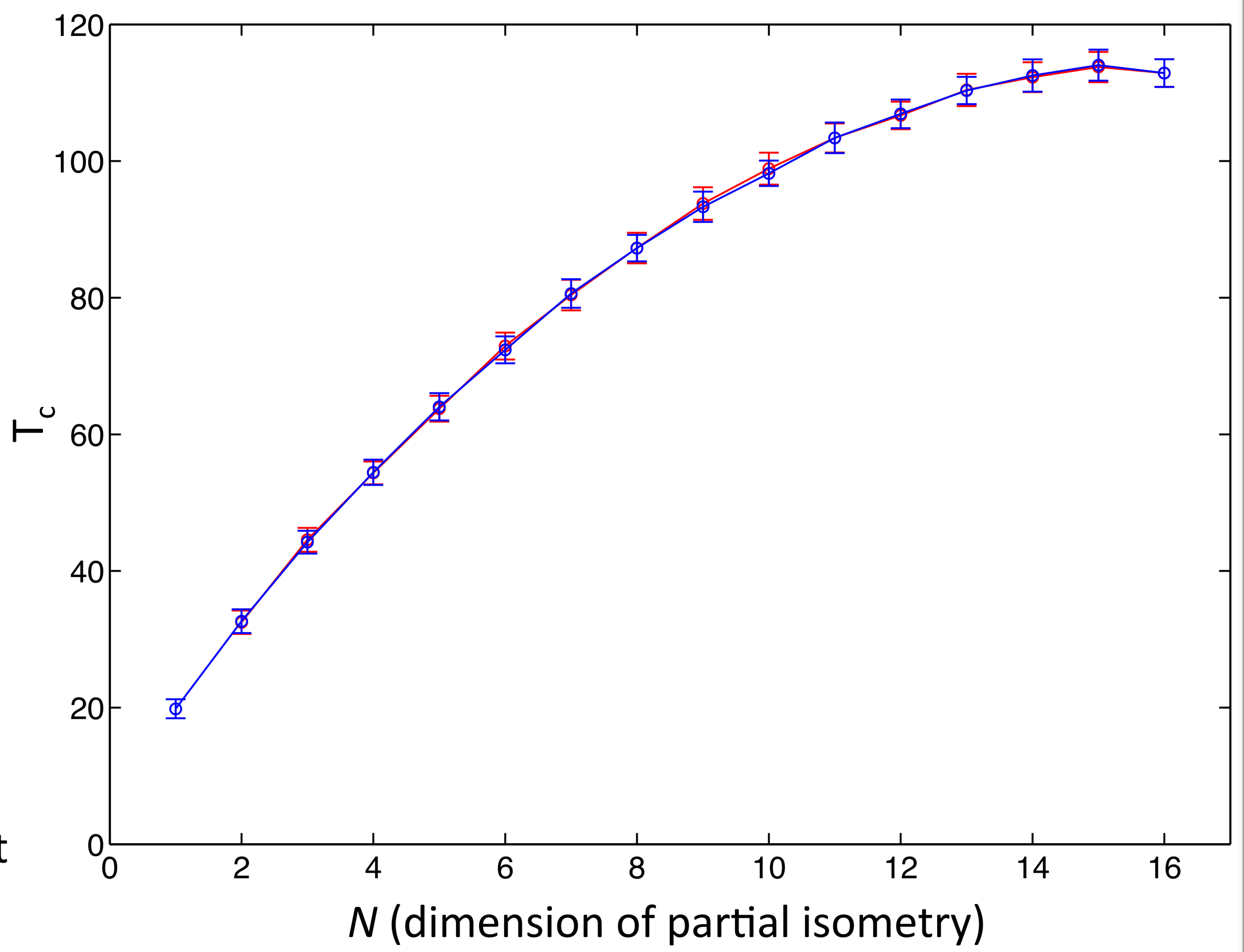



Fig. 6: Critical times for partial isometries with block diagonal structure (red) and normal structure (blue.) The uncertainty bars represent one standard deviation for the critical times for the range of phases.

# Partial Isomtry Control

The target unitary *V* is still required to be in SU(d) but our optimization only searches for the first *n* columns of *V* and leaves the others free.

$$V = \Bigg( \underbrace{\begin{matrix} \uparrow & & \uparrow \\ |\psi_1\rangle & \cdots & |\psi_n\rangle \\ \downarrow & & \downarrow \end{matrix}}_{\text{Constrained in minimization}} \; \underbrace{\begin{matrix} \uparrow & & \uparrow \\ |\psi_{n+1}\rangle & \cdots & |\psi_d\rangle \\ \downarrow & & \downarrow \end{matrix}}_{\text{Unconstrained in minimization}} \Bigg)$$

Therefore, we can specify any global phase on the first *n* columns since the next *d-n* columns will work to compensate so that $V \in \mathrm{SU}(d)$

For example, if *V* is block diagonal

$$V = W_n \oplus X_{d-n}, \; W \in \mathrm{U}(n) \text{ and } X \in \mathrm{U}(d-n)$$

Then in order *V* to be in SU*(d)*

$$\det(V) = \det(W) \times \det(X) = 1$$

Which means we can select any global phase for *W* since the unconstrained part of the optimization, *X*, can compensate so that the product of the determinants is still one.

# Open Questions

- What is the relationship between global phase and critical time for a partial isometry?
- What is the difference between phase control, used in these numerical studies, and amplitude control, studied in most other literature?
- How does the parameter regime affect the critical times?

# References


[1] T.-S. Ho, J. M. Dominy, and H. Rabitz, Phys. Rev. A **79,** 013422 (2009).
[2] S. Machnes *et al.*, Phys. Rev. A **84**, 022305 (2011).
[3] S.T. Merkel, P.S. Jessen, and I.H. Deutsch, Phys. Rev. A, **78**, 023404 (2008).
[4] K.W. Moore Tibbetts *et al.*, Phys. Rev. A **86**, 062309 (2012).
[5] H. Rabitz, M. Hsieh, and C. Rosenthal, Phys. Rev. A **72**, 052337 (2005).
[6] C.R. Riofrio, Ph.D. thesis, University of New Mexico (2011).
[7] T. Schulte-Herbruggen, A. Sporl, N. Khaneja, and S.J. Glaser, Phys. Rev. A **72**, 042331 (2005).