Component Test Specification Based on the Karhunen-Loeve Expansion

Prasad Raghavendra
NOKIA, 12278 Scripps Summit Drive
San Diego, California 92131

Thomas L. Paez
Department 1544, MS 0557, Sandia National Laboratories, PO Box 5800,
Albuquerque, New Mexico 87185

ABSTRACT

Real mechanical shocks are nonstationary random processes, and it is becoming feasible to model these sources based on practical amounts of measured data. An efficient framework for modeling nonstationary random processes is the Karhunen-Loeve (KL) expansion. It simulates a random source as a mean function plus random deviation from the mean. We have developed an efficient technique for modeling non-Gaussian random processes in the KL framework, and we use it to characterize an electronic component's shock response to the random process. The random process is, in turn, used to create a shock test specification for this electronic component. Numerical results are presented in connection with the random process and test specifications. The electronic component is a speaker and is assembled inside a mobile phone. The test specification will be used in qualifying and selecting a robust component design (or manufacturer) for use inside a mobile phone.

INTRODUCTION

Most environments to which structural dynamic systems are subjected are realizations of random processes, i.e., environments that cannot be predicted precisely in advance of their measurement. However, while random vibration environments are treated as stochastic, mechanical shock environments usually are not, even though most shock environments are signals that come from a source that has a formal, probabilistic description. Measures of shock environments like the shock response spectrum are themselves random processes because they are simply well-defined transformations of random processes. Some valuable capabilities in the representation and analysis of randomly excited structural dynamic systems are the facilities to model a random source and then generate realizations of the source.

A compact framework for modeling general random processes is the Karhunen-Loeve (KL) expansion (Traina, et al. [7], Ghanem and Spanos [1], Masri, et al. [2], Paez and Hunter [3], [4]). The KL expansion models a random process as a mean function plus a random deviation from the mean. The random deviation from the mean is the product of normalized shape functions, amplitude constants, and randomizing variates. The random variates come from a multi-variate random source. The random variates in the KL expansion are guaranteed uncorrelated, but not statistically independent. So if the random process is Gaussian distributed, then the vector random variable has a joint Gaussian distribution. If the random process is not Gaussian distributed, then the vector random variable has some distribution other than a joint Gaussian distribution. In the latter case it is usually difficult to characterize the joint distribution of the vector random variable, and therefore, it is difficult to generate realizations of the random process.

However, an empirical/numerical construct is available for the description of joint probability density functions (PDF) known only through data they produce. This is the kernel density estimator (KDE), and it is useful for characterizing joint probability density functions in up to a few dimensions. The KDE is described, for example, in

Silverman [6]. It is useful for describing the joint PDFs of vector random variables, where the random variables are dependent and non-Gaussian. But even though the expression of the KDE in terms of measured data is simple and direct, it presents a form that is not convenient for generating realizations of its random vector.

The Markov chain Monte Carlo (MCMC) method is an efficient numerical technique for the generation of multi-variate realizations from a vector random source, when the joint PDF of the source is known. It is described, for example, in Press [5]. It uses a Markov chain random sampling scheme to develop multi-variate random vector realizations, and then determines whether or not the realizations are acceptable samples of the target source.

This investigation develops (1) the KL expansion as an efficient means for modeling general random processes, (2) the KDE as a framework for modeling the joint PDF of the jointly distributed random variables in the KL expansion, and (3) the MCMC construct to generate realizations of the underlying random vector in the KL expansion representation of the random process of interest. Once these things are done, a practical problem and its data are described, the simulation technique developed here is applied to it, and a test specification for a structural component is developed in terms of shock response spectra.

KARHUNEN-LOEVE EXPANSION

The KL expansion models a random source. The random source is assumed to be sampled at discrete values of a parameter (like time or frequency), and every sample has finite duration and the same number of points. Denote the random process $\boldsymbol{X}_{rp} = \{X_j, j = 0,...,n-1\}$; it may be nonstationary, or a finite segment of a stationary random process. Assume that there are M measured realizations of the random process, and they are denoted $\boldsymbol{x}_m, m = 1,...,M$. Each signal, $\boldsymbol{x}_m$, is an $n \times 1$ vector of real values and a realization of the random process $\boldsymbol{X}_{rp}$. The entire collection of realizations may be gathered in the matrix of column vectors $\boldsymbol{X}_0 = [\boldsymbol{x}_1 \quad \boldsymbol{x}_1 \quad ... \quad \boldsymbol{x}_M]$. Our objective is to generate n_r additional random process realizations using the KL expansion, a KDE, and MCMC sampling.

The KL expansion requires an estimate of the random process mean. It is

$$\bar{\boldsymbol{x}} = \frac{1}{M}\sum_{m=1}^{M}\boldsymbol{x}_m \tag{1}$$

where the sample mean is an $n \times 1$ vector. Next, define a matrix of mean-normalized realizations

$$\boldsymbol{X} = [(\boldsymbol{x}_1 - \bar{\boldsymbol{x}}) \quad (\boldsymbol{x}_1 - \bar{\boldsymbol{x}}) \quad ... \quad (\boldsymbol{x}_M - \bar{\boldsymbol{x}})] \tag{2}$$

The matrix $\boldsymbol{X}$ has dimension $n \times M$. Use the matrix of normalized realizations to estimate the stochastic process auto- and cross-covariances.

$$\hat{\boldsymbol{C}}_{\boldsymbol{X}\boldsymbol{X}} = \frac{1}{M-1}\boldsymbol{X}\boldsymbol{X}^T \tag{3}$$

The dimension of $\hat{\boldsymbol{C}}_{\boldsymbol{X}\boldsymbol{X}}$ is $n \times n$, and the ith row-j[th] column element denotes the estimated covariance between the random variables X_i and X_j, the ith and j[th] elements of the random process $\boldsymbol{X}_{rp}$.

Compute the eigenvalue decomposition of the estimated covariance matrix.

$$\hat{\boldsymbol{C}}_{\boldsymbol{X}\boldsymbol{X}} = \boldsymbol{V}\boldsymbol{W}\boldsymbol{V}^T \tag{4}$$

where the matrix $\boldsymbol{V}$ has dimension $n \times n$, and the columns of $\boldsymbol{V}$ are the eigenvectors of the estimated covariance matrix, and the diagonal matrix $\boldsymbol{W}$ has dimension $n \times n$, and contains the eigenvalues of the estimated covariance matrix. The eigenvectors are normalized so as to be orthonormal, i.e., $\boldsymbol{V}^T\boldsymbol{V} = \boldsymbol{I}$, where $\boldsymbol{I}$ is the $n \times n$ identity

matrix. The eigenvalues contained in $\boldsymbol{W}$ are nonnegative and are usually arranged in descending order of magnitude. The autocovariance estimate matrix can usually be accurately approximated using only a fraction of the columns of $\boldsymbol{V}$ and terms in $\boldsymbol{W}$. We retain only the largest diagonal terms of $\boldsymbol{W}$ in $\boldsymbol{w}$, and the corresponding columns of $\boldsymbol{V}$ in $\boldsymbol{v}$. Then the approximation

$$\hat{\boldsymbol{C}}_{\boldsymbol{XX}} \cong \boldsymbol{v}\boldsymbol{w}\boldsymbol{v}^T \tag{5}$$

can be written. When n_0 terms are retained in the approximation, the dimension of $\boldsymbol{v}$ is $n \times n_0$, and the dimension of $\boldsymbol{w}$ is $n_0 \times n_0$. The number of terms, n_0, to be retained in the approximation is usually gauged by cumulatively summing terms along the diagonal of $\boldsymbol{W}$, and zeroing all remaining terms when the accumulated sum divided by the sum of all diagonal terms reaches a pre-established value.

We can now use the sample mean $\overline{\boldsymbol{x}}$, and the factors of the approximate eigenvalue expansion $\boldsymbol{v}$ and $\boldsymbol{w}$, to write the approximate KL expansion of the stochastic process.

$$\boldsymbol{X}_{rp} \cong \boldsymbol{v}\boldsymbol{w}^{1/2}\boldsymbol{U} + \overline{\boldsymbol{x}} \tag{6}$$

where $\boldsymbol{U}$ is an $n_0 \times 1$ vector of mean-zero, unit-variance uncorrelated random variables. In this context, the shapes contained in the columns of the matrix $\boldsymbol{v}$ are sometimes called the principal shapes of the KL expansion. The diagonal elements of the matrix $\boldsymbol{w}^{1/2}$ are called the amplitudes of the KL expansion.

The realizations of $\boldsymbol{U}$ that correspond to the stochastic process realizations $\boldsymbol{X}_0 = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_1 & \dots & \boldsymbol{x}_M \end{bmatrix}$ can be evaluated through inversion of Eq. (6).

$$\boldsymbol{u}^{(real)} = \boldsymbol{w}^{-1/2}\boldsymbol{v}^T\boldsymbol{X} = \begin{bmatrix} \boldsymbol{u}_1^{(real)} & \boldsymbol{u}_2^{(real)} & \dots & \boldsymbol{u}_M^{(real)} \end{bmatrix} \tag{7}$$

where $\boldsymbol{X}$ is the matrix of mean-normalized realizations of the random process. The sample mean of the $\boldsymbol{u}_m^{(real)}, m = 1,...,M$, is a zero vector, and the sample covariance (or correlation) is the $n_0 \times n_0$ identity matrix. Because the parameters of the KL expansion of Eq. (6) are based on estimates of the random process mean and covariance, the KL expansion is an estimate of the underlying random process. Because only a portion of the components of $\boldsymbol{W}$ and $\boldsymbol{V}$ are included in $\boldsymbol{w}$ and $\boldsymbol{v}$, the KL expansion is only an approximation of the estimate of the random process.

Equation (6) indicates that in order to generate realizations of the random process, $\boldsymbol{X}_{rp}$, it is necessary to generate realizations of the $n_0 \times 1$ random vector $\boldsymbol{U}$. To generate realizations of the random vector $\boldsymbol{U}$, we must first know the joint PDF of the random variables in the vector $\boldsymbol{U}$.

KERNEL DENSITY ESTIMATOR

Every random vector $\boldsymbol{U}$ has a joint probability distribution. One description of the probability distribution is a joint PDF governing the behavior of the random variables in $\boldsymbol{U}$, namely, $U_i, i = 1,...,n_0$. In some instances, the theoretical form for the distribution is known, however, in most practical situations, it is not. When sufficient realizations from the random vector are available, an empirical expression for the joint PDF can be written; it is the kernel density estimator (KDE). Let $\boldsymbol{u}_m^{(real)}, m = 1,...,M$, be the ensemble of realizations of $\boldsymbol{U}$ expressed in Eq. (7). Each $\boldsymbol{u}_m^{(real)}$ is an $n_0 \times 1$ vector of real numbers. In terms of these vectors the KDE is expressed

$$\hat{f}_{\boldsymbol{U}}(\boldsymbol{u}) = \frac{1}{M} \sum_{m=1}^{M} \frac{1}{\left(2\pi\varepsilon^2\right)^{n_0/2}} exp\left[-\frac{1}{2\varepsilon^2}\left\|\boldsymbol{u} - \boldsymbol{u}_m^{(real)}\right\|^2\right] \qquad -\infty < \boldsymbol{u} < \infty \tag{8}$$

where ε is a smoothing parameter, and the symbol $\|\cdot\|$ refers to the operation of the norm – the square root of the sum of the squares of the elements of the vector that is the argument. Equation (8) is essentially a superposition of M, n_0-variate, Gaussian PDFs with diagonal covariance matrix, and with each of the PDFs centered on a data vector, $u_m^{(real)}$. Optimal selection of the smoothing parameter is discussed in Silverman [6].

MARKOV CHAIN MONTE CARLO

The objective is to generate realizations of the random process $X_{rp} = \{X_j, j = 0,...,n-1\}$, and Eq. (6) shows that this can be accomplished, approximately, by generating realizations of the random vector U, whose empirical PDF is given by Eq. (8). We propose to generate samples from U using the Markov chain Monte Carlo (MCMC) approach, and specifically, the Metropolis-Hastings (MH) implementation of MCMC (Press, [5]). Metropolis-Hastings sampling is a Bayesian technique for generating realizations of a multivariate random variable (a random vector) with prescribed PDF.

Metropolis-Hastings generates realizations of a random vector, U, by specifying an initial realization $u_0^{(MH)}$. When the vector $u_0^{(MH)}$ is used in Eq. (8) as the argument, the quantity $\hat{f}_U\left(u_0^{(MH)}\right) = L_U\left(u_0^{(MH)}\right)$ can be interpreted as the likelihood that the random vector U would produce the realization $u_0^{(MH)}$. Metropolis-Hastings next generates another realization of U – a trial realization - by adding to $u_0^{(MH)}$ an $n_0 \times 1$, vector random deviate δ. The characteristics of the random variable δ are chosen so that the quantity

$$u_{trial} = u_0^{(MH)} + \delta \tag{9}$$

yields realizations at least some of which lie in the vicinity of the vectors $u_m^{(real)}, m = 1,...,M$. Next, the likelihood $L_U(u_{trial})$ of u_{trial} is computed, and used with $L_U\left(u_0^{(MH)}\right)$ to form the acceptance probability

$$p_{acc} = min\left(\frac{L_U(u_{trial})}{L_U\left(u_0^{(MH)}\right)}, 1\right) \tag{10}$$

The quantity p_{acc} is the probability of acceptance of the trial realization, u_{trial}, as a member of the ensemble of the random vector U. Based on Eq. (10), when the likelihood, $L_U(u_{trial})$, of the trial realization, u_{trial}, is greater than or equal to the likelihood, u_{trial}, of the current realization, $u_0^{(MH)}$, the acceptance probability is one – the realization u_{trial} is certain to be accepted as a realization of U. When the likelihood of the trial realization, $L_U(u_{trial})$, is less than the likelihood of the current realization, $L_U\left(u_0^{(MH)}\right)$, the acceptance probability is less than one, but typically nonzero – the realization u_{trial} is accepted as a realization of U with the probability $L_U(u_{trial})/L_U\left(u_0^{(MH)}\right)$. Even relatively unlikely "outlier" realizations can, on occasion, be accepted as realizations of U. To establish whether or not the trial realization, u_{trial}, is accepted as a member of the ensemble of the random vector U, we choose, at random, a number, r_0, from a source that is uniformly distributed on the interval $[0,1]$. If r_0 is equal to or less than p_{acc} we accept the trial realization, u_{trial}, as the next realization of U in a sequence; if not we reject it. In equation form

$$u_1^{(MH)} = \begin{cases} u_{trial} & r_0 \leq p_{acc} \\ u_0^{(MH)} & r_0 > p_{acc} \end{cases} \tag{11}$$

This procedure is repeated an arbitrary number of times to build up a collection of vectors each of which is a likely member of the ensemble of the random vector $\boldsymbol{U}$. The vector realizations of $\boldsymbol{U}$ generated using the MH scheme are denoted $\boldsymbol{u}_j^{(MH)}, j = 1,...,n_{MH}$.

Numerous practical matters need to be treated before the MH procedure can be implemented in a specific case. Among others, the distribution of the vector innovation random source $\boldsymbol{\delta}$ needs to be developed. This is an art. The covariance of $\boldsymbol{\delta}$ needs to be small enough that most of the trial vectors are not rejected (because the innovations are too far from the collection $\boldsymbol{u}_m^{(real)}, m = 1,...,M$). This should not be too difficult, because the $\boldsymbol{u}_m^{(real)}, m = 1,...,M$, are known to have zero-mean and unit-variance, and to be uncorrelated. Further, when it is not certain that the starting realization $\boldsymbol{u}_0^{(MH)}$ is in the vicinity of the vectors $\boldsymbol{u}_m^{(real)}, m = 1,...,M$, then the first part of the generated sequence of vector random variates $\boldsymbol{u}_j^{(MH)}, j = 1,...,n_{MH}$, is discarded, and the latter part, only, of the sequence is retained. However, in the present case, for the reasons mentioned above, it is considered safe to use a starting vector, $\boldsymbol{u}_0^{(MH)}$, that is the $n_0 \times 1$, zero vector, or a starting vector that equals any of the vector realizations in the collection $\boldsymbol{u}_m^{(real)}, m = 1,...,M$.

Once the realizations of the random vector $\boldsymbol{U}$ are obtained, they can be used with Eq. (6) to approximately define an ensemble of realizations of the random process $\boldsymbol{X}_{rp}$. The MH realizations are given by

$$\boldsymbol{X}^{(MH)} \cong \boldsymbol{v}\boldsymbol{w}^{1/2}\boldsymbol{U}^{(MH)} + \overline{\boldsymbol{x}} \tag{12}$$

where

$$\boldsymbol{U}^{(MH)} = \left[\boldsymbol{u}_1^{(MH)} \quad \boldsymbol{u}_2^{(MH)} \quad \cdots \quad \boldsymbol{u}_{n_{MH}}^{(MH)}\right] \tag{12a}$$

The matrix $\boldsymbol{X}^{(MH)}$ has dimension $n \times n_{MH}$, and each of its $n \times 1$ columns is an approximate realization of the random process.

NUMERICAL EXAMPLE

The data chosen for this study represent the shock motion at the support point of a speaker in a cell phone exposed to a random drop from a height of 28 inches. The phone was dropped as though it got pushed off a standard table top, therefore, the orientation of the phone and the speaker inside, at the time of impact, is random. The impact velocity of the phone center of gravity is approximately 147 in/s. It had multiple bounces (clatter) on hitting the floor, therefore, it is practically impossible to determine the changes in velocity and displacement at the measurement point in the direction of measurement.

Figure 1 shows the measured acceleration environments for an ensemble of 18 tests. Peak acceleration magnitudes range from a low of 1143 g to a high of 2748 g. The mean of the shocks is shown by the red, dashed line.
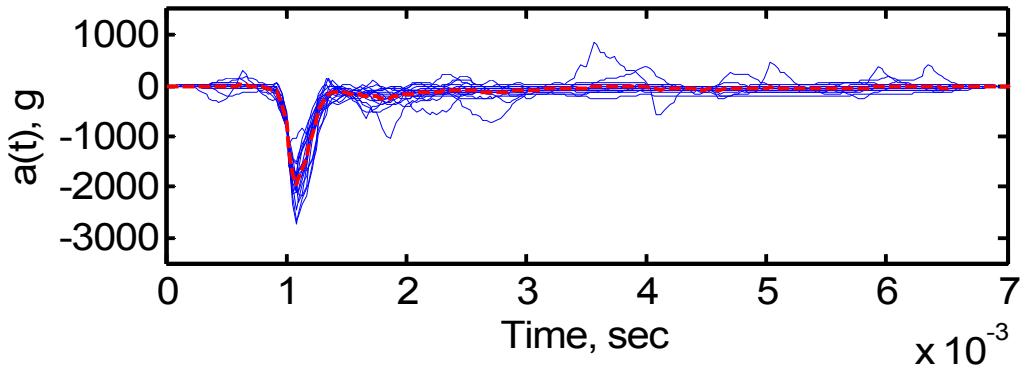
Figure 1. Realizations of shock time histories (blue); mean of shocks (red).

The KL expansion of the source of the random shocks was estimated, based on the measured ensemble, and the parameters v and w were established. In this application, ten components were retained in the KL expansion, i.e., $n_0 = 10$. Ten terms were retained in the representation because the sum of the first ten terms in W divided by the sum of all terms in W equals ninety-five percent, and this is the accuracy sought in the representation. The principal shapes of the KL expansion, v, are plotted in Figure 2. The amplitudes of the KL expansion, $diag\left(w^{1/2}\right)$, are plotted in Figure 3.
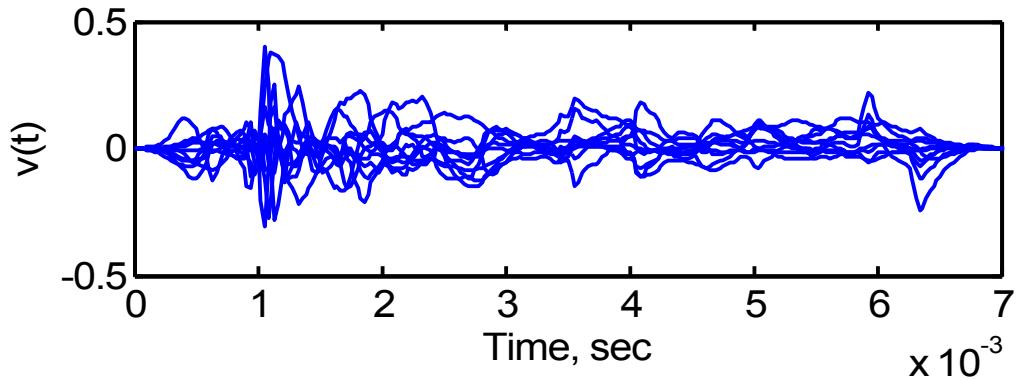
Figure 2. Principal shapes of the estimated KL expansion of the shock random process.
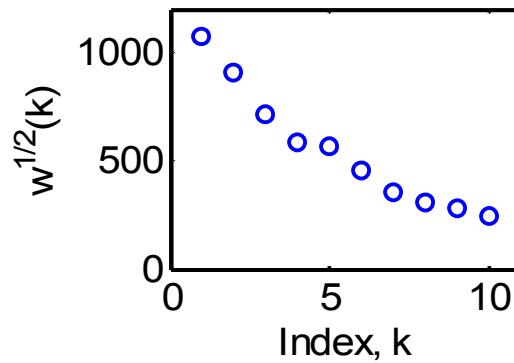

Figure 3. Amplitudes of the estimated KL expansion of the shock random process.

In view of the fact that the KL expansion includes ten components, the random vector U has dimension 10×1, and because the approximation to the KL expansion of the shock random process is based on 18 measured realizations, $M = 18$, and we can evaluate the vectors $u_m^{(real)}, m = 1,...,18$. Of course, the vectors cannot be plotted in all their dimensions simultaneously because the plot would require ten dimensions. We can, however, plot the realizations in two of their dimensions at a time using scatter plots. Figure 4 shows the 18 realizations of

the random variable U_2 (the second random variable in the random vector U) plotted versus the 18 realizations of the random variable U_1. Figure 5 is similar, plotting realizations of U_3 versus realizations of U_1.
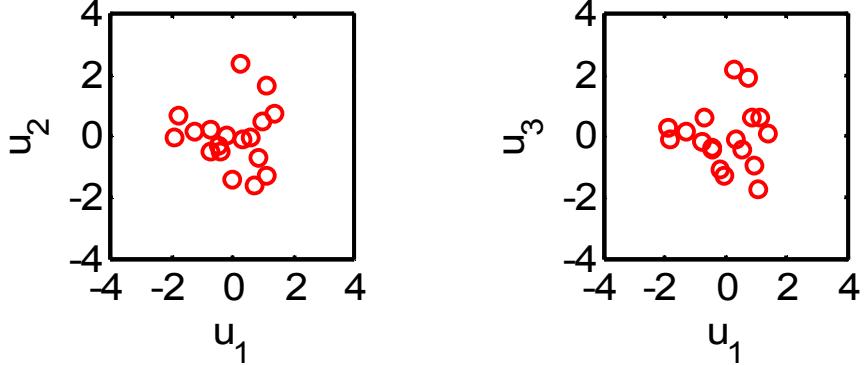
Figures 4 and 5. Realizations of U_2 versus realizations of U_1, and realizations of U_3 versus realizations of U_1.

The MCMC process was used to generate realizations $u_j^{(MH)}, j = 1,...,n_{MH}$, of the of the random vector U. Two-hundred and forty two realizations were generated, i.e., $n_{MH} = 242$, and the results are shown in Figures 6 and 7, in the style of Figures 4 and 5. The realizations generated via the Metropolis-Hastings implementation of MCMC are shown by the blue dots, and the realizations from the identified KL model are shown as red circles. As before, Figure 6 plots realizations of U_2 versus realizations of U_1, and Figure 7 plots realizations of U_3 versus realizations of U_1. The two figures show that the MCMC-generated random variable realizations appear to satisfactorily represent the realizations that come directly from the KL model.
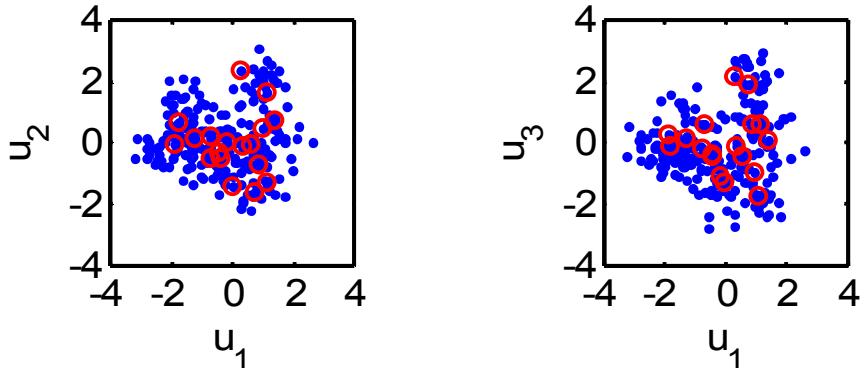
Figures 6 and 7. Realizations of U_2 versus realizations of U_1, and realizations of U_3 versus realizations of U_1. Realizations from the identified KL model are shown as red circles, and MCMC-generated realizations are shown as blue dots.

The realizations that correspond to the full vector realizations $u_j^{(MH)}, j = 1,...,n_{MH}$, which are shown by the blue dots in Figures 6 and 7, were used in Eq. (12) to obtain realizations of the shock random process. The realizations are shown by the blue curves in Figure 8. The figure also includes the 18 measured realizations of the shock random process in red.
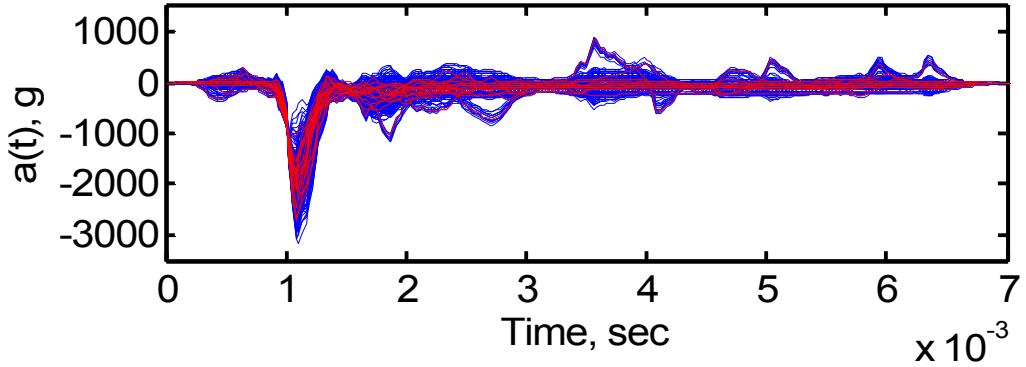
Figure 8. Realizations of the shock random process. MCMC-generated realizations (blue). Realizations from the KL model (red).

Though 242 MCMC realizations were generated for this example, an arbitrary number can be generated for user-defined purposes.

Analyses and test specifications related to the random shock source are practically unlimited given that realizations of the shock random process are available. For example, if, as in this case, a test specification is required, a criterion that measures the severity of the shock realizations can be defined, and the generated realizations can be ranked according to the measure. The single shock measure that falls on the probability limit of the specification defines the test specification. For example, consider the case where a ninety percent conservative environment is sought and the measure used to specify the shock test is the shock response spectrum (SRS), and the quantity used to rank the environments is the peak value of the SRS. Then to obtain the SRS test specification we first compute the SRS of the MCMC-generated shocks in Figure 8. The SRS are shown by the blue curves in Figure 9. (The SRS of the original shocks are also shown, for reference.)
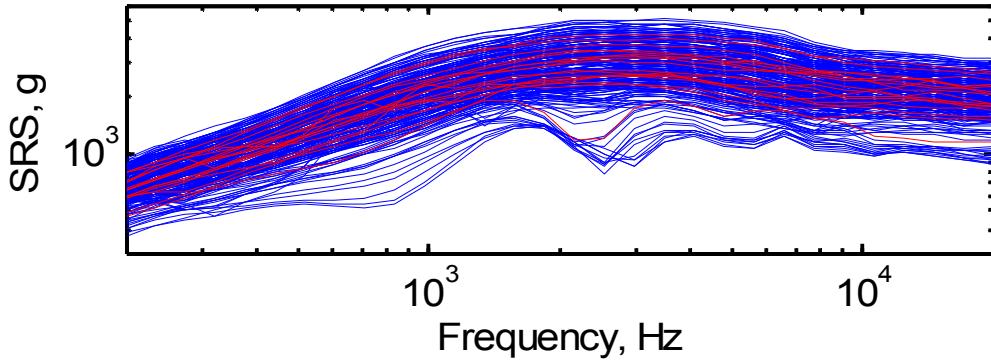

Figure 9. SRS of the MCMC-generated shock realizations from Figure 8 (blue) and the SRS of the original shocks (red).

The peak values of the MCMC-generated SRS are evaluated and the curve corresponding to the ninetieth percentile peak is chosen to specify the test. The SRS of the MCMC-generated shocks are shown in Figure 10 along with the SRS of the specified test.
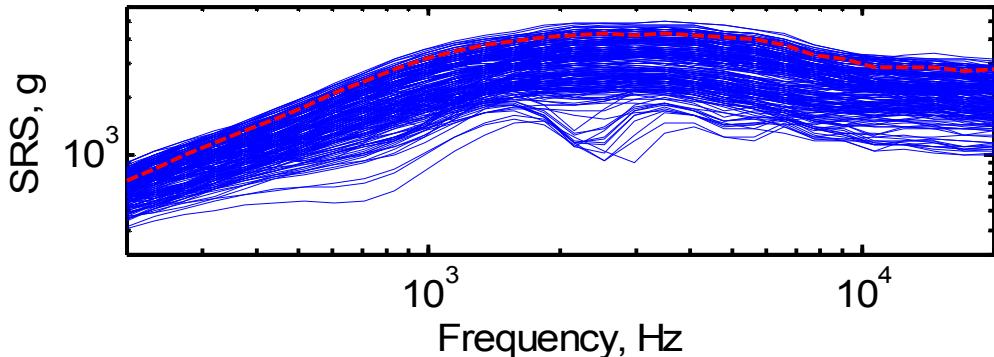
Figure 10. SRS of the MCMC-generated shock realizations (blue) and the SRS of the specified test (red).

The test specification can be used with any appropriate laboratory technique to generate shock test excitations for testing the cell phone speakers. It can be inferred, in an SRS sense, that the test level is ninety percent conservative.

In a similar manner a test specification could be developed for any component in any system. For example, we can write test specifications for hard drives, speakers, LCD modules, and cameras, inside computers, hand-held devices, and cell phones.

CONCLUSIONS

A method for generating random process realizations via the Markov Chain Monte Carlo (MCMC) technique is described here. The method uses measured random process realizations to approximate a Karhunen-Loeve (KL) expansion for the random process. Next, it uses a kernel density estimator to characterize the joint probability density function of the underlying random vector in the KL expansion. Finally, the method uses the Metropolis-Hastings implementation of the MCMC technique to generate vector random variable realizations from the random source that is the underlying random vector in the KL expansion.

A specific example is given here, but the technique is useful for the generation of random process realizations from any random source where experimentally measured realizations are available.

ACKNOWLEDGMENT

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